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THE SIMPLEST THREE-PERSON NON-COOPERATIVE GAMES

by

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Abstract

This paper lists some classes of 2x2x2 games. All have the restriction that there are only two payoff levels per player. In the first group of games exactly one player wins at every outcome, in the second group exactly one player loses, and in the third group no more than one player wins. Some of the games illustrate theorems about non-cooperative games, and some provide counterexamples to false hypotheses.

Three non-cooperative solution concepts, the Nash equilibria, Selten's uniformly perfect equilibria, and Harsanyi and Selten's tracing procedure solution are calculated for each game and some unusual aspects of the outcomes selected are discussed. The method of trying out various solution concepts on a standard set of games is an effective way of learning their properties, and complements the method of characterizing the solution concepts by means of axioms and theorems.

1. Introduction

For any class of games we can ask: What are the simplest, smallest games it contains? By listing these games we can try out different solution concepts, look for possible theorems, and generally get a feeling for what it is like to play games of this type.

Here we will list the simplest 3-person non-cooperative games, selected by a criterion of simplicity to be explained. Some features of individual games will be noted and three solution concepts will be discussed: the theory of Nash equilibria, (Nash, 1951), Selten's theory of uniformly perfect equilibria, (Selten, 1975), and Harsanyi and Selten's solution concept involving the tracing procedure, (Harsanyi and Selten, 1980; Harsanyi, 1975). These procedures will be applied to the games on the list and the results will be discussed.

Several lists have appeared in literature giving the simplest members of various classes of games. Von Neumann and Morgenstern (1945) and Shapley (1962) listed simple n-person characteristic function games, "simple" in the technical sense of having only two payoff levels for each coalition. Rapoport and Guyer (1966) presented a taxonomy of two-person 2 x 2 non-cooperative games with ordinal payoffs, and Mayberry (1968) determined the types of two-person cooperative games. As far as we know no one has listed n-person non-cooperative games.

2. One-winner games

The following conditions are imposed:

1) each game has only three players,

2) each player has only two strategies, neither of which is strongly or weakly dominated,

3) each player has only two possible payoff levels: winning (payoff = 1) and losing (payoff = 0),

4) at every outcome exactly one player wins,

5) no game on the list can be generated from any other by permuting players, strategies or any combination of players and strategies.

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The first four conditions ensure that the games are simple, while the fifth avoids games that are essentially equivalent to other games on the list. The assumption in condition 3) that payoffs equal 0 or 1 does not restrict the generality of the results since any game with two levels of payoff can be transformed to an equivalent game with payoffs of 0 and 1.

It turns out that there are exactly ten different games satisfying 1) - 5). They will be called the one-winner games.

Some of their properties can be deduced immediately. For example, it follows from 3) and 4) that all are zerosum. Conditions 2) and 3) imply that every player must have at least two winning outcomes, since otherwise one of the strategies would be weakly dominated. None of the games can be completely symmetrical in players since there are eight possible outcomes and this number is not divisible by three.

The ten one-winner games are shown in Figure 1 using the following notation. The strategy space of the three players is shown as the unit cube. Points inside the cube are associated with mixed strategies, e.g., a point with coordinates $(2/3, 2/3, 2/3)$ indicates a mixed strategy in which each player uses the second strategy with probability $2/3$. Nash equilibria are shown as heavy lines and dots, and the tracing procedure solution is indicated by an "X". The set of perfect equilibria are listed in Table 1. If a dot or the end of a line falls partway between two vertices in Figure 1, it can be assumed that "partway" is exactly halfway unless otherwise noted.

The three players are labelled R, C and L, abbreviating "row-chooser", "column-chooser" and "layer-chooser". Each vertex of the cube is labelled according to which player wins at that outcome. For example "R" stands for the payoff vector $(1,0,0)$.

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The one-winner games were generated by hand as follows. The row-chooser R was chosen as a reference and the possible ways in which winning outcomes for R could be assigned to the vertices of the cube were listed. Eliminating equivalent patterns and assignments involving dominated strategies there are only the nine possibilities shown in Figure 2.[1] The possible patterns for the column player C were then combined with these keeping in mind the restrictions imposed by 1) - 4). Then the possible patterns for L were added. Finally the games on the list were compared and duplicates were eliminated. The whole procedure was repeated for verification.

We will point out some special properties of certain games on the list, e.g., their symmetries, unusual features of their solutions, or simple ways of describing the rules. Unless stated otherwise it can be assumed the game is asymmetrical.

Game A1.

Symmetries: R and C are in equivalent positions. Each of their strategies are alike. Player L's two strategies are alike. The unique equilibrium point with this symmetry is $(1/2,1/2,1/2)$.

The game can be realized by having R and C choose "Heads" or "Tails". Player L tries to guess whether R's and C's choices will match or not (i.e., both be "Heads" or "Tails", or be different) and wins by guessing correctly, but if L is wrong R wins by matching C or else C wins by avoiding being matched by R.

Game A3.

The interior point has coordinates $(2/5(5\sqrt{5}), (-1+\sqrt{5})/2, (-1+\sqrt{5})/2)$ which are approximately $(.724, .618, .618)$.

Game A5.

Symmetries: the three players are distinctive, but for each player, the two moves are alike. The unique equilibrium point with this symmetry is $(1/2,1/2,1/2)$.

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The game can be realized in the following way. Each player chooses "Heads" or "Tails". Players C and L try to guess R's move. If C alone guesses it then C wins but if both guess correctly L wins, while player R wins by avoiding their guesses.

We might imagine two carnivorous animals seeking their prey which has hidden in one of two caves. One of the two predators is stronger so if both choose the correct cave, that one will be successful.

Game A5 has the special property that it is invariant if the labels of each pair of moves are switched concurrently. In terms of our realization above, this means that neither cave plays any special role, and all that matters is the relationship among the choices of caves. Games B5, C14, C16 and C18 appearing in Sections 3 and 4 share this feature.

Game A6.

Symmetries: players L and C are alike. Player R's two moves are alike. Player C's first move corresponds to L's second move, and C's second move corresponds to L's first. Two equilibrium points have this symmetry: $(1/2,1,1)$ and $(1/2,0,0)$.

Each player must choose "Heads" or "Tails". The total number of heads is counted. If it is 0 or 3, then R wins. If it is 1, C wins, and if it is 2, L wins.

Game A6 is a member of a general class of games defined as follows. There are m strategies for each player, and we can define m strategy-types, such that each player has one strategy of each type and the payoffs depend only on the number of strategies chosen by the players that fall into each type, not on who chooses which strategy. Unanimity games (Harsanyi and Selten, 1980) fall into this class as do the following other games on our lists: B6, C14, C16, C18.

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The game shows one aspect of the behaviour of Nash equilibria that strikes us as noteworthy. If we eliminate all Nash equilibria that do not reflect the symmetries of the game, two are left: one involving strategies (.5,0,0) and the other (.5,1,1). At each of these equilibria R has the power to shut out either C or L completely and give the win to the alternative player, yet this power is of no benefit to R -- at neither equilibrium point does R gain. This is because the game is played without binding threats or agreements so that any verbal interchange among the three players carries no weight. Player R has the power to choose the winner, but cannot make threats or extract promises in order to benefit from this power.

Game A8.

Symmetries: R and L are in equivalent positions and both of C's moves are alike. The two equilibria having this symmetry are (1,1/2,1) and (0,1/2,0).

Game A10.

Symmetries: R and L are alike. R's first move corresponds to L's first move and likewise the second moves correspond. Player C's two moves are alike. All the equilibria, (r,1/2,1-r) for $r \in [0,1]$, have this symmetry.

§ 3. One-loser games

Condition 4, which requires that exactly one player win, can be replaced by

4') Exactly one player loses for every outcome.

The list of one-loser games is shown in Figure 3. The games look exactly like the one-winner list since conditions 1) - 4) are still satisfied by a particular game if winners are declared to be losers

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and vice versa, but the notation in the game matrices must now be rein-
terpreted so that "R", "C", and "L" mean that players R, C and L _lose_, rather
than _win_, at that outcome.

Any completely mixed strategy n-tuple that was an equilibrium in a
one-winner game will also be an equilibrium in the corresponding one-loser
game, so the coordinates of the interior points are the same as those given in §1.

The set of Nash equilibria tends to be smaller in the one-loser case --
seven of the ten one-loser games have unique Nash equilibria, whereas
none of the one-winner games had unique equilibria.

Games B1, B3, B5, and B6.

The symmetries and coordinates of the interior equilibria of these
games are the same as those of the corresponding one-winner games.

Game B7.

This game is noteworthy because it has exactly two Nash equilibria.
Wilson (1971) and Harsanyi (1973) have shown that almost all games have
a finite and odd, or infinite number of equilibria. Because of the equal-
ities that hold in its payoff matrix, B7 lies outside their class of "almost
all" games, and it shows that their theorem cannot be extended to claim that
all games with finite number of equilibria have an odd number of equilibria.

§4. Zero-and-one-winner games

We now replace condition 4) by 4"):

   4") No more than one player wins for every outcome, and there is
        some outcome at which no one wins.

The games on list C will be different from those on list A since we
are requiring that all lose at some outcome.

This results in the eighteen games shown in Figure 4. Some inter-
esting games appear and the sets of Nash equilibria become more elaborate.

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The notation "R", "C", "L" means that R, C or L wins and the new symbol "-" in the game matrix means that no one wins.

Game C1.

The interior Nash equilibrium point has strategies $(2/3,1/2,1/2)$. The logarithmic tracing procedure solution selects the strategy triple $((4-\sqrt{2})/7,1,1)$, R's probability being approximately .369.

Game C2.

The set of interior Nash equilibria has equation $c = 1/2$, $\ell = (1-r)/r$. The tracing procedure selects the point $(.6,2/3,2/3)$.

For two-person games Vorobev (1958) and Kuhn (1961) have shown that the set of Nash equilibria is the union of a finite number of convex sets, a fact helpful in the design of algorithms to find all equilibria (Winkels, 1979). Game C2 shows that this theorem cannot be extended to 3-person games, since some of its equilibria form a continuously bending curve. In this respect it is unique among the games on our lists.

Game C4.

The tracing procedure leads to the strategy triple $(\sqrt{2}-1,0,0)$.

Game C5.

The tracing procedure solution gives all players zero. In terms of the welfare of the three players this is a bad solution, in fact the worst possible one, since its payoff vector is weakly dominated by <u>every other</u> possible one in the game.

Game C8.

The interior Nash equilibrium has strategies $(1/2,2/3,1/2)$.

Game C9.

The interior Nash equilibrium has strategies $((\sqrt{5}-1)/2,(3-\sqrt{5})/2,(3-\sqrt{5})/2)$ or approximately $(.618,.382,.382)$.

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Game C10.

The tracing procedure leads to $(1,(4-\sqrt{2})/7,0)$ where $(4-\sqrt{2})/7 \cong .369$.

Game C12.

Symmetries: R and L are in equivalent positions and C's two moves are alike. The interior equilibria $(r,1/2,1-r)$ for $r \in [0,1]$, have this symmetry.

Game C14.

Symmetries: R and L are in equivalent positions, R's first move corresponding to L's second move and vice versa. Player C's moves are alike. The unique equilibrium showing this symmetry is the central point $(1/2,1/2,1/2)$.

To play C14 each chooses "Heads" or "Tails;" and Player C tries to guess R's move, R tries to guess L's move and L tries to guess C's move. If a player guesses correctly without being guessed, then that player wins.

In the case of finite two-person games, Chin et al. (1974) have shown that if the set of equilibria is convex then the equilibria are exchangeable, that is, each player can choose any strategy appearing in any equilibrium point and the pair of strategies will again yield an equilibrium. This does not hold for three-person games as C14 shows. Its equilibria are convex but are not exchangeable, since for example, $(.2,.5,.7)$ and $(.7,.5,.2)$ are both equilibria but $(.2,.5,.2)$ or $(.7,.5,.7)$ are not.

Chin et al. showed that their theorem could not be extended to three-person games by giving a method for constructing a class of games that were counterexamples. Game C14 is a simple member of their class.

Game C15.

Symmetries: C and L are in equivalent positions. For each player, the two moves are alike. The unique equilibrium point showing this symmetry is the central point $(1/2,1/2,1/2)$.

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Game C16.

Symmetries: completely symmetrical in players and moves. The only equilibrium showing this symmetry is the point $(1/2, 1/2, 1/2)$.

We can regard each player as choosing "Heads" or "Tails". A player wins by being the unique player to choose a head or tail.

Game C18.

Symmetries: players C and L are in equivalent positions. For each of the three players the two moves are alike. The only equilibrium with this symmetry is $(1/2, 1/2, 1/2)$.

The game can be realized by having R choose "Heads" or "Tails".and C and L try to guess the choice. If no one guesses correctly R wins. If exactly one is correct then that player wins, but if both guess correctly no one wins.

This game is somewhat like the predator-prey game A6 but in this case neither of the predators has an advantage over the other. If they both succeed in catching the prey a one-half share will not be enough to keep them alive so that everyone loses.

§5. The Three Solution Concepts

The three concepts will be discussed now by giving their definitions, the methods of calculation and some critical comments on the results of applying them to our games. The solutions form a nested sequence: the Nash equilibrium set contains the set of uniformly perfect equilibria which contains the tracing procedure solution. The three will be discussed in this order.

Nash equilibria

A strategy n-tuple is defined to be a Nash equilibrium if no player gains by changing to another strategy while the other players hold their strategies fixed. The Nash equilibria of the games on the lists can be determined graphically by drawing a cube to represent the players' strategy sets, as in Figures 1-4, and marking Row's set of best reply strategies, which forms a 2-space. This is the set of strategy triples in which Row's choice is optimal given the other two strategies in the triple. Similar diagrams are drawn for the Column and Layer players. The intersection of the three best reply sets is found and this is the set of Nash equilibria of the game.

A difficulty with the concept of Nash equilibria is that it generally does not yield a unique outcome. This is true for our games -- of the thirty-eight on the list only six have unique Nash equilibria. If we eliminate equilibria that do not show the symmetry of the game then seven more games have unique equilibria, making a total of thirteen games in all. Of the thirty-two games with multiple equilibria, thirty-one have an infinity of equilibria.

Uniformly Perfect Equilibria

Selten (1975) observed that some of the Nash equilibria arising from games in extensive form are flawed in that they suggest irrational behaviour at

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some nodes of the game tree. Players can adopt strategies involving poor decisions at these nodes because they are never reached in the course of rational play, so that being disposed to make irrational choices there involves no loss to the player.

Selten developed a suggestion of Harsanyi to introduce a small amount of uncertainty into the players' beliefs about which nodes can be reached, so as to force players to use strategies that give rational behaviour everywhere. He modified the game to a new uniformly perturbed one where no strategy can be chosen with probability less than some small value p. A sequence of values of p with limit zero generates a sequence of games whose limit is the original game. The Nash equilibria of the sequence of games also form sequences with limits but the limits are typically a proper subset of the equilibria of the original game. None of the irrational equilibria appear as limits since for every game in the sequences all its nodes can be reached with some small probability. The limit set as p goes to 0 comprises the uniformly perfect equilibria.

The uniformly perfect equilibria of our games can be determined geometrically, in the same way as the Nash equilibria. The best reply diagrams are drawn with the provision that no strategy is used with probability less than some small value p, and the set of equilibria as p → 0 are determined.

The uniformly perfect equilibria of our games are in Table 1. It is clear that any completely mixed Nash equilibrium will be uniformly perfect. The requirement of uniform perfectness reduces the number of equilibria: of the 32 games with nonunique Nash equilibria, the set of uniformly perfect equilibria is a proper subset of the Nash equilibria for 27 games. For six of these games the uniform perfect equilibrium is unique.

Although requiring uniform perfectness decreases the number of equilibria, it is not clear that the remaining ones are more rational than the eliminated ones. For the games on the lists, the uniformly perfect equilibria do not seem to have any obvious benefit to recommend them. In our view the requirement of perfectness should be modified. For most principles of rational choice, their justification can be repeated with specific reference to any situation to which they apply. For example, to convince two players to use a Nash equilibrium we could cite Nash's arguments using the parameters of the very game they are playing. This is not true for the concept of uniform perfectness, however, since it was developed for extensive form games and applies to normal form games only by "analogy".

Originally Selten (1965) defined a perfect equilibrium as one that induced an equilibrium at every subgame of an extensive form game. He later noted (1975) that this definition was too weak, since equilibria still appeared involving irrational play at some unreached nodes. The stronger definition involving the limit of a sequence of games was proposed and this was shown to rule out the inappropriate equilibria and thus not be too weak. However in our view it is too strong, so we would prefer an intermediate definition such as the sequential equilibria of Kreps and Wilson (1982).

The logarithmic tracing procedure

The solution proposed by Harsanyi and Selten (1980) is the most complicated of the three, but also the strongest in that it always selects a unique outcome for finite games.

We will give only the outline of their procedure. First a series of operations are performed on the game to eliminate dominated strategies and

otherwise undesirable strategies. The game is then disassembled if possible into parts closed with respect to best replies. (If this is not possible other procedures are followed.) These components, called _basic games_, are analyzed by the logarithmic tracing procedure, which selects a unique perfect equilibrium point as the solution of the game.

It can be shown that all the games on our lists are basic, so the only operative step is the one involving the logarithmic tracing procedure. This is intended to model the mental processes of the players as they incorporate into their strategy selection, the notion that their opponents are intelligent and are trying to outwit them. First a new game is constructed in which the players are paid off as if their opponents moved randomly and unstrategically, according to a probability distribution that Harsanyi and Selten specify. A term is introduced into the payoff function of this game so that each player has a unique optimal strategy. The payoff function is then continuously modified until it matches the original game, and the path of equilibria is traced. If new equilibria enter during the modification they are ignored, unless they lie on the path. The endpoint is an equilibrium in the final game and represents the solution.

Applying Harsanyi and Selten's method to the class of basic games with two moves per player leads to a family $G_p(t,e)$ of _auxiliary_ games where $p \in (\frac{1}{2},1]$, $t \in [0,1]$ and $e \geq 0$. If q_i designates the probability assigned by player i to strategy 1, then $G_p(t,e)$ is defined as the game in which player i receives payoff

$$t\, H_p^i(\bar{q}) + (1-t)\, H_p^i(\tfrac{1}{2},\ldots,q_i,\ldots\tfrac{1}{2}) + e\,(1-t)\, \log[q_i(1-q_i)].$$

This is the weighted sum of three components:

1) $H_p^i(\bar{q})$, defined to be player i's payoff in the uniformly perturbed version of the original game, with perturbation parameter p.

2) $H_p^i(\,,\ldots,q_i,\ldots\,)$, the payoff if the others use an equiprobable mix of their strategies.

3) $\log[q_i(1-q_i)]$, a strictly convex function of player i's strategy.

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The weight t represents the players' increasing awareness of their opponents' strategic behaviour. The parameter e introduces the logarithmic term, whose purpose is to break ties among strategies and ensure a unique path, and p makes the equilibria uniformly perfect.

It is shown that there is a unique analytic path of equilibria in $G_p(t,e)$ as t goes from 0 to 1, for small positive e and p. The limit of the endpoint is determined as e goes to 0, and the limit of these limits is determined as p goes to 0.

To calculate this point, we used the fixed-point algorithm of Merrill (1972), as coded by Saigal (1974), and modified for the purpose of tracing a path of fixed points rather than calculating a single value.[2]

Three noteworthy properties of the solutions found will be mentioned. The first is the result in game C5, in which it selects the only point weakly payoff-dominated by every other outcome in the game. This strikes us as rather perverse behaviour for a rational solution concept. On the one hand it is clear that non-cooperative solutions can sometimes be payoff-dominated by some other points, as in the case of the Prisoners' Dilemma, but in that game the players have an outcome even worse than the solution hanging over their heads, and can rationalize the solution as a lesser evil. However in C5 the players have nothing to lose by ignoring the advice of the tracing procedure and have at least the logical possibility of gaining.

A second property is illustrated by game C18. In this game the totally mixed equilibrium strategy $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ is chosen. It can be shown that this choice is unstable with respect to the payoffs. Slight alterations in the payoffs cause the tracing procedure to choose either $(\frac{1}{2},0,1)$ or $(\frac{1}{2},1,0)$. By selecting only uniformly perfect equilibria, the procedure is robust with respect to the assumption of perfect rationality. The players can make small mistakes in strategy choice and the equilibrium stays approximately constant. However

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it is not robust with respect to assumptions about their payoffs as this example shows[3]. In defense of the procedure it should be noted that the unstable solution $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ is a consequence of a deeper requirement -- a solution should be unique and have the same symmetry as the game. This is a very persuasive principle for specifying solutions, and thus game C18 should perhaps be regarded as a demonstration of the difficulty of finding a solution invariant with respect to payoff deviations.

A third feature of the tracing procedure is evident from games in which the solutions have "irregular" coordinates, such as C1 where column 1 is played with probability .369. These values arise from the use of the logarithmic term in the auxiliary games. If another function equally suitable were used, such as q_i^{-1} $(1-q_i)^{-1}$, the strategy prescribed for column would change.

Ideally a special reason for using the logarithmic function might be found, but if not we could take the philosophical position that a solution concept does not have to be uniquely justified in every aspect. It should be generally consistent with our notions of rationality, and lead to the selection of a single outcome, but at some point arbitrary elements must be introduced.

Figure 2 includes configurations with at least two R's since Row must win at two outcomes or more to avoid having weakly dominated strategies, and also no more than four R's since the other players may not have weakly dominated strategies.

That the number of configurations is nine can be verified by enumerating analytically the number of ways of marking the vertices of a cube, with equivalence induced by rotations or reflections, with one pair of opposite faces, the two rows, distinguished. Using Polya's enumeration formula the cycle index is

$$\frac{1}{16} (x_1^8 + 9x_2^4 + 4x_4^2 + 2x_2^2 x_1^4).$$

If we let the coefficient of r^k be the number of ways of placing exactly k R-markers, then the generating function for the number of ways is

$$r^8 + r^7 + 5r^6 + 5r^5 + 10r^4 + 5r^3 + 5r^2 + r + 1.$$

We can generate the twenty configurations corresponding to the coefficients of r^2, r^3 and r^4. Eliminating ones with dominated strategies gives the nine shown in Figure 2.

Merrill's original algorithm constructs a homotopy between a simple function and the function of interest, estimates a fixed point, and then restarts using a new homotopy, a finer search grid, and the previous estimate as a starting point. The modification used here eliminates the restart. After getting a sufficiently accurate estimate of a fixed point it changes the parameters of the problem slightly, and allows the algorithm to shift to the new fixed point.

The parameter sequence was generated using a series of values $(p_1, p_2, \ldots)$, $(t_1, t_2, \ldots)$, and $(e_1, e_2, \ldots)$ with limits 0, 1 and 0,

respectively, and constructed the parameter vectors (p_1, t_1, e_1), (p_1, t_1, e_2), (p_1, t_2, e_2), (p_2, t_2, e_2), etc. Since $G_p(t,e)$ is continuous for $(p,t,e) \in (\frac{1}{2},1) \times [0,1) \times (0,\infty)$, then if $\lim_{p \to 0} \lim_{e \to 0} \lim_{t \to 1}$ (path of equilibria) exists, it will be independent of the choice of a sequence (p_i, t_i, e_i), if the latter is in this domain and has the limit $(0,1,0)$.

The algorithm gives only an approximate limit, but the exact values given in Sections 2 - 4 can be calculated when some of their coordinates and the approximate location of others are known. The three simultaneous nonlinear equations that determine the path of the equilibria can be replaced by approximations that are solvable.

Footnote 3.

Harsanyi and Selten recognize that this type of behavior occurs and reject the idea that equilibria should be continuous in payoffs.

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Figure captions:

Figure 1. One-winner games A1 to A10. Solid lines and dots: Nash equilibria. "X": logarithmic tracing procedure.

Figure 2. The nine possible configurations for wins by the row player.

Figure 3. One-loser games B1 to B10.
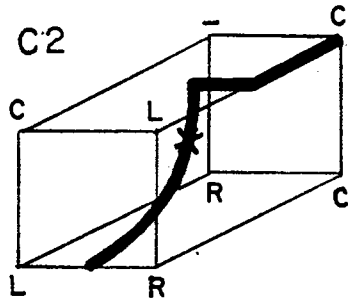
Figure 4. Zero-and-one winner games C1 to C18.

A1

A2

A3

A4

A5

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A7

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A9

A10

B1

B2

B3

B4

B5

B6

B7

B8

B9

B10

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C6

C12

C18
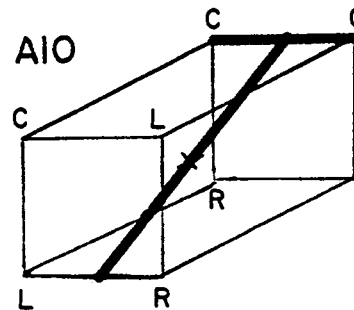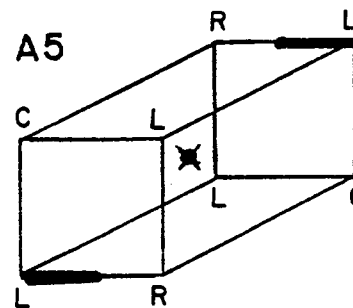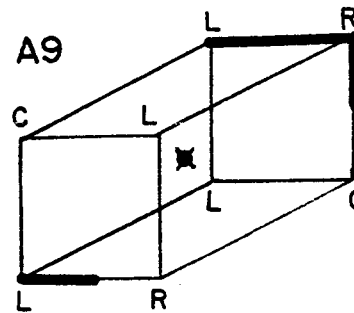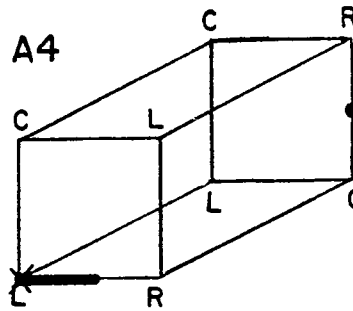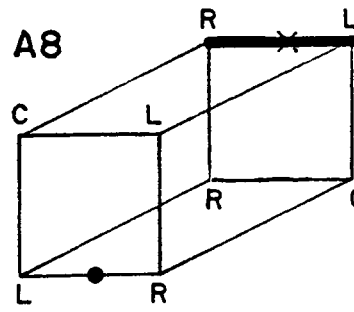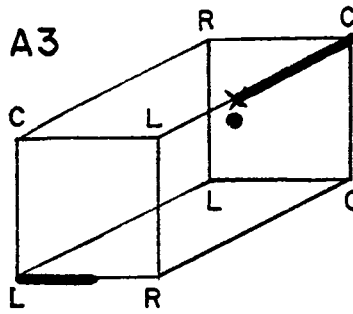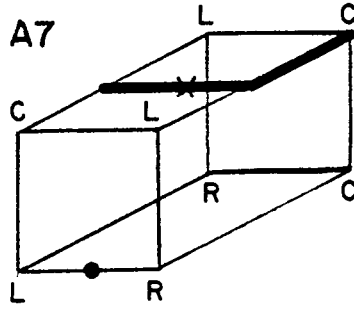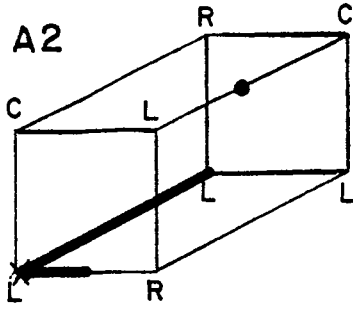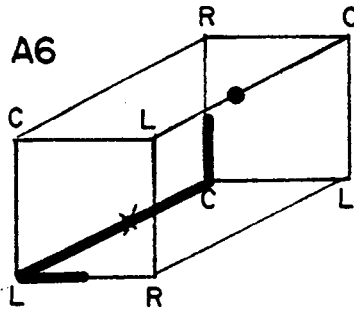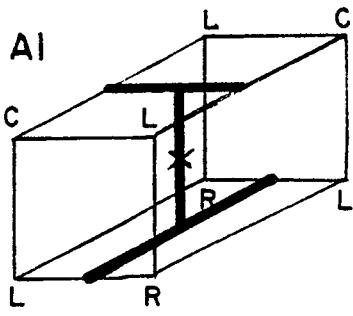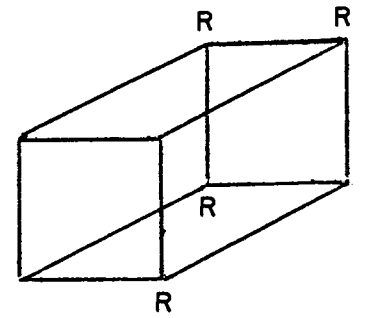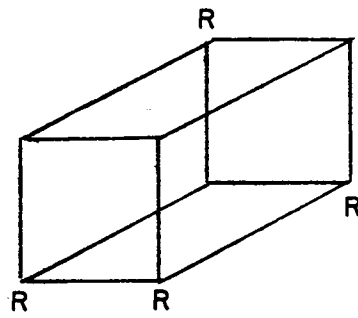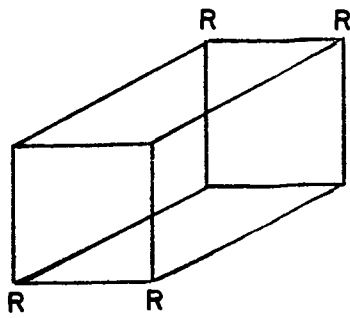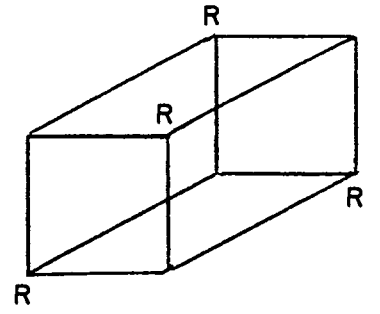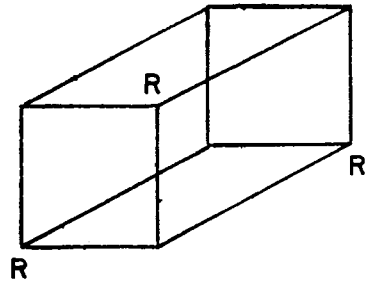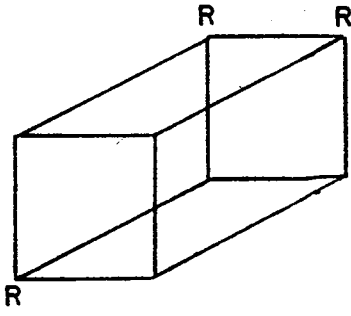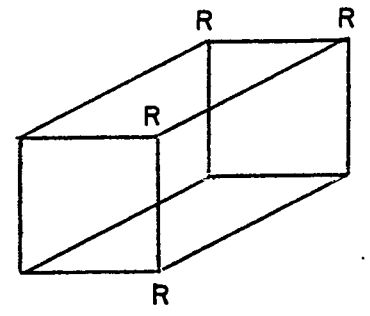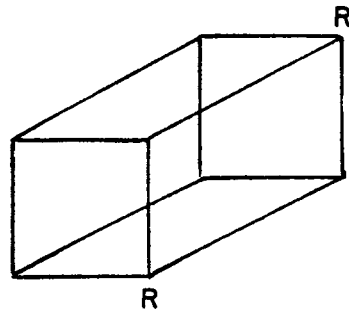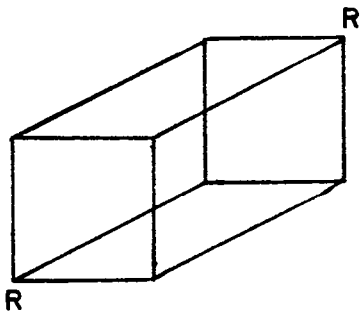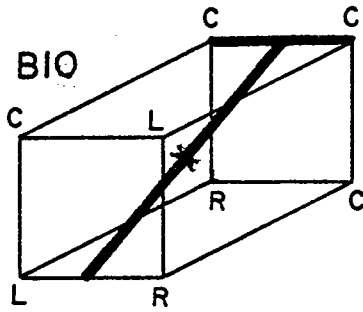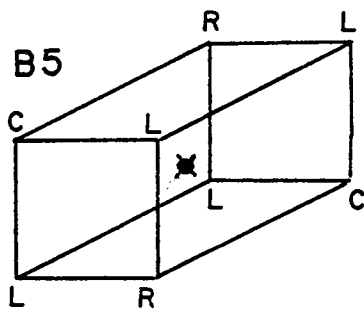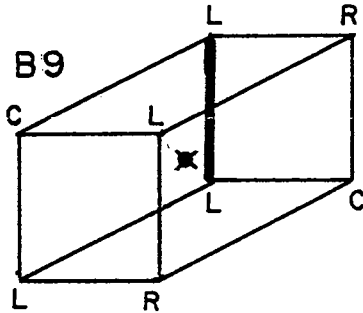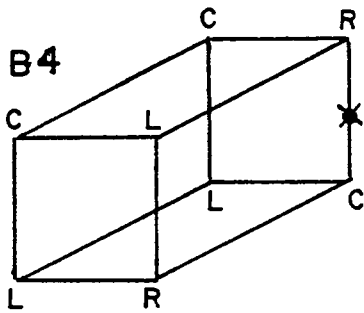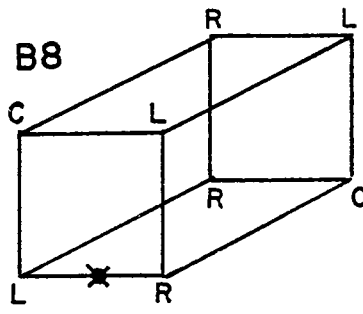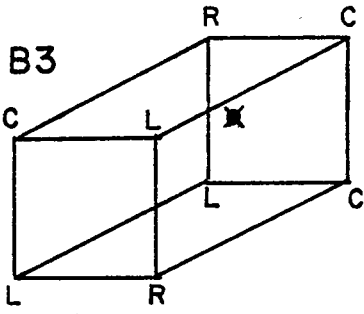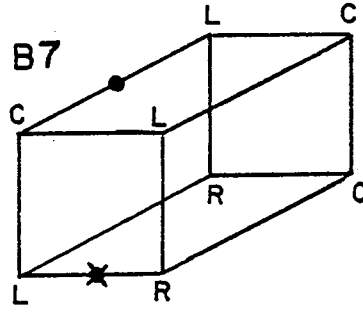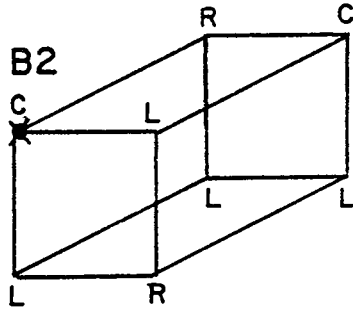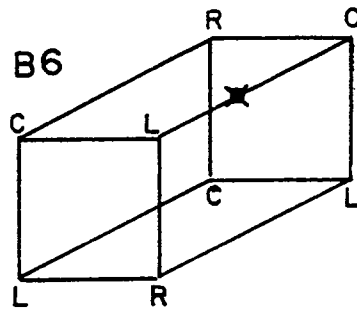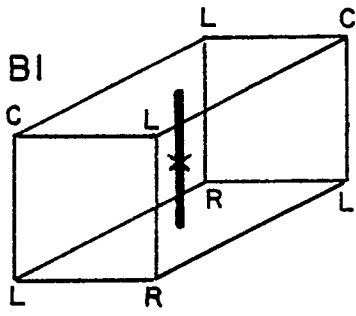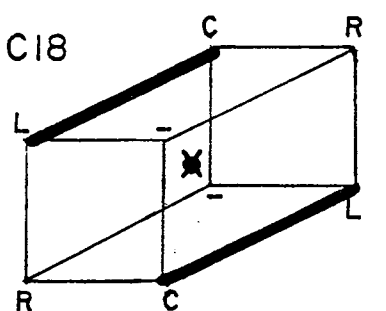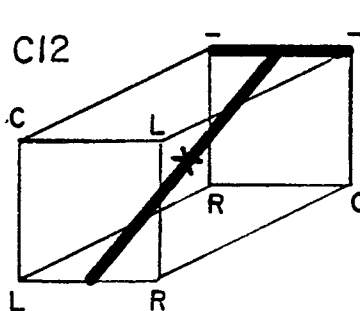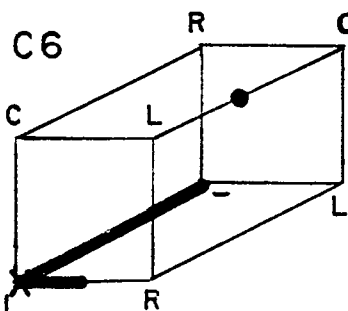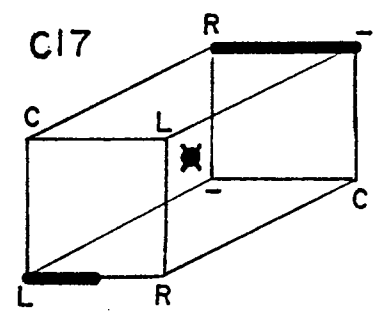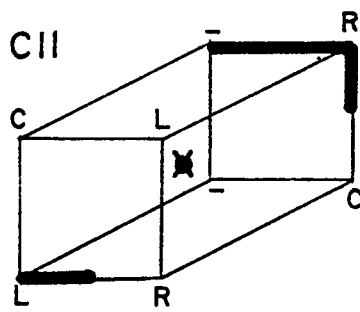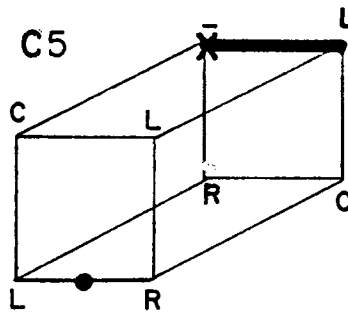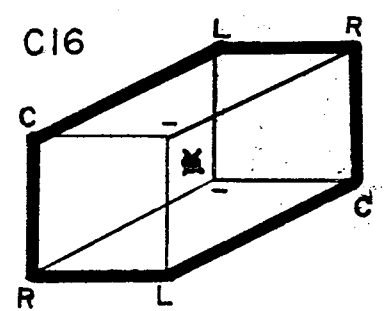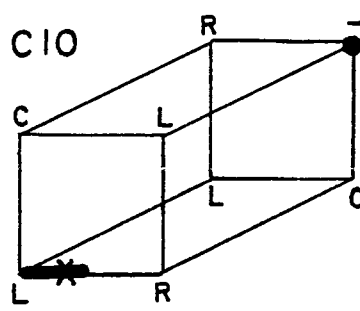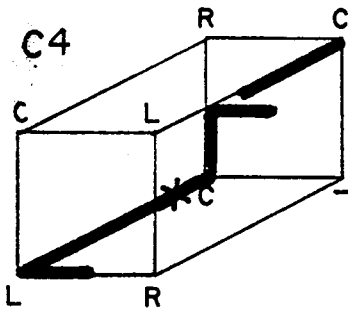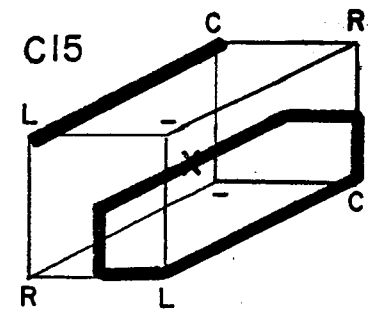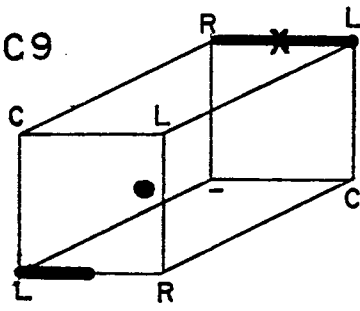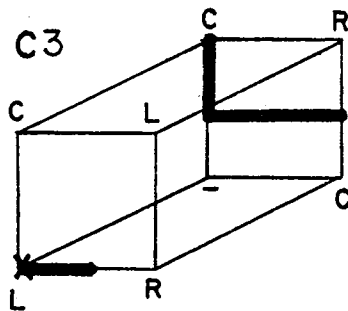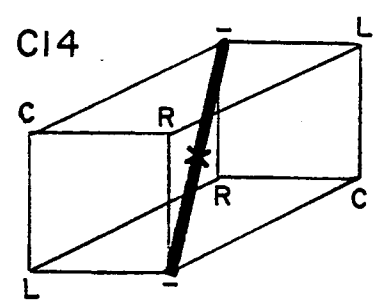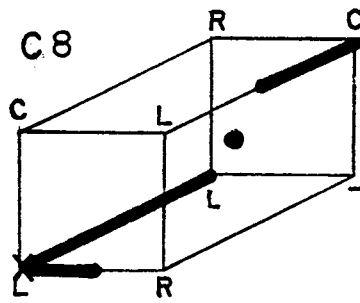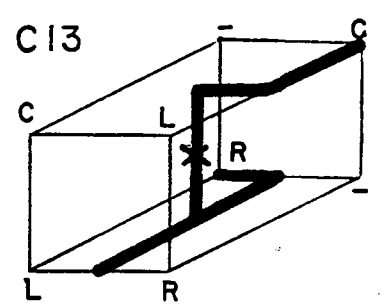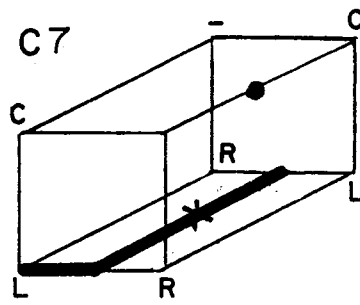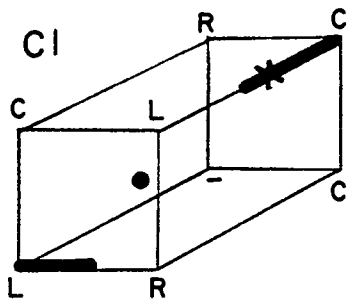
One-winner games

A1: $(\frac{1}{2},\frac{1}{2},\ell)$, $\ell \in [0,1]$

A2: $(r,0,0)$, $r \in [\frac{1}{2},1]$

A3: $(2/(5-\sqrt{5}),(-1+\sqrt{5})/2,(-1+\sqrt{5})/2)$

    $(r,1,1)$, $r \in [0,\frac{1}{2}]$

A4: $(1,0,0)$

A5: same as set of Nash equilibria

A6: $(r,0,0)$, $r \in [0,1]$

A7: $(\frac{1}{2},\frac{1}{2},1)$

A8: $(0,c,1)$, $c \in [0,1]$

A9: $(0,c,1)$, $c \in [\frac{1}{2},1]$

    $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$

    $(1,c,0)$, $c \in [0,\frac{1}{2}]$

A10: $(r,\frac{1}{2},1-r)$, $r \in [0,1]$

One-loser games

B1: $(\frac{1}{2},\frac{1}{2},\ell)$, $\ell \in [0,1]$

B2: same as the Nash equilibrium

B3: same as the Nash equilibrium

B4: same as the Nash equilibrium

B5: same as the Nash equilibrium

B6: same as the Nash equilibrium

B7: $(1,\frac{1}{2},0)$

B8: same as the Nash equilibrium

B9: $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$

B10: $(r,\frac{1}{2},1-r)$, $r \in [0,1]$

Zero-and-one-winner games

C1: $(r,1,1)$, $r \in [0,\frac{1}{2}]$

    $(2/3,\frac{1}{2},\frac{1}{2})$

    $(0,0,1)$

C2: $(\frac{1}{2},r,(1-r)/r)$, $r \in [\frac{1}{2},1]$

C3: $(1,0,0)$

C4: $(r,0,0)$, $r \in [0,1]$

    $(r,1,1)$, $r \in [0,1]$

C5: $(0,c,1)$, $c \in [0,\frac{1}{2}]$

C6: $(r,0,0)$, $r \in \left[\frac{1}{2},1\right]$

C7: $(\frac{1}{2},\frac{1}{2},0)$

C8: $(r,1,1)$, $r \in \left[0,\frac{1}{2}\right]$

    $(\frac{1}{2},2/3,\frac{1}{2})$

    $(r,0,0)$, $r \in \left[\frac{1}{2},1\right]$

C9: same as set of Nash equilibria

C10: $(1,c,0)$, $c \in \left[0,\frac{1}{2}\right]$

C11: $(0,c,1)$, $c \in \left[\frac{1}{2},1\right]$

    $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$

    $(1,c,0)$, $c \in \left[0,\frac{1}{2}\right]$

C12: $(r,\frac{1}{2},1-r)$, $r \in \left[0,1\right]$

C13: $(\frac{1}{2},\frac{1}{2},\ell)$, $\ell \in \left[0,1\right]$

C14: same as set of Nash equilibria

C15: $(r,1,0)$, $r \in \left[0,1\right]$

    $(r,\frac{1}{2},\frac{1}{2})$, $r \in \left[0,1\right]$

    $(r,0,1)$, $r \in \left[0,1\right]$

C16: same as set of Nash equilibria

C17: $(0,c,1)$, $c \in \left[\frac{1}{2},1\right]$

    $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$

    $(1,c,0)$, $c \in \left[0,\frac{1}{2}\right]$

C18: same as set of Nash equilibria

Table 1: Uniformly perfect equilibria for the thirty-eight games.