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THE ECONOMICS OF MATING, RACING AND RELATED GAMES

by

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. Introduction

In many multi-agent dynamic contexts resources allocated now to some investment activity generally affect the probability of future states experienced by other agents. For example, the care taken by a particular automobile driver now determines the probability that some other driver will be involved with him in an accident in the near future. Assuming non-cooperative behavior, an external effect exists in these situations unless property rights are properly defined. In the example, all drivers have an incentive to drive with proper care if the driver at fault in any accident is required to compensate for damages. In these games, another issue also exists. Is the number and composition of the participants optimal under conditions of free entry? Will the equilibrium associated with the assignment of property rights that induces efficient investment decisions also emit signals that attract the right numbers and types of participants?

Two different examples have recently appeared in the literature. The first is a model of matching equilibrium formulated by Diamond [80] and by Mortensen [79]. The second is a model of innovative competition introduced by Loury [79] and by Lee and Wilde [78]. Similar dynamic externalities are shown to exist in the two models. In each case, the external effect is internalized by an assignment of property rights analogous to fault liability in the accident example. However, the incentives for entry associated with an equilibrium given this assignment do not yield an efficient "market structure" in either example for different reasons. Comparative study of these two examples suggest a general model of related dynamic games. This general model is formulated and analyzed. The conclusions drawn from the examples are shown to hold more generally.

In the Loury-Lee-Wilde model of innovative competition, the R&D investments made by all agents determine the probability distribution over time to discovery of some specific new product or process. In the case of exclusive patent rights, there is too much competition in the sense that the non-cooperative investment in R&D made by each agent is excessive. In this paper, it is shown that the source of the externality can be viewed as follows. No competitor takes account of the loss in the prospect that the others suffer in the event that he will be the inventor. If the successful inventor were required to compensate the others for the value of seeking the discovery, the non-cooperative investment decisions are efficient. However, a non-cooperative equilibrium solution to the game defined by this allocation rule is shown to attract too many players.

In the Diamond-Mortensen model of matching there are two types of agents, say firms and workers. A surplus accrues to any matched pair which induces individuals of each type to seek out a member of the other. How the surplus is divided once a match is formed affects the incentive of an unmatched individual of each type to invest in search effort. In turn, the search effort determines the instantaneous probability at which each individual finds a partner. Given an equal division of the surplus, every unmatched agent invests too little in search because the benefit accruing to the partner once a match is formed is ignored. All unmatched agents will invest in search at socially efficient rates if the agent responsible for making each match is awarded its total value less the other's forgone value of continuing to search. However, if the probability of contacting an agent of the opposite type depends on their relative numbers, then the composition by type attracted to the process is sub-optimal. Specifically, no individual agent takes account of the effect of his participation on the contact probabilities of other agents.

These two examples suggest a general model which includes them as special cases. In the general formulation, each player's current action affects the probability that he will perpetrate some event involving others in the near future. The incentive of each agent to take action is determined by the manner in which the aggregate capital value of each event is allocated among the agents involved ex post. A non-cooperative solution to the game defined by the following contingent allocation scheme exists and every solution is Pareto efficient. The perpetrator of each event receives the total capital value of the event less a compensation paid to all other agents whose play is terminated by the event equal to the value of their continued participation under the scheme. In general, multiple equilibria exist and some of these may be perverse. Under conditions that guarantee uniqueness, the composition of players by type maximizes social surplus if and only if the probability of being the perpetrator of an event and the expected compensation required in each event are both independent of the distribution of agents by type.

The paper is organized as follows. The first section introduces some of the ideas to be explored by considering the problem of assigning accident liability. The innovation race and the mating game are treated in sections 2 and 3 respectively. Sections 4 and 5 contain the general formulation and theorems which apply to the entire class of games of this type. The final section summarizes the conclusion drawn from the analysis

1. Driving

The purpose of the section is to introduce some of the issues studied in the paper in the context of the easily understood problem of assigning accident liability. Let $i = 1,2,\ldots,n$ devote the set of automobile drivers. Initially, the total number is regarded as given. Assume that the frequency of accidents due to the negligence of each driver is a random variable with a Poisson distribution. Specifically, the instantaneous probability that driver i will cause an accident involving another driver is $\lambda_i dt$ where λ_i is the expected number of such accidents per unit time. Let us suppose that λ_i is proportional to speed which is beneficial in the absence of an accident. Let $b_i(\lambda_i)$ denote the current benefit flow associated with driving at a speed λ_i. We assume

$$b_i'(x) > 0,\ b_i''(x) < 0,\ b_i(0) = 0, \text{ and } b_i'(0) = \infty. \tag{1}$$

In other words, the current joy of driving increases but at a diminishing marginal rate with speed. Finally, let L_i denote the expected value of damage suffered by driver i when involved in an accident.

By assumption, the other driver involved in any accident is a random draw from the set of all other drivers. Hence, the probability that driver i will have an accident during an instant of length dt is $\lambda_i dt + \frac{1}{n-1}\sum_{j\neq i}\lambda_j\ dt$. It is equal to the probability that driver i will ram someone else plus the probability that he will be hit by one of the other $n - 1$ drivers. The probability that any driver will be involved in more than one accident during an instant is negligible. Consequently, if there is no provision for compensation, the total instantaneous benefit of driving at speed λ_i less the expected damage due to accidents is

$$b_i(\lambda_i)dt - [\lambda_i + \frac{1}{n-1} \sum_{j\neq i} \lambda_j]dt\ L_i.$$

Equivalently, the per period expected net benefit flow expressed as a function of all the drivers' accident propensities is

$$\pi_i^o(\lambda_1 \ldots, \lambda_n) = b_i(\lambda_i) - (\lambda_i + \frac{1}{n-1} \sum_{j\neq i} \lambda_j)L_i \qquad i=1,\ldots,n. \qquad (2)$$

Given the speeds chosen by the other drivers, each driver might be expected to choose his own to maximize expected net benefits. The joint solution to these choice problems is a non-cooperative equilibrium of the n person game defined by the payoff functions $\pi_i^o(\cdot)$. Formally, a non-cooperative solution is a vector of accident propensities $(\lambda_i^o, \ldots, \lambda_n^o) \geq 0$ which solves the problems

$$\max_{\lambda_i \geq 0} \pi_i^o(\lambda_1^o, \ldots, \lambda_i, \ldots, \lambda_n^o), \ i=1,\ldots,n.$$

By virtue of (1) and (2), the unique solution is defined by

$$b_i'(\lambda_i^o) = L_i, \qquad i = 1,\ldots,n. \qquad (3)$$

The marginal current benefit flow required to reduce the driver's own accident frequency by one equals his private expected loss per accident.

Under this 'driver beware' liability rule, it is clear that too many accidents occur per period. Each driver takes account of own loss caused by his negligence but ignores the loss incurred by those he runs into. To formalize this point, consider an alternative game in which the driver at fault is required to compensate the other for loss as well as pay his own damages. In this case, the driver's benefit per period less the expected loss per period due to the possibility of an accident is

$$\pi_i^*(\lambda_1,\ldots\lambda_n) = b_i(\lambda_i) - \lambda_i[L_i + \frac{1}{n-1}\sum_{j\neq i} L_j], \tag{4}$$

since no loss is sustained in accidents where the fault lies with the other driver under this scheme. A non-cooperative solution to the driving game solves

$$b_i'(\lambda_i^*) = L_i + \frac{1}{n-1} \sum_{j\neq i} L_j \quad , \qquad i=1,\ldots,n. \tag{5}$$

As the negligent driver pays all damages, the forgone current marginal benefit attributable to reducing his own accident frequency equals the aggregate expected loss incurred in an accident which is his fault. Requiring compensation internalizes the external effect present in the no fault case. As the externality is a diseconomy, equations (1), (3) and (5) imply

$$\lambda_i^o > \lambda_i^*. \tag{6}$$

The equilibrium accident frequencies when no compensation is required are too large.

Like this one, the models considered in the sequel are formulations of dynamic investment games in which the probabilities of future events involving more than one agent depend on the current actions taken by all. In each case a contingent compensation scheme exists that internalizes what would otherwise be an external effect. The practical feasibility of such a scheme generally requires that the perpetrator of each pertinent event and the value of the external effect attributable to the event can be relatively easily varified ex post. In this example, the negligent driver must be accurately named and the damage suffered by the other driver accurately assessed on average. Although we know that doing so is a practical problem, the existing law in most of the U.S. imposes liability on the driver judged at fault which suggests that it is not insurmountable. Nevertheless, the verification problem and other possible complications may prevent the adoption of what would otherwise be a socially optimal incentive scheme in the examples which follow and others that the reader might be able to imagine.

2. The Race to Innovate

There are n firms indexed by $i=1,\ldots n$ seeking the same discovery of capital value B. Time to discovery is a random variable jointly determined by the R&D activities of all n firms. The instantaneous probability that firm i will make the discovery at date t is $\lambda_i(t)dt$ where λ_i is a measure of the R&D intensity pursued by firm i. The corresponding R&D investment of firm i is $c_i(\lambda_i(t))$ where

$$c_i'(x) > 0, \quad c_i''(x) > 0, \quad c_i'(0) = c_i(0) = 0 \text{ and } c_i'(\infty) = \infty, \quad i=1,\ldots,n. \tag{7}$$

All firms invest continuously until the discovery is made by one of them.

The n competitors play a dynamic or super game in which $\lambda_i(t) \geq 0$ is the action of firm i at date t. The game ends when one of them makes the discovery. A non-cooperative solution to the game is an action time path, a strategy for each firm, that maximizes the firm's discounted expected future profits at every date given the strategy pursued by other firms. Following Loury [79] we assume an exclusive patent right; the inventor obtains the lump sum B or its future income equivalent at the discovery date.

Consider a typical instant of time $[t,t + dt]$. Given that it has not yet been made, the probability that firm i will make the discovery during the instant is $\lambda_i(t)dt$. The probability that one of the others makes the discovery instead during the instant is $\sum_{j\neq i} \lambda_j(t)dt$. Agent i receives B in the first event but nothing in the second. If no one makes the discovery during the instant, then the game continues. During the instant, the firm makes an investment equal to $c_i(\lambda_i(t))dt$. Let $v_i(t)$ denote the value of play at time t. Since it equals the expected discounted future value of making the discovery less the discounted stream of R&D expenditures to the date of discovery,

$$v_i(t) = e^{-rdt}[\lambda_i(t)dtB \quad + (1 - \sum_{j=1}^{n} \lambda_j(t)dt)v_i(t+dt)] - c_i(\lambda_i(t))dt$$

where $r > 0$ denotes the positive interest rate.

Because the future value of play, $v_i(t+dt)$, depends on the future action time paths of all the agents, non-cooperative behavior and Bellman's principle of dynamic optimality implies that the agent's optimal action at t maximizes the right side given the other agent's current action and the own value of continuing. Let $v_i^o(t)$ denote the value of the non-cooperative joint strategy $\lambda^o(t) = (\lambda_1^o(t)\ldots,\lambda_n^o(t))$. Given that all agents play non-cooperatively in the future, we have

$$v_i^o(t) = \max_{\lambda_i \geq 0} \{e^{-rdt}[\lambda_i dt(B-v_i^o(t+dt)) + v_i^o(t+dt) - \sum_{j\neq i} \lambda_j^o(t)dtv_i^o(t+dt)] - c_i(\lambda_i)dt\}.$$

Hence, by rearranging terms appropriately and by taking limits of the result as $dt \to 0$, one obtains

$$\lim_{dt\to 0}\left[v_i^o(t)(1-e^{-rdt})/dt - e^{-rdt}[v_i^o(t+dt)-v_i^o(t)]/dt\right]$$

$$= rv_i^o - dv_i^o/dt = \max_{\lambda_i \geq 0} \pi_i^o(\lambda_n^o,\ldots,\lambda_i,\ldots,\lambda_n^o,v_i^o(t)),\ i=1,\ldots,n. \tag{8}$$

where

$$\pi_i^o(\lambda,v_i) = \lambda_i(B-v_i) - c_i(\lambda_i) - \sum_{j\neq i} \lambda_j v_i \tag{9}$$

is the expected profit per period attributable to R&D activity.

Each agent's strategy in a non-cooperative equilibrium solves the problem on the right side of (8). Under the assumption, it is the unique solution to

$$c_i'(\lambda_i^o) = B - v_i^o, \qquad i=1,\ldots,n \tag{10}$$

given $B \geq v_i^o$. Specifically, the marginal cost of making the discovery is equal to the capital gain, the value of discovery less the value of the prospect of continuing to look for it. Notice that the joint equilibrium action $\lambda^o(t)$ is the same at every date if and only if the corresponding joint value of play $v^o(t) = (v_1^o(t),\ldots,v_n^o(t))$ is stationary. Hence, every steady state solution to the differential equation system defined by (8)-(10) is a joint value vector corresponding to a joint stationary equilibrium action. It can be shown that at least one exists and that $0 < v_i^o < B$ for all i characterizes every steady state.

For a related model, Loury [79] argues that rivalry among the firms induces each to invest too much in R&D. Because every competitor fears that the others will win the race, the same result holds here. The dynamic externality present is revealed by the fact that (9) implies $\partial\pi_i^o(\cdot)/\partial\lambda_j = -v_i < 0, j\neq i$. No firm takes account of the fact that an increase in its own R&D intensity reduces the chance that each of the others will be the inventor. Consequently, a smaller joint vector of R&D intensities exist that all prefer.

Loury suggests a limited patent life as a means of correcting this distortion. By analogy with the previous example, the reader can probably suggest an alternative. Require that the inventor compensate the others for the lost value of the prospect of being the inventor. Under this rule, firm i receives $B - \sum_{j\neq i} v_j$ if the investor and v_i if not at the time of discovery. We show that the non-cooperative solution to the game defined by this contingent allocation rule yields an efficient vector of R&D investments and maximizes the joint wealth of all the competitors.

Since $\lambda_i(t)dt$ is in the instantaneous probability of discovery by firm i and each player is indifferent between continuing and discovery by one of the others,

$$v_i(t) = e^{-rdt}[(\lambda_i(t)dt\ (B - \sum_{j\neq i} v_j(t+dt))$$
$$+(1-\lambda_i(t)dt)v_i(t+dt)] - c_i(\lambda_i(t))dt.$$

Let $v_i^*(t)$ denote the equilibrium value of play to agent i and $\lambda_i^*(t)$ denote the equilibrium strategy. An argument analogous to that applied above implies that $\lambda_i^*(t)$ at date t maximizes the right side given non-cooperative behavior in the future. Hence,

$$rv_i^* - dv_i^*/dt = \max_{\lambda_i \geq 0} \pi_i^*(\lambda_i, v^*), \qquad i=1,\ldots,n \tag{11}$$

where $v^* = (v_1^*,\ldots,v_n^*)$ is the joint value of play and

$$\pi_i^*(\lambda_i, v) = \lambda_i[B - \sum_{j=1}^{n} v_j] - c_i(\lambda_i) \tag{12}$$

is the expected profit per period given action λ_i and the joint value of continued play v under the contingent compensation rule. Each agent's equilibrium action maximizes π_i^* given $v = v^*$, it solves the problem on the right side of (11). Specifically, it maximizes the aggregate flow of capital gains net of costs attributable to current R&D activity.

A comparison of (12) with (9) reveals the change in incentives induced by the requirement to compensate. Since $\partial\pi_i^*(\cdot)/\partial\lambda_j \equiv 0$, $j \neq i$, there is no dynamic externality. The resulting efficient equilibrium R&D activities solve

$$c_i'(\lambda_1^*) = B - \sum_{j=1}^{n} v_j^*, \qquad i=1,\ldots,n. \tag{13}$$

The marginal cost of making the discovery is equal to the aggregate capital gain attributable to discovery, its value less the aggregate value of continuing the race.

A stationary equilibrium is defined by (13) and any steady state solu- v^* to the system of differential equations defined by (11). Because a steady state aggregate value of play is any solution to

$$r \sum_{i=1}^{n} v_i^* = \max_{\lambda \geq 0} \sum_{i=1}^{n} \left[\lambda_i (B - \sum_{j=1}^{n} v_j^*) - c_i(\lambda_i) \right], \tag{14}$$

one and only one stationary equilibrium R&D intensity vector $\lambda^* = (\lambda_1, \ldots, \lambda_n^*)$ exists. To prove uniqueness, simply notice that the right side of (14) is a non-negative, continuous and strictly decreasing function of the sum Σv_j. Hence (14) has one and only one solution and it satisfies $0 < \Sigma v_i^* < B$. The corresponding value of λ_i^* is determined by (13) for all i.

Equation (14) also implies that the aggregate value of play in equilibrium, total ex ante wealth, is maximized by λ^* over the set of all stationary joint action vectors. Specifically, $\sum_{i=1}^{n} v_i^* \geq \sum_{i=1}^{n} v_i^o$. Since $v_i^o > 0$, (10) and (13) imply

$$\lambda_i^* < \lambda_i^o, \quad i=1,\ldots,n. \tag{15}$$

Every firm pursues R&D less intensity given the requirement to compensate.

Loury [79] also claims that too many competitors enter the innovation race when the winner receives an exclusive patent. Here we show that it is not the infinite patent life which is responsible for the distortion that Loury identifies. Although the contingent compensation scheme internalizes the dynamic externality present given an infinite patent life, too many firms still enter because no one takes account of the fact that the value of play to each falls with the total number. Formally, every entrant receives the expected wealth per firm rather than the marginal contribution to aggregate expected wealth.

Consider the special case in which all competitors face the same R&D cost structure is; i.e. $c_i(x) = c(x)$ for all i. Let $\lambda^* = \lambda_i^*$ denote the common stationary equilibrium R&D intensity and let $v^* = v_i^*$ denote the common value of play in equilibrium. By virtue of (11)-(13) these solve

$$c'(\lambda^*) = B - nv^* \tag{16.a}$$

and

$$rv^* = \max_{\lambda \geq 0} [\lambda(B-nv^*) - c(\lambda)]. \tag{16.b}$$

Regarding n as a continuous variable, a differentiation of both relationships yields

$$\frac{d\lambda^*}{dn} = - r/(r+n\lambda^*v^*)c''(\lambda^*) < 0 \tag{17.a}$$

since

$$\frac{dv^*}{dn} = - \lambda^*v^*/(r+n\lambda^*v^*) < 0. \tag{17.b}$$

The R&D intensity as well as the value of play of each firm decrease with the total number of competitors.

A potential competitor enters the race if and only if the private capital value of playing $v^*(n)$ is at least as large as the entry cost a. Hence, the equilibrium number of competitors is the solution to

$$v^*(n) = a. \tag{18}$$

However, the aggregate expected wealth generated by all players in the race is equal to the aggregate value of play; i.e.,

$$W^*(n) = nv^*(n)$$

Hence, the socially optimal number of competitors in that which equates the marginal contribution of each player to aggregate value,

$$\frac{\partial W^*}{dn} = v^* + n\frac{dv^*}{dn} = \left(\frac{r}{r+n\lambda^*}\right)v^* < v^*,$$

to a. Since free entry implies $v^* = a$, this condition is never satisfied. Indeed, since the private return to entry always exceeds the social return, too many firms enter the race. Of course, in principle, this distortion can be corrected by charging an appropriate entry fee.

The conclusion that compensation of the losers in the innovation race yields efficient R&D investment rates is crucially dependent on two assumptions. First, the invention or discovery sought must be unique. As we show later, no compensation is required to obtain efficiency if a discovery made by one firm leaves the value of continued R&D activities by others materially unaffected. Second, the compensation scheme must be costlessly implementable. Since the value of seeking a discovery depends on the private cost structure faced by the firm in question, this assumption does not hold in practice. It may well be that an exclusive patent is the best allocation mechanism available when each firm's cost structure is regarded as private information.

3. The Mating Game

Individual agents of two types, denoted as $i = 1,2$ seek to form pairs. The total capital value of a match involving an agent of each type is $B > 0$. The value is divided among the members of the pair ex post. The number of agents of type i is n_i, $i = 1$ and 2, and each agent of type i contacts other agents at a probabilistic rate $\lambda_i(t)$ at date t subject to the cost $c_i(\lambda_i)$ per unit time. The cost functions have the following properties:

$$c_i'(x) \geq 0,\ c_i''(x) > 0,\ c_i'(0) = c_i(0) = 0 \text{ and } c_i'(\infty) = \infty, \quad i = 1 \text{ and } 2. \tag{19}$$

Assume that the agent contacted by another is a random draw from the set of all agents. Hence, the probability that an agent of type i contacts an agent of the opposite type during the instant $[t,t+dt]$ is $\lambda_i(t)dt\, n_j/(n_1+n_2)$. The probability that the same agent is contacted by some agent of the opposite type is $n_j\lambda_j(t)dt/(n_1+n_2)$. The probability that the agent will meet another of the opposite type is the sum of these two instantaneous probabilities. Let $v_i(t)$, $i=1$ and 2, denote the value of the prospective meeting to an agent of type i at date t. It is the expected present value of the agent's share of the value of the match once formed less the discounted stream of costs incurred prior to the meeting data. Following Diamond [80], we assume that the total capital gain or surplus $B - v_1(t) - v_2(t)$ given that a meeting takes place during the instant $[t,t+dt]$ is shared equally by the two agents involved.

There are several possible interpretations of this simple model of matching. It could be viewed as a representation of a housing market in which the two agent types are buyers and sellers, as a labor market in which employers and workers search for one another, or as a marriage market populated by men and women seeking a mate. In the first case B is the difference between the typical buyer's demand price and the typical seller's supply price for the house in the absence of search. In the latter two cases, B is the present value of a future stream of income or benefits accruing only to a matched pair. In each context, the purpose of the model is to permit an analysis of the time required to form a matched pair.

Under the allocation rule, an agent of type i realizes the lump sum or a future income flow equivalent equal to $v_i(t+dt) + 1/2\ (B-v_1(t+dt)-v_2(t+dt))$ if the agent either contacts or is contacted by some agent of the opposite type during the instant [t,t+dt]. Otherwise the agent continues to search which has capital value equal $v_i(t+dt)$ by definition. During the instant, the cost flow $c_i(t)dt$ is incurred. Since the value of search at date t is the expected present value of the agent's capital value at the end of the instant less the cost of search incurred during the instant, we have

$$v_i(t) = e^{-rdt}\frac{n_j}{n_1+n_2}\left[\lambda_i(t)dt + \lambda_j(t)dt\right]\frac{1}{2}(B-v_1(t+dt) - v_2(t+dt))$$

$$+ e^{-rdt}v_i(t+dt) - c_i(\lambda_i(t))dt$$

where r is the common discount rate. The optimal non-cooperative action at t, denoted as $\lambda_i^o(t)$, maximizes $v_i(t)$ given the current action of the other agent type and the future values of continued play of both types. If all play non-cooperatively now and in the future, then the equilibrium value of play $v_i^o(t)$ satisfies

$$v_i^o(t) = \max_{\lambda_i \geq 0}\left\{e^{-rdt}\frac{n_j}{n_1+n_2}\quad[\lambda_i dt + \lambda_j^o(t)dt]\frac{1}{2}(B-v_1^o(t+dt) - v_2^o(t+dt))\right.$$

$$\left. + e^{-rdt}v_i^o(t+dt) - c_i(\lambda_i)dt\right\}$$

where $\lambda_j^o(t)$ is the non-cooperative of the other type at date t and $v^o(t+dt)$ is the future joint equilibrium value of play. By rearranging terms and taking limits appropriately, one obtains the simultaneous system of differential equations in the values of play which follows:

$$rv_i^o - dv_i^o/dt = \max_{\lambda_i \geq 0} \pi_i^o(\lambda_i, \lambda_j^o, v^o), \text{ i=1 and 2, } j \neq i \tag{20}$$

where

$$\pi_i^o(\lambda, v) = \frac{n_j}{n_1+n_2} (\lambda_i + \lambda_j)\frac{1}{2}[B-v_1-v_2] - c_i(\lambda_i)$$

is the expected flow of private benefits attributed to the possibility of meeting another of the opposite type net of the search cost incurred per unit of time.

The optimal action of an agent of type i at date t is the solution to the problem on the right side of (20). Under the assumptions,

$$c_i'(\lambda_i^o) = \frac{n_j}{n_1+n_2}\frac{1}{2}(B-v_1^o-v_2^o), \qquad i=1,2 \text{ and } j \neq i \tag{21}$$

defines the non-cooperative strategy of every agent given $B \geq v_1^o + v_2^o$. The contract rate of every agent is such that the marginal cost of contacting another is equal to the agent's share of the surplus attributable to a match weighted by the probability that the agent contacted is of the opposite type. A stationary strategy is a time path of the joint action $\lambda^o(t)$ which is constant over time. By virtue of (21) a stationary solution to the game is associated with every steady state solution to the differential equation system involving v_1^o and v_2^o defined by (19)-(21). One can easily show that $0 < v_1^o + v_2^o < B$ in any such steady state.

The dynamic externality present in this formulation of the mating game is revealed by (20). Specifically,

$$\frac{\partial \pi^o}{\partial \lambda_j} = \frac{n_j}{n_1+n_2} \frac{1}{2}[B-v_1^o-v_2^o] > 0, \ j \neq i$$

given any stationary equilibrium. No agent takes account of the fact that an increase in one's own contact rate reduces the expected time to match for some agent of the opposite type. Hence, expected time required to find a mate is too long for both types. Formally, a larger pair of contact rates exists which yield a greater payoff to every agent at every date than that associated with the equilibrium pair.

The externality present in the matching process is internalized by the following alternative allocation rule. Let the contacting agent receive the entire value of the match less a compensation paid to the contacted agent equal to the latters forgone value of continued search. Inotherwords, an agent of type i receives the lump sum (or its future income stream equivalent) $B - v_j$ if the contacting agent. Since every agent is indifferent between forming a match as the contacted agent and continuing to search for a partner, we have

$$v_i(t) = e^{-rdt} \frac{n_j}{n_1+n_2} \lambda_i(t)dt[B_i - v_j(t+dt))dt.$$

$$+ \quad e^{-rdt}\left(1 - \frac{n_j}{n_1+n_2} \lambda_i(t)dt\right) v_i(t+dt) - c_i(\lambda_i(t)dt.$$

Let $\lambda^*(t) = (\lambda_1^*(t), \lambda_2^*(t))$ denote the equilibrium joint action associated with the equilibrium joint value of play at date t, $v^*(t) = (v_1^*(t),\ v_2^*(t))$. An argument analogous to that presented above implies

$$rv_i^* - dv_i^*/dt = \max_{\lambda_i \geq 0} \pi_i^*(\lambda_i v_i^*), \quad i=1,2, \qquad j \neq i \tag{22}$$

where

$$\pi_i^*(\lambda_i, v) = \frac{n_j \lambda_i}{n_1+n_2} [B - v_1 - v_2] - c_i(\lambda_i) \tag{23}$$

is the expected payoff of the dynamic game at every date under the contingent allocation rule. The latter equals the expected capital gain flow attributable to the possibility of contacting an agent of the opposite type in the near future less the current search cost flow incurred. Since λ_i^* solves the problem on the right side of (22),

$$c_i'(\lambda_i^*) = \frac{n_j}{n_1+n_2}\,[B-v_1^*-v_2^*], \qquad i=1,2. \tag{24}$$

The equilibrium contact rate is such that the marginal cost of contacting another is equal to the capital gain obtained as the contacting agent weighted by the probability that the agent contacted is of the opposite type. Since the current expected payoff is independent of the contact rate chosen by the opposite type, $\partial\pi_i(\cdot)/\partial\lambda_i \equiv 0$, the equilibrium pair λ^* is efficient. No other exists that yields higher values of search to both types.

One can also show that a unique joint action exists. Since (24) implies that $dv_1^*/dt + dv_2^*/dt = 0$ given a stationary equilibrium action λ^*, the corresponding sum of values $v_1^* + v_2^*$ solves

$$r(v_1^* + v_2^*) = \max_{(\lambda_1,\lambda_2)\geq 0}\left[\left(\frac{n_2\lambda_1+n_1\lambda_2}{n_1+n_2}\right)(B-v_1^*-v_2^*) - c_1(\lambda_1) - c_2(\lambda_2)\right] \tag{25}$$

by virtue of (21) and (23). Uniqueness follows by virtue of the fact that the right side of (25) regarded as a function of $v_1 + v_2$ is non-negative decreasing and continuous on R+. Furthermore, the solution satisfies $0 < v_1^* + v_2^* < B$; i.e. the joint exante wealth of a pair is positive but is less than the total capital value when matched.

Equation (25) also implies that the equilibrium stationary action λ^* maximizes the joint ex ante wealth of the typical unmatched pair. This fact can be used to show that the equilibrium contact rate of each type exceeds that obtained when the surplus attributable to a match is shared equally by the partners; i.e.

$$\lambda_i^* > \lambda_i^o, \qquad i = 1 \text{ and } 2. \tag{26}$$

Formally, if the contact rate pair $\lambda = (\lambda_1, \lambda_2)$ is the joint action of every date, then the corresponding sum of the values of search $v_1(\lambda) + v_2(\lambda) = w(\lambda)$ solves

$$rw(\lambda) = \left(\frac{n_1\lambda_1}{n_1+n_2} + \frac{n_2\lambda_2}{n_1+n_2}\right)(B-w(\lambda)) - c_1(\lambda_1) - c_2(\lambda_2)$$

independent of the allocation rule or equivalently

$$w(\lambda) = \frac{(n_1\lambda_1+n_2\lambda_2)B - (n_1+n_2)[c_1(\lambda_1)+c_2(\lambda_2)]}{r(n_1+n_2) + n_1\lambda_1 + n_2\lambda_2} \tag{27}$$

By virtue of (25) $w(\lambda^*) \geq w(\lambda)$ for all $\lambda \geq 0$ and specifically, $w(\lambda^*) \geq w(\lambda^o) = v_1^o + v_2^o$. Because the distortion is an external economy when the surplus is shared equally, we also know that $\partial w(\lambda)/\partial\lambda > 0$, i=1 and 2, at $\lambda = \lambda^o$. Furthermore, strictly convex search costs together with (21) and (24) imply either $\lambda^o \geq \lambda^*$ or $\lambda^o < \lambda^*$. Since $w(\lambda)$ is decreasing in both of its arguments in the region $\lambda \geq \lambda^*$ by virtue of (27), $\lambda^o \geq \lambda^*$ is ruled out. It follows then that the expected time required to find a mate is shorter under the contingent compensation scheme.

Because the probability that the agent contacted is of the opposite type is equal to the fraction of all agents who are of the opposite type, the equilibrium values of search for both types depends on the distribution of agents by type. These dependencies are implicit in the fact that the equation system

$$rv_1^* = \max_{\lambda_1>0}\left[\frac{n_2\lambda_1}{n_1+n_2}(B - v_1^* - v_2^*) - c_1(\lambda_1)\right]$$

$$\tag{28}$$

$$rv_2^* = \max_{\lambda_2\geq 0}\left[\frac{n_1\lambda_2}{n_1+n_2}(B-v_1^* - v_2^*) - c_2(\lambda_2)\right]$$

implicitly define both v_1^* and v_2^* to be functions of (n_1, n_2), the number of agents of the two types. A complete differentiation of (28) reveals that

$$\frac{\partial v_i^*}{\partial n_i} < 0 \text{ and } \frac{\partial v_i^*}{\partial n_j} > 0, \; i = 1,2 \text{ and } j \neq i. \tag{29}$$

Inotherwords, the value of search of each agent decreases with the number of participants of the same type and increases with the number of the opposite type. The reason is clear. An increase in the number of the same type reduces the probability that a contacted agent is of the opposite type while an increase in the number who are of the opposite type increases that probability.

Under conditions of free entry, the number of participants of each type is endogenous. In particular, agents of a given type enter until the private value of search is driven down to the cost of entry plus the present value of the future stream of benefits forgone as a consequence of entry. Letting $v_1^*(n_1,n_2)$ and $v_2^*(n_1,n_2)$ denote the functions defined by (28), the equilibrium market structure is that agent distribution which satisfies

$$v_i^*(n_1,n_2) = a_i, \qquad i = 1,2 \tag{30}$$

where a_i is the total capital cost of participation for agents of type i. Is this structure optimal? The answer is no in general.

The gross wealth of all those that participate is the aggregate value of search defined by

$$W(n_1,n_2) = n_1 v_1^*(n_1,n_2) + n_2 v_2^*(n_1,n_2)$$

Hence, net wealth or social surplus, $W(n_1,n_2) - n_1 a_1 - n_2 a_2$, is maximum only if the marginal contribution of an agent of type i to gross wealth

$$\frac{\partial W(\cdot)}{\partial n_i} = v_i^* + n_i \frac{\partial v_i^*}{\partial n_i} + n_j \frac{\partial v_j^*}{\partial n_i}, \quad i=1,2 \text{ and } j\neq i$$

is equal to a_i. Since the last two terms, although of opposite sign, do not cancel in general and $v_i^* = a_i$ under conditions of free entry, the condition holds only by accident. The distortion present arises because no agent takes account of the fact that his entry will lower the value of search to all participants of the same type and will raise the value of search to all agents of the opposite type. Equivalently, the distortion is due to the fact that an agent's private return to search is the average rather than marginal contribution of his type to gross wealth.

Given the established efficiency properties of equilibrium under the contingent compensation schemes, one might properly ask why allocation rules that embody its features are not observed in labor, housing and marriage markets where search is regarded as important. There are several possible answers to the question. First, search costs are trivial and, hence the welfare loss involved is insignificant. Second, in the marriage market at least there is no common currency for the purpose of compensating a reluctant mate. Third, even when an appropriate currency exists, a significant degree of within type heterogeneity is present. Given heterogeneity, the total value of a match B depends on the identities of the agents involved as does the value of search. The verification of both necessary to operate an appropriate compensation scheme poses a serious revelation of preferences problem. Fourth, in those cases where the total benefit attributable to a match is not realized as a lump sum but is instead a stream of future benefits, the equilibrium is not perfect. Because the contacting agent generally receives the more favorable share of the total, across matches agents of indentical type realize different streams ex post. Hence, in the absence of indenture or no-divorce laws, contacted agents of either type have an incentive to search for an alternative while matched even though sociallly suboptimal. To the extent that some combination of these reasons other than the first precludes the adoption of the appropriate contingent allocation scheme, the analysis suggests an argument for subsidizing search effort.

4. The Existence and Efficiency of Compensated Equilibria

It is apparent that the examples studies above are members of a general class of dynamic games with an endogenous player structure. Each player's action affects the probability that he will be the perpetrator of some event involving others in the near future. The way in which the total capital value of each such event is allocated among the agents involved ex post affects each agent's incentive to take prior action.

In this section a general class of games with these features is formulated. All three examples are special cases of the class of games. A contingent allocation scheme with the following properties is analyzed. The perpetrator of each event receives its total capital value less a compensation paid to every other agent whose play is terminated by the event equal to the agent's value of continuing to play.

Let $N = \{1,\ldots,n\}$ denote the set of players. Although the play of some individuals is terminated from time to time, the set N is regarded as fixed overtime. There ate two interpretations. First, each individual whose play terminates is immediately replaced by an identical individual. Inotherwords, an infinite number of each player type exist but only a finite subset choose to participate at any moment. Second, the analysis pertains to a steady state of a birth and death process in which the distribution of players by type is stationary.

Player i takes an action $\lambda_i \geq 0$ at date t which determines the probability $q_i\lambda_i(t)dt$ of perpetrating some event during the instant $[t, t+dt]$. Let $\lambda=(\lambda_1,\ldots, \lambda_n)$ denote the joint action of all the players. The conditional probability that the play of agent $j \varepsilon N$ is terminated by an event perpetrated by agent i is denoted as p_{ij}. In general, both the parameter vector $q=(q_1,\ldots,q_n)$ and matrix $P=[p_{ij}]$ depend on the player structure, the distribution of players by type denoted as $m=(n_1,\ldots,n_\ell)$. Finally, the aggregate capital value of any event perpetrated by agent i is denoted as B_i and the cost (or benefit) per unit time of the action persued is denoted as $c_i(\lambda_i)$.

An event is an accident involving two drivers, a discovery made by any competitor, and a meeting between two agents of the oppositve type respectively in the three examples. The perpetrator respectively is the driver at fault, the inventor, and the contacting agent. In the accident model, $p_{ij}=0$ for all i and j; no agent terminates play in the event of any accident. In the innovation race $p_{ij}=1$ for all i and j; the game ends for all when anyone makes the discovery. There are two agent types in the mating game ($\ell=2$). The play of both agents is terminated in each event and the contacted agent is a random draw from the set of all who are of opposite in type; $p_{ii}=1$ $p_{ij}=0$ if $i\neq j$ are of the same type and $p_{ij}=1/m_k$, k=1 or 2, if $i\neq j$ are of the opposite type and i is not of type k. In both the driving example and the innovation race, the probability of being the perpetrator is independent of the player structure; i.e. $q_i=1$ for all i. However, the probability of contacting an agent of the opposite type is proportional to their relative numbers; i.e. $q_i=n_k/(n_1+n_2)$ given i is not of type k= 1 or 2. In the driving game $-B_i=L_i+\frac{1}{n-1}\sum_{j\neq i}L_j>0$ is the expected capital loss incurred in an accident involving agent i where L_j, $j\varepsilon N$, is the value of the damage to the automobile driven by agent j. The flow of benefits derived from driving is some function $b_i(\lambda_i)=-c_i(\lambda_i)\geq 0$ of his own action. In both the innovation race and the mating game $B_i=B>0$ and $c_i(\lambda_i)\geq 0$.

The general model permits variations on each theme. In the driving model $q_i=n$ corresponds to a situation in which the probability of an accident increases with driver density. In general $p_{ij} \geq 0$ can be interpreted as the probability that driver j will suffer a fatal injury in an accident caused by drive i. The contingent compensation rule then requires driver i to pay a death benefit. In the innovation race, $p_{ij}=0$ for all i and j if an infinite potential supply of equally valuable discoveries exists. Alternatively, $p_{ii}=1$ and $p_{ij}=0$ for all $i \neq j$ characterized a specification in which each discovery is firm specific. There is no rivalry in either of these cases. The mating game is the dating game when $p_{ij}=0$ for all i and j. Finally, Diamond's [80] version of the mating game specifies $q_i=1$, each agent's probability of contacting another of opposite type is independent of the distribution of agents by type.

The contingent compensation rule has the following properties. The perpetrator of any event receives its aggregate capital value (or incurrs the associated capital loss) B_i expost. In addition, the perpetrator compensates every other agent whose play is terminated by the event a lump sum equal to that agent's expected capital value of continued play. Hence, if during the instant [t,t+dt] agent i perpetrates an event, his expected exante worth is $B_i - \sum_{j \neq i} p_{ij} v_j(t+dt) + (1-p_{ii}) v_i(t+dt)$ at the end of the instant where $v_j(t+dt)$ is the value of continued play to agent j. It equals the aggregate capital value (or loss) of the event less compensation plus the capital value of continued play weighted by the probability of continuing given the event. If the same agent does not perpetrate an event during the instant, then his end of instant capital value is $v_i(t+dt)$ because either he continued to play or he has been compensated for the capital value of his continued play. Since the probability that the agent will perpetrate an event during the instant [t,t+dt] is $q_i\lambda_i(t)dt$ (approximately), the beginning of instant capital value of play is

$$v_i(t) = e^{-rdt}[q_i\lambda_i(t)dt\ (B_i - \sum_{j=1}^{n} p_{ij}\ v_j(t+dt) + v_i(t+dt)) \tag{31}$$

$$+ (1-q_i\lambda_i(t)dt)\ v_i(t+dt)] - c_i(\lambda_i(t))dt$$

$$i=1,\ldots,\ n$$

where $\lambda_i(t)\geq 0$ is the current action of agent i.

Because every agent's end of instant value of play depends only on the future time path of the joint action, the optimal non-cooperative current action of agent i maximizes the right side of (31) by virtue of Bellman's principle of dynamic optimality. By rearranging (31) appropriately and then taking limits as $dt\to 0$, one finds that

$$\lim_{dt\to 0} v_i(t)[1-e^{-rt}]/dt - \lim_{dt\to 0} e^{-rdt}[v_i(t+dt) - v_i(t)]/dt \tag{32}$$

$$= rv_i - \frac{dv_i}{dt} = \max_{x\geq 0} [q_i x\ (B_i - \sum_{j=1}^{n} p_{ij}\ v_j) - c_i(x)] \quad i=1,\ldots,n.$$

along any non-cooperative solution path to the dynamic game. Of course, the best replay at data t for agent i $\lambda_i(t)$ is the solution to maximization problem defined on the right side of (32). Inotherwords, the non-cooperative strategy maximizes the difference between the expected aggregate flow of capital gain attributable to the current action and its cost flow.

The general class of games of interest satisfy the following assumptions:

A.1 $r>0, q=(q_1,\ldots,q_n) > 0$ and $0 \leq P = [P_{ij}] \leq [1]$ where [1] denotes the nxn matrix of ones.

A.2 $c_i(x)$ is monotonic, differentiable and strictly convex. In addition, $c_i(0) = c_i'(0) = 0$.

A.3 $B_i c_i(x) \geq 0$ for all $x \geq 0$.

Recall $B_i>0$ and $c_i(x)\geq 0$ in the innovation race and the mating game. However, in the driving example - $B_i>0$ is the total expected property damage given that agent i caused the accident and $-c_i(\lambda_i)$ is the benefit flow associated with driving at "speed" λ_i. A.3 reflect these facts.

In the sequel we study stationary non-cooperative solutions to the general dynamic game, solution paths with the property that the joint action is the same at every data for all agents; i.e. $\lambda(t) = \lambda$ for all t. One can show that a stationary action is the optimal non-cooperative strategy for each agent given that all the others pursue a stationary strategy. Furthermore, if all agents pursue a stationary strategy then the joint value of play v(t) is the same at every date under the assumptions. Hence, we have the following equilibrium concept.

<u>Definition 1</u>: A stationary equilibrium is a joint action $\lambda^* \varepsilon R_+^n$ and a joint non-negative value $v^* \varepsilon R_+^n$ such that λ_i^* solves

$$rv_i^* = \max_{x\geq 0} [q_i x (B_i - \sum_{j=1}^{n} p_{ij} v_j^*) - c_i(x)] \qquad i=1,\ldots,n. \qquad (33)$$

In equilibrium, every agent maximizes the expected income attributable to his current action given that all agents continue to do so in the future. An equilibrium can have any of the following properties.

Definition 2: A stationary equilibrium is said to be

(a) trivial if and only if no agent acts $\lambda^{*}=0$.

(b) interior if and only if the sub vector of strictly positive elements of λ^{*} is an equilibrium of the game played by the subset of agents for whom $\lambda_i^{*} > 0$.

(c) efficient on the set of stationary joint action strategies if no other $\lambda \varepsilon R_{+}^{n}$ exists such that its associated value v dominates v^{*}.

(d) symmetric if and only if $\lambda_i^{*} = \lambda_j^{*}$ for all agents i and j of the same type.

No one acts in a trivial equilibrium. The trivial action by any agent is equivalent to not playing in an interior equilibrium. An efficient equilibrium is not Pareto dominated by any other stationary strategy. Finally, agents of the same type take the same action in a symmetric equilibrium.

Theorem 1: A stationary equilibrium exists. Every equilibrium is non-trivial, interior, and efficient.

Proof: Existence. Let f: $R_{+}^{n} \rightarrow R_{+}^{n}$ denote the vector mapping defined by

$$f_i(v) = \frac{1}{r} \max_{x \geq 0} [q_i x(B_i - \sum_{j=1} p_{ij} v_j) - c_i(x)] \qquad (34)$$

$$i=1,\ldots n.$$

f(v) is a continuous decreasing function which maps R_{+}^{n} into the compact convex subset $X \subset R_{+}^{n}$ defined by

$$X = \prod_{i=1}^{n} [0, V_i] \qquad (34.a)$$

$$0 < V_i = f_i(0) = \frac{1}{r} \max_{x \geq 0} [q_i x B_i - c_i(x)] < \infty, \; i \varepsilon N \qquad (34.b)$$

under the assumptions. Hence, f(·) has a fixed point by Brouwer's theorem and every fixed point $v^{*} = f(v^{*})$ is a solution to (33).

The equilibrium action λ^* associated with each fixed point v^* is the unique solution to the collection of optimization problems defined by the right side of (33). The following is used frequently in the sequel.

Corollary: $\lambda_i^* = 0 \iff v_i^* = 0 \quad \forall\ i \in N.$

Non trivial: Suppose $\lambda^* = 0$. The corollary and (4) imply the contradiction $0 = f(0) = V > 0$, $V = [V_1, \ldots, V_n]$.

Interior. Let $N(\lambda^*) = \{i \varepsilon N | \lambda_i^* > 0\}$ and let λ_+^* denote the sub vector of λ^* composed of all the positive elements of λ^*. By virtue of the corollary and (33), we have

$$rv_i^* = \max_{x \geq 0} \left[q_i x (B_i - \sum_{j \varepsilon N(\lambda^*)} p_{ij} v_j^*) - c_i(x) \right] \forall\ i \varepsilon N(\lambda^*)$$

and λ_+^* is the solution to these optimization problems. Inotherwords, λ_+^* is an equilibrium solution to the game played by the non-empty subset of players $N(\lambda^*)$.

Efficient. Suppose that a $\lambda \varepsilon R_+^n$ exists such that $v > v^*$ where v is the joint value of play given that $\lambda(t) = \lambda$ for all t. One can show that v is a solution to the linear system of equations,

$$rv_i = q_i \lambda_i [B_i - \sum_{j=1}^{n} p_{ij} v_j] - c_i(\lambda_i), \qquad i=1,\ldots,n.$$

Therefore, $v > v^* > 0$, $p \geq 0$ and (33) imply the contradiction

$$rv_i^* \geq q_i \lambda_i [B_i - \sum_{j=1}^{n} p_{ij} v_j^*] - c_i(\lambda_i)$$

$$\geq q_i \lambda_i [B_i - \sum_{j=1}^{n} p_{ij} v_j] - c_i(\lambda_i) = rv_i \qquad \text{for all } i \varepsilon N.$$

Of course, this result establishes existence and efficiency of equilibrium for all the examples studied above.

5. The Optimal Player Structure

When does the player structure obtained under conditions of free entry maximize the aggregate expected value of play net of entry costs and opportunity costs of play? To answer this question one needs to formulate the optimization problem.

Given a specified set of players N and a particular equilibrium to their game (λ^*, v^*), the equilibrium aggregate value of play is $W^* = \sum_{i=1} v_i^*$. Because there are multiple equilibrium in general W^*, is not unique given N. This problem might be resolved by associating with N that equilibrium which maximizes the aggregate value of play. Let

$$W(N) = \sup_{\lambda \varepsilon \Omega(N)} \sum_{i=1}^{n} v_i^* \tag{35}$$

denote the maximal aggregate value of play on the set of equilibria $\Omega(N)$ corresponding to the player set N.

The socially optimal player structure, then, is that set of players N which maximizes W(N) net entry and opportunity costs. The following example, apply dubbed the "boxing tournament", illustrates why this definition of a socially optimal player structure is inappropriate in general.

At any moment of time there are two boxers in the ring who fight until one knocks the other out. The loser of each bout is eliminated from the tournament but is immediately replaced by an identical opponent who takes on the winner. The tournament continues forever according to this rule. An event is a knockout and the perpetrator of the event is the winner. Hence, n=2 and

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let r = 1 and assume identical agents. Specifically, $B_i = B$, $q_i = 1$ and $c_i(x) = x^2/2$, i=1 and 2. Boxer i receives $B - v_j^*$ if the winner and v_i^* is the loser where B is the total prize per fight. The consolation price v_i^* is, of course, equal to the value of entering the ring and is endogeneously determined in any compensated equilibrium.

In this case (33) implies that an equilibrium corresponds to a simultaneous solution (v_1^*, v_2^*) to the system

$$v_i = f_i(v_j) = \max_{x \geq 0}[x(B-v_j) - x^2/2] \tag{36.a}$$

$$= \begin{cases} \frac{1}{2}(B-v_j)^2 & \text{if} \quad v_j \leq B \\ 0 & \text{if} \quad v_j \geq B \end{cases}$$

The equilibrium actions contingent on the values is

"When B = 2, three equilibria exist. They are $(v_1, v_2) = (3,0)$, $(0,3)$, and (v^*, v^*) where $v^* = 3 - \sqrt{5}$ labeled E_1, E_2 and E respectively in Figure 1. At E_i, boxer i is guaranteed the entire prize if he loses. Consequently, boxer $j \neq i$ has no incentive to resist so that boxer i must fight to eventually win the prize. Equilibrium E is symmetric; both make the same effort; the odds are even."

Figure 1: Identical Boxers (B=2)

Notice that the solution to (35) is E_1 and E_2 even though only one boxer fights in either equilibrium. Clearly no promotor could finance the prize money if either represented behavior, let alone make a profit. To maximize interest in attendance and, hence, to maximize his profit, the promoter would enforce the symmetric equilibrium. However, that equilibrium ($v_1^* = v_2^*$) minimizes the total value of the game v_1+v_2 to the boxers since $f_1(v_2) = f_2^{-1}(v_2)$ is convex.

The example suggests that one might want to restrict the set of equilibria in the optimal player structure problem to be the symmetric subset. However, one can also show that multiple equilibria exist when the two fighters are not identical. Suppose that one type is more able than the other. Specifically, type 1 boxers are more likely to win given the same amount of effort; i.e., $c_i(x) = c_i x^2/2$ and $c_1 < c_2$. In this case

$$v_i = f_i(v_j) = \max_{x\geq 0} [\ x(B-v_j) - c_i x^2/2\ ]$$

$$\begin{cases} (B-v_j)^2/c_i 2 & \text{if } v_j \leq B \\ 0 & \text{if } v_i \geq B \end{cases}, \quad i=1 \text{ and } 2,\ j\neq i \tag{37.a}$$

and

$$\lambda_i = \max [\ 0,\ (B-v_j) / c_i]\ ,\ i = 1 \text{ and } 2. \tag{37.b}$$

Figure 2 illustrates the case of $B>2$ and $1=c_1 < c_2 = B/2$. There are two equilibria satisfying the requirement $v_i^* \geq 0$ labled E and E_2. In equilibrium E_2 the more able boxer doesn't resist. However, E_2 is the only solution to (35) even though the promoter would enforce equilibrium E. Notice that both equilibria are strictly interior but (35) still choses E_2 if c_2 is made slightly larger. We must conclude then, that our proposed definition of a socially optimal player structure is misleading in general except in the case of a unique symmetric equilibrium to the investment game.

Figure 2: Non-identical Boxers ($B > 2$)

The following result provides conditions on P that guarantee both uniqueness and symmetry of equilibrium.

Lemma. Let $A = rI + DP$ denote an n x n matrix where r is a positive scalar, I is the n x n identity matrix and D is an n x n non-negative diagonal matrix and P is a non-negative n x n matrix. The principal minors of A are positive for all D if and only if the principal minors of P are all non-negative.

Proof. Given D diagonal, it follows that any principal minor matrix of A of order k, denoted as A_k, is

$$A_k = rI_k + D_k P_k, \quad k = 1,\ldots,n,$$

where I_k is the identity matric of order k, and D_k and P_k are the principal minor matrices of D and P respectively obtained by deleting the same common rows and columns deleted to obtain A_k.

Since the principal minors of DP are non-negative under the hypotheses, an argument used by Nikaido [1960, Theorem 20.8] establishes that $|A_k| > 0$ for all k. Specifically, it can be shown that

$$|A_k| = r^k + \emptyset(r)$$

where $\emptyset(r)$ is a polynomial of order k-1 whose coefficients are the various principal minors of $D_k P_k$.

To prove necessity, suppose $|P_k| < 0$. Let D(m) denote a diagonal matrix with all principal diagonal elements equal to the real number m. It follows that for all $m>0$

$$D_k^{-1}(m)\, A_k = r\, D_k^{-1}(m) + P_k$$

and, consequently

$$|D_k^{-1}(m) A_k| = \left|\frac{1}{m}\right|^k |A_k| = \left|\frac{r}{m} I + P_k\right|$$

Hence, monotonicity and continuity of $\left|\frac{r}{m}I + P_k\right|$ in m together with

$$\lim_{m\to\infty} \left(\frac{1}{m}\right)^k |A_k| = |P_k| < 0 ,$$

imply $|A_k| \leq 0$ for all large enough values of m.

<u>Theorem 2</u>: If the principal minors of P are all non-negative, then there is one and only one stationary equilibrium.

Proof. The system (33) can be represented as

$$g(v) = r\,(\,v - f(v)\,) = 0$$

where $f\colon R_+^n \to R_+^n$ is the vector map defined by (34).

Every solution v^* lies in the n-dimension compact rectangle X defined by (34.a) and (34.b). The Jacobian matrix is

$$\frac{\partial g(v)}{\partial v} = J(v) = rI + D(\lambda^*(v))P$$

where $\lambda^*(v)$ is the equilibrium function mapping joint values to joint actions defined by the solution to the right side of (34) and $D(\lambda) = \text{diag } q_i\lambda_i$. Since $D(\lambda^*(v))$ is a non-negative diagonal matrix on X, the Lemma implies that all principal minors of J(v) are positive everywhere on X. Hence, at most one solution exists by virtue of an univalence theorem due to Gale and Nikaido (1965). Since we have already established that f(v) has a fixed point in proving Theorem 1, the assertion follows.

At this point let us note that all examples considered except the boxing tournament satisfy the condition. It is known that P has non-negative principal minors if either (i) P is dominant diagonal or (ii) the symmetric part of P, 1/2(P'+P) where P' denotes the transpose, is positive semi-definite. In the driving example P is the null matrix. In the innovation race P is the matrix of all ones which satisfies (ii). P is dominant diagonal in the mating game. Since $|P| = -1$ in the boxing tournament the condition fails and there are many equilibria. The hypothesis also implies symmetry.

Theorem 3: Every stationary equilibrium is symmetric if $p_{ii} \geq p_{ij}$ (equivalently $p_{jj} \geq p_{ji}$) for all i and j of the same type.

Proof. Given that i and j are of the same type, Definition 1 implies that $\lambda_i^* = \lambda_j^*$ if and only if $p_{ii}v_i^* + p_{ij}v_j^* = p_{jj}v_j^* + p_{ji}v_i^*$ or equivalently $(p_{ii}-p_{ij})(v_i^*-v_j^*) = 0$. Hence, it suffices to show that $v_i^* = v_j^*$ under the hypothesis. Equation (33) implies

$$rv_j^* = \max_{x\geq 0}\left[q_j x(B_j - p_{jj}v_j^* - p_{ji}v_i^* - \sum_k p_{jk}v_k^*) - c_j(x)\right]$$

$$= \max_{x\geq 0}\left[q_i x(B_i - p_{ii}v_j^* - p_{ij}v_i^* - \sum_k p_{ik}v_k^*) - c_i(x)\right]$$

given that i and j are of the same type. Hence

$$[r + q_i\lambda_i^*(p_{ii}-p_{ij})][v_j^*-v_i^*] = 0$$

which completes the proof if $p_{ii} \geq p_{ij}$.

Given that P has non-negative principal minors, λ^* is unique and symmetric by virtue of Theorems 2 and 3. In this case, the unique value of play is the same for every player of the same type. Let $N_k \subset N$ denote the subset of players of type $k = 1,\ldots,\ell$ and let $m = (n_1,\ldots,n_\ell)$ denote the distribution of players by type. Let $w_k^* = v_i^*$ for all $i \varepsilon N_k$ denote the common value of play to those of type k. The aggregate value of play given N as defined by (35) can be expressed as a function

$$W(m) = \sum_{i \varepsilon N} v_i^* = \sum_{k=1}^{\ell} n_k w_k^* \tag{38}$$

of the distribution of agents by type.

An agent of type k enters the game if and only if the private value of play w_k^* is at least as large as the cost of entry for the type. Hence, the market structure obtained under conditions of free entry is optimal if and only if the private value of play for each type is equal to the marginal contribution to the aggregate value of a player of type k; i.e.,

$$\frac{dW(m)}{dm} = w^* \tag{39}$$

where $w^* = (w_1^*,\ldots,w_\ell^*)$. By virtue of (38) this condition is satisfied in general if and only if w^* is independent of the distribution of players by type.

Theorem 4: Given that all principal minors of P are non-negatives, condition (39) holds if and generally only if (i) the probability of being the perpetrator (q_i) and (ii) the expected compensation required when the perpetrator ($\sum_{j=1}^{n} p_{ij}v_j$) are both independent of the distribution of players by type (m).

Proof. That (i) and (ii) are sufficient is implied by (33). Specifically v_i^* is independent of m for all i given (i) and (ii). That (i) is necessary when (ii) holds follows from the mating example. That (ii) is necessary when (i) holds follows from the racing example.

Notice that the driving example, $q_i = 1$ and $p_{ij} = 0$, satisfies all the conditions. So does the mating game if the probability of contacting another agent is independent of that relative number, $q_i = 1$, and so does the racing example if discovery does not end the play of other agents, $p_{ij} = 0, j \neq i$.

6. Conclusions

The class of dynamic games studied in this paper can be described as follows. Every player takes an action now that affects the probability of some event inthe near future involving others. How the aggregate gain or loss associated with each event is allocated ex post among the agents involved affects the ex ante incentive of each agent to take prior action. Given an allocation scheme, the problem of determining the actions of all the agents can be formulated as a multi-person non-cooperative dynamic game. In general, solutions are not efficient in the Pareto sense. However, a contingent allocation scheme exists such that every stationary solution to the game defined by the scheme is efficient. The scheme allocates the total value of each event to the perpetrator less a compensation paid to every other agent whose play is terminated by the event equal to the value of continued play.

If the solution to the dynamic game defined by the contingent compensation scheme is unique and symmetric, then the player structure obtained under conditions of free entry maximizes aggregate wealth rate of entry and opportunity costs of play if and only if the following conditions are satisfied: (i) The probability that any agent will perpetrate an event is independent of the player structure. (ii) The compensation that any agent must pay given that he is the perpetrator is independent of the player structure. A sufficient condition for uniqueness and symmetry is presented which the three principal examples - driving, racing and mating - all satisfy. The driving example also satisfies both (i) and (ii). However, (i) fails in the case of the mating game and (ii) fails in the case of the innovation race. Finally, the boxing tournament example establishes that multiple and non-symmetric solutions can exist in general. Furthermore, the example illustrates that the aggregate net wealth criterion is not an appropriate social welfare measure in such cases.

The distortion present when conditions (i) or (ii) fail are congestion effects that are present whether the contingent allocation scheme obtains or not. In the mating game, no individual agents takes account of the fact that his entry will increase the probability an agent of the opposite type will contact an agent of his type and decreases the probability that an agent of his type will contact one of the opposite type. Because his entry increases the value of play to agents of the opposite type but decreases the value of play to agents of the same type, whether there are too many or too few agents of a given type attracted to the game is indeterminant in general. In the innovation race, no agent takes account of the fact that his entry reduces the probability that any one of the others will make the discovery. Since the value of play to each decreases with the number of competitors, too many are attracted to the game.

In general the implementation of the contingent allocation and compensation mechanism shown here to yield Pareto optimal solutions to the dynamic game requires the following information. First, the perpetrator of each event must be accurately identified on average. Second, the aggregate gain or loss attributable to each event must be assessed. Third, the agents whose play is terminated and their values of continued play must be determined. One or more of these requirements may be impossible to fulfill in a particular case. Hence, the determination of allocation schemes that are efficient in a constrained sense given that one or more of the requirements fail would be an interesting research problem.

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Figure 1 Identical Boxers
($B = 2$)

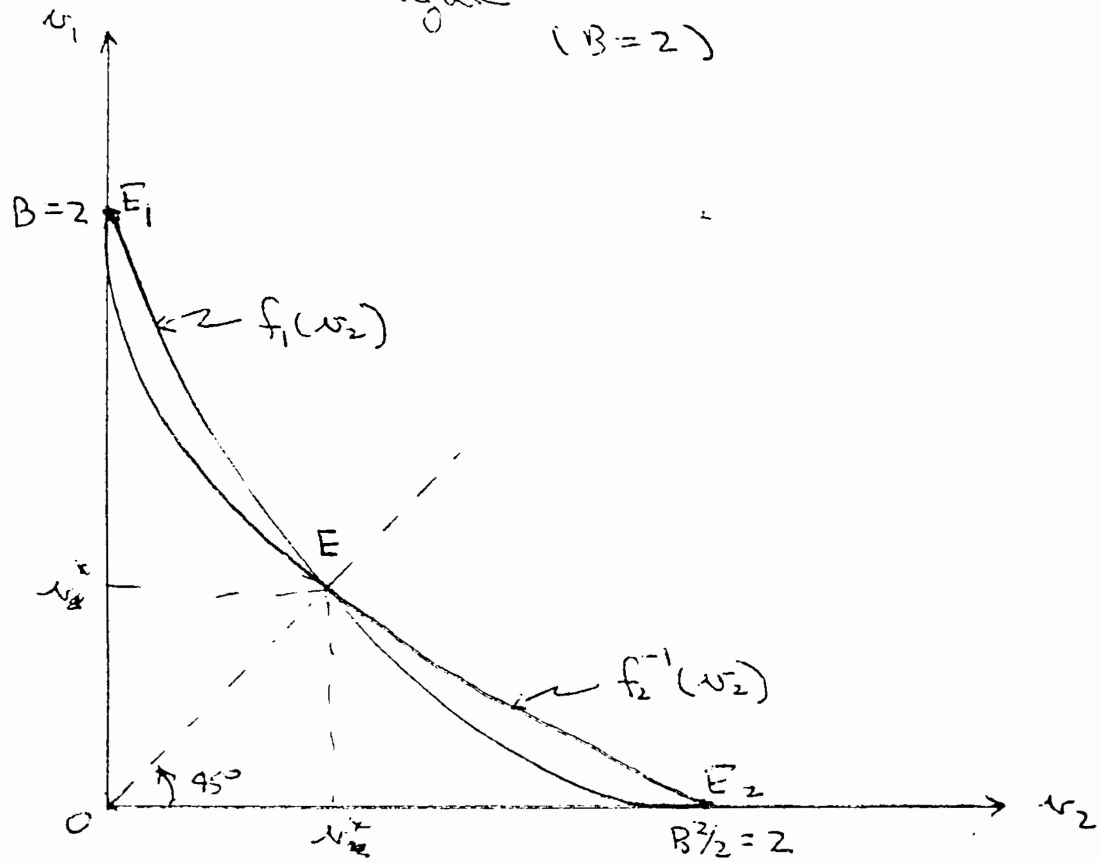

Figure 2: Non-identical Boxers
($B > 2$)