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Duopoly, Conjectural Variations and Supergames*

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Abstract

Cournot's assumption of zero conjectural variation in a duopoly game has been extensively criticized in the economic literature. Recently notions of equilibrium with non-zero conjectural variation have been suggested. However, these notions were largely defined for one shot simultaneous move games and in such setups the idea of variation or reaction to other players' changes in outputs is meaningless. This paper supplies a game theoretic foundation for notions of conjectural variations other than zero in the context of infinitely repeated games. It is demonstrated that in the infinitely repeated duopoly game, large families of conjectural variations Nash equilibria exist and that their performance agrees with economic intuition. The results presented here coincide with and generalize those of Smithies and Savage in their treatment of the one shot duopoly game using a differential equation and calculus of variations approach.

DUOPOLY, CONJECTURAL VARIATIONS AND SUPERGAMES

Ehud Kalai and William Stanford

The central focus of this paper is on equilibrium in a duopoly game. Our model follows the work of Cournot (1), in that it is characterized by a homogeneous product and pure competition on the buyers' side of the market. We consider only the case of duopoly, where the decision variable for each firm is the amount of the homogeneous good to produce.

Beginning perhaps with Bowley (2), the study of the oligopoly problem has been intimately related to the idea of conjectured reaction functions and their derivatives. These derivatives capture the notion of each firm's beliefs about its rival's response to changes in the firm's output level. These beliefs should enter into the firm's decision process for its output. This dependence gives rise naturally to the concept of reaction functions for the firms, each reacting optimally to the conjectured response of the other. The firm's belief or conjecture about the slope of its rival's reaction function was named the conjectural variation by Frisch (3).

Thus constant conjectural variations, for example, are associated with linear reaction functions. Conjectural variations of zero are the Cournot assumption. In this case, each firm maximizes myopically under the assumption that its rival's output will remain fixed. Kamien and Schwartz (4) have effectively criticized the Cournot assumption on several grounds, and have studied the implications of nonzero conjectural variations. Bresnahan (5), among others, has also studied this problem.

Difficulties remain, however, with the definition of conjectural variations. As outlined above, the idea has much intuitive content, but its foundation would be more secure if some precise context of action and reaction for the firms were presecribed. The purpose of the present paper is to build a secure game theoretic foundation for the notion of conjectural variations and to present some results on equilibrium in the duopoly game.

It appears that conjectural variations have no meaning in a static, single-period setting. If the firms are to choose production levels simultaneously just once, we see the necessity for action, but there is no opportunity for reaction. The Stackelberg (6) leader-follower model allows for a degree of reaction, but it is unclear what singles out one of the firms as leader and the other as follower.

Thus, we naturally turn to a dynamic setting where the firms meet in competition repeatedly. Friedman (7), among others, has considered dynamic reaction function models, and supergame models. The equilibrium concept he uses in his reaction function models is not, however, that of the usual Nash equilibrium. For him, an equilibrium is a set of reaction function for the firms to which there corresponds a set of "conjectured" reaction functions such that the actual functions are best replies to the conjectured functions and at some output vector, both the values and slopes of the actual and conjectured functions are the same. This goes to the heart of "consistent" conjectures, an idea whose implications have been analyzed by Kamien and Schwartz (4), Bresnahan (5), Lainter (8), Boyer and Moreaux (9), and others. Friedman's supergames treatment employs the so-called "grim" strategies to show the existence of supergame Nash equilibria which Pareto dominate the repeated play of the stage game Cournot equilibrium outputs.

In 1940, Smithies and Savage (10) applied the calculus of variations and mixed difference and differential equation theory to the dynamic duopoly problem. They considered the case of identical constant conjectural variations in a finite horizon, undiscounted, continuous time model. Our supergame results will be seen to parallel theirs as we consider the identical constant conjectural variations case. We then extend our consideration to asymmetric linear and non-linear conjectural variations.

The Model

In this work, we consider the supergame consisting of infinitely many repititions of the stage duopoly game. We assume downward sloping linear demand, and identical constant marginal costs for the two firms. The object of each firm is to maximize the sum of discounted profits by choosing at each stage its own output level of the product. We assume that the firms have the same discount parameter, $0 \leq \alpha < 1$. Thus, the profit function for firm 1 at each stage is:

$$\pi_1(q_1,q_2) = q_1(A-B(q_2+q_1)),$$

and for firm 2,

$$\pi_2(q_1,q_2) = q_2(A-B(q_2+q_1)),$$

where $A > 0$, $B > 0$.

It is easy to show that in the stage game, the usual Cournot equilibrium output for each firm is A/(B3), and the monopoly output in this situation is A/(B2). Thus if the firms acted identically in concert to produce the monopoly output, each would produce A/(B4).

At each stage, the firms are allowed to choose quantities $q_i \varepsilon R^+$. A strategy for i in the supergame is a set of functions

$$\{q_{i,t}\}_{t=1}^{\infty} \text{ where } q_{i,1} \varepsilon R^+ \text{ and for } t \geq 2,\ q_{i,t}: \prod_{j=1}^{t-1} [R^+ \times R^+] \to R^+.$$

Thus a supergame strategy is a choice of output at every stage, where each choice is possibly dependent on the outcomes of the preceding games and where both firms know all the choices made by each in the past. The set of supergame strategies of i will be denoted Q_i. Q is the set of pairs of strategies, $Q = Q_1 \times Q_2$.

A Nash equilibrium in the supergame is a strategy pair $q \varepsilon Q$ with the property that for firm 1, for example, a deviation (i.e., playing some $\{\hat{q}_{1,t}\}_{t=1}^{\infty}$ instead of $\{q_{1,t}\}_{t=1}^{\infty}$) cannot strictly increase the firm's sum of discounted profits, given that firm 2 plays $q_{2,t}$ at all stages. Of course, for q to be a Nash equilibrium, we must also be able to make the analogous statement for firm 2.

We begin the study of conjectural variations by considering simple strategies in the supergame discussed above. In particular, instead of depending on the entire history of outcomes, the Tit for Tat strategy defined below will depend only on the outcome at the previous stage.

Definition 1: The supergame strategy defined by $q_{1,1} \varepsilon R^+$ and $q_{1,t} = q_{2,t-1}$ is called the Tit for Tat strategy for player 1 (with initial output $q_{1,1}$).

The Tit for Tat strategy for player 2 can be analogously defined. (Note that there is a slight abuse of notation here, for the RHS of the defining equations ($q_{2,t-1}$) is taken as 2's actual output at stage t-1 rather than its entire stage t-1 game strategy as the notation would indicate.)

Now if 2, for instance, believed that 1 were playing Tit for Tat, then it would be natural to say that the conjectural variation is one, for the slope of 1's reaction function $q_{1,t} = q_{2,t-1}$ is believed by 2 to be one (with respect to changes in 2's output).

Definition 2: The conjectural variations are defined to be one if each firm believes that the other is playing Tit for Tat.

Theorem A: The strategy pair

$$q_{i,1} = \frac{A}{B(3+\alpha)} \qquad i = 1,2$$

$$q_{1,t} = q_{2,t-1}\ ,\ q_{2,t} = q_{1,t-1}$$

is a Nash equilibrium in the duopoly supergame. That is, the initial quantity $\bar{x} = \frac{A}{B(3+\alpha)}$ supports an equilibrium in the supergame if the conjectural variations are one.

Proof: Theorem A is a Corollary to Theorem D.

It is perhaps worthwhile to point out that this strategy pair gives rise to the same output at each stage, namely the initial output $\bar{x} = \frac{A}{B(3+\alpha)}$ for each firm. Also, as α varies between zero and one, this stationary output varies between the stage game Cournot output and half the stage game monopoly output for each firm. In other words, the more heavily the firms weight future considerations, the more closely their combined output will approximate the joint profit maximizing output.

We can now consider extensions which will move toward a definition of constant conjectural variations, which should capture the idea that the conjectured reaction functions are linear.

Definition 3: The conjectural variations are the constant c if the beliefs of the firms about their rival's behavior are consistent with a strategy pair of the form

$$q_{1,1} = x \ , \ q_{2,1} = x$$
$$q_{1,t} = x + c(q_{2,t-1} - x), \qquad q_{2,t} = x + c(q_{1,t-1} - x)$$

As required, the conjectured reaction functions are linear in the firm's output. There are certainly other ways to define linear reaction functions, but this specification seems to offer a favorable combination of tractability and intuition. In particular, if c is positive, each believes that a deviation from the specified output x will elicit a response reminiscent of the Tit for Tat response seen earlier. In general, Tit for Tat carries with it strong implications of punishment for unseemly behavior. This appears to be an effective, if not rational attitude in many situations.

The companion thoerem for this definition is Theorem B, which can easily be seen to generalize Theorem A.

Theorem B: Let $\bar{x} = \frac{A}{B(3+\alpha c)}$. If $0 \leq c \leq 1$, then the strategy pair

$$q_{1,1} = \bar{x} \ , \ q_{2,1} = \bar{x}$$

$$q_{1,t} = \bar{x} + c(q_{2,t-1} - \bar{x}) \ , \qquad q_{2,t} = \bar{x} + c(q_{1,t-1} - \bar{x})$$

is a Nash equilibrium in the duopoly supergame. That is, the initial quantity $\bar{x} = \frac{A}{B(3+\alpha c)}$ supports an equilibrium in the supergame if the conjectural variations are the constant c.

Proof: Theorem B is a corollary to Theorem D.

As in the case of Theorem A, this strategy pair gives rise to the same output at each stage, with each firm repeatedly producing $\bar{x} = \frac{A}{B(3+\alpha c)}$. The intuition behind this equilibrium appears to be strong. Note that, for small conjectural variations c, this equilibrium output will closely approach the Cournot equilibrium output for the firms. Presumably, since small conjectural variations are consistent with the assumption of relatively weak retaliation for increased output, we can think of both firms perceiving the opportunity to produce at levels much larger than the equally shared monopoly outputs. This departs from the usual Cournot assumption of fixed output for rivals to the extent that c exceeds zero.

In Definition 4, we consider the case of asymmetric constant conjectural variations.

<u>Definition 4</u>: The conjectural variations are the constants c_1 and c_2 if the beliefs of the firms about their rival's behavior are consistent with a strategy pair of the form

$$q_{1,1} = x_1, \; q_{2,1} = x_2$$

$$q_{1,t} = x_1 + c_1(q_{2,t-1} - x_2), \qquad q_{2,t} = x_2 + c_2(q_{1,t-1} - x_1)$$

All of the remarks following definition 3 also apply here. In addition suppose, for example, that $c_1 > c_2$. In this case, firm 2 supposes firm 1 is a "tough" opponent relative to firm 1's belief about 2. If we look for initial quantities $\bar{x}_1$ and $\bar{x}_2$ which support the above strategies as an equilibrium in the supergame, we might expect that firm 1 will benefit from this fundamental imbalance in the sense that $\bar{x}_1 > \bar{x}_2$, resulting in greater profits for firm 1 at each stage, over the entire infinite horizon of the game. Intuition also dictates that we should see $\bar{x}_1$ increasing in c_1, and decreasing in c_2. Also, we require that such $(\bar{x}_1, \bar{x}_2)$ quantities should generalize the quantities given in Theorems A and B. the quantities in Theorem C, below, possess all of these characteristics.

Theorem C: Let $\bar{x}_1 = \frac{A(1+\alpha c_1)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]}$, $\bar{x}_2 = \frac{A(1+\alpha c_2)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]}$,

where $c_1, c_2 \varepsilon [0,1]$. Then the strategy pair

$$q_{1,1} = \bar{x}_1 , \; q_{2,1} = \bar{x}_2$$

$$q_{1,t} = \bar{x}_1 + c_1(q_{2,t-1} - \bar{x}_2), \qquad q_{2,t} = \bar{x}_2 + c_2(q_{1,t-1} - \bar{x}_1)$$

is a Nash equilibrium in the duopoly supergame. That is, the initial quantities $(\bar{x}_1, \bar{x}_2)$ support an equilibrium in the supergame if the conjectural vairations are the constants c_1 and c_2.

Proof: Theorem C is a corollary to Theorem D.

As in the earlier cases, this strategy pair gives rise to the same output at each stage, with firm 1 repeatedly producing $\bar{x}_1$, and firm 2 repeatedly producing $\bar{x}_2$. If we take $c_1 = c_2$, the result of Theorem B is obtained. If $c_1 = c_2 = 1$, then Theorem C reduces to Theorem A.

Some results concerning the behavior of $\bar{x}_1$ and $\bar{x}_2$ are given in Proposition 1.

Proposition 1:

1. $\bar{x}_1$ increases in c_1, and decreases in c_2 $\quad \forall\alpha$.
2. If $c_1 > c_2$, then $\bar{x}_1 > \bar{x}_2$ $\quad \forall\alpha$.
3. $A/(2B) \leq \bar{x}_1 + \bar{x}_2 \leq (2A)/3B$ $\quad \forall\alpha, c_1, c_2$.
4. If $c_1 \leq 2c_2$, then $\bar{x}_1$ decreases in α .
5. If $c_1 \leq 2c_2$, then $\bar{x}_1 \leq A/(B3)$ $\quad \forall\alpha$.

6. If $|c_1 - c_2| \leq \frac{1-\alpha^2}{2\alpha}$, then $A/(4B) \leq \bar{x}_1$.

7. If $\alpha \leq 1/2$, then $A/(4B) \leq \bar{x}_1$ $\forall c_1, c_2$.

8. If $c_2 \leq c_1$ then $A/(4B) \leq \bar{x}_1$ $\forall \alpha$.

9. If $c_2 \leq 1/2$ then $A/(4B) \leq \bar{x}_1$ $\forall \alpha, c_1$.

Proof: All proofs are immediate or involve only the obvious differentiation and/or elementary manipulations.

Result 3 shows that the combined equilibrium output is always between the monopoly output and the combined Cournot equilibrium output.

Result 6 shows that for relatively large α, if the conjectural variations are close together, the output for each firm exceeds half the monopoly output.

Result 7 shows, on the other hand, that for small α, independent of the conjectural variations, the output for each firm exceeds half the monopoly output.

Results 4-9 all give sufficient conditions for their conclusions. That none of these conditions are necessary is easy to show by counter-examples. Also, there exist triples (α, c_1, c_2) such that $\bar{x}_1 < \frac{A}{4B}$ and $\bar{x}_2 > \frac{A}{3B}$. For example, if $\alpha = 0.9$, $c_1 = 0.1$, and $c_2 = 0.7$, then $\bar{x}_1 \approx 0.242\frac{A}{B}$ and $\bar{x}_2 \approx 0.362\frac{A}{B}$.

Finally, we consider the case of greatest generality, where asymmetric nonlinear conjectural variations are allowed.

Definition 5: Let $f_i : R^1 \to R^1$ be differentiable , i=1,2. The conjectural variations are the functions f_1' and f_2' if the beliefs of the firms about their rivals' behavior are consistent with the strategy pair:

$$q_{1,1} = x_1, \; q_{2,1} = x_2$$

$$q_{1,t} = x_1 + f_1(q_{2,t-1} - x_2),\ q_{2,t} = x_2 + f_2(q_{1,t-1} - x_1).$$

Theorem D: Let $f_i : R^1 \to R^1$ be twice differentiable, and $c_i = f_i'(0)$ for $i=1,2$. Let

$$\bar{x}_1 = \frac{A(1+\alpha c_1)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]},\quad \bar{x}_2 = \frac{A(1+\alpha c_2)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]}.$$

Assume:

(1) $f_i(0) = 0 \quad i=1,2$

(2) $0 \leq c_i \leq 1 \quad i=1,2$

(3) $0 \leq f_1'(q-\bar{x}_2) \leq 1/\sqrt{\alpha},\ 0 \leq f_2'(q-x_1) \leq 1/\sqrt{\alpha}$ for $q \varepsilon [0, A/B]$.

(4) $0 \leq f_1''(q-\bar{x}_2)\ ,\ 0 \leq f_2''(q-\bar{x}_1)$ for $q \varepsilon R^+$.

Then the strategy pair

$$q_{1,1} = \bar{x}_1\ ,\ q_{2,1} = \bar{x}_2$$

$$q_{1,t} = \bar{x}_1 + f_1(q_{2,t-1} - \bar{x}_2)\ ,\ q_{2,t} = \bar{x}_2 + f_2(q_{1,t-1} - \bar{x}_1)$$

is a Nash equilibrium in the duopoly supergame. That is, the initial quantities $(\bar{x}_1, \bar{x}_2)$ support an equilibrium in the duopoly supergame if the conjectural variations are the functions f_1' and f_2'.

Proof: See Appendix.

All of the remarks following Theorem C and Proposition 1 in its entirety apply to the result of Theorem D.

It is perhaps remarkable that the equilibrium outputs depend on the reaction functions only through their slopes at zero. Of course, the restrictions on the f_i and their derivatives are very stringent; these functions are very well behaved away from zero.

With this fairly large family of equilibrium strategies to consider, we might hope to find some which retain their credibility away from the equilibrium path. This is the concept of subgame perfection, which concerns the infinite period subgame which lies ahead following whatever history of play up to period t occurs. The initial period for the subgame is t+1. A strategy pair q induces in the obvious way, a strategy of pair $\hat{q}$ in any subgame. If we require that the induced strategy pair $\hat{q}$ be an equilibrium in any subgame which could possibly lie ahead in every period, then the strategy pair q is called a subgame perfect strategy pair.

Results in this direction are entirely negative.

Theorem E: Let $0 < \alpha < 1$. Except for $f_1 \equiv f_2 \equiv 0$, none of the equilibrium strategy pairs of Theorem D are subgame perfect equilibrium strategy pairs.

Proof: See Appendix.

Conclusion

The above definitions of conjectural variations seem to be natural and intuitive. That they are also fruitful is demonstrated in the theorems. These theorems indicate the possibility of non-cooperative (i.e., non-collusive) behavior which moves the firms in the direction of Pareto optimality by restricting output and thus increasing payoffs.

We view the model of Smithies and Savage and our work as complementary. In the case of identical constant conjectural variations, both account for essentially the same cooperative behavior--in their case at intermediate stages, and in our case over all values of the time parameter. The asymmetry of conjectural variations we have introduced allows for more reasonable conjectured response behavior, based perhaps on differences of reputation or other factors such as the ability of firms to react to a changing environment.

Appendix

First, we prove two lemmas.

Lemma 1. Let $\tau = (\tau_1, \tau_2, \ldots, \tau_t, \ldots)$ where $\tau_t \varepsilon R^+ \forall t$. Suppose $A > 0$, $B > 0$, $x_1 > 0$, $x_2 > 0$, $0 \leq \alpha < 1$. Let $f:[-x_1, \infty) \to R$ have the property that

$$x_2 + f(\tau_t - x_1) \geq 0 \text{ for } \tau_t \varepsilon R^+.$$

Let $G_N(\tau) = \sum_{t=2}^{N} \alpha^{t-1} \tau_t (A - B(\tau_t + x_2 + f(\tau_{t-1} - x_1)))$.

Then there exists K such that $\forall \tau$, $\infty > K > \overline{\lim}\, G_N(\tau) = \underline{\lim}\, G_N(\tau) \geq -\infty$. That is, $\lim_{N\to\infty} G_N(\tau)$ exits, and is bounded above, independent of τ.

Proof: It is easy to see that the terms $a_t = \tau_t(A - B(\tau_t + x_2 + f(\tau_{t-1} - x_1)))$ are uniformly bounded above on $R^+ \times R^+$. This shows $\overline{\lim}\, G_N(\tau)$ is bounded above, independent of τ.

Suppose $\overline{\lim}\, G_N(\tau) > \underline{\lim}\, G_N(\tau) > -\infty$. Let $\delta > 0$ have the property that $\overline{\lim}\, G_N(\tau) = \underline{\lim}\, G_N(\tau) + \delta$. The terms $G_N(\tau)$ must lie within a neighborhood radius $\delta/3$ centered at $\underline{\lim}\, G_N(\tau)$ infinitely often. Also, the terms $G_N(\tau)$ must lie within a neighborhood of radius $\delta/3$ centered at $\overline{\lim}\, G_N(\tau)$ infinitely often. But is clear that the positive terms of the sequence $\{\alpha^{t-1} a_t\}_{t=2}^{\infty}$ have bounded sum. This yields a contradiction. Thus, in case $\underline{\lim}\, G_N(\tau) > -\infty$, we have $\overline{\lim}\, G_N(\tau) = \underline{\lim}\, G_N(\tau)$. The case where $\underline{\lim}\, G_N(\tau) = -\infty$ is proved by similar contradiction.

Lemma 2: Let $\tau = (\tau_1, \tau_2, \ldots, \tau_t, \ldots)$ where $\tau_t \varepsilon R^+ \forall t$. Suppose $A > 0$, $B > 0$, $x_1 > 0$, $x_2 > 0$, $0 \leq \alpha < 1$. Let $f:[-x_1, \infty) \to R$ be differentiable, with $f' \geq 0$ and with the property that $x_2 + f(\tau_t - x_1) \geq 0$ for $\tau_t \varepsilon R^+$. Let $\bar{\tau}_t = \min\{A/B, \tau_t\}$, $\forall t$. Then

$$\bar{\tau}_1(A-B(\bar{\tau}_1+x_2)) + \sum_{t=2}^{\infty} \alpha^{t-1}\bar{\tau}_t(A-B(\bar{\tau}_t+x_2+f(\bar{\tau}_{t-1}-x_1)))$$

$$\geqslant \tau_1(A-B(\tau_1+x_2)) + \sum_{t=2}^{\infty} \alpha^{t-1}(A-B(\tau_t+x_2+f(\tau_{t-1}-x_1))) .$$

Proof: Both infinite sums exist by Lemma 1. It will be sufficient to show that for arbitrary N,

$$\bar{\tau}_1(A-B(\bar{\tau}_t+x_2)) + \sum_{t=2}^{N} \alpha^{t-1}\bar{\tau}_t(A-B(\bar{\tau}_t+x_2+f(\bar{\tau}_{t-1}-x_1)))$$

$$\geqslant \tau_1(A-B(\tau_1+x_2)) + \sum_{t=2}^{N} \alpha^{t-1}\tau_t(A-B(\tau_t+x_2+f(\tau_{t-1}-x_1))) .$$

Let t^* be the smallest t for which $\tau_t > A/B$, and consider changing only τ_{t^*} to A/B.

Case 1: $t^* = 1$. We must show

$$\frac{A}{B}(A-B(\frac{A}{B} + x_2) + \alpha\tau_2(A-B(\tau_2+x_2+f(A/B - x_1)))$$

$$\geqslant \tau_1(A-B(\tau_1+x_2)) + \alpha\tau_2(A-B(\tau_2+x_2+f(\tau_1-x_1))) .$$

Since $0 > A-B(\frac{A}{B} + x_2) > A-B(\tau_1+x_2)$ and $f(\tau_1-x_1) \geqslant f(A/B-x_1)$, the result is immediate.

The cases where $1 < t^* < N$ and $t^* = N$ are very similar. The result follows by a finite induction type argument.

Theorem D: Let $f_i:R^1 \to R^1$ be twice differentiable, and $c_i = f_i'(0)$ for i=1,2.

Let $$\bar{x}_1 = \frac{A(1+\alpha c_1)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]} , \bar{x}_2 = \frac{A(1+\alpha c_2)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]} .$$

Assume:

1. $f_i(0) = 0 \quad i=1,2$.

2. $0 \leq c_i \leq 1 \quad i=1,2$

3. $0 \leq f_1'(q-\bar{x}_2) \leq 1/\sqrt{\alpha}$, $0 \leq f_2'(q-\bar{x}_1) \leq 1/\sqrt{\alpha}$ for $q \varepsilon [0,A/B]$

4. $0 \leq f_1''(q-\bar{x}_2)$, $0 \leq f_2''(q-\bar{x}_1)$ for $q \varepsilon R^+$.

Then the strategy pair

$$q_{1,1} = \bar{x}_1 , \ q_{2,1} = \bar{x}_2$$

$$q_{1,t} = \bar{x}_1 + f_1(q_{2,t-1} - \bar{x}_2) , \ q_{2,t} = \bar{x}_2 + f_2(q_{1,t-1} - \bar{x}_1)$$

is a Nash equilibrium in the duopoly supergame.

Proof: First, we must show that these strategies are well defined. This means $q_{i,t}:R^+ \to R^+$. Clearly, $x_1, x_2 \varepsilon (0,A/B) \subseteq R^+$. To see, for example, that $q_{1,t}:R^+ \to R^+$, note that by properties of convex functions, $f_1(-x_2) \geq -c_1\bar{x}_2$. Also, f_1 is minimized when $q_{2,t-1} = 0$. Thus:

$$\bar{x}_1 + f_1(q_{2,t-1} - \bar{x}_2) \geq \bar{x}_1 - c_1\bar{x}_2 = \frac{A(1+\alpha c_1 - c_1 - \alpha c_1 c_2)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]} \geq 0 .$$

Turning to the body of the proof, it is clear that these strategies, played one against the other, yield the stationary quantity pair $(\bar{x}_1, \bar{x}_2)$ at each stage. So if firm 1, for instance, has a strategy $\hat{q}_1$ which is strictly better than $q_1 = \{q_{1,t}\}_{t=1}^{\infty}$ against $q_2 = \{q_{2,t}\}_{t=1}^{\infty}$, then $\hat{q}_1$ would have to prescribe an output at some stage which is different from $\bar{x}_1$. In contradiction to this, we show that it is optimal for firm 1 (given that firm 2 plays q_2) to produce $\bar{x}_1$ at each stage.

Given 2's strategy q_2, firm 1 confronts the problem:

$$\max_{q_1,q_2,\ldots} \quad q_1(A-B(q_1+\bar{x}_2)) + \alpha q_2(A-B(q_2+\bar{x}_2+f_2(q_1-x_1))) + \alpha^2 q_3(A-B(q_3+\bar{x}_2+f_2(q_2-\bar{x}_1))) + \alpha^3 q_4(A-B(q_4+\bar{x}_2+f_2(q_3-\bar{x}_1))) + \cdots, \text{ where } q_t \varepsilon R^+ \ \forall t.$$

(Note that we have dropped the first subscript from the $q_{1,t}$'s to simplify notation.) By Lemma 2, it will be sufficient to consider as the feasible set for the above maximization problem the set $\prod_{t=1}^{\infty} [0,A/B]$. That is, we need only consider $q_t \varepsilon [0,A/B]$ $\forall t$.

For any vector of outputs $\tau = (\tau_1,\tau_2,\ldots,\tau_t,\ldots)$, where $0 \leq \tau_t \leq A/B$, let

$$F_N(\tau) = \tau_1(A-B(\tau_1+\bar{x}_2)) + \alpha\tau_2(A-B(\tau_2+\bar{x}_2+f_2(\tau_1-\bar{x}_1)))$$

$$+\alpha^2\tau_3(A-B(\tau_3+\bar{x}_2+f_2(\tau_2-\bar{x}_1)))$$

$$+$$

$$\vdots$$

$$+\alpha^{N-1}\tau_N(A-B(\tau_N+\bar{x}_2+f_2(\tau_{N-1}-\bar{x}_1))), \text{ for } N \geq 2 .$$

Also, define $\bar{F}_N(\tau) = F_N(\tau) - \alpha^N Bf_2'(0)\bar{x}_1\tau_N$, $\bar{q} = (\bar{x}_1,\bar{x}_1,\ldots,\bar{x}_1,\ldots)$, and $F_\infty(\tau) = \lim_{N\to\infty} F_N(\tau)$.

We will show that $F_\infty(\bar{q}) \geq F_\infty(\tau)$ fo all τ, where $0 \leq \tau_t \leq A/B$ $\forall t$. So suppose for some fixed τ, we have $F_\infty(\tau) > F_\infty(\bar{q})$. Let $\delta > 0$ have the property that $F_\infty(\bar{q}) + \delta = F_\infty(\tau)$. By continuity of f_2, there exists a real number m such that $|\tau_t(A-B(\tau_t+\bar{x}_2+f_2(\tau_{t-1}-\bar{x}_1)))| \leq m$ for all $(\tau_{t-1},\tau_t)\varepsilon[0,A/B] \times [0,A/B]$. Choose N_1 such that $n \geq N_1$ implies $m\sum_{i=n}^{\infty}\alpha^i \leq \delta/8$, and N_2 such that $n \geq N_2$ implies $\alpha^n Bf_2'(0)\bar{x}_1\tau_N \leq \delta/8$ for $\tau_N\varepsilon[0,A/B]$. Let $N = \max\{N_1,N_2\}$.

Now, for $n \geq N$, it is clear that

$$|F_\infty(\bar{q})-F_n(\bar{q})| \leq \delta/8 \ , \ |F_\infty(\tau)-F_n(\tau)| \leq \delta/8$$

$$|F_n(\bar{q})-\bar{F}_n(\bar{q})| \leq \delta/8 \ , \ |F_n(\tau)-\bar{F}_n(\tau)| \leq \delta/8 \ .$$

By the Triangle inequality, we thus have

$$|F_\infty(\bar{q})-\bar{F}_n(\bar{q})| \leq \delta/4 \ , \ |F_\infty(\tau)-\bar{F}_n(\tau)| \leq \delta/4, \text{ for } n \geq N. \ *$$

By assumption, we have $F_\infty(\bar{q}) = F_\infty(\tau)-\delta$. By * and the Triangle inequality:

$$|\bar{F}_n(\bar{q})-\bar{F}_n(\tau)+\delta| \leq |\bar{F}_n(\bar{q})-(F_\infty(\tau)-\delta)| + |F_\infty(\tau)-\bar{F}_n(\tau)| \leq \delta/2 \ . \quad **$$

Suppose we can show $\bar{F}_N(\bar{q})-\bar{F}_n(\tau) \geq 0$ for all n. This would give an immediate contradiction to the assumption and establish the theorem, since by **, we would have $\bar{F}_n(\bar{q})-\bar{F}_n(\tau)+\delta \leq \delta/2$, or $\bar{F}_n(\bar{q})-\bar{F}_n(\tau) \leq -\delta/2 < 0$, for $n > N$.

The remainder of the proof consists of showing $\bar{F}_n(\bar{q}) \geq \bar{F}_n(\tau)$ for all n, for all $\tau \varepsilon \prod_{t=1}^{\infty} [0,A/B]$. Recall

$$\begin{aligned}\bar{F}_n(\tau) &= \tau_1(A-B(\tau_1+\bar{x}_2)) \\ &+ \alpha\tau_2(A-B(\tau_2+\bar{x}_2+f_2(\tau_1-\bar{x}_1))) \\ &+\alpha^2\tau_3(A-B(\tau_3+\bar{x}_2+f_2(\tau_2-x_1))) \\ &+ \\ &\vdots \\ &+\alpha^{n-1}\tau_n(A-B(\tau_n+\bar{x}_2+f_2(\tau_{n-1}-\bar{x}_1))) - \alpha^n Bf_2'(0)\bar{x}_1\tau_n \ .\end{aligned}$$

First order conditions are given by:

$$-2B\tau_1+A-B\bar{x}_2-\alpha B\tau_2 f_2'(\tau_1-\bar{x}_1) = 0$$

$$-2B\tau_t+A-B\bar{x}_2-\alpha B\tau_{t+1} f_2'(\tau_t-\bar{x}_1)-Bf_2(\tau_{t-1}-\bar{x}_1) = 0 \qquad t=2,\ldots,n-1$$

$$-2B\tau_n+A-B\bar{x}_2-\alpha B\bar{x}_1 f_2'(0)-Bf_2(\tau_{n-1}-\bar{x}_1) = 0$$

So it must be shown that $\tau_1 = \tau_2 = \dots = \tau_n = \bar{x}_1 = \frac{A(1+\alpha c_1)}{B[(2+\alpha c_2)(2+\alpha c_1)-1]}$

solves

$$-2B\tau_1 - \alpha B c_2 \tau_2 = B\bar{x}_2 - A$$

$$-2B\tau_t - \alpha B c_2 \tau_{t+1} = B\bar{x}_2 - A \qquad t=2,\dots n-1$$

$$-2B\tau_n - \alpha B c_2 \bar{x}_1 = B\bar{x}_2 - A$$

if $\bar{x}_2 = \frac{A(1+\alpha c_2)}{B[(2+\alpha c_2)(2+\alpha c_1)-1]}$. It is clearly enough to show this for the first equation only. To this end, it is straightforward to verify that

$$-B[2+\alpha c_2]\,\frac{A(1+\alpha c_2)}{B[(2+\alpha c_2)(2+\alpha c_1)-1]} = \frac{BA(1+\alpha c_2)}{B[(2+\alpha c_2)(2+\alpha c_1)-1]} - A\,.$$

So it remains to check second order conditions for a maximum. For this, it will be sufficient to show that the associated Hessian matrix, H, evaluated at every $(\tau_1,\dots,\tau_n)$ is negative definite, where $0 \leq \tau_y \leq A/B$ for $t=1,\dots,n$.

The first row of H is given by

$$(-2B\ -\alpha B\tau_2 f_2''(\tau_1-\bar{x}_1))\quad(-\alpha B f_2'(\tau_1-\bar{x}_1))\quad 0\quad \underbrace{0\ .\ .\ .\ 0}_{n-2\text{ zeros}}$$

the t^{th} row of H is

$$\underbrace{0\dots0}_{\text{t2zrs}}\ \ (-\alpha^{t-1}Bf_2^1(\tau_{t-1}-\bar{x}_1))(-\alpha^{t-1}\ 2B-\alpha^t\tau_{t+1}f''(\tau_t-\bar{x}_1))(-\alpha^t Bf'(\tau_t-\bar{x}_1))\ \underbrace{0\dots0}_{n-(t+1)\text{ zeros}}$$

the n^{th} row of H is

$$\underbrace{0 \; . \; . \; . \; 0}_{n\,-\,2\text{ zeros}} \quad (-\alpha^{n-1} B f_2'(\tau_{n-1}-\bar{x}_1)) \quad (-\alpha^{n-1} 2B) \; .$$

Let d_t be the determinate obtained by considering the first t rows and t columns of H, for a fixed but arbitrary $(\tau_1,\ldots,\tau_n)$. It is well known that H is negative definite if and only if d_t has the sign of $(-1)^t$ for $t=1,\ldots,n$. For $t=1,2,\ldots,n-1$, let $\beta_t = \tau_{t+1} f_2''(\tau_t-\bar{x}_1)$, $\gamma_t = f_2'(\tau_t-\bar{x}_1)$. Let $\beta_n = 0$. By standard theorems from determinate theory,

$$d_t = (-1)^t B^t \alpha^{t(t-1)/2} \begin{vmatrix} 2+\alpha\beta_1 & \gamma_1 & & & & & \\ \alpha\gamma_1 & 2+\alpha\beta_2 & \gamma_2 & & & 0 & \\ 0 & \alpha\gamma_2 & 2+\alpha\beta_3 & \gamma_3 & & & \\ & & & \ddots & & & \\ & & & & \gamma_{t-2} & 2+\alpha\beta_{t-1} & \gamma_{t-1} \\ & 0 & & & & \alpha\gamma_{t-1} & 2+\alpha\beta_t \end{vmatrix}$$

for $t=1,2,\ldots,n$.

Let $\bar{d}_t$ be the above determinate. If $\bar{d}_t > 0$ for all t, we will be done. We have $\bar{d}_1 = 2+\alpha\beta_1 \geq 2$, since $\beta_i \geq 0$ for all i.

$$\bar{d}_2 = (2+\alpha\beta_2)\bar{d}_1 - \alpha\gamma_1^2 > \bar{d}_1 \geq 2, \text{ since } \alpha\gamma_1^2 \leq 1 \; .$$

For $t \geq 3$, it is easy to see that

$$\bar{d}_t = (2+\alpha\beta_t)\bar{d}_{t-1} - \alpha\gamma_{t-1}^2 d_{t-2} \; .$$

We have $\bar{d}_2 > \bar{d}_1 > 0$. Suppose for $t \geq 3$, we have $\bar{d}_{t-1} > \bar{d}_{t-2} > 0$. Then

$$\bar{d}_t = (2+\alpha\beta_t)\bar{d}_{t-1} - \alpha\gamma_{t-1}^2\bar{d}_{t-2} \geq (2+\alpha\beta_t)\bar{d}_{t-1}-\bar{d}_{t-2}$$

$$= (1+\alpha\beta_t)\bar{d}_{t-1}+(\bar{d}_{t-1}-\bar{d}_{t-2}) > (1+\alpha\beta_t)\bar{d}_{t-1} \geq d_{t-1}$$

By induction, $\bar{d}_t$ strictly increases in t. Since $\bar{d}_1 > 0$, the theorem is proved.

Theorem: Let $0 < \alpha < 1$. Except for $f_1 \equiv f_2 \equiv 0$, none of the equilibrium strategy pairs of Theorem D are subgame perfect equilibrium strategy pairs.

Proof: Assume $f_1 \not\equiv 0$.

Suppose at time t, firm 1 produces $\bar{x}_1$ and firm 2 produces a large output M, to be specified below. Then in the subgame beginning at time t+1, the following outputs are realized by the induced strategies:

Period	Firm 1	Firm 2
t+1	$\bar{x}_1+f_1(M-\bar{x}_2)$	$\bar{x}_2$
t+2	$\bar{x}_1$	$\bar{x}_2+f_2(f_1(M-\bar{x}_2))$
t+3	$\bar{x}_1+f_1(f_2(f_1(M-\bar{x}_2)))$	$\bar{x}_2$
t+4	$\bar{x}_1$	$\bar{x}_2+f_2(f_1(f_2(f_1(M-x_2))))$
⋮	⋮	

By Lemma 1, firm 1's total payoff beginning at time t+2 is bounded above, independent of M. It is easy to see that $\lim_{M\to\infty} f_1(M-\bar{x}_2) = \infty$. Thus by choosing M large, the total output at time t+1 can be made arbitrarily large. This implies that firm 1's payoff at time t+1 can be made so small that firm 1's total payoff is negative. But firm 1 can achieve a positive payoff by producing $\bar{x}_1$ at t+1 and then following its induced strategy. This proves the theorem.

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