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VECTOR MEASURES ARE OPEN MAPS

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ABSTRACT

N nonatomic vector measures are shown to be open maps from the σ-field on which they are defined to their range, where the σ-field is equipped with the pseudo-metric of the symmetric difference with respect to a given scalar measure.
We prove the following:

**Main Theorem:** Let $\mu_1, \ldots, \mu_n$ be nonatomic, $\sigma$-additive, finite measures on a measurable space $(\Omega, \Sigma)$, and let $\lambda$ be a nonnegative measure. Then the vector measure $\mu = (\mu_1, \ldots, \mu_n)$ is an open map from $\Omega$ to the range of $\mu$, where $\Sigma$ is equipped with the topology induced by the pseudo-metric $d_\lambda$ defined by

$$d_\lambda(S, T) = \lambda(\{S \setminus T \cup T \setminus S\}),$$

and the range of $\mu$ is equipped with its relative topology in $R^n$.

Let us introduce the following notations. For $S$ in $\Sigma$ we denote by $\overline{S}$ the complementary set $\Omega \setminus S$. The symmetric difference of $S$ and $T$, $(S \Delta T)$, is denoted by $\Delta$. The Euclidean norm in $R^n$ is denoted by $\| \cdot \|_2$, and the scalar product of $x$ and $\xi$ in $R^n$ is denoted by $\langle \xi, x \rangle$. By the *relative boundary* of a closed set $K$ in $R^n$ we mean the set of all points in $K$ which are not in the relative interior of $K$. The *face* of a convex set $K$ in the direction $\xi$, i.e., the set $F(\xi) = \{x \in K : \langle \xi, x \rangle = \max \langle \xi, y \rangle \}$, is convex if all the points on the relative boundary of $K$ are extreme, or alternatively if for each $\xi \in R^n, F(\xi)$ is either $K$ or a singleton. For a scalar measure $\lambda$, we denote by $|\lambda|$ the sum of the positive and the negative parts of $\lambda$. For a vector measure $\mu = (\mu_1, \ldots, \mu_n)$, $|\mu|$ is the sum $\sum_{i=1}^{n} |\mu_i|$. For each $S$ we define $R(\mu, S) = \{ \mu(T) | T \subseteq S \}$. Clearly $R(\mu, S) + R(\mu, \overline{S}) = R(\mu, \Omega)$. By Lyapunov's Theorem [1], $R(\mu, S)$ is a convex and compact set.

A convenient way to describe $R(\mu, \Omega)$ is as follows. Let $f_1$ be the Randon-Nikodym derivative of $\mu_1$ with respect to $|\mu|$. Then $\mu(S) = \int_S |f_1| \, d|\mu|$ and for $\xi \in R^n, \langle \xi, \mu(S) \rangle = \int_S \langle \xi, f_1 \rangle \, d|\mu|$. Obviously $\mu(S) \in F(\xi)$ if and only if $\{ t | \langle \xi, f_1(t) \rangle > 0 \} \subseteq S \subseteq \{ t | \langle \xi, f_1(t) \rangle > 0 \}$ almost everywhere with respect to $|\mu|$. It follows then, that $R(\mu, \Omega)$ is strictly convex if and only if, the set $\{ t | \langle \xi, f_1(t) \rangle = 0 \}$ is of $|\mu|$-measure zero for all
supporting hyperplanes $\xi$ which do not contain $R(\mu, I)$, or alternatively if for each subspace $V$ of $R^n$ of dimension smaller than that of $R(\mu, I)$, the set
$$\{t, f(t) \in V\}$$
if of $\nu$-measure zero.

We can prove now:

**Lemma 1** There is a decomposition $R(\mu, I) = \bigcup_i R(\mu, S_i)$ such that $\bigcup_i S_i$ is a partition of $I$ and $R(\mu, S_i)$ is strictly convex for each $i$.

**Proof:** The decomposition is built in $n$ stages. In the stages $1, \ldots, k-1$ a family of disjoint sets $S_{ij}$, $1 \leq j \leq k-1$, $1 \leq i < j$, if defined ($i, j$ is possibly $= 0$) such that $R(\mu, S_{ij})$ is strictly convex and of dimension $j$. Moreover, for each $k-1$ dimensional subspace of $R^n$, $V$, the set $\{t \in \bigcup_i S_{ij}, f(t) \in V\}$ is of $\nu$-measure zero. In the $k$-th stage we define the sets $S^k_i$, $1 \leq i \leq k$ which are all the subsets of $\bigcup_i S_{ij}$ of the form $\{t | f(t) \in V\}$ which have positive $\nu$ measure, where $V$ is a $k$-dimensional subspace of $R^n$. The disjointness of the sets $S^k_i$ can be guaranteed since the intersection of such two sets is a set of $t$'s for which $f(t)$ belongs to a subspace of dimension less than $k$. The strict convexity of $R(\mu, S^k_i)$ follows similarly.

Q.E.D.

Let us call a vector measure $\mu = (\mu_1, \ldots, \mu_n)$ monotonic if each $\mu_i$ is either nonnegative or nonpositive. We will show now that it suffices to prove the Main Theorem for monotonic $\mu$ with strictly convex range $R(\mu, I)$. Indeed, there is a partition $I = \bigcup_i I_i$ such that the restriction of $\mu$ to each $I_i$ is monotonic. We can decompose, furthermore, each $I_i$
according to Lemma 1, to get eventually a partition \( I = \bigcup_{i \in I} S_i \) and a decomposition \( R(u, I) = \bigcup_{i \in I} R(u, S_i) \) such that for each \( i \), \( u \) is monotonic on \( S_i \) and \( R(u, S_i) \) is strictly convex. For \( \epsilon > 0 \) and \( S \in I \) denote

\[
Q(S, \epsilon) = \{ T \mid \| T - S \| < \epsilon \} \quad \text{and} \quad \bar{Q}(S, \epsilon) = \{ T \mid \| T - S \| \leq \epsilon \}.
\]

It is easy to verify that the family of sets \( Q(S, \epsilon) \) where \( S \) ranges over \( I \) and \( \epsilon \) ranges over the positive reals, is a basis to the topology induced by \( d_1 \) on \( I \). Moreover \( u(Q(S, \epsilon)) = \bigcup_{S \in I} u(Q_i(S, \epsilon)) \).

But \( u(Q(S, \epsilon)) \subseteq R(u, S_i) \), \( R(u, S_i) \) is strictly convex and the restriction of \( u \) to \( S_i \) is monotonic. Therefore by proving the Main Theorem for monotonic \( u \) with strictly convex range we prove that \( u(Q(S, \epsilon)) \) is relatively open in \( R(u, S_i) \) which says that \( u(Q(S, \epsilon)) \) is relatively open in \( R(u, I) \).

We assume now that \( u \) is monotonic and that \( R(u, I) \) is strictly convex. We start by proving the following lemma.

**Lemma 2.** If \( x_0 = u(S_0) \) then for each \( 1 \leq i < n \) and \( \epsilon > 0 \) the set \( u(S_i \setminus Q(S_0, \epsilon)) \) contains a set \( \{ x \mid x \in R(u, I), \| x - x_0 \| < \delta \} \) for some \( \delta > 0 \).

We first prove the lemma in the case that \( x_0 \) is in the relative interior of \( R(u, I) \), using Lemma 3.

**Lemma 3.** If \( x_0 = u(S_0) \) is in the relative interior of \( R(u, I) \), then the intersection of the relative interiors of \( R(u, S_0) \) and \( R(u, \bar{S}_0) \) is not empty.

**Proof of Lemma 3:** Indeed, if this intersection is empty then there exists a hyperplane which separates the two sets and for at least one of them, say \( R(u, S_0) \), contains only points from its relative boundary. Since \( 0 \in R(u, S_0) \cap R(u, \bar{S}_0) \) we conclude that there exists \( \delta \in \mathbb{R}^n \) such that
\[ \langle x, x \rangle > 0 \] for \( x \in R(\mu, S_0) \) and \( \langle x, x \rangle < 0 \) for \( x \in R(\mu, S'_0) \) and moreover let some \( x \) in the relative interior of \( R(\mu, S_0) \), \( \langle x, x \rangle > 0 \). Now let \( S \in I \) and denote 
\[ S_1 = S \cap S_0, \quad S_2 = S \cap S'_0. \]
We have:
\[ \langle x, x \rangle < 0 < \langle x, u(S_0^1) \rangle = \langle x, u(S_0) \rangle = \langle x, u(S_1) \rangle \]
and therefore:
\[ \langle x, u(S) \rangle = \langle x, u(S_1) \rangle + \langle x, u(S_2) \rangle < \langle x, u(S_0) \rangle. \]
This inequality holds for each \( S \) in \( I \) and moreover, for some \( \delta \) the inequality is strict which shows that \( u(S_0^1) \) is in the relative boundary of \( R(\mu, I) \), contrary to our assumption.

Q.E.D.

**Proof of Lemma 2:** Assume first that \( x_0 \) is in the relative interior of \( R(\mu, I) \). Let \( E_0, E_1 \) and \( E_2 \) be the linear spaces spanned by \( R(\mu, I) \) and \( R(\mu, S_0) \) respectively, and denote by \( E_0, E_1 \) and \( E_2 \) the intersection of the unit ball in \( \mathbb{R}^3 \) with \( E_0, E_1 \) and \( E_2 \) respectively. Since any \( v \in R(\mu, S_0) \cap R(\mu, S'_0) \) we find, using Lemma 3, a point \( w \) which belongs to the relative interiors of both \( R(\mu, S_0) \) and \( R(\mu, S'_0) \) and for which \( \| v + w \| < \frac{\delta}{2} \). Choose now \( 0 < n < \frac{\delta}{2} \) such that \( w + nB_1 \subseteq R(\mu, S_0) \) and \( w + nB_2 \subseteq R(\mu, S'_0) \). Clearly \( E_0 = E_1 + E_2 \) and therefore we can choose \( 0 < \delta < \frac{\delta}{2} \) such that
\[ \delta B_0 \subseteq n(B_2 + B_1) = n(B_2 - B_1). \]
Now let \( x \in R(\mu, I) \) with \( \| x - x_0 \| < \delta \) and denote \( z = x - x_0 \). Since \( x \in \delta B_0 \) there exist \( z_1 \in nB_1 \) and \( z_2 \in nB_2 \) such that
\[ z = z_1 + z_2. \]
There exist also \( S_1 \subseteq S_0 \) and \( S_2 \subseteq S'_0 \) such that \( u(S_1) = w + z_1 \) and \( u(S_2) = w + z_2 \). Define \( S = (S_0 \cap S_1) \cup S_2 \). We have
\[ u(S) = u(S_0^1) + u(S_2) = x_0 - z_1 + z_2 - x_0 + z = x, \]
and using the monotonicity of \( u \),
\[ d(u(S), u(S_0^1)) = d(u(S_0^1) + u(S_2), 0) = d(2w + z_1 + z_2, 0) < \frac{\delta}{2} + 2n < 2 \cdot \frac{\delta}{4} + 2n < \epsilon. \]
We continue now to prove Lemma 2 for \( x_0 \) on the relative boundary of \( R(\mu, I) \).

Consider a sequence \( x_n = u(S_n^1) \) such that \( x_n \rightarrow x_0 \). We will show that
\[ u(S_n \cap S_0^1) + 0 \]
which is more than we need to complete the proof of Lemma 3.
let \( T_n = S \cap S_0 \) and \( T'_n = S_n \setminus S_0 \). Since the sequences \( \mu(T'_n) \) and \( \mu(T''_n) \) belong to the compact sets \( R(\mu, S_0) \) and \( R(\mu, S_0) \), we can assume without loss of generality that \( \mu(T'_n) + \mu(T') \) and \( \mu(T''_n) + \mu(T'') \) where \( T' \subseteq S_0 \) and \( T'' \subseteq S_0 \). It follows that \( \mu(T' \cup T'') = \mu(S_0) \) and since \( R(\mu, I) \) is strictly convex \( T' = \emptyset \) and \( T'' = \emptyset \) almost everywhere with respect to \( \mu \), which shows that

\[
\mu(S_n \setminus S_0) = \mu(S_0) - \mu(T'_n) - \mu(T''_n) \geq 0.
\]

Q.E.D.

To complete the proof of the Main Theorem we have to show that \( d_A \) can replace \( d_{|\mu|} \) in Lemma 2. There is a partition \( I = S_1 \cup S_2 \) of \( I \) such that the restriction of \( \lambda \) to \( S_0 \) is continuous with respect to \( |\lambda| \) and \( |\lambda|(S_2) = 0 \).

Define \( A(S, \varepsilon) = \{ T \mid T \subseteq S, d_A(T, S) < \varepsilon \} \) for \( i = 1, 2 \), and

\[
A(S, \varepsilon) = \{ T_1 \cup T_2 \mid T_i \in A_i(S, \varepsilon), i = 1, 2 \}.
\]

Clearly \( \mu(A(S, \varepsilon)) \) is open in the topology induced by \( d_{|\mu|} \) on the \( \sigma \)-field

\[
\{ T \mid T \subseteq S, T \subseteq S \}.
\]

correspondence by Lemma 2 \( \mu(A(S, \varepsilon)) \) is relatively open in \( R(\mu, I) = R(\mu, I) \).

**Corollary:** The projection \( \pi : \mathbb{R}^{n+1} \to \mathbb{R}^n \) on the first \( n \) coordinates is an open map from \( R((\mu_1, \ldots, \mu_{n+1}), I) \) onto \( R((\mu_1, \ldots, \mu_n), I) \).

**Proof:** Denote \( \mu = (\mu_1, \ldots, \mu_{n+1}) \). Then \( \pi = (\mu_1)_{n+1}^{-1} \). The result follows from the continuity of \( \pi \) and the fact that \( \mu^{-1} \) is an open map.

Finally let us remark that the Main Theorem in an extension of Lemma 2 in [2]. This lemma states for a nonnegative vector measure \( \mu \), that for each \( x \in \text{R}(\mu, I) \) there exists \( S \) with \( \mu(S) = x \) such that any neighborhood of \( S \) (with respect to \( d_\mu \)), is mapped by \( \mu \) to a neighborhood of \( x \). The Main Theorem is used in [3] where the weaker result of [2] is not enough.
REFERENCES

