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THE ULTIMATE OF CHAOS RESULTING FROM WEIGHTED VOTING SYSTEMS

by

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1 Introduction

When a group is to rank order n alternatives, it is common for them to adopt some sort of weighted voting system. Namely, n numbers w(1), w(2), . . . ,w(n) are selected and each individual of the group assigns to each of the alternatives one of these numbers. The number of points cast for each alternative is then totaled. The n alternatives are ordered according to their total point values where the group ordinal ranking of the n alternatives is given either by this ordering or by the reversal of this ranking. For example, for a plurality vote w(1) = 1 while w(i) = 0. The more votes an alternative receives the higher is its group ranking. There are voting systems, notably in some athletic events, where lower values are assigned to higher ranked choices, and the smaller the total assigned to an alternative the higher is its group ranking.

Perhaps most people instinctively realize that the choice of the weights can alter the outcome of the election. For example, for 4 alternatives, should one advocate the use of the weights 5,3,2,1 or the weights

4,2,2,1 and how will the choice affect the outcome? (Several years ago several countries employed different weighting systems to determine which country was the "real" victor of the Olympics. As one might expect, the "conclusions" depended upon the choice of the adopted system.) On the other hand, unless there is a drastic difference between two weighted voting systems, it would seem reasonable to expect only slight differences between the two resulting rankings. This need not be the case. In a recent paper [2], P. Fishburn established the following surprising result. Suppose for n > 2 alternatives there are two different monotone weighted voting systems. (This is a system where larger values of weights are assigned to more favored choices.) Then for any ranking of the alternatives, there exist examples of voter preferences whereby if the voters use the first weighted system then the outcome will be the given ranking of the alternatives, but when they use the second weighted system, the group outcome is the exact reversal! For example, if the four alternatives are a, b, c, and d, then there exist examples of voters so that if they vote with the first system given above then the outcome is a>b>c>d while by using the second system the outcome is d>c>b>a. (The inequality between alternatives, such as d>c, implies that d is preferred to c.)

In this paper we show that the situation can be much worse than implied by the Fishburn article. We do this by extending Fishburn's result in the following three ways:

1) The weighted voting systems need not be monotone

and the same results hold

2) The specified outcomes need not be reversals of each other. More specifically, let A(1) and A(2) correspond to ANY two rankings of the n>2 alternatives. Then for any two different weighted voting systems there exist examples of voter's preferences so that when they use the first system the outcome is A(1), yet if they use the second system then the outcome is A(2)! Ranking A(1) need not be the reversal of A(2).

3) The result holds for more than two voting systems. Suppose we are given j completely different weighted voting systems (what it means for systems to be "completely different" will be defined later). Let A(1), . . . ,A(j) correspond to ANY j rankings of the n alternatives. Then, there exist examples of voters' preferences so that if the voters use the ith weighted voting system, the outcome will be A(i)!

These results are similar in flavor to a result we obtained earlier (Saari[3]) concerning n>2 alternatives and weighted voting systems. In that paper we started with ANY ranking A of the n alternatives. Next, some one alternative is removed and the (n-1) remaining alternatives are reranked into any ranking B. Then for any weighted voting system on the n alternatives and for any weighted voting system on the (n-1) alternatives, it was shown that there exist examples of voters such that when they vote on the n alternatives the outcome is A, yet when they use the second system to rank the (n-1) alternatives the outcome is B. Notice, ranking A and ranking B need not have any relationship to each other. In particular, ranking B need not be a restriction of A. In this current paper, this result will be generalized even further. Start with n>2 alternatives and then in (n-2) steps successively discard an alternative. Let A(j) be an arbitrary ranking of the j alternatives which remain in the (n-j)th step, j = 2, ..., n. For each j, select a weighted voting system to rank the j alternatives. Our new result asserts that there exist examples of voters such that when they are to rank the set of j alternatives using the designated voting system, the outcome is A(j)! Again, as in the original result, these rankings need not have any relationship with one another. No restrictions are imposed upon the weighted voting systems. In particular, the voting system adopted for smaller number of alternatives could be the restriction, or what is proposed as being a natural "projection" of a voting system used for a larger number of alternatives and the conclusion asserting that the resulting group outcomes may have no relationship between each other still holds. This means that this type of paradox cannot be explained away by arguing that an "incorrect" restriction or projected system was used on the smaller set of alternatives, because this result implies that such a "projected" system does not exist.

Finally, we combine both theorems to obtain a theorem which implies that the ultimate of chaos can result from weighted voting systems. Start with sets of 2, 3, ..., n alternatives which are selected in the fashion given above. On the set of j alternatives, select (j-1) "completely different" weighted voting systems. On the set of j alternatives, let A(i,j), i=1, . . ., j-1; j=2, 3,

. . .,n, be a ranking of the set of j alternatives -- there is no restriction on the choice of these rankings. The theorem asserts that there exists examples of voters such that for any choice of i and j, when these voters rank the set of j alternatives using the ith weighted voting system, the resulting ranking is A(i,j). In the choice of the weighted voting systems, we impose a condition that the (j-1) systems selected to rank j alternatives are "completely different" (see Definition 2), but no conditions are imposed between a pair of voting systems selected to rank j and k alternatives.

As one might expect, there is an extensive literature concerned with the theme of weighted voting systems. Some of these references are given in my earlier paper [3], but perhaps a more authoritative source would be Fishburn's paper where he cites and discusses the appropriate articles which predate his result and which contribute to this development.

The approach used here and in our earlier paper appears to be new in the social choice literature. The difficult combinatorics associated with showing this type of result is avoided by posing the problem in a geometric setting. The result then follows from the geometric properties of Euclidean Spaces and the properties of open mappings between them. Furthermore, in the social choice literature examples disproving "what should be" are often constructed by use of the Condorcet triplets. These are three rankings of three alternatives which are arranged in a cyclic order such as a>b>c; b>c>a; and c>a>b. There is a natural generalization of these rankings to what we call "Condorcet n-tuples"; it turns out that their properties play a role in our approach. In fact, one of the goals of this paper is to outline what are the geometric properties associated with these n-tuples which permit them to be the building blocks for "counter-examples". This discussion is at the end of Section 5 where we briefly consider the effects on these paradoxes which result from imposing a restriction on voters' preferences.

In the next section we describe the geometric setting for the problem and state the basic theorems. Section 3 will be devoted to the first part of the proof of these results; the remainder of the proof is in Section 4. In Section 5 we will state some extensions of our basic results. One extension allows indifference among alternatives to be admitted in the group outcome; another permits a wider range of voting systems--systems such as approval voting where for a given ordinal ranking, a voter has several possible ways in which to vote. There is an interesting corollary corresponding to these more general systems which results from the fact that each voter has several different ways of voting; all of which are compatible with the voter's ranking of the alternatives. This means that it is possible to have the same general voting system with the same voters, but any election could yield any one of several arbitrarily different group outcomes; the particular outcome would depend upon the choice made by the voters as how to cast one from their selection of votes.

2. FORMULATION AND STATEMENT OF THEOREMS

Assume there are n>2 alternatives denoted by a(1), a(2), ..., a(n). We start this section by providing a geometric representation for the n! possible ordinal rankings of these alternatives.

Let each coordinate axis of n-dimensional Euclidean space correspond to an alternative. Then a n-vector corresponds (monotonically) to a cardinal ranking of the alternatives when the value of each coordinate is interpreted as representing the intensity of preference for the cooresponding alternative. Thus x(i) > x(j), where x(k) denotes the kth coordinate value, implies that alternative a(i) is preferred to a(j). Similarly, the hyperplane x(i) = x(j) represents indifference between the two alternatives and it divides n-space into two regions each of which denotes the appropriate strict preference between a(i) and a(j). All possible hyperplanes of this type divide n-space into n! cones where each open cone represents a unique ordinal ranking of the a(i)'s which does not admit indifference among alternatives. We will call such an ordinal ranking of the alternatives a "strict ranking". The hyperplanes, and the intersection among hyperplanes, represent ordinal rankings where indifference among alternatives are permitted. All possible complete, transitive, ordinal rankings are thus represented by some region. For example, the line given by scalar multiples of vector e=(1/n)(1,...,1) corresponds to the ranking of complete indifference among all of the alternatives. This line is the intersection of all of the above described hyperplanes.

The above regions which represent the ordinal rankings are equivalence classes of cardinal rankings of the alternatives. The dimension of this representation can be reduced by one in the following manner: Let P(n) be the intersection of the positive orthant of n-space with the hyperplane given by $\sum_{i=1}^{n} v(i) = 1$. P(n) is a (n-1) dimensional simplex, and its intersection with the above construction provides a representation of the ordinal ranking regions on this simplex. P(3), along with its ranking regions, is given in figure 1. We shall use P(n) to represent both the simplex and the simplex divided into the ordinal ranking regions. Notice that the ordinal ranking of complete indifference is always given by point e so we will call e the "complete indifference point".

Also notice that the open regions are symmetrically located about e and that the hyperplanes give a baricentric division of P(n).

FIGURE 1 SHOULD BE PLACED ABOUT HERE

It is in this way we see that the set of transitive ordinal rankings is a (n-1) dimensional set. (This can be established with a more abstract representation for the ordinal rankings, one which is not dependent upon a cardinal representation of the alternatives. However, such abstraction is not required for our present purposes.) This higher dimensionality and the various symmetry groups it permits play an important role in what follows. Indeed, it appears that this higher dimensionality resulting from n>2 is the cause of several of the different anomalies and paradoxes described in the social choice literature. (Saari[4])

Next, we shall describe a class of weighted voting systems. Although the definition is given in terms of vectors, it is what one would expect: either the voters cast the larger of assigned weights for more favored alternatives, or they cast the smaller. The group outcome is determined by the sum cast for each alternative and the ranking is in accordance with the method of assignment

DEFINITION 1 Assume there are n>1 alternatives. A "weighted voting system" which ordinally ranks the n alternatives is one which satisfies the following:

1. Assign n real numbers w(i), i=1,2,...,n, which are not all the same. The n! ways these numbers, or weights, can be permuted to define n-vectors form the n! weight vectors W(k), k=1,...,n!

2. Each weight vector is assigned to one and only one of the n! strict ordinal rankings of the n alternatives. A monotone method is one where each assigned vector lies in the closure of its open ranking region (in n-space) in that algebraically larger weights are assigned to more favored alternatives. A reversed system is one where the reversal of the assigned vector lies in the closure of its open ranking region, that is, algebraically smaller weights are assigned to more favored alternatives. The reversed vector is obtained by multiplying the given vector by the scalar (-1). The assignment of weight vectors must define either a monotone or a reversed system.

3. Each voter casts the weight vector assigned to the region which reflects the voter's strict ordering of the n alternatives. The vector sum is taken of all of the cast weight vectors. For a monotone system, the ranking region in which this sum lies defines the group ordinal ranking. For a reversed system, the group ranking is determined by the location of the reversed sum vector.

The first condition is fairly obvious. If all the weights were the same, then the group outcome would always be complete indifference—this would not create a very interesting system! The requirement that the voting systems must be either monotone or reversed is imposed for convenience of exposition as the results obtained in this paper do not depend upon this condition. Moreover, while there may be applications for systems not satisfying this condition, I do not know what they are. Indeed, the only justification for introducing such a wider class of systems which I currently could conceive would be to show that the type of result presented here does not require any of the usual "Pareto" or "monotonicity" conditions common to the social choice literature. The reader interested in generalizing to this more general setting can find the basic approach outlined in [3].

In sentence 2, the statement that either the assignment process for the monotone or for the reversed system has the weight vectors in the closure of the appropriate ranking class can be easily verified by using the fact that either the weight vector or its reversal corresponds to a particular cardinal ranking of the alternatives. The same approach shows that the weight vectors are on the boundary of these open regions if and only if at least two of the assigned weights have the same values.

It is obvious that adding a fixed multiple of e to all of the weight vectors will not effect the outcome of the election. Therefore we can and will assume that all the weights are non-negative. Furthermore, since the outcome is also invariant with respect to positive scalar multiples of these vectors, WE ASSUME THAT ALL WEIGHT VECTORS LIE IN P(n). This scalar invariance holds true for the sum of the cast vectors, so we scale the sum and assume that it too lies in P(n). This scaling is achieved by dividing the sum vector by the number of voters. In this way the group outcome can be viewed as being a convex combinations of the weight vectors. When we use this interpretation, we impose the restriction that the scalars are all fractions--the common denominator, or some multiple of it corresponds to the number of voters while the numerators, or their appropriate multiple, correspond to the number of voters casting a particular weight vector.

Next, we define what we mean when we say that there are "j completely different weighted voting systems". First the formal definition will be presented, and then we will discuss it.

DEFINITION 2. Let n>1 alternatives be given. Assume that j weighted voting methods are given where for the ith system W(i) is the weight vector corresponding to the ranking a(1)>a(2)>...>a(n). The j voting systems are said to be "completely different" or "value independent" if the j+1 vectors W(1), ..., W(j), e are linearly independent.

The ordinal ranking a(1)> ...>a(n) was chosen for convenience primarily because for a monotone method it results in the weight vector W(i) corresponding to a non-increasing ordering of the weights. However, it will become clear from the proof that any strict ordinal ranking could have been used to reach an equivalent definition. (This is not true had we admitted systems which were neither monotone nor reversed.)

When j=2, the above definition is equivalent to the one used by Fishburn [2] to say that two monotone methods differ. We will use this definition even should one system be monotone and the other a reversed system. By using the arguments given above about the effects of adding multiples of e, it is clear that should any two systems not be different, even is one is reversed and the other is a monotone system, then the results in any election must agree. Therefore, should n=4, then the Borda weight vector W(1)=(4,3,2,1) and the reversed system with weight vector (0,1,2,3) are the same.

In our geometric setting vector e plays a special role because it is orthogonal to the simplex P(n), thus multiples of it do not project onto any of the other regions of P(n). If W is a weight vector, then the vector

W-e lies in the translated simplex P(n) - e and it is orthogonal to e. Consequently, if two weighted voting systems do not differ, then their weight vectors, which correspond to the same strict ordinal ranking, must lie on the same line in P(n)-e passing through point 0. (This follows from the fact that in any region of P(n), one of the weight vectors lies on the hyperplane defined by e and the other weight vector. By what we have shown, this two dimensional hyperplane intersects P(n) in a line passing through e.) The converse is obviously true. Therefore, when we have a system of completely different weighting systems, we can assume that all of the weight vectors for all of the systems lie on the same sphere in P(n) with center e.

If i weighted voting systems are not completely different, then it is easy to see that the outcome of some one system is always given by a fixed linear combination of the (i-1) other systems; thus if the systems were not completely different, the type of result described in the Introduction would not be possible. (Indeed, this observation can be used to show that the theorems which follow are "best possible" for the types of paradoxes which are presented.) Since P(n)-e has dimension (n-1), it follows immediately that there can be at most (n-1) systems in a collection of completely different weighted voting systems. Also, it follows that one can always find (n-1) completely different systems. If n=2, then there can be only one system.

The above statements which derive immediately from linear algebra and from the fact that the rotational symmetries of $P(n)$ have e as the center point, play an important role in the proof of the following theorem

THEOREM 1. For $n>1$ alternatives, assume given a collection of j completely different weighted voting systems. Let $A(1), A(2), \ldots, A(j)$ be any j strict ordinal rankings. There exist examples of voters' preferences such that when the voters use the ith voting system, the group outcome is $A(i)$, $i=1, \ldots, j$.

For $n=2$ the proof of this theorem is obvious as in this case $j=1$, so for what follows we shall assume that $n>2$. If we restrict the voting systems to be monotone, $j=2$ and $A(2)$ to be the reversal of $A(1)$, we obtain Fishburn's Theorem.

To further illustrate this Theorem we consider a special case where $n=4$, where one of the voting systems is a plurality vote, one is the usual Borda method, and the third system has each voter casting one vote for each of the bottom two alternatives. The outcomes for the corollary were selected to demonstrate what can occur.

COROLLARY 1.1 For $n=4$ alternatives, let the weight vectors for three voting methods be $W(1)=(1,0,0,0)$, $W(2)=(4,3,2,1)$, and $W(3)=(0,0,1,1)$. (System 3 is a reversed system.) Let three ordinal rankings of the alternatives be $A(1)=(a(1)>a(2)>a(3)>a(4))$,

$A(2)=(a(4)>a(2)>a(3)>a(1))$; and $A(3)=(a(3)>a(4)>a(1)>a(2))$. There exist examples of voters' preferences so that when the voters use system $W(i)$, the outcome is ranking $A(i)$, $i=1,2,3$.

This theorem shows that for completely different systems, there need not be any relationship whatsoever among the outcomes. The next theorem states that the same holds as one decreases the number of alternatives. This implies that one should not interpret the group outcome as a linear ordering.

THEOREM 2. For $n>2$ alternatives, let $S(j)$, $j=2,3,\ldots,n$, be a subset of j of the alternatives where set $S(j)$ is a subset of $S(k)$ if $j<k$. For each j select a weighted voting system. Let $A(j)$ be some strict ranking of the alternatives in set $S(j)$. Then there exists examples of voters such that for all choices of j, when the voters use the designated voting system on $S(j)$, the outcome is $A(j)$.

The final theorem given in this section combines both of these results.

THEOREM 3. Let sets $S(j)$, $j=2,3,\ldots,n$, be as described in Theorem 2. For j alternatives select and then arbitrarily impose an order on $(j-1)$ completely different voting systems, $j=2,\ldots,n$. Let $A(i,j)$, $i=1,\ldots,j-1$, be any strict ranking of the alternatives in $S(j)$. Then there exist examples of voters such that for all choices of i and j, when the voters rank the elements of $S(j)$ by using the jth [illegible]

To illustrate this theorem, we use only two systems, the Borda count and plurality voting.

COROLLARY 3.1. Let S(j) consist of the alternatives a(1), ..., a(j), j = 3, ..., n. There exist examples of voters such that when they rank the elements of S(j) by use of the Borda count (weight vector (j, j-1, ..., 1)) the group outcome is a(1)>a(2)> ... >a(j) when j is an even integer, and it is the reversal of this ranking when j is a odd integer. On the other hand when the same group of voters rank the elements of S(j) by use of a plurality vote (weight vector (1, 0, ..., 0)) the outcome always is the reversal of the Borda outcome.

The title of this paper refers to all three of these theorems, but in particular to Theorem 3. By chaos, we mean complete disorder or lack of perdictability of the system. The choice the word "chaos" is selected to reflect both this generic usage of the word as well as to provoke comparisons with its technical usage coming from dynamical systems. In the dynamical systems literature, chaos is used to mean the existence of a subsystem of the regular dynamical system which is highly random. The statement of the theorems in this current paper fulfill this requirement. (As an example how this applies to Newton's method for finding zeros of polynomials, see [5].)

3 PROOF OF THEOREMS

The proofs of all three theorems are similar. First we define mappings which describe a scaling of the vector sums cast by the voters. The mappings will be defined in the order the theorems are stated. Next, we shall assert that these mappings are open mappings. From this assertion, it will be shown how the conclusions of the theorems follow. The proof of the crucial assertion that the mappings are open will be proved in Section 4.

Theorem 1.

Assume that A(1), ..., A(j), j<n, are j strict orderings of the n alternatives. Also assume given j completely different voting methods where the corresponding weight vectors are W(1), ..., W(j). There are n! strict ordinal rankings among the n alternatives. Arbitrarily impose an order on these rankings and let W(k,i), i=1, ..., j, k=1, ..., n! denote the weight vector from the ith system

which corresponds to the kth ranking of the alternatives. Let $f: P(n!) \rightarrow (P(n) x \ldots x P(n))$ (this is a j-fold Cartesian product of P(n) with itself) be defined as

$$3.1 \quad f(c(1), \ldots, c(n!)) = (\sum_k c(k)W(k,1), \ldots, \sum_k c(k)W(k,j))$$

Since the W(k,i)'s all lie in P(n) and since the c(k)'s are non-negative numbers which sum to unity, each of the j sums given in the definition of f are convex combinations of the appropriate weight vectors for that system. Therefore each sum lies in P(n); this implies that f is well defined. Function f can be interpreted in the following way: If the c(i)'s are rational numbers with a common denominator, then the denominator can be viewed as being the total number of voters and the numerator of c(k) is the number of voters whose preferences correspond to the kth ranking of the alternatives.

We claim that when all of the c(i)'s equal (1/n!), the image of f is (e, e, ..., e). This choice for the c's corresponds to point e in P(n!), so the claim is that f(e)=(e,...,e). To see why this is so, notice that each alternative is ranked in mth place in precisely (n-1)! of the n! possible orderings of the alternatives. This is true for each m=1,...,n. Now, for any fixed voting system, the ranking of a(i) determines which weight is in the ith coordinate position. Therefore, each of the weights appears in the ith coordinate position in precisely (n-1)! of the n! weight vectors {W(k,1), k=1, ..., n!}. So for this choice of c's, the ith coordinate of the Ith component of the range of f is the sum of the weights for the Ith system times (n-1)!/n!. Since the weights sum to unity, each coordinate in this sum has value 1/n. This completes the proof of the claim.

Fundamental to the proof of this theorem is the following lemma:

Lemma 1. For j completely different weighted voting systems, function f is an open mapping. That is, f maps open sets in P(n!) to open sets in the range.

The proof of this crucial lemma will be given in the next section.

Because e lies on the boundary of all of the open ranking regions Z(i) of P(n), it follows from the definition of the product topology that point (e,...,e) is a boundary point of Z(1)x...xZ(j) where the ith factor Z(i) in this j-fold Cartesian product is an arbitrary open ranking region of P(n).

Consider an open neighborhood U of point e in P(n!). According to the Lemma, set f(U) is an open set V in P(n)x...xP(n) which contains the point (e,...,e). This means that the intersection of V with the j-fold product of the open ranking regions corresponding to A(1),...,A(j), respectively, is a non-empty open set. Because the set of vectors with rational components is dense in P(n!), it follows that there is a vector d in U with the property that f(d) is in this intersection.

Vector d is used to create the asserted example of voters. The total number of voters corresponds to the common denominator of the components of d, while the numerator of the kth component corresponds to the number of voters possessing the kth ranking of the alternatives. By construction when these voters use the ith voting system, the outcome is strict ranking A(i). Only the lemma remains to be proved.

Theorem 2

Assume that the n alternatives are a(1), a(2), ..., a(n) and that set S(j) consists of the first j alternatives. Let A(j), j=2,3, ..., n, be some ranking of the elements of set S(j). Assume that W(j) is the weight vector which defines the weighted voting system to be used to rank the elements of S(j).

In our proof of this theorem, we will need to be more careful in the labeling of the n! strict rankings of the n alternatives. Toward this end, note that on set S(j) there are j! strict ordinal rankings. Each of these rankings are related to (j+1) rankings on the set S(j+1); these are the rankings which correspond to the (j+1) different ways alternative a(j+1) can be positioned within the given ranking. So, for any ranking B(j) on set S(j), call the (j+1) rankings on set S(j+1) which are obtained in this fashion as belonging to the "lift of B(j)". (See [3].)

As in the proof of Theorem 1, we will be defining a series of maps, each of which is intended to capture the voting procedures described in the statement of the theorem. Start with a two-vector on P(2); one of the components is intended to represent the proportion of voters preferring alternative a(1) to a(2) while the other component corresponds to the proportion having the preference a(2)>a(1). Next, take a vector from P(3)xP(3). The three vector corresponding to the first factor P(3) is intended to represent the three elements in the lift of (a(1)>a(2)); each component denotes the fraction of these voters choosing a particular strict ranking in the lift. In the same way, the vector coming from the second factor P(3) denotes the S(3) fractional splitting of the voters preferring a(2) to a(1). Continue this construction from one level to the next; corresponding to each ranking in P(j), we associate a vector from P(j+1). The components of this vector describe the proportions of those voters which have this same P(j) ranking but which possess the various rankings in the lift. Therefore going from the jth to the (j+1)th stage requires a j! fold product of P(j+1) with itself. This gives a (j+1)!! vector where each component corresponds to one of the strict rankings of S(j+1). The mappings given below are defined in accordance with this description.

For each j, j=2,3,...,n, define a mapping g(j) from

$$P(2) \times \left(P(3)\right)^{2!} \times \ldots \times \left(P(j)\right)^{(j-1)!} \longrightarrow P(j)$$

as

3.2 g(j;D(1,2); D(1,3),D(2,3); ...;D(1,j), ... ,D((j-1)',j)) =

$$\sum d(i,1,2)\left(\sum d(y,m,3)\left(\ldots\left(\sum d(k,p,j)W(k,j)\right)\ldots\right)\right)$$

Here, vector D(i,j) = (d(1,i,j), d(2,i,j), ...,d(j,i,j)) is an element of P(j). The order of the summations given in this equation are defined as follows: Each i in D(i,j) corresponds to one of the (j)' ordinal rankings on S(j), call it B(i,j). In Equation 3.2 each coefficient of D(i,j) is a scalar multiple of a summation containing j+1 of the coordinates of vector D(k,j+1). The appropriate choice and indicing of these j+1 terms, d(t,k, j+1), is to correspond to those rankings in the lift of B(i,j). The middle index is to identify the ranking which defines the lift, while the first index describes which element in the lift is being considered. In the innermost summation, each of the coefficients are also multiplied by the weight vector for the appropriate ranking in the lift coming from the proceeding level.

The same argument given above for function f holds to show that g(j,e,e, ...,e) = e. Note, vector e can lie in a different dimensional space with each appearance in this equation.

Because g is linear, it follows that function g(j) is an open mapping if its image contains an open set. To see why this is so, we shall show that this occurs even with the restriction that D(k) = e for k<j, and the (j-1)! different D(i,j)'s remain variables. With this restriction, g(j) is a function of (j)! variables. This mapping is open if and only if its Jacobean Dg is of maximum rank. However, with this restriction, the Jacobean turns out to be equivalent to the Jacobean of one of the component maps of fuction f given in Equation 3.1. Thus the open mapping property of g(j) follows from Lemma 1.

Let V be an open neighborhood of e in P(n). By the continuity of c(n) and the condition that (e,... ,e) is mapped to e, the inverse image of V with respect to g is an open neighborhood U containing (e,e, ... ,e). This and the open mapping property of g (hence g is surjective) implies there is a choice for vectors D(i,n), say H(i,n), such that g(e,e, ...,H(1,n), ... ,H((n-1)!,n)) lies in the non-empty intersection of set V with the open ranking region corresponding to ranking A(n). (Recall, ranking A(n) is the arbitrarily selected ordering of the elements of S(n).) In fact, the inverse image of this new open set in P(n) yields an open neighborhood of point (e,e, ...,e;H(1,n), ...,H((n-1)!,n)). Let U(2)xU(3)x ... xU(n) be a still smaller open set in the domain which is contructed by the (n-2) fold Cartesian product of open sets coming from the factors which define the domain of g. Here U(i), i=2, ..., n-1, is an open neighborhood of (e, ...,e) in the (i-1)! product of P(i) with itself, and U(n) is an open neighborhood of (H(1,n), ..., H((n-1)!,n)) in the (n-1)! fold product of P(n) with itself. Since U(n) is an open neighborhood, we can assume that the coordinates of the point H(i,n) are all rational numbers.

We now continue this argument backwards from one level of alternatives to the next level containing one fewer alternative. Suppose that at level (j+1) we have that the vectors D(i,k), k=1, ...,j, lie in the product U(1)x ... xU(j) which is a neighborhood of point (e,... ,e). Fixing the variables at the first (j-1) levels to be e, we have that g(j) takes open set U(j) to an open neighborhood of e in P(j). This open neighborhood has a non-empty intersection with the open ranking region corresponding to A(j). The inverse image of this open set created by this intersection contains a still smaller product set of open sets of (e, ... ,e) in the (k-1)! fold product of P(k), k=2, ..., j-1, and an open subset of U(j). In this open subset of U(j) select a point with rational components (H(1,j), ..., H((j-1)!,j)). Continue this argument at the j-1 level using the smaller product set of open sets just derived.

According to the construction of set U(2)x. . .x U(n), and the subsequent product subsets of projections of this set, g(j: H(1,2), . . . ,H(1,j), . . .,H((j-1)!,j)) lies in the

open ranking region of P(j) which corresponds to ranking A(j), j=2, . . . ,n. Next we show how to find the appropriate rational equivalent representation of points H(1,2); . . .H((n-1)!,n) so that it provides the example of voters asserted in the statement of the theorem.

The desired rational equivalent representation of these points will be obtained though a finite iterative process.

1. Rewrite vector H(1,2) so that the two components have a common denominator.

2. Each component, h(i,k,j), of H(k,j) corresponds to some ranking, B(i,k,j), of the set S(j), j=2, . . . ,n-1. One of the H(j+1) vectors corresponds to the lift of B(i,k,j), rewrite the coefficients of this vector so that their common denominator is a multiple of the numerator of h(i,k,j).

3. Rewrite each component of the H(j) vectors, j=2, . . . ,n, so that the numerator of each component is the same

as the common denominator of the elements of the appropriate H(j+1) vector corresponding to the lift of B(i,k,j)

After this process is completed, the common denominator of H(1,2) yields the total number of voters in the example. The numerators of the components of the H(i,j) vectors specify the number of voters holding the corresponding ranking of the alternatives in S(j). By construction, when these voters consider j alternatives using the designated voting system, the result if equivalent to evaluating function g(j) at these H points. By construction, this outcome is A(j). This completes the proof of Theorem 2.

Theorem 3.

The proof of this theorem is obtained by simple modifications and combination of ideas from the above two proofs. Assume the j-1 rankings of the alternatives in S(j) have been made and the (j-1) completely different voting methods have been selected. Define a series of mappings G(j) which have the same domain as g(j) but which have range the (j-1) fold Cartesian product of P(j) with itself, j = 2, ..., n. The j-1 different functions defining G(j) are essentially the same as g(j), they differ only in the choice of the j-1 different weight vectors selected to correspond to the different voting systems. The argument showing that g, or f, take (e, ..., e) to e extends directly to G(j). The fact that when all the variables below the ith are fixed at e and G(i) is still an open mapping follows from Lemma 1 and a Jacobean argument similar to the one used for g(i). Thus we have that there is an open set in the domain of G(n) containing e's for all but the nth level which is mapped into the appropriate ranking region in the appropriate factor of the n-1 fold Cartesian product. The same reduction argument as presented in the proof of Theorem 2 further restricts these open sets so that G(j) will have the appopriate image. The rational representation argument is the same as given above.

Thus, the proof of all three theorems are completed once we establish the validity of Lemma 1. This will be done in the next section.

4 PROOF OF LEMMA 1

Function f, given in the statement of Lemma 1, is a linear map where the dimension of the range is j(n-1). For n>2, this dimension is less than that of the domain which is (n! -1). (Here we use the fact that j(n) To prove the lemma, it suffices to show that the Jacobean map Df, which maps the tangent space of P(n!) to the tangent space of the j fold product of P(n) with itself, is surjective. If the weight vectors are W(1), W(2), ...,W(j), then the column vectors of Df are the transpose of the vectors [W(i,1), ... ,W(i,j)] where i ranges from 1 to n! and this index designates which of the n! rankings of S(n) is being considered.

This matrix representation of Df treats f as a mapping from n! dimensional space rather than from the subspace P(n!). Consequently, when the rank of Df is determined, it is equivalent to finding j(n-1) columns which are linearly independent with respect to those scalars which define vectors in the tangent space TP(n!). Since the normal vectors to P(n!) or TP(n!) are scalar multiples of e, the condition imposed on the scalars is that they define vectors orthogonal to e; that is, the sum of these scalars must equal zero.

In order to remove this constraint on the scalars, extend f from P(n!) to n!-space. It is clear that if f is an open mapping on this new domain, it is an open mapping on the restriction to P(n!). The only impact this extension of f has on the range is that the range now includes scalar multiples of any element from the range of the original function f. Thus, this new range is P(n)x ... xP(n)xR where R denotes the real line. In this new setting, f is a linear mapping if and only if Df has rank j(n-1)+1, but since the tangent space is n!-space, no longer are there any constraints on the scalars when this rank condition is being verified.

The same argument used to show that f(e) = (e, ...,e) carries over to demonstrate that a linear combination of the columns of Df yields a column vector which is the transpose of (e ... ,e). Therefore, the problem of showing that Df has maximal rank is the same as finding j(n-1) choices of k such that the resulting set of j(n-1) vectors {W(k,1)-e, ... ,W(k,j)-e } is linearly independent; there are no constraints on the choice of the scalars.

Since we will be considering vectors of the type {(W(1,k)-e, ... ,W(j,k)-e)}, we simplify the notation by denoting such a vector on (P(n) -e)x ... x(P(n)-e) as Z(k). If A is a linear map mapping P(n)-e back into itself, then we extend A to the product space by defining A(Z(k)) to be {A(W(1,k)-e), ... ,A(W(j,k)-e)) }. The linear mappings in which we are interested are the transposition maps M(l,m) which map n-space back onto itself but which interchange the lth and the mth coordinates. It is easy to show that these maps map (P(n) -e) back into itself.

These transposition maps are the mappings which take one Z vector onto another, and for any two Z vectors, there is some combination of transposition mappings which takes one onto the other. (This is because the transposition maps generates the permutation maps.) Therefore, the linear space spanned by the vectors Z(k), k=1,2, . . .,n), is invariant with respect to these mappings, where by being invariant we mean that the linear space is mapped back into itself. We need to show that this space generated by the Z's has dimension j(n-1). The way we do this is to classify the invariant spaces with respect to certain classes of mappings.

DEFINITION 3. Let G be some set of linear mappings which map a vector space VS back into itself. Then L(G) is the set of linear subspaces of VS which are invariant with respect to all the mappings in G.

It is an elementary exercise to show that L(G) is the intersection of the sets L(g) where g ranges over the elements of G. Therefore, the problem is to determine what subspaces are invariant with respect to each linear map g in G. One approach, of course, is to determine the eigenvalues and the corresponding eigenspaces of each mapping g. By using the eigenvectors as a basis for VS, it is fairly easy to compute the invariant subspaces. This is the approach we shall use to analyze the following two examples, these examples are designed to complete the proof of Lemma 1. A different approach can be found in Lemma 2.

Example 1. Let G be the set of all transposition matrices M(k,l), on n-space. We will show that L(G) consists of n-space, (P(n) - e), the space spanned by e, or the zero-dimensional space 0. Once this fact is established, the proof of Lemma 1 follows for j=1. This is because since a weight vector cannot be a multiple of e, the L(G) invariant subspace spanned by (W(k,i) - e), k=1,...,n), must contain (P(n) - e). Thus, establishing the claim of this example completes the proof of Theorem 2.

In the proof of this claim, it will be useful to have a basis for the linear space spanned by (P(n) - e). This is easy to determine because the n vertices of the simplex P(n), e(i), i=1,...,n, are linearly independent. (Here, vector e(i) is the unit vector which has zero for all components but the ith, and that component is unity.) Therefore, the n-1 vectors e(1) - e(i), i=2,...,n, are linearly independent and they lie in this translated, space. Consequently, they will serve as a basis for P(n) - e

The eigenvalues of M(k,l) are simple eigenvalue -1 and eigenvalue 1 with multiplicity (n-1). The eigenspace corresponding to -1 is the space spanned by the vector having zero in all components except the kth and the lth. For these components, the values are the same, but they differ in sign. Denote this vector by (e(k), -e(l)). The eigenspace corresponding to eigenvalue 1 is the n-1 dimensional space orthogonal to (e(k), -e(l)). From this eigenspace decomposition, it follows that any element of L(M(k l)) is a linear subspace which is either orthogonal to the distinguished eigenvector, or it includes it.

Let V be an element of L(G), and project V onto the subspace spanned by {e(k), -e(j)}. Since the span of the space generated by (P(n) - e) consists of vectors of this type, it follows that should this projection be 0 for for all choices of k and j, then V is either 0 or the one-dimensional space spanned by e. Therefore, assume for some choice of k and j that this projection is non-trivial. By what we have shown above, this means that V contains this vector. Let l be any other index. According to the definition of L(G), subspace V must be invariant with respect to M(k,l) and M(j,l). But M(k,l)({e(k), -e(j)}) = {e(l), -e(j)} and M(j,l)({e(k), -e(j)}) = {e(k), -e(l)}. By continuing this argument, it is clear that V contains (P(n) -e). This means that either it is this space, or it is n- space. In either case, this completes the proof of the claim.

Example 2. Let the vector space be the j-fold Cartesian product of n-space with itself. The set of mappings we will consider are the same as in Example 1, the transposition mappings from n-space back into itself. Here we will use the mappings according to the convention described above: namely, any mapping is applied to all j factors of this j-fold product. (This is equivalent to choosing the diagonal of the j-fold product of set G with itself.) Rather than determining all of the elements of L(G), we will classify only enough of the elements so that we can prove Lemma 1.

For M(i,k), the eigenvalue -1 has multiplicity j while the eigenvalue 1 has multiplicity j(n-1). A basis for the j-dimensional eigenspace corresponding to eigenvalue -1 can be constructed with the products of j-1 zero vectors (from n-space) with vector {e(j), -e(k)}; for each of the j vectors, this non-zero n-vector component corresponds to a different factor of the j-fold product. The orthogonal complement of this space is the j(n-1) dimensional eigenspace for eigenvalue 1. Any subspace invariant with respect to M(j,k) must either be

orthogonal to the (-1) eigenspace, or it must contain a linear subspace coming from it.

Let V be an element of L(G). Now, it is fairly easy to show that if the dimension of V lies between j(n-1) and jn, then V is the j-fold product of n-space and (P(n) -e). The number of factors (in this Cartesian product) which correspond to the last space equals the difference between jn and the dimension of V. Therefore, assume that the dimension of V is less than that of j(n-1). Furthermore, because our goal is the completion of the proof of Lemma 1, we will be concerned with the special case where V is a subspace of the j-fold Cartesian product of (P(n) - e).

If the projection of V onto the i-fold product of the space spanned by {e(k), -e(l)} is trivial for all choices of k, l, then V is orthogonal to our desired product space. So, assume for some choice of indices that this projection is non-trivial. If this projection contains the i-dimensional eigenspace corresponding to eigenvalue -1, then the argument given for Example 1 applies to show that V contains the i-fold product of (P(n) -e).

Next, assume that this projection of V on the M(k,l) -1 eigenspace is a d-dimensional space where D(i;k,l), i=1, . . , d < i is the basis for the subspace. Each vector D(i;k,l) is determined by the coeficients in its representation in terms of the distinguished eigenvectors. The coefficients define a dxi matrix. Using standard linear algebra tools, in particular the techniques related to row reduction of matrices, we can assume that the basis was selected so that the matrix is in a row reduced form. Furthermore, in order to simplfy the exposition, assume that the indicing is such that the first d columns yields a dxd identity matrix. Now, for any other index, m, we have M(m,l)(D(i;k,l)) = D(i;m,l), a vector which is determined by the same coefficients. In this way we have the basis for the projection of V onto the various subspaces. The argument used in Example 1, and the basis of (P(n) -e), apply to show that the first d components of V are given by the d-fold Cartesian product of (P(n) - e) with itself. Indeed, by this choice of the basis vectors, it follows directly that the subspace determined by these D's has dimension d(n-1) and it has the graph representation (J(1), J(2), ...,J(d), J(d+1), ...,J(j)) where J(i), i= 1,...,d, is an arbitrary element of (P(n) - e) and where J(k), k= d+1, ...,j is a specific linear combination of the first d vectors.

We are now prepared to return to Lemma 1. The weight vectors coming from the j completely different voting systems define vector Z(k), k= 1, ..., n!, which lies in a L(G) invariant subspace V of the j-fold product of (P(n)-e). These subspaces are described above. Since the n components of Z(1) are linearly independent, V can only be the total product space, which has dimension j(n-1). This completes the proof of Lemma 1, and consequently, this completes the proofs of the three theorems.

5 SOME EXTENSIONS AND THE CONDORCET N-TUPLE

The purpose of this section is to extend the three theorems of Section 2 so that they apply to most weighted voting systems. Then, we conclude this section and the paper by briefly discussing the effects of imposing restrictions on voters' preferences; some restrictions still admit the conclusions of the theorems while others will admit only a subset of these results.

In the statement of the three theorems, we required the group outcomes to be strict rankings of the n, or j<n, alternatives. This condition was used in two places in the proof; one was for convenience of exposition and the other was necessary to derive the general result. The reason we imposed this condition is that strict rankings define open regions in the ranking space P(n). We used this and the surjectivity of mappings f and g(j) to show that the inverse image of the appropriate ranking region is a non-empty subset of the domain. This use of the "strict rankings" is for convenience because even if the group ranking contained indifference between alternatives,

the surjectivity of the mappings would ensure that the inverse image of the ranking regions are non-empty. Although these inverse images no longer are open sets (they are parts of linear subspaces which have the same codimension as the ranking region in P(n)), straight-forward arguments employing the continuity and surjectivity of these mappings can be used to show that there exist points in the domain which satisfy all but possibly one of the properties described in the proofs. The one property which does not necessarily follow (without having the flexibility of an open set in the domain) is that it is possible to choose appropriate domain points which have rational components.

The reason a problem occurs at this point is that by admitting indifferences, the domain points must now satisfy linear equations which involve the weights. If the weights are rationally independent scalars, then the appropriate rational solution in the domain may not exist. On the other hand, I have never encountered a weighted voting system used in the "real world" where the weights are irrational numbers! Consequently, even though the conditions imposed on the following theorem are far more strict than necessary to obtain the conclusion, the results most likely apply to all practical voting systems.

THEOREM 4. In addition to the conditions imposed on the weighted voting systems in Theorems 1,2,3, assume that all of the weights are rational numbers. Then, the respective conclusions of these theorems hold even if the group rankings admit indifference among alternatives.

We will illustrate this theorem with the following statement which follows from Theorem 2 and Theorem 4.

COROLLARY 4.1. Assume given five alternatives a,b,c, d and f. Then there exist examples of voters so that when they rank a and b by means of plurality vote the outcome is complete indifference; when they rank a,b, and c with the Borda count, the outcome is a>c>b; when they rank the four alternatives a, b, c, and d with a plurality vote, the outcome is b=d > c>a; yet when they rank all five alternatives with the reversed system having weight vector (1, 2, 3, 4, 5), the outcome is a> f = d > b = d

A second way in which these theorems can be extended is to allow the voters to admit indifferences in their preferences. Of course, this is only meaningful should the voting system admit weight vectors which reflect this indifference among certain alternatives. This means that we are introducing additional weight vectors to correspond to all of the classes in the ranking regions of P(n); not just the open ranking regions. But, if we are going to introduce additional weight vectors, we might as well handle other situations where more weight vectors are available. In particular, I am referring to voting systems such as approval voting where each voter casts the weight of unity for each approved alternative, or cardinal preference voting where each voter determines the weight to be cast for each alternative in accordance with the proportions of intensity of preference among the alternatives. What characterizes these systems is that there may be more than one weight vector corresponding to each ranking region of P(n).

Definition 4. Assume there are n>1 alternatives. A general weighted voting system is one where at least one weight vector is assigned to each of the open ranking regions of P(n). Not all of these weight vectors assigned to an open ranking regions are to be scalar multiples of e. Furthermore, it is required that the assignment of the weight vectors to the open regions are either all monotone or reversed, as defined in Definition 1.

THEOREM 5. For Theorems 1,2,3,and 4, assume that the described weighted voting systems are replaced with general weighted voting systems admitting a subsystem of weight vectors satisfying both Definition 1 and the conditions of the respective Theorems. Then the conclusions of the respective theorems apply.

To prove this theorem, one modifies the definition of the mappings f and g(n) to admit the additional possibilities. Then, the conclusion of Theorem 5 follows because we have established that it holds for a restriction imposed upon the same mappings.

A general voting system can contain more than one weighted voting system which satisfy the conditions of Definition 1. An example of this would be a common voting method to determine the membership of a committee--a voter is told to vote for no more than a certain number of candidates. For example, suppose there are five candidates and you can vote for no more than three of them. This means that contained within the general system are three "completely different" weighted voting systems defined by the weight vectors (1, 0, 0, 0, 0), (1, 1, 0, 0, 0) or (1, 1, 1, 0, 0). Consequently, from Theorems 1, 4 and 5, for any three rankings of the five alternatives, there exists examples of voters where the outcome could be any of these rankings, and the ranking actually selected depends on how the voters cast their possible ballots. This is an illustration of the following Corollary of these three theorems.

COROLLARY 5.1. Let $n > 1$ alternatives be given. Assume given a general voting system which includes as subsystems j completely different weighed voting systems. Let A(1), A(2), ..., A(j) be j rankings of the n alternatives. Then there exist examples of voters where any of these outcomes can occur when they use the general voting system to rank the alternatives, the particular outcome depends upon how the voters choose to cast one of their several possible weight vectors.

We now turn from the question of extensions to the issue of restrictions of the possible voters' preferences. Implicit in the statement of the theorems and explicit in their proofs is the requirement that there are no restrictions on voters' preferences. A natural question would be to determine what types of restrictions will still admit the conclusions of all these theorems and what types will not. From a technical point of view, the answer to this question will add further insight into the reasons for these various voting paradoxes.

Since the proof of all three theorems depends upon the proof of Lemma 1, this is the first place this question should be examined. Phrased in terms of this lemma and the tool used to prove it, the restrictions on voters' preferences defines a new set G of mappings which are to be kept invariant, and we wish to find when is and when isn't L(G) the appropriate full space. The second technical condition we need to prove the theorem is the symmetry one which requires that some interior point of the domain is mapped to e of the range. What we illustrate next is an important symmetrical set of preferences which satisfy both conditions. As a result, it is should not be surprising that variants of them are found in several "counter-examples" or as the building blocks for "paradoxes" in the social choice literature.

Because they are natural extensions of the set of three rankings on three alternatives used by Condorcet, we call them "Condorcet n-tuples" (See [3,4].)

DEFINITION 5 Assume there are n>1 alternatives. Let B be some ranking of the alternatives b(1)>...>b(n) where each b(i) is some alternative a(k). The following set of n rankings of the alternatives is called the Condorcet n-tuple generated by B. {B; b(n)>b(1)>...>b(n-1); b(n-1)>b(n)>b(1)>...>b(n-2); ...; b(2)>b(3)>...>b(n)>b(1)} To get from one ranking to the next in this n-tuple, each alternative is lowered one position in the ranking while the least favored alternative now becomes the most favored alternative.

If n=2, then the Condorcet 2-tuple is(a>b; b>a) where a and b are the two alternatives. Since the Condorcet n-tuple is defined in a cyclic fashion, it seems resonable to expect the associated weight vectors to share a similar property. To see that they do, let P be the permutation matrix ((p(i,j))) where p(i,j) = 1 if j= i+1 for i=1,...,(n-1) and if i=n, j=1. Otherwise p(i,j)=0. If W is the column weight vector associated with ranking B, then a simple computation shows that vector P(W) is the column weight vector associated with the next ranking in the Condorcet n-tuple. In general, to obtain the weight vector for one ranking from the weight vector for the previous ranking, you just apply matrix P. Thus any one of weight vectors is given by the appropriate power of P acting on vector W as P((P(W)...)). So, if $W=(w(1),...,w(n))$, then the other weight vectors are $(w(2),...,w(n),w(1))$; $(w(3),...,w(1),w(2))$; ...; $(w(n),w(1),...,w(n-1))$.

Our goal is to show that in certain situations the n weight vectors for a Condorcet n-tuple are linearly independent. This would give us a i=1 version of Lemma 1 where a constraint is imposed upon the possible voters' rankings. Although the approach we use may appear different from that of the proof of Lemma 1, it has similarities in that it depends upon the eigenvalues and eigenvectors of P(i,j).

Lemma 2: Let $W=(w(1),...,w(n))$ be a weight vector for n>1 alternatives. Let $p(W,z)$ be the polynomial

5.1 $$p(W,z)=w(1)+w(2)z+w(3)z^{2}+ ... +w(n)z^{(n-1)}$$

assume that $p(W,r)$ does not equal zero for any r which is an nth root of unity. Then the Condorcet n-tuple generated by vector W is linearly independent.

Proof of Lemma 2: Let W(i) be the ith vector in the Condorcet n-tuple generated be vector W. With this notation, W(1)=W. The vector sum $\sum d(k)W(i) = 0$, where $k=(n+1-i) \bmod n + 1$, is equivalent to the matrix equation (D)C(W)=0. Here matrix C(W) is the circulant matrix (see, for example, Davis[11]) where the ith row is given by vector W(k) (k is determined as given in the previous line) and row vector D is given by (d(1),...,d(n)). Vector D must be the zero vector if and only if the determinant of matrix C(W) is zero which is true if and only if any of the eigenvalues of this symmetric matrix

are equal to zero. One of the eigen values of this matrix is unity with corresponding eigenvector e, while the other eigenvalues are given by p(W,r) where r is one of the nth roots of unity. By hypothesis, all of the eigenvalues are non-zero, so the vectors in the Condorcet n-tuple are linearly independent. This completes the proof of the lemma. (See [1].)

There are weight vectors which do not satisfy these conditions, such as W=(1,1,0,0). For this vector, p(W,-1)=0, and it is trivial to show that the corresponding Condorcet 4-tuple is not linearly independent. On the other hand, from this system sets of three weight vectors can be found which are linearly independent. It is easy to show that if n=3, then all weight vectors satisfy these conditions.

In reference [3], we showed that there always is a Condorcet (n+1) tuple contained within the lift of a

Condorcet n-tuple. We use this in the statement of the next theorem which asserts that even should we impose a restriction that the voters' preferences can only lie in n of the n! possible ranking classes, if enough symmetry is admitted, then the conclusions of Theorem 2 still hold.

THEOREM 6. Assume there are n>1 alternatives, and assume the conditions of Theorem 2 with the additional restriction on the weight vectors that if r is a jth root of unity, then it is not a zero of the (j-1)th degree polynomial p(W(j),z). Then there exist examples of voters with the property that when j alternatives are being considered, each voter's preferences are one of the j possible rankings of some particular Condorcet j-tuple and still the conclusions of Theorem 2 hold. If in addition, the weights are all rational numbers, then the corresponding conclusion of Theorem 4 apply.

In the proof of this theorem, Lemma 2 is substituted for that of Lemma 1. It follows from the cyclic behavior of these vectors in a Condorcet n-tuple that their sum is n(e), so the second technical condition is also satisfied. (This is because in the vector sum each weight appears once and only once in each coordinate position.)

A similar restriction on voters' preferences will not yield the conclusion of Theorems 1 and 3. By comparing a dimension count on the domain and the range, it is clear that mapping f cannot be surjective. Notice that by admitting additional voters' preferences, the rank of this mapping can increase. The full strength of Theorems 1 and 3 do not apply until this rank has dimension $\frac{1}{2}(n-1)$. On the other hand, once the rank has exceeded (n-1) some "paradoxes" are admitted. We have not attempted to classify these paradoxes in terms of admitted preference classes.

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