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SEARCH EQUILIBRIUM IN A SIMPLE MULTI-MARKET ECONOMY

by

DALE T. MORTENSEN*

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* Department of Economics
Northwestern University
Evanston, Illinois

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INTRODUCTION

In what has become "General Competitive Analysis" two different relative prices for the exchange of the same two commodities is ruled out by assumption. This assumption can be interpreted in two ways. Either all exchanges take place in a single all inclusive market; e.g. as in Debreu [4], or price differentials vanish as a consequence of arbitrage; e.g. Walras [28]. In the latter case a system of markets in which the same two commodities are traded (either indirectly or directly) on more than one market acts as one. From the point of view of positive economic theory, the first interpretation is untenable; it requires a degree of centralized exchange activity which is never observed and rarely approximated. The second interpretation is really a theorem. It would appear to hold trivially only if the activities called arbitrage were costless.

Recent developments in the theory of price search deal with one form of arbitrage. In these works [10, 13, 15] the searching trader is faced with different and/or uncertain prices for the same commodity. This feature of the environment is described by a probability or frequency distribution. The traders are assumed to search these price opportunities at random. If the commodity exchanged is an asset, i.e. a contract to exchange a service for either money or another service for a specified period of time, then it can be shown that the trader will not exchange at some sampled prices. The analysis suggests that a portion of what has traditionally been regarded as unemployed resources may instead represent "speculative balance" or "search unemployment." Moreover, the extent of "search unemployment" has been shown to depend on the nature of the assumed price distributions as well as other parameters of the decision problem faced by the searcher.

Rothschild [24] has criticized these and related works [15, 21, 27] on the grounds that the analysis is partial at best. In particular, he takes the authors to task for leaving unexplained that which is supposed to motivate search -- the price differentials. Since search, as one of the arbitrage activities, is a principal determinant of the nature of the price distribution, one cannot rely on any comparative static result which is based on the assumption that the distribution will remain unchanged. Obviously, the criticism is to the point. It serves to motivate this paper.

In an economy in which trading is decentralized in the sense that the same commodities are exchanged on more than one market at the same time, resources must be expended for the purpose of acquiring price information. In addition, time is required not only in search but in the process of effecting trades. Because of these transaction costs, the extent of search by any one trader is limited. Consequently, price differentials may persist.

The principal purpose of this paper is to formalize these ideas. An important and related secondary purpose is to provide a justification for the assumptions of search theory -- namely, that the searching traders regard the set of price opportunities as stationary probability distributions and search as if sampling prices according to some random rule. Finally, we also show that the differences across markets in the prices of the same commodity are small if the number of searching traders is sufficiently large even when the extent of search by individual traders is severely limited.

The conceptualization under consideration is a simple pure exchange economy comprised of searching trader, non-searching traders and a system of markets. In each market and in every trading period the same two perishable commodities are exchanged among some specified subset of traders. Non-searching traders exchange in the same market in every period while each searching trader chooses a market to search at the beginning of each period. In any period the price on every market is that which will clear that market ex post, after the market to be searched is determined for every searching trader. The market clearing conditions implicitly defines a mapping from the set of all possible trader combinations to the price space.

Because some traders don't search at all while others only search one market per period, search is limited. However, price differentials affect the choice of market by searching traders. But, because prices are determined ex post, these differentials are not known with certainty at the moment of decision. They depend on the manner in which the searching traders distribute themselves among the markets within the period.

A formal statement of this conceptualization is included in the next section of the paper. In Section II, the distribution of price in each market is shown to depend on the search strategies used by all searching agents. 1/ The notion of search equilibrium is defined in Section III. It is a set of probability distributions on prices, one for each market, which are generated by optimal search given that all agents know the distribution which their collective actions generate. Existence is established by demonstrating that this concept is equivalent to the concept of a Nash equilibrium solution to an n-person game. 2/ In Section IV we show that in a non-trivial class of cases every search equilibrium is approximately competitive when the number of searching agents is large relative to the number of markets. The last section of the paper is devoted to interpretation and to a discussion of other applications of the approach used in this paper.

I. THE STRUCTURE OF THE MODEL

Consider an economy in which two non-storable commodities are exchanged in m distinct markets. A market is an abstract entity in the model; one which is defined only by the set of traders who are allowed to exchange with one another in a specified period. We distinguished between two kinds of agents searching and non-searching traders. Searching traders can trade in any one of the markets subject to restrictions specified later. A non-searching trader exchanges in the same market in every period by assumption. Searching traders are further subdivided into types. Two traders are of the same type if and only if they have the same preferences and the same endowment.

The set $M = \{j \mid j = 1,2,\ldots,m\}$ represents the set of all markets and $N = \{i \mid i = 1,\ldots,n\}$ denotes the set of all searching traders. The set $\{N_1,\ldots,N_k,\ldots,N_\ell\}$ is a partition of $N; N_k \subset N$, $k \in \{1,\ldots,\ell\}$ is the subset of searching traders of type k. The cardinality of N_k is n_k. Hence

$$n = \sum_{k=1}^{\ell} n_k.$$

At the beginning of any period, each agent receives an endowment of the two commodities. The endowment and the preferences of every trader are stationary. By assumption, preferences are restricted to the trader's own commodity space. The excess demand function for the first commodity of any searching trader $d_i(p)$ satisfies

$$u_k(d_i(p), - pd_i(p)) = \underset{a \in R}{\text{Max}}\; u_k(a,- pa),\; i \in N_k, \text{and } k \in \{1,\ldots,\ell\}$$

for every price, p, of the first commodity in terms of the second. The function u_k is the utility index common to all searching traders of type k.

Assumption 1: The preference ordering for every trader type, $k\epsilon\{1,\ldots,\ell\}$, is representable by a strictly increasing, strictly quasi-concave utility function $u_k: R^2 \to R$ in C^2.

Under Assumption 1 the excess demand function $d_i(p)$ is in C^1 as well as the indirect utility function defined as follows:

$$\varphi_i(p) = u_k(d_i(p), -pd_i(p)),\ i\epsilon N_k \text{ and } k\epsilon\{1,\ldots,\ell\} \tag{1}$$

In any multi-market economy potential traders must find one another. The process by which they do so is called search. A search is a match between a particular searching trader and a market. We restrict every searching trader to one and only one search per period. Let

$$x_i = (x_{i1},\ldots,x_{ij},\ldots,x_{im}) \in R^m,\ i \in N$$

denote the search strategy of trader $i \in N$. Searching traders choose to search according to some feasible strategy. In this paper we allow mixed or stochastic strategies;i.e. we allow each searching trader to sample the market to be searched in a manner consistent with a chosen probability distribution defined on the set of markets M.

Assumption 2: For every $i \in N$, $x_i \in S$ where

$$S = \{x_i \in R^m \mid \sum_{j\epsilon M} x_{ij} = 1, x_i \geq 0\} \tag{2}$$

Let p_j, $j \in M$, denote the price paid for the first commodity in terms of the second in market j. If a trader i exchanges in market j, the utility of such an exchange is $\varphi_i(p_j)$. For reasons which will be clear later, the trader does not know the price in any market prior to search with certainty. Instead he holds some subjective expectation. In this paper expectations take the form of a set of probability measures; each element of the set is defined on the price in a different markets. Let $\hat{\eta}_{ij}$ denote the subjective probability measure held by trader i and defined on the price space corresponding to market j. The price space corresponding to market j is denoted by (Λ_j, α_j) where Λ_j is the set of possible prices in R and α_j is a σ-field of Borel sets of Λ_j. The triplet $(\Lambda_j, \alpha_j, \hat{\eta}_{ij})$ then, represents a subjective probability space such that $\hat{\eta}_{ij}(P)$, $P \in \alpha_j$, is the probability that the price in market j is an element of P under the expectation of trader i.

The (subjectively) expected net utility attributable to an exchange in market j is

$$\int_{\Lambda_j} \varphi_i(p)\, \hat{\eta}_{ij}(dp) \ .$$

Hence, the expected utility associated with any feasible search strategy $x_i \in S$ is defined as follows:

$$\Psi_i(x_i, \hat{\eta}_i) = \sum_{j \in M} x_{ij} \int_{\Lambda_j} \varphi_i(p) \hat{\eta}_{ij}(dp), i \in N \tag{3}$$

where $\hat{\eta}_i$ is the vector of subjective probability measures held by trader i; i.e.

$$\hat{\eta}_i = (\hat{\eta}_{i1}, \ldots, \hat{\eta}_{ij}, \ldots, \hat{\eta}_{im}).$$

The vector represents the expectations of trader i with regard to the vector $(p_1, \ldots, p_j, \ldots p_m)$ of all prices.

We assume that every trader selects a search strategy which maximizes his expected net utility given his expectation. Let

$$x = (x_{ij}) = (x_1, \ldots, x_i, \ldots, x_n) \in S^n$$

denote a joint search strategy. Consider a particular possible joint strategy $x^0 \in S^n$.

<u>Definition 1:</u> A feasible joint search strategy $x^0 \in S^n$ is optimal given expectations if and only if

$$\Psi_i(x_i^0,\hat{\eta}_i) = \max_{x_i \in S} \Psi_i(x_i,\hat{\eta}_i) \quad \forall \quad i \in N.$$

Because Ψ_i is linear in x_i and S is the simplex in R^m, an optimal joint search strategy given expectations exists and the set of optimum is convex and compact. As specified the optimal strategies of individual traders are not interrelated because there need not be a relationship among expectations of the various traders. However, if expectations are "realistic" in some sense the optimal strategies will be interrelated.

As a consequence of search every searching trader will be matched to one and only one market in any specified market period. Let z_{ij} equal unity if trader i searches market j and equal zero otherwise. The vector

$$z = (z_{ij}) = (z_1,\ldots,z_i,\ \ldots,z_n)$$

is an integer element of S^n. Its value in any period specifies the location with respect to markets of all searching traders. We refer to z as the <u>joint search outcome</u>.

The elements of the joint search strategy x and the joint search outcome z which relate to market j we denote as follows:

$$x^j = (x_{1j},\ldots,x_{ij},\ldots,x_{nj})$$
$$z^j = (z_{1j},\ldots,z_{ij},\ldots,z_{nj}).$$

Clearly, both are elements of

$$\Omega = \{y \in R^n \mid 0 \leq y_i \leq 1\} \quad (4)$$

but the search outcome in market j is an integer element.

The excess demand expressed by the traders in N who search market j in any specified period can be expressed as $\sum_{i \in N} z_{ij} d_i(p)$ given price p. Let $g_j(p)$ denote the aggregate excess supply, given p, for the first commodity of all non-searching traders who exchange in market j only. Hence the total excess demand *ex post* is

$$\sum_{i \in N} z_{ij} d_i(p) - g_j(p).$$

We assume that the price in market j is determined *ex post* and equals that which clears its own market.

Assumption 3: For every $j \in M$, p_j clears the market *ex post*; i.e.

$$\sum_{i \in N} z_{ij} d_i(p_j) = g_j(p_j).$$

The final assumption implies the existence of unique market clearing price in every market given any possible search outcome.

Assumption 4: For every $j \in M, g_j(p)$ is a continuous differentiable strictly increasing function on

$$\Lambda_j = \{p \in R \mid \underline{w} \leq p \leq \overline{w},\ 0 \leq \underline{w} < \overline{w} < \infty\} \quad (5)$$

such that $g_j(\overline{w}) \geq 0$ and $g_j(\underline{w}) \leq 0$. Moreover, for every $i \in N, d_i(p)$ is strictly decreasing in Λ_j and $d_i(\overline{w}) \leq 0$ and $d_i(\underline{w}) \geq 0$.

By virtue of Assumption 1, $d_i(p)$ is also continuous and differentiable for all $i \in N$.

Our assumptions formalize the following ideas. The price in each market in a given period is determined by the excess demands and supplies only of the traders who exchange in said market in the specified period. The price p_j is competitive relative to this set of traders, but it may vary from period to period due to changes in the composition and size of the set of searching traders. The composition of the latter set depends on the choices made by the searching traders with respect to search strategies. In general, the joint search strategy x is a vector of probability distributions and the joint search outcome z is a random variable whose distribution is determined by x. Consequently, there will be variations in price in each market across periods and within a period across markets even if x is stationary.

As the comments above suggest, a multi-market system of the type formalized above will not act as one. In particular, a common market clearing price does not prevail in all markets in all periods. This is so for at least two related reasons. First, the searching agents may not have perfect information with regard to the price behavior in any market j, e.g. his expectation may differ with reality. The second, search is limited by assumption. It is limited in the sense that some agents don't search at all while others search only once per market period. These assumptions we make in an effort to take account of the fact that search is an activity which consumes time and other resources. The purpose of this paper is to characterize and study the nature of "equilibrium" within the context of a model which takes these facts into account.

II. SEARCH STRATEGIES AND RANDOM PRICES

Because the search strategy of any trader in N can be mixed, the search outcome in market j, z^j, is in general a random variable. Because the market clearing price p_j depends on the search outcome z^j, p_j is also a random variable given an arbitrary possible joint strategy x. In this section we derive the probability distribution induced on z by x.

The derivation is based on the fact that

$$\Pr\{z_{ij}=1\}=x_{ij} \qquad \forall (i,j) \in N \times M,$$

an implication of Assumption 2. Since x_{sj} and x_{tj}, $s \neq t$, are independent,

$$\Pr\{z^j\} = \begin{cases} \prod_{i\in N} x_{ij}^{z_{ij}} (1 - x_{ij})^{1-z_{ij}} & \text{if } z^j \in I(\Omega) \\ 0 & \text{otherwise} \end{cases} \qquad (6)$$

where $I(\Omega)$ represents the subset of integers in Ω. Let β denote the σ-field of Borel sets of Ω. Then

$$\mu(Z;x^j) = \sum_{z^j \in I(Z)} \Pr\{z^j\}, \quad Z \in \beta \qquad (7)$$

is the probability that the outcome is an element in some set Z; i.e. it is a probability measure defined on the search outcome space (Ω,β). Any $x \in S^n$, then, determines the entire vector of probability measures (all defined in (Ω,β)) which follows:

$$(\mu(\cdot\,;x^1),\ldots,\mu(\cdot\,;x^j),\ldots,\mu(\cdot\,;x^m)).$$

These are marginal measures from the joint measure on the joint outcome space. Since the joint strategy x is a vector of individual strategies and each individual strategy is an element in S, the measures are interrelated; i.e. z^s and z^t, $s \neq t$, are not stochastically independent. [3/]

The market clearing condition

$$\sum_{i \in N} z_{ij}\, d_i(p_j) = g_j(p_j)$$

implicitly defines a mapping from Ω to the price space Λ for every $j \in M$. We denote this mapping as h_j and call it the price rule in market j. By virtue of Assumptions 1 and 4 and the Implicit Function Theorem, it is a continuous differentiable function. Let α_j be the σ-field of Borel sets generated by h_j; i.e., for every $P \in \alpha_j$ there exists a $z \in \beta$ such that $z = h_j^{-1}P$.

Given a particular joint search strategy and the price rules in each market, the probability measures $\mu(\cdot;x^j)$, $j = 1,\ldots,m$, induces a measure on the price space in each market. In the case of any $j \in M$ the measure is defined as follows:

$$\eta_j(P;x^j) = \mu(h_j^{-1}P;x^j),\ P \in \alpha_j \tag{8}$$

where $h_j^{-1}P$ is the image of P in β. We have then a unique probability space $(\Lambda_j, \alpha_j, \eta_j(\cdot;x^j))$ for every $j \in M$ associated with any $x \in S^n$.

In addition, we can derive a conditional measure for every trader-market pair given the search strategy. The conditional measure is the probability that p_j is an element of a specified $P \in \alpha_j$ given that agent i searches market j. By virtue of (6) and the fact that $Pr\{z_{ij} = 1\} = x_{ij}$,

$$\Pr\{z^j \mid z_{ij} = 1\} = \frac{\Pr\{(1, <z_i^j>)\}}{\Pr\{z_{ij}=1\}}$$

$$= \begin{cases} \prod_{t\neq i} (x_{tj})^{z_{tj}}(1 - x_{tj})^{1-z_{tj}} & \text{if } (1, <z_i^j>) \in I(\Omega) \\ 0 & \text{otherwise} \end{cases}$$

where $<z_i^j>$ denotes the $n-1$ element vector formed by deleting z_{ij} from z^j and $(1,<z_i^j>)$ is a vector in Ω such that $z_{ij} = 1$. Given the analogous interpretation of $<x_i^j>$ and $(1, <x_i^j>)$, the relationships above and equation (7) imply

$$\mu_i(Z;x^j) = \mu(Z;(1,<x_i^j>)),\ Z \in \beta \tag{9}$$

where $\mu_i(\cdot\,;\,x^j)$ denotes the conditional measure on search outcomes. Hence, the conditional probability that p_j is an element of $P \in \alpha_j$ given search by trader i can be defined as follows:

$$\eta_{ij}(P;x^j) = \mu_i(h_j^{-1}P;x^j) = \mu(h_j^{-1}P;(1,<x_i^j>)) \tag{10}$$

$$= \eta_j(P;(1,<x_i^j>)) .$$

In words, the conditional probability depends only on the strategies of other traders.

In the previous section we introduced a subjective probability measure $\hat{\eta}_{ij}(\cdot)$ for every pair $(i,j) \in N \times M$. This measure can be interpreted as the belief of trader i concerning the conditional distribution of p_j given search by i. In otherwords, expected price behavior differs from actual to the extent that $\hat{\eta}_{ij}(\cdot)$ differs from $\eta_{ij}(\cdot\,;\,x^j)$. The subjective distribution might be an estimate of the true distribution based on the observations of the price seen while searching in the past. If this is the case, we would expect $\hat{\eta}_{ij}$ to converge in some sense to $\eta_{ij}(\cdot;x^j)$ over time. Weak convergence would imply

$$\int_{\Lambda_j} \varphi_i(p)\, \hat{\eta}_{ij}(dp) = \int_{\Lambda_j} \varphi_i(p)\, \hat{\eta}_{ij}(dp;x^j) \tag{11}$$

in the limit since $\varphi_i(p)$ is continuous by virtue of Assumption 1. 4/ When expectations have this property we say they are consistent.

Definition 2: The expectations of all searching agents are consistent if and only if (11) holds for all $(i,j) \in N \times M$.

III. SEARCH EQUILIBRIUM: DEFINITION AND EXISTENCE

The price rules h_j, $j = 1,\ldots,m$, are such that the price in any market tends to be higher in markets searched by more "buyers" than that in markets searched by relatively more "sellers" given Assumption 4. Moreover, the indirect utility function $\varphi_i(p)$ is such that $\varphi_i'(p) \gtrless 0$ as $p \gtrless v_i$ where v_i is the solution to $d_i(v) = 0$ by virtue of Assumption 1. Consequently, each trader in N tends to search either markets which he expects offer a relatively higher price (as a seller) or those in which a relatively lower price (as a buyer) is expected. 5/ If traders learn in the sense that their own subjective probability measure on every price converges weakly to the true measure, then all of these forces tend to generate an equilibrium. The nature and the existence of this equilibrium is the subject matter of this section.

Because mixed or stochastic search strategies are allowed and these, in general, induce randomness in prices, the usual concept of competitive equilibrium is inappropriate. Following the lead of other authors [8,11] we speak of an equilibrium vector of probability measures. For equilibrium we require optimal search and consistent expectations.

Definition 3: A search equilibrium is the vector of probability measures on prices of the form

$$(\Pi_1(\cdot;x^{1*}),\ldots,\Pi_j(\cdot;x^{j*}),\ldots,\Pi_m(\cdot;x^{m*}))$$

generated by a joint x^* which is optimal under consistent expectations. 6,7/

Competitive equilibrium as traditionally defined is the special case in which prices in all markets are the same with certainty. Our concept of equilibrium is more closely related to that of Hildebrand [11] and others [8,9] who have written on general equilibrium under uncertainty. However, the equivalence of search equilibrium as we have defined it and the concept of a solution to an n-person non-cooperative game introduced by Nash [17,18] is most insightful and useful. The game structure arises when expectations are consistent because, on the one hand, the actual probability measure on each price space is generated by the choices of all searching agents while, on the other, the criterion of each trader is defined in terms of these probability measures.

Given consistent expectations equations (3),(10) and (11) imply that the expected utility attributable to any possible strategy can be expressed for each $i \in N$ as a function of the trader's own search strategy x_i and the strategies of all other searching traders $<x_i>$. In particular,

$$\Psi_i(x_i,\hat{\Pi}_i) = \sum_{j \in M} x_{ij} \int_{\Lambda} \varphi_i(p)\Pi_j(dp\,;(1,<x_i^j>)) \equiv \theta_i(x_i,<x_i>), \quad i \in N \tag{12}$$

Here

$$\langle x_i \rangle = (\langle x_i^1 \rangle, \ldots, \langle x_i^j \rangle, \ldots, \langle x_i^m \rangle) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

since $\langle x_i^j \rangle$ is the vector x^j with x_{ij} deleted. It follows, thus, from Definitions 1 - 3 that $x^* \in S^n$ generates a search equilibrium if and only if it satisfies

$$\theta_i(x_i^*, \langle x_i^* \rangle), = \max_{x_i \in S} \theta_i(x_i, \langle x_i^* \rangle) \quad \forall\ i \in N. \tag{13}$$

But, (13) defines a Nash equilibrium point to the n-persons non-cooperative game described by the payoff functions θ_i, $i = 1, \ldots, n$ defined on the joint strategy space $S^n = \underset{i \in N}{X} S$. Given the appropriate change in variable,

$$\theta_i(x_i, \langle x_i \rangle) = \sum_{j \in m} x_{ij} \int_\Omega \varphi_i(h_j(z)) \mu(dz; (1, \langle x_i^j \rangle))$$

by virtue of equations (12) and (10). Since the measure μ is a continuous function of x^j $\forall\ j \in M$ (see (6) and (7)), the payoff function is continuous in the joint strategy x and linear in its own strategy x_i. Moreover, the joint strategy space S^n is the n-fold product of the simplex S in R^m. These facts are consistent with the hypothesis of Nash's [17,B] existence theorem. By virtue of the equivalence of x^* to a Nash equilibrium point, we know that a search equilibrium exists.

In the next section we analyze the model under the assumption that the set of searching agents is large. Our results, which take advantage of the law of large numbers, require that all traders of the same type pursue the same strategy. The following result establishes the meaningfulness of such an analysis.

Definition 5: A feasible joint search strategy x is symmetric if and only if $x \in Q = \{x \in S^n \mid x_s = x_t \ \forall (s,t) \in N_k \times N_k \text{ and } k \in \{1,\ldots,\ell\}\}$.

Theorem 1: A search equilibrium generated by a joint symmetric search strategy exists.

Proof: Our proof is an extension of Rosen's [23]. Consequently, it holds for any game which satisfies his hypothesis; i.e. a Nash equilibrium point exists such that all agents with the same payoff function pursue the same strategy if the game satisfies Rosen's hypothesis.

Let $\varphi^k(p)$ denote the indirect utility function common to all agents in N_k and let

$$\theta_i(x_i,\langle x_i\rangle) = \theta^k(x_i,\langle x_i\rangle) = \sum_{j\in M} x_{ij} \int_\Lambda \varphi^k(p)\, \eta_j(dp\,;(1,\langle x_i^j\rangle))$$

denote the corresponding payoff function of trader $i \in N_k$. This function which is defined on S^n, takes on the same value for all $i \in N_k$ given any $x \in Q$; i.e.,

$$\theta_s(x) = \theta^k(x_s,\langle x_s\rangle) = \theta_t(x) = \theta^k(x_t,\langle x_t\rangle) \quad \forall\ x \in Q$$

given $(s,t) \in N_k \times N_k$. The function

$$\rho(z,x) = \sum_{k=1}^{\ell} \sum_{i\in N_k} \theta^k(z_i,\langle x_i\rangle)$$

maps $S^n \times S^n$ into R. Moreover, because every payoff function is continuous in z_i and $\langle x_i\rangle$ and is concave in z_i, ρ is continuous in z and x and concave in z. Hence, both of the correspondences defined below are upper semi-continuous.

$$\Gamma_1 x = \{y \mid \rho(y,x) = \max_{z \in S^n} \rho(z,x)\}$$

$$\Gamma_2 x = \{y \mid \rho(y,x) = \max_{z \in Q} \rho(z,x)\}$$

Since S^n is a convex compact metric space and Q is a convex subspace of S^n, Q is also convex and compact. Therefore, Γ_1 is an upper semicontinuous correspondence which maps compact convex space S^n into itself and Γ_2 is an upper semicontinuous correspondence which maps S^n into its compact convex subspace Q. By virtue of Kakutani's [13] fixed point theorem a $x^* \in S^n$ exists such that $x^* \in \Gamma_1 x^*$ and a $x^o \in Q$ exists such that $x^o \in \Gamma_2 x^o$. Rosen shows that any fixed point of Γ_1 is a solution to (13). It suffices to show, then, that one of the fixed points of Γ_1 is in Q. This we do by exploiting the fact that Γ_2 has a fixed point in Q.

Suppose that no $x \in Q$ exists satisfying $x \in \Gamma_1 x$. Then for every $x \in Q$ and for som $(s,t) \in N_k x N_k$, $k \in \{1,\ldots,\ell\}$, a point $\bar{x}_t \in S$ exists such that $\bar{x} = (x_1,\ldots,x_s,\ldots,\bar{x}_t,\ldots,x_n) \in S^n$ and $\rho(\bar{x},x) > \rho(x,x)$ for every $x \in Q$. Because the strategy spaces of s and t are identical, we can construct

$$\bar{\bar{x}} = (x_1,\ldots,\bar{x}_s,\ldots\bar{x}_t,\ldots,x_n) \in Q$$

by setting $\bar{x}_s = \bar{x}_t$ given any $x \in Q$. Of course, $\rho(\bar{x},x) > \rho(x,x)$ and the fact that s and t are of the same type imply

$$\theta^k(\bar{x}_t,<x_t>) = \theta^k(\bar{x}_s,<x_s>) > \theta^k(x_t,<x_t>) = \theta^k(x_s,<x_s>)$$

since $x \in Q$ and $\bar{x}_s = \bar{x}_t$. This inequality and the definition of ρ imply

$$\rho(\bar{\bar{x}},x) - \rho(\bar{x},x) = \theta^k(\bar{x}_s,<x_s>) - \theta^k(x_s,<x_s>) > 0.$$

Hence, for some $x^o \in \Gamma_2 x^o$ an $\bar{x}^o \in S^n$ an $\bar{\bar{x}}^o \in Q$ exist such that

$$\rho(\bar{\bar{x}}^o, x^o) > \rho(\bar{x}^o, x^o) > \rho(x^o, x^o).$$

But, this result contradicts the fact that

$$\rho(x^o, x^o) \geq \rho(x, x^o) \; \forall \; x \in Q.$$

The economics of the foregoing argument are of some interest. In economic language, if no symmetric equilibrium strategy exists, then for a trader of some type there exists a strategy which is better than that used by traders of his type given any symmetric strategy. But if it is better for one it is better for any other agent of the same type as well. This fact implies that no solution exists to a game restricted solely to symmetric strategies. But this cannot be true because the set of joint symmetric strategies is also compact and convex. [7/] So, even though our game is not restricted to symmetric strategies, a Nash equilibrium exists which is symmetric.

IV. COMPETITITVE SEARCH EQUILIBRIUM

Prices in all markets are market clearing *ex post* and search is optimal *ex ante*. It is in this sense that a search equilibrium can be considered distributions of "equilibria" prices. But by the traditional definition these prices are not competitive equilibria. Received theory includes the assumption, often implicit, that any system of markets dealing in the exchange of the same two commodities will act as one. The only justification for such an assumption is that optimal search behavior and other forms of arbitrage reduce price differentials to negligible quantities. Since search is explicit in our analysis, it is a theorem to be derived. In this section we investigate under what conditions and in what sense such a theorem holds.

Consider the following definition:

Definition 6: A search equilibrium is competitive if and only if for every $\epsilon > 0$

$$\Pr\{ \; | p_j - p^* | \geq \epsilon\} = 0$$

where p^* solves

$$\sum_{i \in N} d_i(p^*) = \sum_{j \in M} g_j(p^*). \tag{14}$$

Assumption 4 implies that p^*, the competitive equilibrium price, is unique.

Because every market clears and each searching trader searches one and only one market per period, one can easily verify that any price common to all markets is the competitive equilibrium price. Hence, if a search equilibrium is such that the same price prevails in every market with certainty, then that search equilibrium is competitive.

One's intuition suggests a stronger result. Namely, any search equilibrium generated by optimal search given consistent expectations in which prices are certain *ex ante* is competitive. The intuitive appeal of this conjecture can be traced to the following argument. Consistent expectations as we have defined them and certain prices *ex ante* imply that all searching traders know all prices prior to search. Hence, every trader will either search a market offering the highest price as a seller or a market offering the lowest price as a buyer. In either case, some market must have a non-zero excess demand, a contradiction, unless the prices are the same in all markets.

This argument is incorrect in general for two reasons. First, non-searching traders may be willing to demand the quantities which searching traders wish to sell in high price markets and to supply the quantities which the searching traders desire in low price markets because by assumption the non-searching traders cannot take advantage of the price differential between any two markets. Second, the price in any market depends on who searches it. Consequently, the unconditional price distribution and the conditional given search by some trader are not the same in any finite economy. But, it is the conditional distribution which is relevant to each traders search decision. For this reason it need not be true that every searching agent will search only the two extreme prices under certainty.

Consider a case in which there are three identical searching agents who are always buyers; i.e. $d_1(p) = d_2(p) = d_3(p) > 0$ on the interior of Λ, and two identical markets; i.e., $g_1(p) = g_2(p)$ on Λ. The following strategies generate a search equilibrium: $x_1 = (1,0)$, $x_2 = (1,0)$ and $x_3 = (0,1)$. The prices are different and certain. Indeed, $p_1 > p_2$ since there are more buyers in the first market than in the second. Given x_2 and x_3, trader 1 is indifferent between the two markets because the conditional price will be the same in either case. For the same reasons x_2 is optimal given consistent expectations. Finally, trader 3 knows that the conditional price in the second market is lower than the first given x_1 and x_2. Since he too is a buyer, the utility of exchange in market 2 is higher.

It is of interest to note that no competitive equilibrium exists in the example. The price rule is such that $1\frac{1}{2}$ buyers must search each of the two markets with certainty. Because buyers are indivisable, this requirement cannot be fulfilled. Note as well that the particular search equilibrium illustrated is one in which the searching traders do not pursue identical strategies even though they are of the same type.

The problems illustrated by the example can be interpreted, heuristically, as the consequence of search costs. Presumably, no trader would restrict exchange to one market if search were costless and price differentials existed. Moreover, if the benefits implicit in the existing price differentials outweigh the additional costs, each searching agent would search more than one market per period and, by doing so, would be able to spread his excess demand appropriately among several markets. Indeed, if search were costless, indivisability would not be a problem.

In other words, we can interpret our model as a special case of a more general formulation in which search costs are explicitly introduced. Obviously, we need the general formulation in order to acquire a complete understanding of the impact of search costs. Nevertheless, our special assumptions illustrate two important points. First, a search equilibrium can exist in which prices differ even when the differences are known. Second, competitive equilibrium need not exist

Although no competitive equilibrium exists in our example, a search equilibrium is generated by random search with equal probability. In particular, given $x_i = (\frac{1}{2},\frac{1}{2})$, $i = 1,2,3$, the <u>ex ante</u> price distribution is the same in both markets as is the conditional distribution given search by any one of the three searching traders. Hence, given search strategies of any two, there is no better strategy open to the remaining trader than sampling at random. Moreover, since the expected outcome in either market is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, actual prices include the competitive equilibrium price. If we were to increase the number of searching traders, the competitive price would prevail with certainty in the limit as a consequence of the law of large numbers. Note that the latter conclusion would hold even if the two markets were different, since in the limit the impact of the non-searching traders on the price would be negligible. Hence both problems discussed above are unimportant in our example if the number of searching traders is sufficiently large and if traders search as if sampling at random.

The remainder of the section represents an attempt to generalize this insight. For this purpose we need some additional notation. First, let $f_k(p)$ be excess demand function common to all traders of type k; i.e.

$$f_k(p) = d_i(p) \quad \forall\ i \in N_k,\ k_2 \in \{1,\ldots,\ell\} \tag{15}$$

Let $y^j = (y_{1j},\ldots,y_{kj},\ldots,y_{\ell j})$ denote the vector of proportions of searching agents of each type who search market j; i.e.,

$$y_{kj} = \frac{1}{n_k} \sum_{i \in N_k} z_{ij}, \quad k = 1,\ldots,\ell. \tag{16}$$

Let

$$\sigma_k = n_k/n, \quad k = 1,\ldots,\ell, \tag{17}$$

represent the share of all traders who are of type k. The market clearing condition, equation (14), can be expressed as follows

$$\sum_{k=1}^{\ell} \sigma_k y_{kj} f_k(p_j) = \frac{1}{n}\, g_j(p_j),\ j \in M \tag{18}$$

in terms of the new notation.

In the sequel we compare economies which differ only with respect to the size of the set of searching traders. The competitive equilibrium price given n searching traders p_n^* is that price which clears all markets simultaneously, i.e.

$$\begin{aligned}
&\sum_{j\in M} \sum_{k=1}^{\ell} \sigma_k y_{kj} f_k(p_n^*) \\
&= \sum_{k=1}^{\ell} \sigma_k f_k(p_n^*) \sum_{j\in M} y_{kj} \qquad (19)\\
&= \sum_{k=1} \sigma_k f_k(p_n^*) = \sum_{j\in M} \frac{1}{n} g_j(p_n^*)
\end{aligned}$$

since $\sum_{j\in M} y_{kj} = 1$ for all $k \in \{1,\ldots,\ell\}$.

Corresponding to any sequence of economies is a sequence $\{p_n^*\}$ of competitive equilibrium prices. Because f_k is continuous for all k and g_j is bounded for all j on Λ, the sequence converges to p^*, which solves

$$\sum_{k=1} \sigma_k f_k(p^*) = 0 , \qquad (20)$$

as $n \to \infty$. The limiting competitive equilibrium price p^* depends only on the excess demands of the searching traders because the non-searching traders are inconsequential. In the sequel we show that the price in all markets is p^* with certainty in the limiting economy given any symmetric search equilibrium.

By virtue of Assumption 4 equation (18) implicitly defines a continuous single valued price rule $h_{jn}: \Omega \to \Lambda$ for every $j \in M$ and every positive integer n where now

$$\tilde{\Omega} = \{y \in R^{\ell} \mid 0 \leq y_k \leq 1\}. \qquad (21)$$

Let $h_j^*: \Omega \to \Lambda$ denote the function implicitly defined as follows:

$$\sum_{k=1}^{\ell} \sigma_k y_k f_k(h_j^*(y)) = 0, \quad y \neq 0 \tag{22a}$$

$$g_j(h_j^*(y)) = 0, \quad y = 0 . \tag{22b}$$

Note that $h_j^*(y)$ is continuous except at $y = 0$. The following result is useful in the sequel.

Lemma 1: For every $j \in M, h_{jn}(y) \to h_j^*(y)$ for every fixed $y \in \Omega$. Moreover, on every compact set $Y \subset \Omega - \{0\}$ the sequence of functions $\{h_{jn}\}$ converges uniformly to h_j^*.

Proof: Because the argument is identical for all $j \in M$ we drop the market subscript in the proof. Since $h_n(0) = h(0) = w$, where w is the unique solution to $g(w) = 0$, we have $h_n(0) \to h^*(0) = w$. Because $f_k, k=1,\dots,\ell$, are continuous and g is bounded on Λ by virtue of Assumption 4,

$$\Sigma\, \sigma_k y_k f_k(\lim_{n\to\infty} h_n(y)) = \lim_{n\to\infty} \frac{1}{n}\, g(h_n(y)) = 0.$$

Since the solution to (23a) is unique;

$$\lim_{n\to\infty} h_n(y) = h^*(y), \quad y \neq 0$$

as asserted.

To prove uniform convergence on any compact subset Y of the open subset $\Omega - \{0\}$, we appeal to Dini's theorem. [9/] Because h_n and h^* are both continuous on Y and $\{h_n(y)\}$ converges to $h(y)$ for every fixed $y \in Y$, Dini's theorem applies if the distance $d(h_n(y), h^*(y))$ is decreasing in n as it tends to zero. By Assumption 1, f_k is also differentiable on Λ for all Λ

and g is continuous and differentiable on Λ by Assumption 4. Moreover, $f_k' < 0$ and $g' > 0$. Consequently, if we let

$$\Delta h_n(y) = h_{n+1}(y) - h_n(y)$$

then

$$\sum_k \sigma_k y_k [\Delta h_n f_k' + f_k(h_n)] = \frac{1}{n+1}[\Delta h_n g' + g(h_n)]$$

where $f_n' \ \forall k$ and g' are evaluated at some point on the closed interval bounded by $(h_{n+1}(y), h_n(y))$. This equality and (18) imply

$$\Delta h_n [\sum_k \sigma_k y_k f_k' - \frac{1}{n+1} g'] = (\frac{1}{n+1} - \frac{1}{n})g(h_n)$$

for every fixed $y \in Y$. Alternatively,

$$\Delta h_n(y) = \Delta h_{n+1}(y) - h(y) = \frac{g(h_n(y))}{ng' - (n+1)n\sum_k \sigma_k y_k f_k'}$$

The denominator is positive for all n. Hence, $\Delta h_n(y) \gtreqless 0$ as $g(h_n(y)) \gtreqless 0$. Since $g(h_{n+1}(y)) > g(h_n(y)) > 0$ if $\Delta h_n(y) > 0$ and $g(h_{n+1}(y)) < g(h_n(y)) < 0$ if $\Delta h_n(y) < 0$, either $\Delta h_n(y) = 0 \ \forall n$ or $\Delta h_n(y)$ is monotonic in n by induction. Consequently,

$$d(h_n(y), h^*(y))$$

is either zero $\forall n$ or decreases in n.

We can represent any feasible symmetric joint search strategy as a vector $q \in S^\ell$ where

$$q_{kj} = x_{ij} \quad \forall i \in N_k, \quad k \in \{1,\ldots,\ell\} \text{ and } j \in M \tag{23}$$

where q_{kj} is the probability that any searching trader of type k searches market j. Let $q_k = (q_k, \ldots, q_{km}) \in S$ denote the search strategy common to all agents of type k and let $q^j = (q_{1j}, \ldots, q_{\ell j})$ denote the components of q pertinent to market j. The number of searching traders of type k who search market j, n_{kj}, can be defined in a variety of ways as follows:

$$n_{kj} = \sum_{i \in N_k} z_{ij} = n_k y_{kj} = n\,\sigma_k y_{kj} \quad . \tag{24}$$

Because all traders of the same type search market j with the same probability and because these probabilities are independent, the vector $(n_{1j}, \ldots, n_{\ell j})$ is distributed according to the joint binomial, i.e.

$$\Pr\{n_{1j}, \ldots, n_{\ell j}\} = \prod_{k=1}^{\ell} b(n_{kj}, q_{kj}, n_k)$$

where

$$b(n_{kj}, q_{kj}, n_k) = \begin{cases} \binom{n_k}{n_{kj}} q_{kj}^{(n_{kj})} (1 - q_{kj})^{(n_k - n_{kj})} & \text{if } n_{kj} \in \{1, 2, \ldots, n_k\} \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

As earlier β denotes the σ-field of Borel sets of Ω and $Y \in \beta$ is a set of possible proportions. The following probability measure is defined on (Ω, β) given any component q^j from a symmetric search strategy $q \in S^{\ell}$:

$$\mu_n(Y_j q^j) = \sum_{y \in Y} \prod_{k=1}^{\ell} b(n\,\sigma_k y_{kj};\ q_{kj}, \sigma_k n). \tag{26}$$

In the sequel we speak of $(\Omega, \beta, \mu_n(\cdot; q^j))$ $j \in M$ as the probability distribution on the search outcome, y^j, given search strategy q in an economy of size n.

Consider the conditional probability measure given that any one of the searching traders of type k has already searched market j. Since this equals the probability that $n_{kj} - 1$ of the remaining $n_k - 1$ agents in N_k will search market j, the conditional measure is

$$\mu_n^k(Y_; q^j) = \sum_{y \epsilon \{Y | y_{kj} \neq 0\}} b(n_{k-1}; q_{kj}, n_{k-1}) \prod_{t \neq k} b(n_{tj}; q_{tj}, n_t)$$

Because $b(n_{kj}; 0, n_k) = 0$, because

$$b(n_{kj}; q_{kj}, n_k) = \frac{n_k q_{kj}}{n_{kj}} b(n_{kj} - 1; q_{kj}, n_k - 1)$$

and because $y_{kj} = n_k q_{kj}$, we can express the measure as

$$\mu_n^k(Y; q_j) = \begin{cases} \sum_{y \epsilon Y} \frac{y_{kj}}{q_{kj}} \prod_{t=1}^{\ell} b(n\,\sigma_t y_{tj}; q_{tj}, \sigma_{tn}) & \text{if } q_k \neq 0 \\ \mu_n(Y; q^j) & \text{if } q_{kj} = 0 \end{cases} \tag{27}$$

This measure is also defined on (Ω, β).

Finally, we introduce still another probability measure defined on (Ω, β):

$$\mu(Y, q^j) = \begin{cases} 1 & \text{if } q_j \in Y \\ 0 & \text{if } q^j \notin Y \end{cases} \tag{28}$$

Of course, q^j is the expectation of y^j under $(\Omega, \beta, \mu_n(\cdot; q^j))\ \forall n$. We intend to show that both the unconditional and all conditional measures converge weakly to $\mu(\cdot; q^j)$ for all $j \in M$ as $n \to \infty$.

A sequence of probability measures defined on the same space <u>converges weakly</u> to a given measure $(\mu_n \Rightarrow \mu)$ if and only if $\mu_n(Y) \to \mu(Y)$ for all Y except those of measure zero under μ; i.e., $\forall\, Y \in \beta \ni \mu(Y) \neq 0$. Equivalently,

$\mu_n \Rightarrow \mu$ if and only if

$$\int_\Omega f d\mu_n \to \int_\Omega f\, d\mu$$

for every bounded continuous real valued function f on Ω. Because of the special nature of $\mu(\cdot;\, q^j)$ in our case weak convergence in measure and convergence in probability of the random variable y^j to q^j are equivalent concepts. 10/

Lemma 2: Given any sequence $\{q_n\} \subset S^\ell$ converging to some q, $\mu_n(\cdot;q_n^j) \Rightarrow \mu(\cdot;q^j)$ for all $j \in M$. Moreover, $\mu_n^k(\cdot;q_n^j) \Rightarrow \mu(\cdot;q^j)$ for all $k \in \{1,\ldots,\ell\}$ and $j \in M$.

Proof: Suppose that $\mu_n(\cdot;q^j) \Rightarrow \mu(\cdot;q^j)$ for every fixed $q^j \in \Omega$ and that the sequence $\{q_n\} \subset S^\ell$ converges to q. Because $\mu_n(\cdot;q^j)$ is continuous is q^j by (26), for every continuous bounded real valued function f on Ω

$$\int_\Omega f(y)\, \mu_{n_1}(dy;q^j_{n_2}) \to \int_\Omega f(y)\, \mu_{n_1}(dy;q^j) \to \int_\Omega f(y)\, \mu(dy;q^j)$$

as first $n_1 \to \infty$ and then as $n_2 \to \infty$ respectively. Alternatively,

$$\int_\Omega f(y)\, \mu_{n_1}(dy;q^j_{n_2}) \to \int_\Omega f(y)\, \mu(dy;q^j_{n_2})$$

$$= f(q^j_{n_2}) \to f(q^j) = \int_\Omega f(y)\, \mu(dy;q^j)$$

as first $n_2 \to \infty$ and then $n_1 \to \infty$ respectively. Consequently $\{\mu_{n_1}(\,.;q^j_{n_2}) \mid n_1 = n_2 = n\}$ converges weakly to $\mu(\,.;q^j)$ for all $j \in M$. 12/

Because $\mu_n^k(\cdot;q^j)$ is also continuous in $q^j \ \forall\ k$,the same argument applies under the same hypothesis. To complete the proof we establish that $\mu_n(\cdot;q^j) \Rightarrow \mu(\cdot;q^j)$ and $\mu_n^k(\cdot;q^j) \Rightarrow \mu(\cdot;q^j)$ given q^j.

The unconditional random variable y^j is a vector of independent random variables; the representative element is y_{kj}. Because these elements are independent, y^j converges in probability to q^j if y_{kj} converges in probability to q_{kj} for all k. Because the expectation of y_{kj} is q_{kj} and its variance is

$$\frac{1}{n\sigma_k}(1 - q_{kj})q_{kj},$$

Chebyshev's inequality implies the latter condition. Hence, $\mu_n(\cdot;q^j) \Rightarrow \mu(\cdot;q^j)$.

By virtue of (26) and (27), $\mu_n^k(Y;q^j) = \mu_n(Y;q^j)\ \forall\ Y \in \beta$ if $q_{kj} = 0$ and

$$\mu_n^k(Y;q^j) = \mu_n(Y;q^j) + \sum_{y \in Y}\left(\frac{y_{kj}}{q_{kj}} - 1\right) \Pr\{y^j\}$$

otherwise, where

$$\Pr\{y^j\} = \begin{cases} 0 \text{ if } (n\,\sigma_k y_{kj}) \notin I \\ \prod_{k=1}^{\ell} b(n\,\sigma_k y_{kj};q_{tj},n\,\sigma_k) & \text{otherwise.} \end{cases}$$

If $q^j = 0$, then $\mu_n^k(\cdot;q^j) = \mu_n(\cdot;q^j)$ and $\mu_n(\cdot;q^j) \Rightarrow \mu(\cdot;q^j)$ implies $\mu_n^k(\cdot;q^j) \Rightarrow \mu(\cdot;q^j)$. If $q^j \neq 0$, then weak convergence of $\{\mu_n(\cdot;q^j)\}$, convergence in probability of y_{kj} to q_{kj}, and the equation above imply

$$\mu_n^k(Y;q^j) \to \mu(Y;q^j) + E\,\{\frac{y_{kj}}{q_{kj}}\}-1\ \mu(Y;q^j) = 1$$

for every $Y \in \beta$ such that $q^j \in Y$. But, since $\mu(Y;q^j) = 0$ for all other

$\Upsilon \epsilon \beta$, we have shown that $\mu_n^k(\Upsilon;q^j) \Rightarrow \mu_n(\Upsilon;q^j) \quad \forall q^j \neq 0$.

The technical apparatus just constructed provides the means by which to prove that every search equilibrium generated by a symmetric joint strategy is competitive when the number of searching agents if infinite. The result holds because the two problems, which exist in the finite case, vanish in the limit. In particular, Lemma 1 implies that the importance of the non-searching traders disappears in the limit and the impact of any single searching trader is nil when $n = \infty$ by Lemma 2. At the same time randomness induced by any mixed joint search strategy is inconsequential due to the law of large numbers if the joint strategy is symmetric.

To establish the last statement in a formal way, we need to show that the price in every market converges in probability to one point in the price space Λ as $n \to \infty$. Given the price function $h_{jn}: \Omega \to \Lambda$, and any sequence $\{q_n^j\}$, the measure $\mu_n(\cdot;q_n^j)$ induces on (Λ,α) a unique measure $\eta_{jn}(P;q^j)$ defined by

$$\eta_{jn}(P;q_n^j) = \mu_n(h_{jn}^{-1}P;q_n^j) \ \forall \ P \ \epsilon \ \alpha \tag{29}$$

where α is the σ-field of Berel sets of the price space Λ and $h_{jn}^{-1}P$ is the inverse image of P in β. Lemma 1, Lemma 2, and Theorem 5.5 in Billingsley [3 ,p. 34] implies that $\eta_{jn}(\cdot;q_n^j) \Rightarrow \eta_j(\cdot;q^j)$ for every j such that $q^j \neq 0$ given that $q_n \to q$ where

$$\eta_j(P;q^j) = \mu(h_j^{*-1}P;q^j = \begin{cases} 1 \text{ if } h_j^*(q^j) \ \epsilon \ P \\ 0 \quad \text{otherwise} \end{cases} \tag{30}$$

In other words, the price in every market converges in probability to that associated by the price rule with the expected search outcome in the limit if in the limit there is a positive probability that the market will be searched by agents of at least one type.

Clearly, the identical argument implies $\eta^k_{jn}(\cdot;q^j_n) \Rightarrow \eta_j(\cdot;q^j)$ under the same hypothesis where

$$\eta^k_{jn}(P;q^j_n) = \mu^k_n(h^{-1}_{jn}P;q^j_n) \quad \forall\ P \in \alpha \quad . \tag{31}$$

The probability space $(\Lambda,\alpha,\eta^k_{jn}(\cdot;q^j_n))$ is the conditional price distribution given search by a single searching agent of type k. Because any such agent is of measure zero in the limit, the conditional and unconditional price distributions are identical.

Lemma 3: Given any sequence $\{q_n\} \subset S^{\ell}$ converging to q, $\eta_{jn}(\cdot;q^j_n) \Rightarrow \eta_j(\cdot;q^j)$ for all $j \in m$ such that $q^j \neq 0$. Moreover, if $q^j \neq 0$, then $\eta^k_{jn}(\cdot;q^j_n) \Rightarrow \eta_j(\cdot;q^j)$ for all $k \in \{1,\ldots,\ell\}$.

Proof: The result referred to in Billingsley presumes that h_n and h are measurable mapping from a metric space Ω to a metric space Λ. The measures μ_n and μ are defined on (Ω,β). Let $\mu_n h_n^{-1}$ and μh^{-1} denote the induced measures on (Λ,α). A variant of the theorem is as follows: If h_n converges uniformly to h on compact sets of Ω - D_k where D_k is the set of discontinuities of h, if $\mu(D_h) = 0$, and if $\mu_n \Rightarrow \mu$, then $\mu_n h_n^{-1} \Rightarrow \mu h^{-1}$. In all of our cases these three conditions are implies by Lemma 1, $q^j \neq 0$ and Lemma 2 respectively.

One final notational detail is required later. Let $q^*_n \in S^{\ell}$ be a symmetric joint search strategy in an economy with n searching traders which is optimal under consistent expectations. By virtue of Theorem 1 at least one q_n^* exists for every integer n greater than or equal to ℓ, the number of distinct searching trader types. Moreover, because any q_n^* is optimal under consistent expectations and $(\Lambda,\alpha,\eta_{jn}^k(\cdot;q_n^{j*}))$ is the condition distribution of p_j given that some trader of type k searches market j in an economy of size n

$$\sum_{j\in m} q_{kj}^{n*} \int_\Lambda \varphi_i(p)\eta_{jn}^k(dp;q_n^{j*}) = \max_{x_i \in S} \sum_{j\in M} x_{ij} \int_\Lambda \varphi_i(p)\eta_{jn}^k(dp;q_n^{j*}). \quad (32)$$

<u>Theorem 2: If the number of searching agents is infinite $(n = \infty)$, any symmetric joint strategy $q^* \in S^\ell$ generates a search equilibrium if and only if q^* is such that $p^* = h_j(q^{*j})$ for all $j \in M$. Every search equilibrium so generated is competitive.</u>

Proof: Let $q_n = q^*$ $\forall$ n. Since $\{q_n\} \subset S^\ell$ and converges to q^*, Lemma 3 implies that prices are certain and equal to $h_j(q^{*j})$ for all markets j such that $q^{*j} \neq 0$ when $n = \infty$. If $q^{j*} = 0$, then $p_j = h_{jn}(0) = h_j(0) = w$ with certainty $\forall$ n because $\mu_n(Y,0) = \mu(Y,0) = 1$ $\forall$ Y containing $y = 0$. Hence, by the definition of a competitive search equilibrium,it suffices to prove that q^* is optimal under consistent expectations if and only if $p^* = h_j(q^{j*})$ $\forall$ $j \in M$.

For every $i \in N_k$ and $k \in \{1,\ldots,\ell\}$, the expected utility of searching market j is given by

$$\int_\Lambda \varphi_i(p)\eta_{jn}^k(dy;0) = \varphi_i(h_{jn}(0)) = \varphi_i(h_j^*(0)) \quad \text{if } q_n^j = 0$$

and

$$\int_\Lambda \varphi_i(p)\eta_{jn}^k(dy;q_n^j) \quad \text{if } q_n^j \neq 0 .$$

Because the sequence $\{q_n\}, q_n = q^* \varepsilon S^\ell$, converges as $n \to 0$, and $\varphi_i(p)$ is a bounded continuous real valued function on Λ by virtue of Assumption 2, Lemma 3 implies

$$\int_\Lambda \varphi_i(p)\eta^k_{jn}(dy,q^j_n) \to \int_\Lambda \varphi_i(p)\eta_j(dy,q^j_n) = \varphi_i(h^*_j(q^*))$$

for all j such that $q^{j*} \neq 0$. Consequently, the expected utility associated with any feasible search strategy $x_i \varepsilon S$ is

$$\sum_{j\varepsilon m} x_{ij}\varphi_i(h^*_j(q^{j*})) \quad \forall\ i \ \varepsilon\ N$$

in the limit.

Clearly, if $h^*_j(q^{j*}) = p^*$, then $x_i = q^*_k$, $i \ \varepsilon\ N_k$, is dominated by no other feasible strategy, i.e., it is optimal under consistent expectations. Suppose instead that $h_j(q^{j*}) = p^o \neq p^* \ \forall\ j \ \varepsilon\ m$. Because $y_{kj} = q^*_{kj}$ with certainty in the limit and prices are own market clearing, the supposition is not consistent with the uniqueness of the competitive equilibrium price.

Suppose that q^* is optimal but that $p^o_j = h_j(q^{j*})$ differ on M. Because $\varphi'_i(p) \gtreqless 0$ as $p \gtreqless v_i$ where v_i is the unique solution to $d_i(v_i) = f_k(v_i) = 0$, $i \ \varepsilon\ N_k$, by Assumption 2, Assumptions 1 and 4 imply

$$\sum_{j\varepsilon} q^*_{kj}\varphi_i(p_j) = \max\ [\varphi_i(\overline{p}^o), \varphi_i(\underline{p}^o)] \ \forall\ i \ \varepsilon\ N_k \text{ and } k$$

where $\overline{p}^o = \max_{j\varepsilon M} p^o_j$, and $\underline{p}^o = \min_{j\varepsilon M} p^o_j$. Consequently,

$$q^*_{kj} = 0 \quad \forall\ j \in M \ni \underline{p}^o < p^o_j < \bar{p}^o$$

$$d_i(p_j) = f_k(p^o_j) < 0 \quad \forall\ i \in N_k \text{ and } j \in M \ni p^o_j = \bar{p}^o \text{ and } q^*_{kj} \neq 0$$

$$d_i(p^o_j) = f_k(p^o_j) > 0 \quad \forall\ i \in N_k \text{ and } j \in M \ni p^o_j = \underline{p}^o \text{ and } q^*_{kj} \neq 0.$$

Because $y^j = q^{j*}$ with certainty and $q^*_{kj} \neq 0$ for some $j \in M$ and every k, at least one $j \in M$ exists such that

$$\sum_{k=1}^{\ell} \sigma_k y_{kj} f_k(p^o_j) \neq 0$$

which contradicts Assumption 3.

The equations of (22) imply that the price rules are the same in the limiting economy except at the origin; i.e. $h^*_s(y) = h^*_t(y)\ \forall\ y \neq 0$ and $(s,t) \in M \times M$. Consequently, if $q_{kj} = \frac{1}{m}$ for all $(k,j) \in N \times M$, then the price is the same in all markets with certainty. The price in p^* by virtue of the market clearing condition.

<u>Corollary:</u> <u>The strategy of sampling randomly with equal probability, i.e. $q = (\frac{1}{m})$, generates a competitive search equilibrium in the limiting economy.</u> Of course, random search with equal probability is not generally optimal under consistent expectations when the number of searching agents if finite. A sufficient condition is that all price rules be identical. This condition holds if the excess supply functions of the non-searching agents are identical across markets.

Obviously, the fact that all search equilibria generated by symmetric search strategies are competitive in the limit is of interest only if the limiting economy approximates economies with large numbers of searching agents.

It is natural to require of an "appromixately competitive" search equilibrium that the probability of large deviations from the competitive equilibrium price in any market be small. In particular, a search equilibrium generated by q_n^* is approximately competitive if given n small positive numbers (ε, δ) exist such that

$$\Pr\{|p_j - p_n^*| \geq \varepsilon\} \leq \delta.$$

Because the sequence of competitive equilibrium prices $\{p_n^*\}$ converges to p^*, this condition is satisfied for large n when the price in every market converges in probability to the limiting competitive equilibrium price.

Of course, any sequence of symmetric strategies optimal, under consistent expectations, $\{q_n^*\}$ generates a sequence of equilibrium probability measures defined on the price in market j $\{\Pi_{jn}(\cdot;q_n^{j*})\}$ for all markets $j \in M$. The collection of these sequences is a sequence of search equilibria. Convergence in probability to the competitive equilibrium price is equivalent to the weak convergence of every $\{\Pi_{jn}(\cdot;q_n^{j*})\}$ to $\{\Pi_j(\cdot;q^{j*})\}$ if q^{*j} is such that $p^* = h_j(q^{j*})$ for all j. Hence, if these conditions hold, we can say that the sequence of search equilibria converges weakly or in probability to the competitive equilibrium as the number of searching traders tends to infinity.

There are two complications, however, First, the optimal strategy q_n^*, for fixed n, is not necessarily unique. Hence, any sequence $\{q_n^*\}$ chosen arbitrarily need not converge. But, lack of convergence is not necessarily meaningful unless there were some way to preselect the sequence of search strategies. This does not appear to be possible. Second, even if $\{q_n^*\}$ does converge to some q^*, Lemma 3 does not guarantee that the prices converge in probability unless $q^{j*} \neq 0$ for all $j \in M$.

These problems can be partially resolved as follows: First, because every $q_n^* \in S^{\ell}$, any arbitrary $\{q_n^*\}$ is a subset of S^{ℓ}. Because S^{ℓ} is a compact subset of a finite dimensional Euclidean space, $\{q_n^*\}$ contains at least one convergent subsequence. Second, convergence of this subsequence to a q^* such that $q^{j*} = 0$ for one or more markets implies that these markets are isolated from the others in the limit. That is, almost no searching trader searches these markets in the limit and few do when the number of searching agents is finite and large. Consequently, if the subsequence of equilibrium search strategies converge to q^* and if this fact implies that the price in every market converges in probability to the competitive equilbrium price except for those markets almost no one searches in the limit, then it is meaningful to say that the corresponding subsequence of search equilibria converges weakly to the competitive equilibrium except for that set of markets not searched almost surely in the limit.

<u>Definition 5:</u> Given a subsequence $\{q_{n_r}^*\}$ of $\{q_n^*\}$ which converges to some q^*, the subsequence of search equilibria generated by $\{q_{n_r}^*\}$ converges weakly to the competitive equilibrium except on the set of markets not searched almost surely if and only if for every $\epsilon > 0$

$$\lim_{n_r \to \infty} \Pr\{|p_j - p_{n_r}^*| \geq \epsilon\} = 0$$

for all $j \in M^* = \{j \in M \mid q^{j*} \neq 0\}$.

Note that the subset of markets M^* is in competitive equilibrium in the limit and for large n all markets in the subset offer prices which are approximately competitive given convergence in this sense.

Theorem 3: Every sequence of search equilibria generated by symmetric search strategies contains a subsequence which converges weakly to the competitive equilibrium except on the set of markets not searched almost surely in the limit.

Proof: We prove the theorem under the hypothesis that $\{q_n^*\}$ converges to q^*. The same argument applies given any converging subsequence with an appropriate change in notation.

By virtue of Lemma 3, both the unconditional and every conditional price in market j converges in probability to $h_j^*(q^{j*})$ for all $j \in M^*$. Hence, the result holds if $p^* = h_j^*(q^{j*})\ \forall\ j \in m^*$. This condition holds if no other strategy dominates q^* in the limit when feasible strategies are restricted to the set of markets searched in limit; i.e. if

$$\sum_{j \in M^*} q_{kj}^* \varphi_i(h_j^*(q^{j*})) = \max_{x_i \in S^*} \sum_{j \in M^*} x_{ij} \varphi_i(h_j^*(q^{j*}))\ \forall\ i \in N_k \text{ and } k \in \{;, \ldots, \ell\} \quad (33)$$

where

$$S^* = \{x_i \in S \mid x_{ij} = 0\ \ \forall\ j \notin M^*\}.$$

(Clearly, S^* is a convex set in R^{m*} where m^* is the cardinality of M^*. Hence the right side of (33) exists.) That (33) is a sufficient condition follows by virtue of the same contradiction used in the proof of Theorem 2. Namely, if q^* is optimal on M^* and if two or more markets in M^* offer different certain prices then at least one of these markets won't clear when $n = \infty$.

To verify (33) we use the fact that for any finite n

$$\sum_{j\in M} q_{kj}^{*n} \int_{\Lambda} \varphi_i(p)\eta_{jn}^{k}(dp;q_n^{j*}) = \max_{x_i\in S} \sum_{j\in M} x_{ij}\int_{\Lambda}\varphi_i(p)\eta_{jn}^{k}(dp;q_n^{j*})$$

$$\geq \max_{x_i\in S^*} \sum_{j\in M^*} x_{ij} \int_{\Lambda} \varphi_i(p)\eta_{jn}^{k}(dp;q_n^{j*})$$

for all $i \in N_k$ and $k \in \{1,\ldots,\ell\}$. Because $\varphi_i(p)$ is continuous and bounded on Λ by virtue of Assumption 1, both sides of the inequality converge by virtue of Lemma 3. Indeed, in the limit

$$\sum_{j\in M} q_{kj}^{*}\varphi_i(h_j(q^{j*})) = \sum_{j\in M^*} q_{kj}^{*}\varphi_i(h_j(q^{j*}))$$

$$\geq \max_{x_i\in S^*} \sum_{j\in M^*} x_{ij}\varphi_i(h_j(q^{j*})).$$

Since the vector q_k^* is an element of S^* the inequality holds as a strict equality.

As we have already noted random search with equal probability is an optimal strategy under consistent expectations if markets are identical in the sense that they have the same price rule. The size or composition of the set of searching agents is immaterial. This is so because $q^s = q^t \ \forall\ (s,t) \in M \times M$ if $x_{ij} = \frac{1}{m}$ for all $(i,j) \in N \times M$. If $h_{sn} = h_{tn}$ as well for all n, then the price distributions are identical on the set of markets. But, if this is true, then all searching traders are indifferent with respect to the market searched. Hence, random search is as good as any other feasible search strategy.

Corollary: If $h_{sn} = h_{tn}$ $\forall\ (s,t) \in M \times M$ and all $n \in \{1,2,\ldots\}$, then the sequence of search equilibria generated by random search with equal probability converges weakly to the competitive equilibrium.

There are conditions under which convergence in our sense implies convergence to the competitive equilibrium on all markets. For example, if $q_n^{j*} = 0$ for all $j \notin M^*$ and all n greater than or equal to some finite integer n_o, then the proof to Theorem 2 implies that $p_j = p^* \; \forall \; j \in M^*$ since in the limit $p_j = h_j^*(0)$ with certainty. More generally, all prices converge to the competitive price if q_n^* tends to q^* in such a way that $h_j(q_n^{j*})$ has a limit for all $j \notin M^*$ by virtue of the same argument. Even if there is no limit for some $j \in M^*$, it is true that the "variance" of $h_j(yj)$ vanishes as $n \to \infty$. In a heuristic sense this fact implies that the variations in p_j^k will be almost predictable in the tail of the sequence of search equilibria. But, if this is true and search is optimal under consistent expectations, then for every n sufficiently large it would appear that an $n_o \geq n$ exists such that all traders of some type will search some market j, not in M^* with probability one; i.e. $q_{kj}^{n_o} = 1$ for some k and some $j \notin M^*$, since the price offered in every market in M^* is approximately competitive. However, if true, this fact contradicts the fact that $q_n^{j*} \to 0 \; \forall \; j \in M^*$. Our conjecture is, then, that weak convergence except on the set of markets not searched is equivalent to weak convergence under our assumptions.

V. SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

The following picture emerges from our analysis: In a multi-market system of the type discussed, search is part of the allocation process. The extent to which the gains in trade are exploited depends on the ways in which traders with complementary preferences and endowments are matched. In an economy operated as a single market this function is performed by the mythical auctioneer. In particular, a matching of trading partners which maximizes gains from trade is implicit in any competitive equilibrium in the standard general competitive market model.

In our model the process by which traders are matched is not centrally directed. It is the joint outcome of the uncoordinated and self interested search behavior of the traders themselves. As we have shown, the joint decision problem can be formulated as a non-cooperative n-person game. Under appropriate informational assumptions, a joint search decision is a Nash equilibrium solution to this game. In finite economies this solution will not generate a competitive equilibrium in general. However, if the number of searching agents is large relative to the number of markets, then the price in all relevant markets is approximately competitive in a non-trivial class of cases.

The last result suggests that the arbitrage process, modeled here as a search process, may well substitute for the directed matching process implicit in the standard model even when, as in our case, search by any individual trader is quite limited. This certainly was Walras' original conjecture.

However, this conclusion should not be jumped to too hastily for a number of reasons.

Note that we have substituted the assumptions of a single market in which the price is always market clearing for an alternative formulation: a system of markets and an associated system of own market clearing prices. One interpretation of our assumption is that there are many auctioneers. Each one in his own market plays all the usual roles.

We could easily change our formulation and by so doing obtain different results. Suppose instead that there is a single non-searching trader associated with each market who assumes the role of setting price. Since in our model after search no trader has an alternative exchange opportunity within the period, the single non-searching trader may well act by setting the price conditional on the search outcome as if a monopolist. The model specified is identical in structure except that the price rule no longer satisfies the competitive market clearing conditions. Hence, a search equilibrium will exist and the prices in all markets will be approximately equal in all markets when the number of searching agents is large. However, the common price is the monopoly price. Analogous results in similar models have been obtained by Phelps and Winter [21] and Diamond [5].

As still another alternative one can interpret the non-searching trader as a dealer who engages in trade at least in part for profit. This interpretation would also require a different price rule since it would seem unlikely that such a dealer would be content with the meager returns due the traditional auctioneer. However, if he were to act similarly, in particular if he were to adjust price from period to period in response to excess demand in his own market, then the model would be similar in spirit to that introduced by Fisher [6,7]. However, Fisher's result, that all prices tend over time to the competitive equilibrium price, will not hold in any finite economy in general.

This is true because the search process derived in this paper does not satisfy the restrictions which Fisher imposes on search. Of course, ones intuition suggests that it will hold as an approximation when the number of searching agents is large.

The following question arises: What is the appropriate price rule when one relaxes the market clearing assumption? An answer to this question requires a richer conceptualization than that given in this paper or in any of the others referred to above. In particular, one must allow the extent of search by each agent as well as its nature be an individual economic decision and model the effect of search, both its extent and nature, on the competitive position of the price setters in order to derive the appropriate rule. Nevertheless, the concept of search equilibrium introduced here or some similar notion of endogeneous stochastic equilibrium should prove useful.[12/]

What about "search unemployment"? In this paper the phenomena does not arise. Indeed, there is no speculation of any kind because commodities are perishable and are traded only on spot markets. This assumption can be relaxed by allowing exchange contracts with duration. Such a contract specifies the price and the duration over which exchange between two traders will take place. When contracts of this type are exchanged in a multi-market system a trader's bid and ask prices will bracket the reservation price, the price at which he is willing to make no exchange given that there is a single certain exchange ratio between any two commodities. In other words, every trader speculates. The concept of search equilibrium can be extended to take this complication into account. One can easily show that search unemployment exists in such an equilibrium.

Of course, one can show that the extent of search unemployment will be slight when the number of searching agents is large because it depends on the magnitude of the randomness induced by search. However, if uncertainty is exogeneous, e.g. endowments or preferences are stochastic, then price differentials will always exist and with it search unemployment. As a topic of future research it would be interesting to model the ways in which the search process and speculation modify the effects of exogeneous randomness on allocations. In particular, we are suggesting a marriage of the type of model developed here and those recently introduced by Hildenbrand [11] and Green [9].

An alternative approach to our analysis of speculation is to allow inventories. An interesting and natural variation on the dealer model discussed above is one in which dealers are the sole inventory holders. By using his inventories so as to maximize his own profit, the dealers may very well simulate the behavior of the traditional auctioneer. 13/

Finally, the value of a storable medium of exchange also suggests itself within the context of our approach. If there were money,then there could be specialization of markets; e.g. the set of markets could be partitioned into subsets some of which exchange goods for money. The possible advantage of such an institutional arrangement is that by searching the money markets the individual trader may be able to obtain a more advantageous exchange ratio more often than that he could obtain on the markets which deal only in goods. But to take advantage of these randomly arising opportunities he would need to hold temporary stores of purchasing power; i.e. "speculative balances." Of course, these ideas are not totally new; e.g. see Ostroy [19]. We are only suggesting that the approach outlined here may be useful in an attempt to formalize them.

FOOTNOTES

1/ The idea that the search process itself induces randomness in prices was suggested to the author by the work of Porter and Mirman [22].

2/ This result is at least heuristically anticipated by S. Salop [25].

3/ This fact plays no important role in the analysis which follows. It might do so if expectation-formation were explicitly taken into account.

4/ See for an analysis of convergence of beliefs in this sense. Sufficient informational conditions for weak convergence are presented.

5/ Obviously, the extent of uncertainty in any two markets and the trader's attitude toward risk also affects his choice between the two in the general case.

6/ If the formation of expectations is a resource using activity, the consistency requirement is too strong. Nevertheless, the condition can be regarded as sufficient.

7/ The equilibrium can also be thought of as the joint distribution on the vector of prices $(p_1,\ldots,p_n)$. All results hold given this definition as well.

8/ This fact is an implication of Rosen's [23] existence theorem.

9/ See Berge [2], p. 106.

10/ See Billingsley [3] p. 12 and pp. 22-25 for the definition and its equivalences.

11/ See Billingsley [3] Theorem 4.2 p. 25 for a justification of this method of proof.

12/ This line of research is currently being pursued by Kenneth Burdett in his Ph.D. dissertation at Northwestern University

13/ Peter Howitt [12] has investigated a general equilibrium model of this type where traders are matched to markets according to prescribed rationing rules.

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