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ON THE CORE AND DUAL SET OF LINEAR PROGRAMMING GAMES

by

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ABSTRACT

We study the relation between the core of a given linear programming game and the dual set of the corresponding linear program. We strenghten a sufficient condition of Owen as to when the later set is obtained as a finite intersection of cores of refined games. Our stronger condition is alos necessary. We give several large classes of linear programming games for which the core qual the dual set even without refinement. We give necessary and sufficient conditions for this equality to hold and show that every totally balanced game can be represented by a linear programming system for which the core equals the dual set.

I. Introduction

Several authors have studied cooperative games with sidepayments which arise from linear programming problems. A general case is considered in Owen, [7]. In his formulation, each player is endowed with a vector of resources from which a given set of outputs can be produced subject to linear constraints. Furthermore, the objective function is linear with the amounts of outputs produced. The value of each coalition is the maximal value of the outputs which can be produced from the resources available to the members of this coalition. We call games which arise in this way LP-games.

A special class of LP-games was studied by Shapley and Shubik, [9], where the linear program involved is the Assignment problem. Lately, Dubey and Shapley [4], and independently Kalai and Zemel, [6], have considered generalization of this idea where the games are derived from general optimization problems having non linear constraints and objective functions. A natural question which arises in such cases is how to allocate the optimal profits (or costs) between the various participants. We deal in this paper with the solution concept of the core. In general, the core of a given game may be empty. However, for LP games, Owen has shown that this does not happen. In fact, there exit allocations in the core of the game whose interpretation is very familiar and appealing. Consider any dual optimal solution for the linear program. The various components of this solution can be viewed as (shadow) prices for the various resources. It turns out that if we pay each player for his resource vector (using the dual prices) than the resulting allocation vector is in the core. We call allocations which arise in this way dual-allocations. In this paper we study the relation between the core and the set of dual allocations of a general LP-game. In general, not every point in the core is a dual allocation. Owen, [7], has shown that by replicating the game infinitely many times the core shrinks to the set of dual allocations. Moreover, he has shown that if the optimal dual solution is a singleton, it is sufficient to replicate the game a finite number of times. Billera and Raanan [2] considered LP-games with a continuum of players and showed that in this case every core allocation is induced by an optimal dual solution. It is interesting to note that for several classes of LP games which where studied in the literature, the core and the dual set coincide even without replication. This is the case for the Assignment Games, [Shapley and Shubik, [9]), Simple Network Games, [Kalai and Zemel, [5]) and Location Games on Trees, (Tamir, [10]).

The main tool in our study is the value function of the linear programming problem involved. Conceptually, this function can be viewed as representing a game with continuum of players of finitely many types (Billera, Raanan [2]) or as a fuzzy game (Aubin, [1]). Value functions of only a subset of the right hand sides were considered by Samet, Tauman and Zang, [8].

The structure of the paper is as follows. In section II we give the necessary definitions and preliminaries. In section III we consider the conditions under which the dual set is obtained as a finite interesection of refined cores. In section IV we describe large classes of LP-games for which the core is equal to the dual set, even without replication. Finally, in section V we give necessary and sufficient condition for this equality to hold and demonstrate that every totally balanced game can be represented as arising from a linear programming problem for which the core of the game and the dual set of the linear program coincide.

II. Definitions and Preliminaries

Let $N = \{1,2,\ldots n\}$ be a set of players. A coalition is a non-empty subset of N. We denote the set of coalitions of N by 2^N. We associate with each coalition $S\varepsilon 2^N$ a 0-1 vector $t^S \varepsilon R^n$ such that

$$t_i^S = \begin{cases} 1 & \text{if } i\varepsilon S \\ 0 & \text{if } i\notin S. \end{cases}$$

Conversely, for every 0-1 vector $t\varepsilon R^n$ we define the coalition S_t by:

$$S_t = \{i \mid t_i = 1\}.$$

We reserve the letter e to denote the vector $(1,1,\ldots,1)\varepsilon R^n$. Thus $S_e = N$.

Assume that each player j is endowed with a vector of resources $b^j = (b_1^j, b_2^j,\ldots,b_m^j)\varepsilon R^m$. Let $B = (b_{ij})$ be the $(m \times n)$ matrix whose j^{th}-column is b^j. For each coalition $S\varepsilon 2^N$, let $b(S) = Bt^S$. Thus b(S) is the vector of total amount of resources available to the coalition S. Let A be an $m \times p$ matrix and $c \varepsilon R^p$. For each coalition S consider the linear program:

$$\begin{array}{lll} P(S): & \text{maximize} & cy \\ & \text{subject to} & Ay \leqslant b(S) \\ & & y \geqslant 0 \end{array}$$

Let $P = \{P(S) \mid S\varepsilon 2^N\}$. We call a system of linear programs of this type a Linear Programming System (LP-System). We assume in the sequel that each of the linear programs P(S) is feasible and bounded[1]. Under these conditions, the optimal objective function values for the various coalitions are well defined. We regard the set of these values as a set function $V_P: 2^N \to R$ i.e. as a characteristic function of a game. We refer to games which arise in this fashion as Linear Programming Games (LP-Games)[2]. In cases where the LP-system associated with a given game is clear from the context, we suppress the subscript P and refer to the game simply as V. This convention is also used for all other constructs which are defined with respect to a given LP system P.

The core of a game V (not necessarily an LP game) is the set

$$\{x \in R^n \mid \sum_{i=1}^{n} x_i = V(N), \quad \sum_{i \in S} x_i \geq V(S) \text{ for every } S \in 2^N\}$$

We denote the core of the game V_P by C_P. In general, the core of a given game may be empty. However for LP-games we have:

Theorem 2.1 (Owen, [7]):

For every LP system P, the core C_p is not empty.

Games with non empty cores are called balanced. For a game V let V^S be the game obtained by restricting V to coalitions contained in S. A balanced game V is called totally balanced if the core of each subgame V^S is not empty. We note that every subgame of an LP-game is itself an LP-game. It follows from Theorem 1 that every LP-game is totally balanced. It was shown in [5] that the converse of this is also true, i.e. that every totally balanced game can be generated by a certain type of LP system. A different LP-system associated with a given totally balanced game is described in

LP-system associated with a given totally balanced game is described in section V.

In general, the task of finding an allocation x in the core of a given game (when this set is not empty) is computationally tedius (one has to solve a linear program with 2^n constraints). However, for an LP-game V we can do much better. Let $u = (u_1,\dots,u_m)$ be an optimal dual solution for the linear program P(N). The components of this vector can be viewed as (shadow) prices of the various resources. Consider the vector x =uB. The n dimensional vector x can be considered as a payoff vecotr which endows each player the value of his resources vector according to the price vector u. We call the vector x, a <u>dual-allocation</u>, and denote by DS_p the set of all dual allocations, i.e.:

$$DS_p = \{x \in R^n \mid x = uB \text{ for some dual optimal vector } u\}.$$

<u>Theorem 2.2</u> (Owen, [7]) :

$$DS_p \subseteq C_p.$$

There are several classes of linear programming systems which yield games with $DS_p = C_p$. However, this is not the case in general. To illuminate the relation between these sets it is instructive to observe their behavior when the players are splitted (or equivalently replicated).

Let P be a given LP-System. The <u>r^{th} refinement</u> of P, denoted P^r, is obtained by splitting each original player (column of B) into r identical players each receiving b^j/r as his initial endowment. We call the set of r identical players which replaces one original player a <u>suit</u>. Let V_{P^r} be the

LP-game generated by the system P^r. Note that every coalition of V_{P^r} which is composed of one representative from each suit must receive an identical amount, $V(N)/r$, in every allocation which belongs to C_{P^r}. It follows that all the members of each suit must receive the same amount in each core allocation. We can thus represent each core allocation for V_{P^r} by a vector $x \varepsilon R^n$ where x_i is the total amount allocated to the i^{th} suit (each member in this suit receiving x_i/r). Let C_P^r be the set of core allocations for the game V_{P^r} represented in this way. Thus, $C^r \subseteq R^n$. Similarly, let $DS_P^r \subseteq R^n$ be the set obtained by compressing the dimension of DS_{P^r} by adding up the (identical) amounts allocated to all r members of each suit. It is straight forward to verify the following relations which are consequences of the appropriate definitions and of Theorem 1:

$$\emptyset \neq DS = DS^r \subseteq C^r \subseteq C$$

Thus, $DS \subseteq \bigcap_{r=1}^{\infty} C^r$. The converse of this statement is also true:

<u>Theorem 2.3</u> (Owen [7]) :

$$DS = \bigcap_{r=1}^{\infty} C^r.$$

Theorem 2.3 can be viewed as a special case of a limit theorem due to Debrew and Scarf, [3]. However, in the linear case, one can sometimes show an even stronger result namely a finite convergence of the core. One sufficient condition for this was given by Owen:

Theorem 2.4 (Owen [7]):

If the linear program P(N) has a unique dual optimal solution, for sufficiently large r

$$DS = C^r$$

Below, we study in more detail the conditions under which such a finite convergence is achieved.

III. Finite Convergence of the Core

It is convenient to study the relation between C, C^r and DS using the following extention of the LP system P. For each $t \in R^n$ consider the linear program:

$$P_t \qquad \text{maximize} \quad cy \qquad \text{subject to} \quad Ay \leq B\,t, \quad y \geq 0$$

We call a system of the type $\{P_t | t \in R^n\}$ an Extended LP-system. To simplify notation, we do not distinguish between a given LP-system and its associated extended system. In particular, we refer to both systems by the same name P.

Let $F_P: R^n \to R \cup \{-\infty\}$ be the function which assigns to each vector $t \in R^n$ the optimal value of P_t. We use the convention $F_P(t) = -\infty$ if the feasible set of the associated linear program is empty. It is well known that F is peacewise linear, concave, and homogenous (of degree one). Also, for every $S \in 2^N$ we have

$$F(t^S) = V(S)$$

Since the function F is concave and homogenous, the above relation ensures that F is finite over the entire non negative orthant, R^n_+. Let K denote the unit cube in R^n. For each positive integer r let

$$GRID^r(K) = \{t \epsilon K | t_i = q_i/r,\ 0 \leq q_i \leq r,\ \text{integer}\}.$$

Clearly for every $r \geq 1$,

$$C^r = \{x \epsilon R^n | xe = F(e),\ xt \geq F(t) \text{ for every } t \epsilon GRID^r(K)\}$$

Note, that this observation, together with the continuity of F imply that Theorem 2.3 can be restated as:

<u>Lemma 3.1</u>

$$DS = \{x \epsilon R^n | \ xe = F(e),\ xt \geq F(t) \text{ for every } t \epsilon K\}$$

Furthermore, it follows from the homogeneity and concavity of F that

$$DS = \{x \epsilon R^n | \ xe = F(e), xt \geq F(t), \text{ for every } t \epsilon R^n\}.$$

Thus, DS is the <u>superdifferential</u> of F at e, i.e., DS is the set of supporting hyperplanes of the graph of F at e.

We are now ready to examine conditions under which DS can be obtained as a finite intersection of cores for refined games.

Theorem 3.2

If the matrices A and B, defining an LP system are rational, then there exists an integer r_o such that

$$DS = C^{r_o}$$

Proof: For each $\varepsilon > 0$, let

$$K(\varepsilon) = \{t \in R^n \mid 1-\varepsilon \leq t_i \leq 1, \ i = 1 \ldots n\}$$

be the cube of side ε located at the upper corner of K. Since F is piecewise linear and concave, the following three conditions hold for sufficiently small ε.

(1) There exists a family of convex sets $C^1, \ldots, C^k$ such that $\bigcup_{i=1}^{k} C^i = k(\varepsilon)$.

(2) F is linear on C^i, $i=1 \ldots k$.

(3) $e \in C^i \quad i=1,\ldots,k$

Let ε be a rational which satisfies (1) - (3) and let $Q^i = \{q_1^i, \ldots q_{\ell_i}^i\}$ be the set of extreme points of C^i. Since ε and the entries in the matrices A and B are rational, it follows that the coordinates of the extreme points q_j^i, $i=1\ldots k$, $j=1\ldots\ell_i$ are rational as well. Thus, there exist positive integers r_{ij} such that $q_j^i \in GRID^{r_{ij}}(K)$.

Let $r_o = \prod_{\substack{i=1\ldots k \\ j=1\ldots\ell_i}} r_{ij}$ let $x \in C^{r_o}$ We have to show that $x \in DS$. By lemma 3.1 it suffices to show that $xt \geq F(t)$ for every $t \in K$. We show first that this relation holds for $K(\varepsilon)$.

Let $t_0 \in K(\varepsilon)$. Then $t_0 \in C^i$ for some $i \in \{1...k\}$. For each extreme point $q_j^i \in Q^i$, we know that $xq_j^i \geq F(q_j^i)$, since $q_j^i \in GRID^{r_0}(K)$. Since xt and F(t) are two affine functions on C_i and $xt \geq F(t)$ for the extreme points of C_i, it follows that this inequality holds for each $t \in C_i$.

To conclude the proof, let $t_0 \in K$. Consider the line interval $[e,t_0]$. There exists an initial segment $[e,t_1]$ of this interval which is within $K(\varepsilon)$. Consider the restriction of the functions F(t) and xt to $[e,t_0]$. We note that the two functions assume the same value at e and that $xt \geq F(t)$ for t in the interval $[e,t_1]$. Since F is concave, it follows that $xt_0 \geq F(t_0)$.

Q.E.D.

Remark. The condition of Theorem 3.2 is not necessary. One can show that a necessary and sufficient conditon for the existance of r_0 such that $DS = C^{r_0}$ is that there exists ε which satisfies (1) - (3) and for which all the extreme points q_i^j are rational. One direction of this claim follows immediately from the proof of Theorem 3.2. We omit the proof of the reverse direction.

Owen's sufficient condition, (Theorem 2.4) can be derived along the lines of the proof of Theorem 2.3. Moreover, we can replace the requirement that there exists a unique dual optimal solution for P(N) by the (weaker) requirement that DS is a singleton:

Corollary 3.3

If there is a unique dual allocation, then for sufficiently large r

$DS = C^r$.

Proof: If DS is a singleton then there exists ε_o such that F is linear in $K(\varepsilon_o)$. Thus for $\varepsilon \leq \varepsilon_o$ the q_i^j are the vertices of the cube $K(\varepsilon)$. For sufficiently large r, we can choose a rational $\varepsilon < \varepsilon_o$ such that the vertices of $K(\varepsilon)$ are in $GRID^r(K)$.

Q.E.D.

IV. Coincidence of the core and the set of dual allocations.

There are several known classes of LP-Games for which DS = C, even without refinement. These include the Optimal Assignment Games of Shapley and Shubik, [9], Simple Network Games, Kalai and Zemel, [5], and Location Games on Tree Networks, Tamir, [10]. We describe the essential features of these classes in the Appendix. Below we study in more detail the conditions under which DS = C. We first describe two classes of games with this property. These classes properly subsume the three classes of games mentioned earlier.

For the first class of games, let the rows of A and B be partitioned into three sets (some of which may be empty) such that the linear system P has the representation

$$
\begin{array}{lll}
P(S) & \text{maximize} & cy \\
 & \text{subject to} & A^1 y \leq B^1 t^S \\
 & & A^2 y \geq B^2 t^S \\
 & & A^3 y = B^3 t^S \\
 & & y \geq 0
\end{array}
$$

(note that the inequalities convention is slightly different from the one introduced in section II). Let

$$A = \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix} , \qquad B = \begin{bmatrix} B^1 \\ B^2 \\ B^2 \end{bmatrix} .$$

The system P is called <u>simple zero-one</u> if the matrix A is composed of zeros and ones and if B is the identity matrix. Under these conditions we can identify the rows of A (or B) with the players of the game V. Thus, each player completely "controls" one of the constraints of the system. Note that in this case DS is simply the set of optimal dual solutions for p(N).

<u>Theorem 4.1</u>

Let P be a simple zero-one system. Then

$$DS_P = C_P.$$

<u>Proof</u>: First, note that every core allocation $x \varepsilon R^n$ must satisfy

$x_i \geqslant 0$ if the i'th constraint is "$\leqslant$".
$x_i \leqslant 0$ if the i'th constraint is "$\geqslant$".

Consider the k^{th} column of A, and let

$$S^k = \{i | a_{ik} = 1\}$$

Then, the (primal) solution $y_i^* = 0$, $i \neq k$, $y_k^* = 1$ is feasible for $P(S^k)$. Thus, x must satisfy

$$\sum_{i \varepsilon S^k} x_i \geq V(S^k) \geq c_k$$

But these conditions, together with the sign convention on the x_i's established previously, are precisely the constraints defining the dual set of P(N). Since $\sum_{i \varepsilon N} x_i = V(N)$ which is the value of P(N), it follows that x is an optimal dual solution.

Q.E.D.

The assignment games, [10], the simple network games [6], (in the path flow formulation) and the location games [10] are all examples of games generated by simple zero-one LP systems. In all three cases, the matrix A possesses some special additional structure. However, as is clear from the proof of Theorem 4.1 the only requirement is that the matrix be zero-one.

It was shown in [6] that DS = C for simple network games also when the arc flow formulation of these games is taken (In the path flow formulation mentioned earlier, there is a variable associated with each path of the network while in the arc flow formulation there is a variable associated with each arc of the network. The arc flow formulation typically involves much fewer variables and is more tractable computationally. For the equivalence between these two formulation see [6]. For details see the appendix). The following class is a generalization of such systems. Consider a system of the form

$$\begin{array}{lll} P(S) & \text{maximize} & cy \\ & \text{subject to} & A^1 y \leq B^1 t^S \\ & & A^2 y \leq 0 \end{array}$$

where the matrix B^1 is the identity matrix. Under these conditions we can identify the rows of A^1 with the players of V. Note that in this case

$$DS = \{x \in R^n \mid \text{for some } w, (x,w) \text{ is an optimal dual solution for } P(N)\}.$$

An interesting necessary and sufficient condition for a given $x \in R^n$ to be contained in DS is given by:

Lemma 4.2

Let $x \in R^n$. Then $x \in DS$ iff.

(i) $x_i \geqslant 0$ $\quad i = 1,\ldots,n$

(ii) $\sum_{i=1}^{n} x_i = V(N)$

(iii) For every primal solution y feasible to P(N) we have $(c - x A^1)y \leqslant 0$.

Proof: The necessity of conditions (i) - (iii) is obvious. For the sufficiency note that the third condition is equivalent to the implication

$$A^2 y \leqslant 0 \implies (c - xA^1)\, y \leqslant 0$$

Thus, by Farkas Lemma, there exists $w \geqslant 0$ such that $w A^2 = c - xA^1$. But this condition, together with (i) and (ii) imply that (x,w) is a dual optimal solution for P(N).

Q.E.D.

Note that for $x \in C$ and for y a primal optimal solution for P(N)

$$(c - A^1 x)y = cy - xA^1 y = V(N) - \sum_{i=1}^{n} x_i\, (A^1 y)_i \geqslant$$

$$\geqslant V(N) - \sum_{i=1}^{n} x_i = 0$$

Using lemma 4.2 we conclude:

Lemma 4.3

Let $x \in C$. Then $x \in DS$ iff the optimal value of the following program is equal to zero:

$$\hat{P}_x \qquad \text{maximize} \quad (c - A^1 x)\, y$$
$$\text{subject to} \quad A^1 y \leqslant e$$
$$A^2 y \leqslant 0$$

It follows that a necessary and sufficient condition for $DS = C$ is that for every $x \in C$ the condition of lemma 4.3 holds. A class of LP systems where this indeed is the case is the following:

Theorem 4.4

Let the matrices A^1, A^2 be such that for every objective vector c there exists an optimal solution for the program

$$\text{maximize} \quad c\, y$$
$$\text{subject to} \quad A^1 y \leqslant e$$
$$A^2 y \leqslant 0$$

with $A^1 y$ a 0-1 vector. Then $DS = C$.

Proof: Let $x \in C$, and consider the problem $\hat{P}_x$. By lemma 4.3 it is sufficient to show that the optimal value of this program is not positive. Let y^* be an optimal solution for this program with the integrality proeprty (i.e. $A^1 y^*$ is a zero one vector). Let $S = \{i \mid (A^1 y^*)_i = 1\}$. Assume, on the contrary, that $(c - x A^1)\, y^* > 0$. Note, that y^* is feasible to $P(S)$ and thus $V(S) \geq cy^*$.

Hence,

$$0 < (c - xA^1)y^* = cy^* - xA^1y^* \leqslant V(S) - \sum_{i\varepsilon S} x_i$$

which contradicts our assumption that $x \varepsilon C$.

Q.E.D.

The stipulations of Theorem 4.4 hold, for instance if A^2 is a totally unimolular matrix which contains (implicilty or explicitly) the non negativity constraints on the variables and where the matrix A^1 is the identity matrix. The path flow formulation of simple network games fall into this category.

VI. Balanced Extentions of Games

The LP systems covered by theorems 5 and 6 do not exhaust the cases for which DS = C. We now turn our attention to complete characterization of the conditions under which this equality holds.

Let P be a given LP with associated game V. Let us consider the following LP system:

$$\tilde{P}(S) \qquad \text{Maximize} \quad \sum_{T \subseteq N} y_T V(T)$$

$$\text{Subject to} \quad \sum_{T \subseteq N} y_T t^T = t^S$$

$$y_T \geqslant 0; \quad T \subseteq N$$

By letting the right hand side of the constraints to be any t we get the extention of the system $\tilde{P}$. Let us denote by H(t) the function $F_{\tilde{P}}(t)$ which associates with each t the value of $\tilde{P}_t$. We shall call H the <u>balanced extenstion</u> of V. Since V is balanced it follows that

$$H(t^S) = V(S)$$

for each $S \subseteq N$, or in other words $V_{\tilde{p}} = V_p$. Moreover, H is the minimal concave, homogeneous function which coincide with V on the vertices of K. Indeed let F be any concave homogeneous function such that $F(t^S) = V(S)$ for each $S \subseteq N$, then

$$H(t) = \sum_{T \subseteq N} y_T^* V(T) = \sum_{T \subseteq N} y_T^* F(t^T) \leqslant F(\sum_{T \subseteq N} y_T^* t^T) = F(t)$$

where y^* in the optimal solution for $\tilde{P}_t$.

Lemma 5.1

$$C_p = \{x \in R^n | xe = H(e),\ xt \geqslant H(t) \text{ for every } t \in K\}.$$

Proof: Observe that the set on the right hand side is $DS_{\tilde{P}}$ by lemma 3.1. Since $V_p = V_{\tilde{p}}$, it suffices to show that $C_{\tilde{P}} = DS_{\tilde{P}}$. By Theorem 2.2 $C_{\tilde{p}} \supseteq DS_{\tilde{p}}$. For the converse inclusion let $x \in C$, $t \in K$, and let y^* be an optimal solution to $\tilde{p}_t$. Then

$$H(t) = \sum_{S \subseteq N} y_S^* V(S) \leqslant \sum_{S \subseteq N} y_S^* xt^S = x \sum_{S \subseteq N} y_S^* t^S = x\, t$$

and $H(e) = V(N) = x\, e$.

Q.E.D.

Theorem 5.2 $C_p = DS_p$ if and only if F coincides with the balanced extension of V_p in a neighborhood of e.

Proof: By lemma 5.1 $C_P = DS_P$ if and only if $DS_P = DS_{\tilde{P}}$. The sets Ds_P and $DS_{\tilde{P}}$ are the supperdifferentials at e of F_P and $F_{\tilde{P}}(= H)$ respectively. Since both F_P and $F_{\tilde{P}}$ are piecewise linear it follows that $DS_P = DS_{\tilde{P}}$ if and only if F_P and $F_{\tilde{P}}$ coincide in a neighborhood of e.

Q.E.D.

It is obvious from the above discussion that DS is related to the linear system P rather then to the associated game V. In general, the game V may arise from several LP-systems each possibly yielding a different DS. It follows immediately from Lemma 5.1, that for every totally balanced game V there exist an LP system (namely $\tilde{P}$) such that $V = V_{\tilde{P}}$ and for which the dual set equals the core.

Remarks

1) We restrict our attention in this paper to games for which V(S) is finite for every S N since this is the usual convention. All the results remain virtually unchanged if we allow some coalitions (but not N) to have $V(S) = -\infty$. In this case all we have to assume is that P(N) is feasible (rather than that all programs P(S), $S \subseteq N$ are).

2) We formulate the game in term of maximization of a certain objective function, (profit). Of course, there is a parallel formulation in terms of minimizing cost.

Appendix

Description of Assignment, Simple Network and Location Games.

In the Assignment games, described by Shapley and Shubik, [9], the set of players is partitioned into two subsets, Q and R. There is a benefit, c_{ij}, which is accrued if player $i \varepsilon Q$ is assigned to player $j \varepsilon R$. A player can be assigned to at most one other player which must belong to the opposite subset. The value of a coaltion is the maximal sum of benefits that can be generated by its members. In the form stated, the problem is of a discrete nature. Nevertheless, it can be described by a linear programming system since the underlying matrix A is known to be totally unimodular. Formally, Let $R(S) = R \cap S$, $Q(S) = Q \cap S$, and consider the problem:

$$P(S) \qquad \begin{array}{ll} \text{Maximize} & \sum_{j \varepsilon R(S)} \sum_{i \varepsilon Q(S)} c_{ij} y_{ij} \\ \text{Subject to} & \\ & \sum_{j \varepsilon R(S)} y_{ij} \leq \begin{cases} 1, & i \varepsilon Q(S) \\ 0 & i \not\varepsilon Q(S) \end{cases} \\ & \sum_{i \varepsilon Q(S)} y_{ij} \leq \begin{cases} 1 & j \varepsilon R(S) \\ 0 & j \not\varepsilon R(S) \end{cases} \\ & y_{ij} \geq 0 \qquad i \varepsilon R(S), \; j \varepsilon Q(S). \end{array}$$

Simple network games, introduced in [5], are defined with respect to a given network G, with a specified source, s, and sink t. There are n arcs in the network, each belonging to a different player and each having a capacity of one unit. There is a profit, c_i (of arbitrary sign) associated with a unit of flow on arc i. For every coalition S (a subset of arcs), V(S) is the maximal value of flow from s to t which can be acheived using to arcs of S only. It is well known, [6], that this problem can be formulated in two equivalent ways: the path flow formulation and the arc flow formulation. For the first, let $Q = \{q_1, \ldots q_r\}$ be a listing of all the distinct simple s-t pathes in the network, each regarded as a subset of edges. Let A be the incidence matrix of Q (with rows corresponding to edges, columns to paths). For each path q_i let d_i be the value of a unit of flow through q_i i.e. $d_i = \sum_{j \varepsilon q_i} c_j$. Then, the linear programming system associated with game V is given by

$$P(S) \qquad \text{Maximize} \quad \sum_{j=1}^{r} c_j \; y_j$$

$$\text{Subject to} \quad \sum_{j=1}^{r} A_{ij} \, y_j \leq \begin{cases} 1 & i \varepsilon S \\ 0 & i \notin S \end{cases}$$

$$y_j \geq 0, \; j=1,\ldots,r$$

Alternatively, let A be the directed edge-node incidence matrix of the network with arcs as columns, nodes as rows (accept for s and t which are omitted), i.e.,

$$A_{ij} = \begin{cases} +1 & \text{if edge j is incident out of node i} \\ -1 & \text{if edge j is incident into node i} \\ 0 & \text{otherwise} \end{cases}$$

Then, game v is described by LP-System:

$$P(S) \quad \text{Maximum} \quad \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} A_{ij} y_j = 0 \quad \text{All nodes accept for } s \text{ and } t.$$

$$y_j \leq \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{if } j \notin S \end{cases}$$

$$y_j \geq 0, \quad j = 1 \ldots n$$

Finally, we consider the location games of Tamir, [10]. Let T = (V,E) a tree network with two specified sets of nodes $Q = \{q_1 \ldots q_n\}$, and $R = \{r_1 \ldots r_m\}$. The players of the game are the members of Q which we regard as customers. the members of R are available sites at which a certain facilities can be constructed. Each customer q_i requires that a facility be located at a distance of no more than d_i units away. The cost of constructing a facility at site r_j is c_j. The game V associates with each coalition S the minimal cost at which this coalition can satisfy the demand of its own members. Let A be defined by

$$A_{ij} = \begin{cases} 1 & \text{if } r_j \text{ is at a distance of no more than } d_i \text{ units away from } q_i \\ 0 & \text{otherwise} \end{cases}$$

and consider the LP system

$$P(S) \quad \text{Minimize} \quad \sum_{j=1}^{m} c_j y_j$$

$$\text{Subject to} \quad \sum_{j=1}^{m} A_{ij} y_j \geq \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

$$y_j \geq 0$$

In general, the system P describes the game V only when we restrict y_j to be a zero-one variable ($y_j = 1$ means a facility will be constructed at r_j). However, theorem 5 is invalid for optimization systems which involve zero-one variables (and in fact the core of such Integer Programming Games may be empty). The point of Tamir result is that for tree network the matrix A is balanced and thus the discrete optimizaiton problem can be replaced by its LP equivalent.

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