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THE CONTINUOUS-TIME MODEL OF MULTIREGIONAL DEMOGRAPHIC GROWTH

by

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ABSTRACT

This paper is the first of a set that will focus on the multiregional generalization of classical single-region mathematical demography. Formal mathematical demography has its origins in the seminal published works of Alfred Lotka, which extended over a period of some forty years beginning in 1907. His fundamental integral equation, which relates the births of one generation to those of the preceding one, leads to several of the basic results of mathematical demography. However, Lotka's model and those of classical mathematical demography in general deal with a single-region population that is assumed to be undisturbed by migration. The principal purpose of this paper is to review and extend some recent efforts to generalize Lotka's integral equation to the case of a multiregional population that experiences internal migration. In so doing we shall focus on the relationships between a multiregional population's fertility, mortality, and mobility and its growth and regional age distributions.

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1. The Multiregional Renewal Equation

The conventional single-region single-sex population projection of mathematical demography in its continuous form is expressed as an integral equation. Beginning with the density of female births, $B(t)$, say, we note that the number of women of ages x to $x + dx$ at time t, who were born since time zero, will be the survivors of those born x years ago, $B(t - x)p(x)dx$, $x \leq t$. At time t, these women give birth to

$$B(t - x)p(x)m(x)dx \tag{1}$$

female children per year. Here $p(x)$ denotes the probability of surviving to age x, and $m(x)dx$ is the probability of a woman x years of age giving birth to a female child during the interval x to $x + dx$.

Integrating (1) over all x and adding $G(t)$ to include births to women already alive at time zero gives the fundamental integral equation

$$B(t) = G(t) + \int_0^t B(t - x)p(x)m(x)dx, \tag{2}$$

in which $G(t)$ may be set equal to zero for $t \geq \beta$, the last age of childbearing and the integration taken over the childbearing ages α through β:

$$B(t) = \int_\alpha^\beta B(t - x)p(x)m(x)dx. \tag{3}$$

On replacing $B(t)$ by Qe^{rt} and $B(t - x)$ by $Qe^{r(t - x)}$ we find the well-known characteristic equation

$$\psi(r) = \int_{\alpha}^{\beta} e^{-rx} p(x)m(x)dx = \int_{\alpha}^{\beta} e^{-rx} \phi(x)dx = 1, \tag{4}$$

where the product $p(x)m(x)$, denoted by $\phi(x)$, is known as the net maternity function.

A continuous model of single-sex population projection also may be defined for the multiregional situation. Consider, for example, the case of a two-region system with regions i and j, say. We define the pair of equations:

$$B_i(t) = G_i(t) + \int_{\alpha}^{\beta} [B_i(t - x)\,{}_{i0}p_i(x) + B_j(t - x)\,{}_{j0}p_i(x)]m_i(x)dx \tag{5}$$

$$B_j(t) = G_j(t) + \int_{\alpha}^{\beta} [B_i(t - x)\,{}_{i0}p_j(x) + B_j(t-x)\,{}_{j0}p_j(x)]m_j(x)dx, \tag{6}$$

where, for example, ${}_{j0}p_i(x)$ denotes the probability of a woman born in region j being alive in region i at age x, $m_i(x)dx$ is the probability of a woman x years of age in region i giving birth to a female child during the interval x to $x + dx$, and

$$G_i(t) = \int_{\alpha-t}^{\beta-t} [k_i(x)\,{}_{ix}p_i(x+t) + k_j(x)\,{}_{jx}p_i(x+t)]m_i(x+t)dx \tag{7}$$

$$G_j(t) = \int_{\alpha-t}^{\beta-t} [k_i(x)\,{}_{ix}p_j(x + t) + k_j(x)\,{}_{jx}p_j(x + t)]m_j(x+t)dx, \tag{8}$$

$k_i(x)dx$ denoting the number of women alive between ages x and $x + dx$ in region i at time $t = 0$.

Expressing (5) and (6) in matrix form, we have the <u>multiregional renewal equation</u>

$$\{\underset{\sim}{B}(t)\} = \{\underset{\sim}{G}(t)\} + \int_{\alpha}^{\beta} \underset{\sim}{M}(x)\underset{\sim}{P}(x)\{\underset{\sim}{B}(t - x)\}dx, \tag{9}$$

where

$$\underset{\sim}{M}(x) = \begin{bmatrix} m_i(x) & 0 \\ 0 & m_j(x) \end{bmatrix} \quad \text{and} \quad \underset{\sim}{P}(x) = \begin{bmatrix} {}_{i0}p_i(x) & {}_{j0}p_i(x) \\ {}_{i0}p_j(x) & {}_{j0}p_j(x) \end{bmatrix}$$

For the case $t \geq \beta$, $\{\underset{\sim}{G}(t)\} = \{\underset{\sim}{0}\}$, and (9) reduces to the homogeneous equation first set out by LeBras (1971):

$$\{\underset{\sim}{B}(t)\} = \int_{\alpha}^{\beta} \underset{\sim}{M}(x)\underset{\sim}{P}(x)\{\underset{\sim}{B}(t - x)\}dx \tag{10}$$

As in the single-region model, the solution of the integral equation in (9) can be found by first obtaining a solution of (10) and then choosing values for the arbitrary constants in that solution so that in addition to satisfying (10), $\{\underset{\sim}{B}(t)\}$ also satisfies (9).

On replacing $B_i(t)$ and $B_j(t)$ in (5) and (6) by $Q_i e^{rt}$ and $Q_j e^{rt}$, respectively, noting that $G_i(t) = G_j(t) = 0$ for $t \geq \beta$, and proceeding as in the single-region case, we obtain the pair of characteristic equations

$$Q_i = \int_{\alpha}^{\beta} e^{-rx}[Q_i \; {}_{i0}p_i(x) + Q_j \; {}_{j0}p_i(x)]m_i(x)dx \tag{11}$$

$$Q_j = \int_{\alpha}^{\beta} e^{-rx}[Q_i \; {}_{i0}p_j(x) + Q_j \; {}_{j0}p_j(x)]m_j(x)dx, \tag{12}$$

or, in matrix form

$$\{\underset{\sim}{Q}\} = \int_{\alpha}^{\beta} e^{-rx}\underset{\sim}{M}(x)\underset{\sim}{P}(x)\{\underset{\sim}{Q}\}dx = [\int_{\alpha}^{\beta} e^{-rx} \underset{\sim}{\Phi}(x)dx]\{\underset{\sim}{Q}\} = \underset{\sim}{\Psi}(r)\{\underset{\sim}{Q}\}, \text{ say,} \tag{13}$$

where $\underset{\sim}{\Phi}(x) = \underset{\sim}{M}(x)\underset{\sim}{P}(x)$ is the <u>multiregional net maternity function</u> and

$\underset{\sim}{\Psi}(r) = \int_{\alpha}^{\beta} e^{-rx}\, \underset{\sim}{\Phi}(x)dx$ is the multiregional characteristic matrix.[1]

We have now reduced our problem from one of solving the integral equation in (10) to that of solving (13) which, unlike (10) is a function of only a single variable, r.

To solve for r in (13), we rewrite the equation as

$$[\underset{\sim}{\Psi}(r) - \underset{\sim}{I}]\{\underset{\sim}{Q}\} = \{\underset{\sim}{0}\}, \tag{14}$$

from which we conclude that $\{\underset{\sim}{Q}\}$ is the characteristic vector that corresponds to the characteristic root of unity of the multiregional characteristic matrix

$$\underset{\sim}{\Psi}(r) = \int_{\alpha}^{\beta} e^{-rx}\, \underset{\sim}{M}(x)\ \underset{\sim}{P}(x)dx, \tag{15}$$

and r is the number for which

$$|\underset{\sim}{\Psi}(r) - \underset{\sim}{I}| = 0. \tag{16}$$

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[1] The multiregional net maternity function defined in (13) is a natural generalization of $\phi(x)$ in (4). However it assumes that inmigrants immediately assume the fertility schedule of the region into which they move. This assumption may be relaxed somewhat by differentiating the fertility schedule of a region by the region of birth of the parent. In the case of our two-region example, we have then

$$\underset{\sim}{\Phi}(x) = \begin{bmatrix} {}_{i0}p_i(x)\,{}_{i0}m_i(x) & {}_{j0}p_i(x)\,{}_{j0}m_i(x) \\ {}_{i0}p_j(x)\,{}_{i0}m_j(x) & {}_{j0}p_j(x)\,{}_{j0}m_j(x) \end{bmatrix},$$

The analysis of the integral equation in (13) remains unchanged.

Alternatively, we may approach the solution of (13) in a manner more reminiscent of the single-region model. Dividing (11) by Q_i and (12) by Q_j, we obtain

$$1=\int_{\alpha}^{\beta} e^{-rx}\left[\frac{Q_i}{Q_i}\ {}_{i0}P_i(x) + \frac{Q_j}{Q_i}\ {}_{j0}P_i(x)\right] m_i(x)dx = \int_{\alpha}^{\beta} \sum_k e^{-rx} \frac{Q_k}{Q_i}\ {}_{k0}P_i(x)m_i(x)dx \tag{17}$$

$$1=\int_{\alpha}^{\beta} e^{-rx}\left[\frac{Q_i}{Q_j}\ {}_{i0}P_j(x) + \frac{Q_j}{Q_j}\ {}_{j0}P_j(x)\right] m_j(x)dx = \int_{\alpha}^{\beta} \sum_k e^{-rx} \frac{Q_k}{Q_j}\ {}_{k0}P_j(x)m_j(x)dx \tag{18}$$

or, in matrix form:

$$\{\underset{\sim}{1}\} = \int_{\alpha}^{\beta} e^{-rx}\underset{\sim}{M}(x)\underset{\sim}{Q}^{-1}\underset{\sim}{P}(x)\underset{\sim}{Q}\{\underset{\sim}{1}\}dx \tag{19}$$

$$= \int_{\alpha}^{\beta} e^{-rx}\underset{\sim}{M}(x)\hat{\underset{\sim}{P}}(x)\{\underset{\sim}{1}\}dx \quad , \quad \hat{\underset{\sim}{P}}(x) \equiv \underset{\sim}{Q}^{-1}\underset{\sim}{P}(x)\underset{\sim}{Q} \quad , \tag{20}$$

$$= \int_{\alpha}^{\beta} e^{-rx}\hat{\underset{\sim}{\Phi}}(x)\{\underset{\sim}{1}\}dx, \qquad \hat{\underset{\sim}{\Phi}}(x) \equiv \underset{\sim}{M}(x)\hat{\underset{\sim}{P}}(x), \tag{21}$$

$$= \hat{\underset{\sim}{\Psi}}(r)\{\underset{\sim}{1}\} \quad , \qquad \hat{\underset{\sim}{\Psi}}(r) \equiv \int_{\alpha}^{\beta} e^{-rx}\hat{\underset{\sim}{\Phi}}(x)dx, \tag{22}$$

where

$$\underset{\sim}{Q} = \begin{bmatrix} Q_i & 0 \\ 0 & Q_j \end{bmatrix}$$

Note that $\hat{\underset{\sim}{P}}(x), \hat{\underset{\sim}{\Phi}}(x)$, and $\underset{\sim}{\Psi}(r)$ are weighted versions of $\underset{\sim}{P}(x), \underset{\sim}{\Phi}(x)$, and $\underset{\sim}{\Psi}(r)$, the weights being appropriate ratios of births in the stable multiregional population.

Rewriting (22) as

$$[\hat{\underset{\sim}{\Psi}}(r) - \underset{\sim}{I}]\ \{\underset{\sim}{1}\} = \{\underset{\sim}{0}\}, \tag{23}$$

we conclude that the solution of (22) may be found by determining that value of r for which

$$|\hat{\underset{\sim}{\Psi}}(r) - \underset{\sim}{I}| = 0. \tag{24}$$

That the value of r which satisfies (24) also satisfies (16) may be seen by observing that the weights are cancelled out in the expansion of the determinant in (24). Why then introduce (22) at all? The reason will become apparent in Section 3 when we discuss alternative numerical procedures for obtaining r. There we shall demonstrate that for each particular set of weights, one can obtain a value of r using the single-region method (and computer program). This leads to an efficient iterative numerical procedure for solving (13).

2. Multiregional Stable Growth

In the preceding section we have seen that $\{\underset{\sim}{Q}\}e^{rt}$ is a solution to (13) provided r is such that:

(i) a characteristic root, $\lambda(r)$, say, of the matrix $\underset{\sim}{\Psi}(r)$ is unity:

$$\lambda(r) = 1; \tag{25}$$

(ii) $$|\underset{\sim}{\Psi}(r) - \underset{\sim}{I}| = 0; \tag{26}$$

(iii) $$\hat{\underset{\sim}{\Psi}}(r)\{\underset{\sim}{1}\} = \{\underset{\sim}{1}\} \tag{27}$$

Moreover, all three of the above conditions were shown to be equivalent in that a value of r that satisfies any one of them also satisfies the other two.

The matrix $\underset{\sim}{\Psi}(r)$ normally has more than one real characteristic root of unity as r ranges from $-\infty$ to $+\infty$. Consequently (25),(26), and (27) may be satisfied by more than one real value of the root r. However, we shall always be interested in the largest, or <u>maximal</u>, real root which we shall denote by r_1. There can be only one such root and in addition to exceeding the value of all other real roots r of (25), (26), and (27), it is also greater than the real part of any complex root of those same three equations.

If, as in almost all empirical situations, $\underset{\sim}{\Psi}(r)$ is a matrix with only positive elements, then by the well-known Perron-Frobenius properties of positive matrices [Gantmacher, Vol. II (1959), pp. 53-66]: [2]

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[2] These properties also apply, in slightly weaker form, to nonnegative matrices $\underset{\sim}{\Psi}(r)$ that are irreducible. $\underset{\sim}{\Psi}(r)$ is a nonnegative matrix by definition.

(i) The matrix $\underset{\sim}{\Psi}(r)$ has a real and positive maximal characteristic root, $\lambda_1(r)$, say, which is a simple root and is greater than the absolute value of any other characteristic root.

(ii) Associated with $\lambda_1(r)$ is a characteristic vector, $\{\underset{\sim}{Q}_1\}$, say, with all of its elements positive. No other such vector of positive elements exists except for multiples of $\{\underset{\sim}{Q}_1\}$.

(iii) The maximal root $\lambda_1(r)$ of the matrix $\underset{\sim}{\Psi}(r)$ decreases in value when an element of $\underset{\sim}{\Psi}(r)$ is decreased (which, as in the single-region model, occurs when r is increased).

Of the real roots r that satisfy (25), (26), and (27), let r_1 denote the r for which $\lambda_1(r) = 1$ and r_k denote the r for which $\lambda_k(r_k) = 1$, for k = 2,3...; that is, $\lambda_1(r_1) = 1$ and $\lambda_k(r_k) = 1$. If $\underset{\sim}{\Psi}(r)$ is a positive matrix, $\lambda_1(r) > \lambda_k(r)$, whence

$$\lambda_1(r_k) > 1 \tag{28}$$

Consequently, $r_1 > r_k$. This can be seen by observing that if $r_k > r_1$, then by the third property of positive matrices listed above $\lambda_1(r_k) < \lambda_1(r_1) = 1$, which is a contradiction of (28). Therefore $r_1 > r_k$, for all k = 2,3,... .

The maximal root $\lambda_1(r)$ is in fact a function that assigns to any value of r the maximal characteristic root of the matrix $\underset{\sim}{\Psi}(r)$. This function is continuous, concave upward throughout, and its values decrease monotonically from $+\infty$ to 0 as its argument increases from $-\infty$ to $+\infty$. Consequently, $\lambda_1(r) = 1$ can occur only once, when $r = r_1$. Such a function is illustrated in the next section (Figure 1) as a graph in which $\lambda_1(r)$ is plotted as the ordinate with r as the abscissa.

As in the single-region model, any complex roots that satisfy (25), (26), and (27), and therefore (13), must occur in complex conjugate pairs. Suppose that $u + iv$ is such a root. Then

$$\underset{\sim}{\Psi}(u + iv)\ \{\underset{\sim}{Q}\} = \{\underset{\sim}{Q}\} \tag{29}$$

where, for example,

$$\begin{aligned} {}_i\Psi_j(u + iv) &= \int_\alpha^\beta e^{-(u+iv)x}\ {}_i\phi_j(x)dx \\ &= \int_\alpha^\beta e^{-ux}[\cos(vx) - i\sin(vx)]\,{}_i\phi_j(x)dx \end{aligned}$$

Equating real and imaginary parts in (29), we find

$$\left.\begin{aligned} \sum_{k=1}^{m} Q_k \int_\alpha^\beta e^{-ux} \cos(vx)\,{}_k\phi_s(x)dx &= Q_s \\ \sum_{k=1}^{m} Q_k \int_\alpha^\beta e^{-ux} \sin(vx)\,{}_k\phi_s(x)dx &= 0 \end{aligned}\right\} \quad (s = 1,2,\ldots,m) \tag{30}$$

It follows that $u - iv$ is also a complex root of (13), whence

$$\underset{\sim}{\Psi}(u - iv)\ \{\underset{\sim}{Q}\} = \{\underset{\sim}{Q}\}. \tag{31}$$

Moreover, we observe that the maximal real root r_1 is greater than u in the complex root $u + iv$. Since $\cos(vx) < 1$ in (30) for some values of x within the range of integration,

$$\sum_{k=1}^{m} Q_k \int_\alpha^\beta e^{-ux}\ {}_k\phi_s(x)dx > Q_s.$$

But

$$\sum_{k=1}^{m} Q_k \int^{\beta} e^{-r_1 x} \, {}_k\phi_s(x)dx = Q_s.$$

Hence $u < r_1$. That is, the maximal real root that satisfies (25), (26), and (27) is larger than the real part of any complex root that also satisfies those same three equations.

Finally, Equation (13) is homogeneous. Consequently its solution vectors are additive. Thus if $r_1, r_2, \ldots$ are roots satisfying (25), (26), and (27), then

$$\{\underset{\sim}{B}(t)\} = \{\underset{\sim}{Q}_k\} e^{r_k t} \qquad (k = 1,2,\ldots)$$

are solutions of (13), as is the general solution

$$\{\underset{\sim}{B}(t)\} = \{\underset{\sim}{Q}_1\} e^{r_1 t} + \{\underset{\sim}{Q}_2\} e^{r_2 t} + \ldots$$

$$= \{\underset{\sim}{Q}_1\} e^{r_1 t} + \sum_{k=2}^{w} \{\underset{\sim}{Q}_k\} e^{r_k t} + \sum_{s=w+1}^{\infty} \{\underset{\sim}{Q}_s\} e^{u_s t}[\cos(v_s t)+i\sin(v_s t)], \quad (32)$$

where the vectors $\{\underset{\sim}{Q}_k\}$, $k = 1,2,\ldots$, may be chosen to satisfy (9). As in the single-region model the birth sequence $\{\underset{\sim}{B}(t)\}$ is increasingly dominated by the maximal real root r_1 as t becomes large. Because $r_1 >$ Real (r_k) for $k = 2,2,\ldots$, all terms after the first in the series set out in (32) become negligible compared with the first for large t, and

$$\{\underset{\sim}{B}(t)\} \cong \{\underset{\sim}{Q}_1\} e^{r_1 t} \qquad (33)$$

We shall see below in Section 5 that exponential births lead to an exponentially growing population with a stable distribution in which each age-by-region subpopulation maintains a constant proportional relationship to the total population and increases at the same constant rate r_1. The influence of the initial population distribution is forgotten as time goes by, a condition known as ergodicity.

Let us conclude this section by considering the problem of evaluating the $\{\underset{\sim}{Q}_k\}$ vectors, k = 1,2,..., to fit a particular initial condition specified by $\{\underset{\sim}{G}(t)\}$. The proof of the fundamental result that we need requires a rigorous and rather complicated matrix generalization of an argument given by Feller (1941). Such an analysis is beyond the scope of this paper. Hence we shall instead adopt a nonrigorous intuitive argument which yields a solution the accuracy of which may be tested against numerical data.

The simplest method for deriving the formula for Q_k in the single-region model is by means of Laplace transforms. Taking Laplace transforms of both sides of (2), after denoting $p(x)m(x)$ by $\phi(x)$, the net maternity function, we have

$$B^*(r) = G^*(r) + B^*(r)\phi^*(r), \tag{34}$$

where

$$\left.\begin{aligned} B^*(r) &= \int_0^\infty e^{-rt} B(t)dt; \\ G^*(r) &= \int_0^\infty e^{-rt} G(t)dt; \\ \text{and} \quad \phi^*(r) &= \int_0^\infty e^{-rt}\phi(t)dt. \end{aligned}\right\} \tag{35}$$

It follows that

$$B^*(r) = \frac{G^*(r)}{1 - \phi^*(r)}, \tag{36}$$

the right-hand side of which may be expanded as follows:

$$B^*(r) = \frac{G^*(r)}{1 - \phi^*(r)} = \sum_{k=1}^{\infty} \frac{Q_k}{r-r_k} , \tag{37}$$

subject to conditions that normally are satisfied im demographic analysis. By applying the usual procedure for determining the coefficients of partial fractions, we have that

$$Q_k = \lim_{r \to r_k} \frac{(r-r_k)G^*(r)}{1-\phi^*(r)} = \left. \frac{G^*(r)}{-d\phi^*(r)/dr} \right|_{r=r_k} , \tag{38}$$

whence

$$Q_k = \frac{\int_0^{\beta} e^{-r_k t} G(t)dt}{\int_{\alpha}^{\beta} xe^{-r_k x} p(x)m(x)dx} \tag{39}$$

Now consider the same argument in matrix form. Taking Laplace transforms of both sides of (9), we have

$$\{\underset{\sim}{B}^*(r)\} = \{\underset{\sim}{G}^*(r)\} + \underset{\sim}{\Phi}^*(r) \{\underset{\sim}{B}^*(r)\} . \tag{40}$$

Consequently

$$\{\underset{\sim}{B}^*(r)\} = [\underset{\sim}{I} - \underset{\sim}{\Phi}^*(r)]^{-1}\{\underset{\sim}{G}^*(r)\}, \tag{41}$$

whence

$$\{\underset{\sim}{B}^*(t)\} = \sum_{k=1}^{\infty} \{\underset{\sim}{Q}_k\} e^{r_k t} , \tag{42}$$

and

$$\{\underset{\sim}{Q}_k\} = \lim_{r \to r_k} (r-r_k)[\underset{\sim}{I} - \underset{\sim}{\Phi}^*(r)]^{-1}\{\underset{\sim}{G}^*(r)\}$$

$$= \left[- \frac{d}{dr} \underset{\sim}{\Phi}^*(r) \right]^{-1} \{\underset{\sim}{G}^*(r)\} \Bigg|_{r=r_k}$$

$$= \underset{\sim}{A}^{-1}\{\underset{\sim}{V}\} \Bigg|_{r=r_k} , \tag{43}$$

where the ijth element of $\underset{\sim}{A}$ is

$${}_iA_j = \int_\alpha^\beta x e^{-rx} {}_i\phi_j(x)dx \tag{44}$$

and the ith element of $\{\underset{\sim}{V}\}$ is

$$V_i = \int_\alpha^\beta e^{-rt} G_i(t)dt \quad . \tag{45}$$

In the special case $r = r_1$, ${}_iA_j$ will be referred to as the mean age of childbearing in region j of mothers born in region i, and V_i will be called the total reproductive value of the (single-sex) resident population of region i. We examine these two measures in greater detail below.

3. Numerical Solution of the Multiregional Renewal Equation

To determine the single real root of (4) one normally adopts the approximation

$$\Psi(r) = \sum_{x=\alpha}^{\beta-5} e^{-r(x + 2.5)} L(x)F(x), \tag{46}$$

in which the integral $\int_0^5 e^{-r(x + t)} p(x + t)m(x + t)dt$ is replaced by the product of $e^{-r(x + 2.5)}$, $L(x)$, computed on a unit radix, and $F(x)$. The summation is over ages x which are multiples of 5. Thus (4) is approximated by

$$\Psi(r) = e^{-12.5r}\ L(10)\ F(10) + \ldots + e^{-47.5r}\ L(45)\ F(45) = 1, \tag{47}$$

where we have assumed childbearing to begin at age $\alpha = 10$ and to end at age $\beta = 50$.

An analogous approach may be followed in the multiregional model. We replace the integral

$$\int_0^5 e^{-r(x + t)} \underset{\sim}{M}(x + t)\underset{\sim}{P}(x + t)dt \tag{48}$$

by the product of $e^{-r(x + 2.5)}$, $\underset{\sim}{F}(x)$, and $\underset{\sim}{L}(x)$ where

$$\underset{\sim}{F}(x) = \begin{bmatrix} F_i(x) & 0 \\ 0 & F_j(x) \end{bmatrix} \quad \text{and} \quad \underset{\sim}{L}(x) = \begin{bmatrix} {}_{i0}L_{.i}(x) & {}_{j0}L_{.i}(x) \\ {}_{i0}L_{.j}(x) & {}_{j0}L_{.j}(x) \end{bmatrix}$$

the matrix $\underset{\sim}{L}(x)$ having been computed on unit radices for both regions. Thus, in the case of our two-region system, we may use the following numerical approximation to the multiregional characteristic matrix:

$$\underset{\sim}{\Psi}(r) = \begin{bmatrix} \sum_{x=\alpha}^{\beta-5} e^{-r(x+2.5)} {}_{i0}L_{.i}(x)F_i(x) & \sum_{x=\alpha}^{\beta-5} e^{-r(x+2.5)} {}_{j0}L_{.i}(x)F_i(x) \\ \sum_{x=\alpha}^{\beta-5} e^{-r(x+2.5)} {}_{i0}L_{.j}(x)F_j(x) & \sum_{x=\alpha}^{\beta-5} e^{-r(x+2.5)} {}_{j0}L_{.j}(x)F_j(x) \end{bmatrix} \tag{49}$$

Table 1 presents age-specific birth rates and data from a two-region life table with which we may calculate $\underset{\sim}{\Psi}(r)$ for United States females residing in California and the rest of the United States in 1958. For example, beginning with $r = 0.020$, we obtain

$$\underset{\sim}{\Psi}(0.020) = \begin{bmatrix} 0.7213515 & 0.0609076 \\ 0.2869147 & 0.9557833 \end{bmatrix} \quad ; \tag{50}$$

a matrix with a maximal characteristic root of $\lambda_1(0.020) = 1.015245$. Next, increasing r to 0.025 yields

$$\underset{\sim}{\Psi}(0.025) = \begin{bmatrix} 0.6375670 & 0.0533113 \\ 0.2508840 & 0.8417467 \end{bmatrix} \tag{51}$$

and $\lambda_1(0.025) = 0.893921$. Since $\lambda_1(0.020) > 1$ and $\lambda_1(0.025) < 1$, r_1 must lie between 0.020 and 0.025. An average of these two values yields $\lambda_1(0.0225) = 0.952632$. Continuing in this manner we ultimately converge on $r_1 = 0.020593$ for which $\lambda_1(r_1)$ is unity to four decimal places, thereby satisfying (25). At this point

$$\underset{\sim}{\Psi}(r_1) = \begin{bmatrix} 0.7108365 & 0.0599501 \\ 0.2823706 & 0.9414467 \end{bmatrix} \tag{52}$$

Note that (26) and (27) are also satisfied:

$$(0.7108365 - 1)(0.9414467 - 1) - (0.2823706)(0.0599501) = 0$$

and

$$\begin{bmatrix} 0.7108365 & \frac{Q_j}{Q_i}(0.0599501) \\ \frac{Q_i}{Q_j}(0.2823706) & 0.9414467 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} ,$$

where $\frac{Q_j}{Q_i} = \frac{1 - 0.7108365}{0.0599501} = 4.82322$ and $\frac{Q_i}{Q_j} = \frac{1 - 0.9414467}{0.2823706} = 0.20733.$

Table 2 and Figure 1 illustrate the behavior of $\lambda_1(r)$ as a function of r. Also detailed are the functions $|\underset{\sim}{\Psi}(r) - \underset{\sim}{I}|$ and $\hat{\underset{\sim}{\Psi}}(r)\{\underset{\sim}{1}\}$.

Equations (25) and (26) both lead naturally to iterative procedures for establishing the value of r_1. The first requires the derivation of the maximal characteristic root of an n by n matrix at each iteration. The second instead requires the calculation of the determinant of an n by n matrix. Either requirement leads to a considerable amount of computation. Consequently, the efficiency of these two iterative procedures is rather low, particularly when compared to the one suggested by (27).

Equation (27) yields, by means of standard single-region methods, an approximation of r_1 that is consistent with the particular system of weights defined by the ratios Q_k/Q_s, for $k,s = 1,2,...$. These ratios are not determinable until one finds r_1, however, and therefore must be approximated along with r_1 by iteration. As an initial approximation the ratios may all be set equal to unity, say, and the corresponding initial approximation of r_1 then may be obtained by a standard single-region method, such as the method of <u>functional iteration</u> described by Keyfitz (1968,p. 111). In the case of our two-region system of California and the rest of the United States,

TABLE 1. The multiregional net maternity function for United States females, 1958: Two-region model of California and the rest of the United States

Region	Age x	$F_i(x)$	${}_{i0}L_i(x)$	${}_{j0}L_i(x)$	${}_i\phi_i(x)$	${}_j\phi_i(x)$
(i) California	10	0.00032	4.16220	0.14050	0.00134	0.00004
	15	0.04959	3.92220	0.18953	0.19451	0.00940
	20	0.12323	3.65597	0.25690	0.45052	0.03166
	25	0.08945	3.33854	0.33460	0.29862	0.02993
	30	0.05262	3.06113	0.39018	0.16109	0.02053
	35	0.02387	2.86151	0.42907	0.06831	0.01024
	40	0.00606	2.70519	0.45559	0.01640	0.00276
	45	0.00030	2.57330	0.47038	0.00078	0.00014
	50	0.00002	2.44204	0.47586	0.00004	0.00001
Region	Age x	$F_j(x)$	${}_{i0}L_j(x)$	${}_{j0}L_j(x)$	${}_i\phi_j(x)$	${}_j\phi_j(x)$
(j) Rest of the U.S.	10	0.00048	0.69909	4.70382	0.00034	0.00225
	15	0.04584	0.92960	4.64493	0.04261	0.21291
	20	0.12567	1.18150	4.56260	0.14848	0.57338
	25	0.09311	1.48124	4.46573	0.13792	0.41582
	30	0.05477	1.73468	4.38419	0.09502	0.24014
	35	0.02825	1.90037	4.30997	0.05369	0.12177
	40	0.00819	2.00486	4.22927	0.01642	0.03463
	45	0.00048	2.05530	4.12971	0.00100	0.00200
	50	0.00001	2.06703	4.00307	0.00003	0.00005

TABLE 2. Values of $\lambda(r)$, $|\Psi(r) - I|$ and $\hat{\Psi}(r)\{1\}$ for a two-region model of United States females, 1958: (i) California and (j) Rest of the United States *

r	$\lambda_1(r)$	$\lvert\Psi(r)-1\rvert$	$\hat{\Psi}(r)\{1\}$: $\sum_k \hat{\Psi}_{ki}(r)$	$\hat{\Psi}(r)\{1\}$: $\sum_k \hat{\Psi}_{kj}(r)$	$\lambda_2(r)$
0.000	1.704195	0.063645	1.696689	1.705700	1.090379
0.001	1.660048	0.041696	1.653113	1.661438	1.063171
0.002	1.617105	0.022633	1.610714	1.618387	1.036676
0.003	1.575331	0.006257	1.56946	1.576551	1.010875
0.0032	1.567114	0.003287	1.561342	1.568272	1.005796
0.0034	1.558942	0.000416	1.553270	1.560080	1.000745
0.0036	1.550815	-0.002358	1.545242	1.551934	0.995720
0.0038	1.542732	-0.005036	1.537257	1.543831	0.990721
0.004	1.534694	-0.007620	1.529315	1.535774	0.985749
0.005	1.495160	-0.019173	1.490252	1.496147	0.961279
0.006	1.456699	-0.028568	1.452238	1.457596	0.937448
0.007	1.419280	-0.035959	1.415243	1.420092	0.914238
0.008	1.382872	-0.041492	1.379240	1.383603	0.891631
0.009	1.347448	-0.045303	1.344199	1.348102	0.869612
0.010	1.312978	-0.047522	1.310095	1.313559	0.848164
0.011	1.279437	-0.048267	1.276901	1.279948	0.827272
0.012	1.246798	-0.047652	1.244591	1.247243	0.806920
0.013	1.215035	-0.045782	1.213140	1.215418	0.787094
0.014	1.184124	-0.042757	1.182525	1.184447	0.767780
0.015	1.154041	-0.038670	1.152723	1.154307	0.748964
0.016	1.124761	-0.033607	1.123710	1.124974	0.730632
0.017	1.096264	-0.027650	1.095464	1.096425	0.712771
0.018	1.068526	-0.020875	1.067965	1.068639	0.695369
0.019	1.041526	-0.013354	1.041191	1.041594	0.678412
0.020	1.015245	-0.005154	1.015123	1.015269	0.661890
0.020592	1.000016	-0.000006	1.000015	1.000016	0.652309
0.020593	0.999990	0.000003	0.999990	0.999991	0.652293
0.020594	0.999965	0.000012	0.999964	0.999965	0.652277
0.021	0.989661	0.003662	0.989741	0.989645	0.645790
0.022	0.964755	0.013037	0.965025	0.964700	0.630102
0.023	0.940508	0.022915	0.940958	0.940417	0.614813
0.024	0.916903	0.033246	0.917522	0.916777	0.599914
0.025	0.893921	0.043982	0.894700	0.893762	0.585393
0.026	0.871544	0.055077	0.872473	0.871355	0.571242
0.027	0.849756	0.066491	0.850827	0.849539	0.557449
0.028	0.828542	0.078184	0.829745	0.828297	0.544006
0.029	0.807884	0.090121	0.809211	0.807614	0.530902
0.030	0.787768	0.102268	0.789212	0.787474	0.518130

* The weights used for computing $\hat{\Psi}(r)$: Q_j/Q_i = 4.82322 and Q_i/Q_j = 0.20733, are derived in Section 3, where they are shown to be associated with the maximal root r = 0.020593

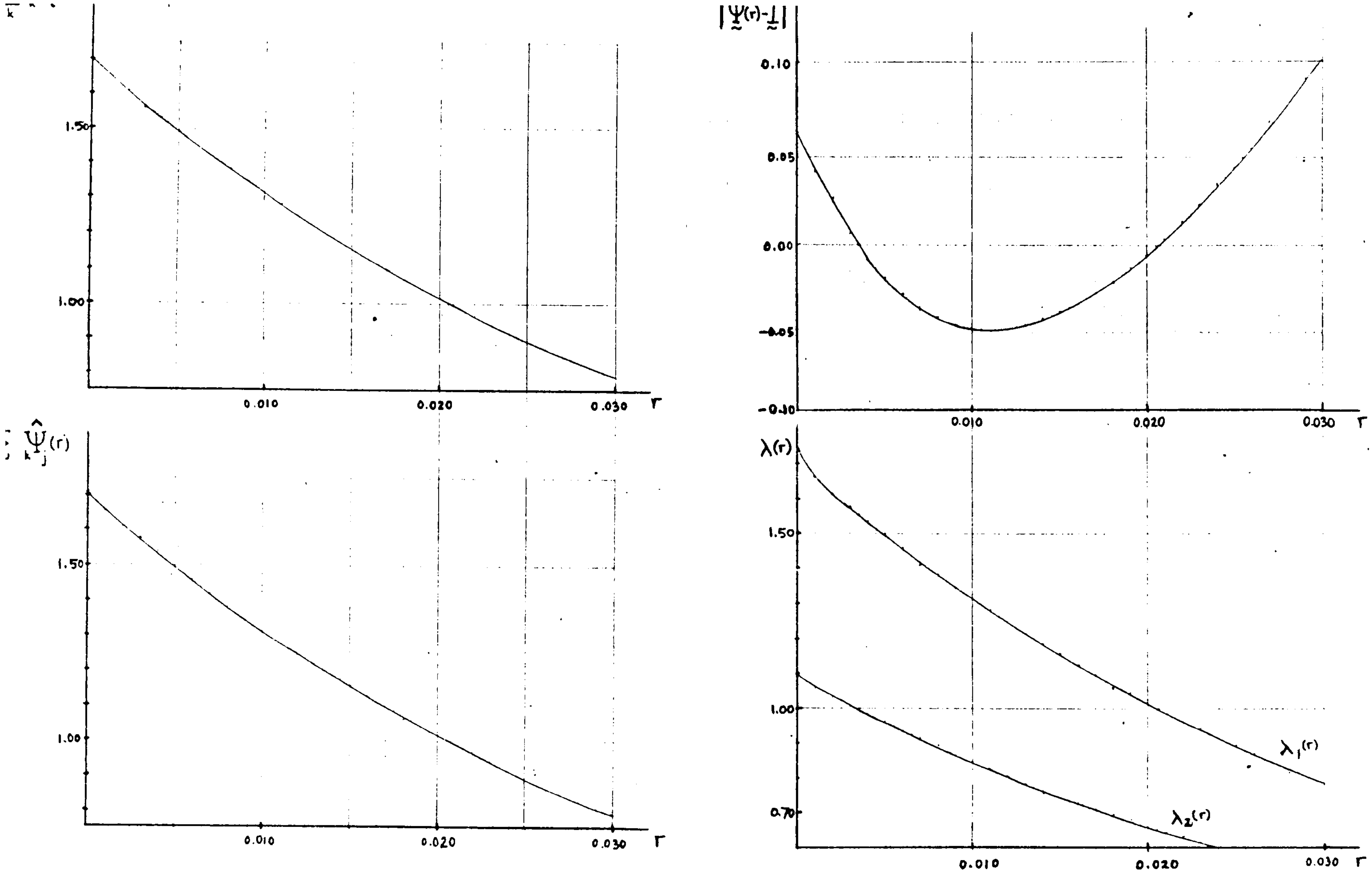

FIGURE 1. Graph of $\lambda(r)$, $|\Psi(r) - I|$, and $\hat{\Psi}(r)\{1\}$ for a two-region model of United States females, 1958: (i) California and (j) Rest of the United States*

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* Source: Table 2.

we find for ${}^{Q}j/Q_i = {}^{Q}i/Q_j = 1$, an initial approximation of $r_1(1) = 0.010210$ if we use the equation relating to California and $r_1(1) = 0.028462$ if instead we rely on the equation for the rest of the United States. With an initial approximation of r_1, we may proceed to obtain an improved approximation of ${}^{Q}j/Q_i$ and ${}^{Q}i/Q_j$ by solving for these two unknowns in (27) to find

$$\frac{Q_j}{Q_i} = \frac{1 - \sum_{x=\alpha}^{\beta-5} e^{-r(x+2.5)} \, {}_{i0}L_{.i}(x)F_i(x)}{\sum_{x=\alpha}^{\beta-5} e^{-r(x+2.5)} \, {}_{j0}L_{.i}(x)F_i(x)} \tag{53}$$

$$\frac{Q_i}{Q_j} = \frac{1 - \sum_{x=\alpha}^{\beta-5} e^{-r(x+2.5)} \, {}_{j0}L_{.j}(x)F_j(x)}{\sum_{x=\alpha}^{\beta-5} e^{-r(x+2.5)} \, {}_{i0}L_{.j}(x)F_j(x)} \tag{54}$$

and computing their values with r set equal to either of the two alternative initial approximations, or preferably, to their average: $r_1(2) = \frac{0.010210 + 0.028462}{2} = 0.019336$. Entering these improved approximations of ${}^{Q}j/Q_i$ and ${}^{Q}i/Q_j$ into (27), we obtain an improved approximation of r_1. Repeating this iterative procedure until two consecutive approximations of r_1 differ by less than a fixed amount, say 0.0001, we find $r_1 = 0.020593$ as before.[3]

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[3] Care must be exercised to ensure that the above iterative procedure converges to the <u>maximal</u> real root and not merely a real root that satisfies (13). In practice this danger can be minimized by setting r equal to an estimate of the single-region solution for the entire multiregional system and then using (53) and (54) to obtain an initial approximation of ${}^{Q}j/Q_i$ and ${}^{Q}i/Q_j$. Alternatively, the iterative procedure may be carried out over region of birth measures instead of region of residence ones, thereby eliminating the use of weighting ratios altogether.

Having outlined methods for the numerical evaluation of r_1, we now turn to the associated problem of numerically evaluating $\{\underset{\sim}{Q}_1\}$. Given a multiregional population, life table, and fertility schedule, all in five-year age groups, say, we may proceed to evaluate (43) for $r = r_1$ by arguments that are the multiregional analogues of those normally used in the single-region case (for example, Keyfitz (1968), p. 264). Assuming once again a two-region system, we begin by evaluating the $i^{\underline{th}}$ element of $\{\underset{\sim}{V}\}$. For this task we need the following numerical approximation of $G_i(t)$:

$$G_i(t) \cong \sum_{x=0}^{\beta-5} \left[K_i(x) \frac{{}_{ix}L_{.i}(x+t)}{L_{i.}(x)} + K_j(x) \frac{{}_{jx}L_{.i}(x)}{L_{j.}(x)} \right] F_i(x+t), \tag{55}$$

where t and x take on values at five-year intervals. Premultiplying both sides by e^{-rt} and averaging the result between two consecutive values of t multiplied by 5, gives

$$\begin{aligned} V_i &= \sum_{t=0} \sum_{x=0} \frac{5}{2} \left[e^{-rt} G_i(t) + e^{-r(t+5)} G_i(t+5) \right] \\ &= \frac{5}{2} G_i(0) + 5e^{-5r} G_i(5) + \ldots + 5e^{-rt} G_i(t) + \ldots \end{aligned} \tag{56}$$

$$= \frac{5}{2} \sum_{x=0}^{\beta-5} K_i(x) F_i(x) + 5 \sum_{x=0}^{\beta-5} \sum_{t=5}^{\beta-x-5} e^{-rt} \left[K_i(x) \frac{{}_{ix}L_{.i}(x+t)}{L_{i.}(x)} + K_j(x) \frac{{}_{jx}L_{.i}(x+t)}{L_{j.}(x)} \right] F_i(x+t)$$

A parallel argument yields an analogous expression for V_j, to complete the formula for the numerical evaluation of $\{\underset{\sim}{V}\}$. Premultiplying $\{\underset{\sim}{V}\}$ by the inverse of

$$\underset{\sim}{A} = \begin{bmatrix} \sum_{x=\alpha}^{\beta-5} xe^{-r(x+2.5)} {}_{i0}L_{.i}(x)F_i(x) & \sum_{x=\alpha}^{\beta-5} xe^{-r(x+2.5)} {}_{j0}L_{.i}(x)F_i(x) \\ \sum_{x=\alpha}^{\beta-5} xe^{-r(x+2.5)} {}_{i0}L_{.j}(x)F_j(x) & \sum_{x=\alpha}^{\beta-5} xe^{-r(x+2.5)} {}_{j0}L_{.j}(x)F_j(x) \end{bmatrix} \quad (57)$$

evaluated at r equal to the maximal real root, provides the required numerical approximation of $\{\underset{\sim}{Q}_1\}$.

4. The Multiregional Net Maternity Function

We now take up the multiregional generalization of several concepts related to the net maternity function defined in (13). To simplify the exposition, we shall continue to focus on a two-region population system with regions i and j, respectively. Thus, we have

$$\underset{\sim}{\Phi}(x) = \begin{bmatrix} {}_i\phi_i(x) & {}_j\phi_i(x) \\ {}_i\phi_j(x) & {}_j\phi_j(x) \end{bmatrix} = \begin{bmatrix} {}_{i0}p_i(x)\, m_i(x) & {}_{j0}p_i(x) m_i(x) \\ {}_{i0}p_j(x) m_j(x) & {}_{j0}p_j(x) m_j(x) \end{bmatrix} \quad (58)$$

where, for example, ${}_{j0}p_i(x)$ denotes the probability that an individual born in region j will be alive in region i at age x, and ${}_j\phi_i(x)$ represents the j-born contribution to the net maternity (or paternity) function of region i.

First, we define the n^{th} moment matrix of the multiregional net maternity function to be

$$\underset{\sim}{R}(n) = \int_\alpha^\beta x^n \underset{\sim}{\Phi}(x)dx = \int_\alpha^\beta x^n \underset{\sim}{M}(x)\underset{\sim}{P}(x)dx. \quad (59)$$

Observe that $\underset{\sim}{\Psi}(0) = \underset{\sim}{R}(0)$, a matrix we shall call the net reproduction matrix. Its typical element ${}_jR_i(0)$ gives the number of children expected to be born in region i to a girl now born in region j, under the current regime of fertility, mortality, and migration.

Next, generalizing two other conventional measures in the single-region model, we define the matrices $\underset{\sim}{\mu}$ and $\underset{\sim}{\sigma}^2$, with elements such as

$$ {}_j\mu_i = \frac{\int_\alpha^\beta x\,{}_j\phi_i(x)dx}{\int_\alpha^\beta {}_j\phi_i(x)dx} = \frac{{}_jR_i(1)}{{}_jR_i(0)} \tag{60} $$

and

$$ {}_j\sigma_i^2 = \frac{\int_\alpha^\beta (x - {}_j\mu_i)^2\,{}_j\phi_i(x)dx}{\int_\alpha^\beta {}_j\phi_i(x)dx} = \frac{{}_jR_i(2)}{{}_jR_i(0)} - \left(\frac{{}_jR_i(1)}{{}_jR_i(0)}\right)^2 , \tag{61} $$

where ${}_j\mu_i$ is the mean age of childbearing of j-born parents who are members of the stationary population of region i, and ${}_j\sigma_i^2$ is the associated variance. As in the single-region case, one may use the numerical values of these measures to graduate the elements of the multiregional net maternity function. For example, we could assume that ${}_j\phi_i(x)/{}_jR_i(0)$ is normally distributed with mean ${}_j\mu_i$ and variance ${}_j\sigma_i^2$ and proceed to graduate this function with the method of moments.

Using the empirical multiregional net maternity function set out in Table 1, we may calculate the matrices $\underset{\sim}{R}(0), \underset{\sim}{\mu}$, and $\underset{\sim}{\sigma}^2$ that are presented at the top of Table 3. Their interpretation is straightforward. For example, under the 1958 regime of growth, a woman born in California is expected to be replaced by 1.192 + 0.495 = 1.687 girl children, of whom 0.495 will be born in the rest of the United States. The mean age of childbearing of the stationary native California population is 25.430 years with a variance of 34.234 years. This may be contrasted with 25.967 years, which is the same measure computed with the single-region model, under the assumption of no internal migration. In turn, this may be compared with the 26.295 years

TABLE 3. Net reproduction rates and the means and variances of the multiregional net maternity function for United States females, 1958: Two-region model of (i) California and (j) the rest of the United States

1. Multiregional model:

$$\underset{\sim}{R}(0) = \begin{bmatrix} 1.192 & 0.105 \\ 0.495 & 1.603 \end{bmatrix} \qquad \underset{\sim}{\mu} = \begin{bmatrix} 25.430 & 27.468 \\ 27.713 & 26.221 \end{bmatrix} \qquad \underset{\sim}{\sigma}^2 = \begin{bmatrix} 34.234 & 37.837 \\ 40.122 & 37.629 \end{bmatrix}$$

2. Separate single-region models (no internal migration):

$$\begin{bmatrix} 1.664 & \\ & 1.712 \end{bmatrix} \qquad \begin{bmatrix} 25.967 & \\ & 26.325 \end{bmatrix} \qquad \begin{bmatrix} 36.092 & \\ & 37.921 \end{bmatrix}$$

3. Consolidated single-region model:

$$1.708 \qquad 26.295 \qquad 37.767$$

which the single-region model yields when applied to the consolidated data, that is, the United States female population.

Finally, Lotka's method for numerically evaluating the intrinsic rate of growth in the single-region model revolves around the solution of the quadratic equation:

$$\frac{1}{2} r^2\sigma^2 - r\mu + \ell n R(0) = 0,$$

which may be solved iteratively as

$$r^* = \frac{\ell n R(0)}{\mu - \frac{r\sigma^2}{2}}$$

where the r^* on the left-hand side is regarded as an improved approximation of the r on the right-hand side. A parallel argument for the multiregional model leads to the solution of the quadratic equation

$$\frac{1}{2} r^2\sigma_k^2 - r\mu_k + \ell n\, R_k(0) = 0, \tag{62}$$

where

$$R_k(0) = \int_\alpha^\beta {}_{\cdot 0}p_k(x)m_k(x)dx = \int_\alpha^\beta \frac{{}_{\cdot 0}\ell_k(x)}{\ell_k(0)} m_k(x)dx$$

$$= \int_\alpha^\beta \left[\frac{{}_{k0}\ell_k(x)}{\ell_k(0)} + \frac{{}_{s0}\ell_k(x)}{\ell_k(0)} \right] m_k(x)dx$$

$$= \int_\alpha^\beta {}_{k0}p_k(x)m_k(x)dx + \frac{\ell_s(0)}{\ell_k(0)} \int_\alpha^\beta {}_{s0}p_k(x)m_k(x)dx$$

$$= {}_kR_k(0) + \frac{\ell_s(0)}{\ell_k(0)}\ {}_sR_k(0)\quad , \tag{63}$$

and, by analogous reasoning,

$$\mu_k = \frac{R_k(1)}{R_0(1)} \tag{64}$$

and

$$\sigma_k^2 = \frac{R_k(2)}{R_k(0)} - \left(\frac{R_k(1)}{R_k(0)} \right)^2 \quad . \tag{65}$$

Evidently, given appropriate values for $\ell_s(0)/\ell_k(0)$, k,s = i,j, k ≠ s, we could compute $R_k(0)$, μ_k, and σ_k^2, for k = i,j, and solve (62) twice to find two separate numerical approximations of r. If the ratios are at their appropriate levels, then the two separate approximations of r should not differ beyond a specified small amount, such as 0.0001. This is because at stability all regional populations of a multiregional system grow at the same constant intrinsic rate of growth, namely, the r value that satisfies the characteristic equation in (13). If the ratios are not at their appropriate levels, then we may obtain improved approximations using (53) and (54) and recompute r once again. By iterating back and forth in this way, we converge, as in Section 3, to the maximal real root, r. Thus, we have yet a fourth method for solving (13). As in the single-region model, this method is equivalent to fitting a normal curve to the net maternity function and yields an approximation for r that is slightly higher than the value found by the other three methods.

A numerical illustration at this point may help to clarify the iterative solution method. Consider the two-region data in Table 3. Setting $Q_i = Q_j = 1$, we begin by calculating

$$\{\underset{\sim}{R}(n)\} = \underset{\sim}{Q}^{-1}\, \underset{\sim}{R}(n)\underset{\sim}{Q}\{\underset{\sim}{1}\} = \hat{\underset{\sim}{R}}(n)\{\underset{\sim}{1}\}, \text{ for } n = 0,1,2 \tag{66}$$

the $k^{\underline{th}}$ element of which is $R_k(n)$, for k = i,j. This gives, for example,

$$\{\underset{\sim}{R}(0)\} = \begin{Bmatrix} 1.296 \\ 2.098 \end{Bmatrix} \quad \{\underset{\sim}{R}(1)\} = \begin{Bmatrix} 33.179 \\ 55.762 \end{Bmatrix} \quad \{\underset{\sim}{R}(2)\} \begin{Bmatrix} 894.359 \\ 1562.811 \end{Bmatrix}$$

and

$$\{\underset{\sim}{\mu}\} = \begin{Bmatrix} 25.595 \\ 26.573 \end{Bmatrix} \quad \{\underset{\sim}{\sigma}^2\} = \begin{Bmatrix} 34.833 \\ 38.619 \end{Bmatrix}$$

Substituting these values into (62) yields $r_i = 0.01021$ and $r_j = 0.02848$. Taking their average as our initial approximation of r, we proceed to compute an improved approximation of Q_j/Q_i and Q_i/Q_j using (53) and (54). These in turn lead to improved approximations of $\{\underset{\sim}{R}(n)\}$ and thereby to r. Repeating this iterative procedure, we converge to the values $r = 0.02060$ and $Q_j/Q_i = 4.823$. At this point,

$$\{\underset{\sim}{R}(0)\} = \begin{Bmatrix} 1.697 \\ 1.706 \end{Bmatrix} \quad \{\underset{\sim}{\mu}\} = \begin{Bmatrix} 26.037 \\ 26.311 \end{Bmatrix} \quad \{\underset{\sim}{\sigma}^2\} = \begin{Bmatrix} 36.175 \\ 37.904 \end{Bmatrix} .$$

5. Relations Under Stability

In a two-region population system, persons aged x in region i at time t were born at time $t - x$ in either region. Hence, for example, we may define the density of persons aged x in region i at time $t, k_i(x,t)$, to be the expected value

$$k_i(x,t) = B_i(t - x)\,{}_{i0}p_i(x) + B_j(t - x)\,{}_{j0}p_i(x). \tag{67}$$

Consequently the total population in region i is

$$\int_0^\omega k_i(x,t)dx,$$

and the proportion of that regional population which is of age x at time t is of density

$$c_i(x,t) = \frac{k_i(x,t)}{\int_0^\omega k_i(x,t)dx} \quad .$$

At stability, births in each region will be increasing geometrically at the rate r, for example, $B_i(t) = Q_i e^{rt}$. Hence

$$c_i(x,t) = \frac{Q_i e^{-rx}\,{}_{i0}p_i(x) + Q_j e^{-rx}\,{}_{j0}p_i(x)}{\int_0^\omega [Q_i e^{-rx}\,{}_{i0}p_i(x) + Q_j e^{-rx}\,{}_{j0}p_i(x)]dx} = c_i(x) \tag{68}$$

an expression for $c_i(x,t)$ that does not contain t.

Notice that by setting $x = 0$ in (5.21), we obtain the birth rate of the stable population in region i:

$$b_i = c_i(0) = \frac{Q_i}{\int_0^\omega [Q_i e^{-rx}\,{}_{i0}p_i(x) + Q_j e^{-rx}\,{}_{j0}p_i(x)]dx} \tag{69}$$

$$= \frac{1}{\int_{o}^{\omega} \sum_{s} \frac{Q_s}{Q_i} e^{-rx} \; {}_{s0}p_i(x)dx} \quad , \tag{70}$$

whence

$$c_i(x) = b_i e^{-rx} \sum_{s} \frac{Q_s}{Q_i} \; {}_{s0}p_i(x). \tag{71}$$

Note that (17) can be obtained by substituting the expression for $c_i(x)$ set out in (71) into the definitional equation for the intrinsic birth rate:

$$b_i = \int_{\alpha}^{\beta} c_i(x) m_i(x) dx \quad , \qquad k = i,j, \tag{72}$$

$$= b_i \int_{\alpha}^{\beta} e^{-rx} \sum_{s} \frac{Q_s}{Q_i} \; {}_{s0}p_i(x) m_i(x) dx,$$

and dividing both sides by b_i gives (17).

Having r and b_k, we may obtain $\Delta_k = d_k + o_k - i_k$ as a residual, since

$$r = b_k - d_k - o_k + i_k = b_k - \Delta_k, \qquad (k = i,j) \tag{73}$$

where Δ_k may be defined to be the intrinsic net "absence" rate in region k, the absences being the result of either death or net outmigration, and

$$d_k = \int_{o}^{\omega} c_k(x) \; \mu_k(x) dx, \tag{74}$$

$$o_k = \int_{o}^{\omega} c_k(x) \; \nu_k(x) dx, \tag{75}$$

$$i_k = r - b_k + d_k + o_k. \tag{76}$$

As in the single-region model, d_k denotes the intrinsic death rate and $\mu_k(x)$ is the force of mortality at age x. We introduce in (75) and (76), the multiregional concepts of the intrinsic outmigration and inmigration rates, respectively, together with the <u>force of outmigration</u>, $\nu_k = \sum_s \nu_{ks}$.

Equation (71) in its single-region form, has been used to infer the birth rate and intrinsic rate of increase from a reliable census age distribution and an appropriate life table [Keyfitz, (1968), p. 184]. Starting with the relation

$$\frac{c(x)}{p(x)} = be^{-rx} \tag{77}$$

we may take natural logarithms to obtain

$$\ln\frac{c(x)}{p(x)} = \ln b - rx. \tag{78}$$

Fitting a straight line $y = \alpha + \beta x$ through the observed points $y = \ln [c(x)/p(x)]$, gives

$$b = e^{\alpha}$$

$$r = -\beta \quad .$$

We may generalize the above to the multiregional case by starting with (71) in the form of

$$\frac{c_k(x)}{\sum_s \frac{Q_s}{Q_k}\, {}_{s0}p_k(x)} = b_k e^{-rx} \quad , \quad k,s = i,j,$$

and then taking natural logarithms of both sides:

$$\ell n \left(\frac{c_k(x)}{\sum_s \frac{Q_s}{Q_k} \, {}_{s0}p_k(x)} \right) = \ell n \; b_k - rx, \qquad (79)$$

which is the multiregional counterpart of (78).

Returning to (71), we re-express the continuous expression for the stable age distribution in region k as

$$c_k(x) = \sum_s b_k e^{-rx} \frac{Q_s}{Q_k} \, {}_{s0}p_k(x) = \sum_s \frac{Q_s}{Q_k} \, {}_s c_k(x), \qquad (80)$$

where, for example,

$${}_j c_i(x) = b_i e^{-rx} \, {}_{j0}p_i(x) = \frac{e^{-rx} \, {}_{j0}p_i(x)}{\int_o^\omega \sum_s \frac{Q_s}{Q_i} e^{-rx} \, {}_{j0}p_i(x) dx} \qquad (81)$$

Collecting such terms into a matrix, we define

$$\underset{\sim}{c}(x) = e^{-rx} \underset{\sim}{b} \, \underset{\sim}{P}(x) = \begin{bmatrix} {}_i c_i(x) & {}_j c_i(x) \\ {}_i c_j(x) & {}_j c_j(x) \end{bmatrix} \qquad (82)$$

where $\underset{\sim}{b} = \begin{bmatrix} b_i & \\ & b_j \end{bmatrix}$, and the associated matrix

$$\underset{\sim}{C}(a) = \int_x^{x+5} \underset{\sim}{c}(a) da = \begin{bmatrix} {}_i C_i(x) & {}_j C_i(x) \\ {}_i C_j(x) & {}_j C_j(x) \end{bmatrix} , \qquad (83)$$

where, for example, ${}_j C_i(x)$ is the proportion of the total population in region i, born in region j and now aged x to $x + 4$.

The numerical evaluation of the various stable growth measures described above is straightforward and follows from the numerical approximations first set out in Section 3. Thus, for example,

$$b_k = \frac{1}{\sum_{x=0}^{\omega-s} \sum_{s} e^{-r(x+2.5)} \frac{Q_s}{Q_k} \, {}_{s0}L_{.k}(x)} , \qquad (84)$$

a result with which we may obtain

$${}_sC_k(x) = b_k e^{-r(x+2.5)} \, {}_{s0}L_{.k}(x) = \frac{e^{-r(x+2.5)} \, {}_{s0}L_{.k}(x)}{\sum_{x=0}^{\omega-s} \sum_{s} e^{-r(x+2.5)} \frac{Q_s}{Q_k} \, {}_{s0}L_{.k}(x)}$$

or, in matrix form,

$$\underset{\sim}{C}(x) = e^{-r(x+2.5)} \, \underset{\sim}{b} \, \underset{\sim}{L}(x). \qquad (85)$$

Next, we recall the matrix of mean ages of childbearing in the stable population defined earlier in (44) and (57):

$$\underset{\sim}{A} = \begin{bmatrix} {}_iA_i & {}_jA_i \\ {}_iA_j & {}_jA_j \end{bmatrix} = \begin{bmatrix} \int_\alpha^\beta xe^{-rx} \, {}_i\phi_i(x)dx & \int_\alpha^\beta xe^{-rx} \, {}_j\phi_i(x)dx \\ \int_\alpha^\beta xe^{-rx} \, {}_i\phi_j(x)dx & \int_\alpha^\beta xe^{-rx} \, {}_j\phi_j(x)dx \end{bmatrix} ,$$

where the numerical approximation of ${}_jA_i$, for example, is

$${}_jA_i = \sum_{x=\alpha}^{\beta-5} xe^{-r(x+2.5)} \, {}_{j0}L_{.i}(x)F_i(x) \quad . \qquad (86)$$

Observe that the mean age of childbearing of the stable population in region k is

$$A_k = \sum_s \frac{Q_s}{Q_k} \, {}_sA_i \ . \tag{87}$$

Finally, as in the single-region model, we define the <u>mean length of generation</u> in region k to be the number of years required to increase the stable population in region k by the net reproduction rate in that region. Whence

$$e^{rT_k} = R_k(0)$$

and

$$T_k = \frac{1}{r} \ln R_k(0) . \tag{88}$$

Table 4 sets out several measures describing relations under stability in the two-region numerical example of United States females residing in California and the rest of the United States.

TABLE 4. Relations under stability for United States females, 1958: Two-region model of (i) California and (j) the rest of the United States

Parameter	California	Rest of the U.S.
r	0.02059	0.02059
b	0.02648	0.02741
Δ	0.00589	0.00682
%	0.1767	0.8233
A_k^*	25.319	25.560
T_k	25.664	25.921

$$* \quad \underset{\sim}{A} = \begin{bmatrix} 24.753 & 26.710 \\ 26.911 & 25.476 \end{bmatrix}$$