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THE MARTINGALE PROPERTY OF ASSET PRICES *

by

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1. Introduction

My goal is to point out the close theoretical link between the extent to which investors are insured and the extent to which assets have the martingale property. If assets are included in an Arrow-Debreu model with complete markets for contingent contracts, then the assets' prices have the martingale property. If investors are not fully insured, then asset prices have the martingale property only by accident. However, one can justify a short-run martingale property, even if there are no insurance markets. If a model included money, then investors would use money to insure themselves nearly perfectly over short periods of time. For this reason, the martingale property would be approximately valid over such short periods.

An asset price is said to have the martingale property if the price of the asset in any period equals its expected discounted future return. The discount rate applied to the future return is either the market interest rate or consumers' pure rate of time preference. In this paper, the discount rate is the pure rate of time preference.

One might assert that the discount rate applied to future returns should include a risk premium. Unless this premium is constant, the martingale property is simply an identity. It turns out that if consumers are not fully insured, then there is no reason to expect the risk premium to be constant.

A simple arbitrage argument provides an intuitive justification of the martingale property. If an asset's price exceeded its discounted expected return, an investor would find it worthwhile to sell the asset and hold money or short-term bonds. Similarly, he would buy the asset if its price were less than the discounted expected return.

This argument is valid only if the expected discounted return really reflects the value of an asset to an investor. That is, the return must be proportional to utility. If the investor is risk averse and not perfectly insured, then return would be proportional to utility only by accident. Thus, the intuitive justification of the martingale property must be based on an assumption that investors are fully insured.

There is a natural measure of the extent to which a consumer or investor is insured. This measure is the constancy of the consumer's marginal utility for the unit of account. For this reason, I focus my analysis on the marginal utility of the unit of account. In an Arrow-Debreu model with complete markets, this marginal utility is constant, for it is simply the Lagrange multiplier associated with the consumer's budget constraint. The alternative to an Arrow-Debreu model is a temporary equilibrium model. In such models, each consumer has a different budget constraint in each period and a Lagrange multiplier associated with each such constraint. The multipliers are the marginal utilities associated with the unit of account used in each period. Asset prices have the martingale property if these marginal utilities are constant over time.

Suppose that money is included in the temporary equilibrium model and is used as the unit of account. Then, consumers should try to use reserves of money to hold constant the marginal utility of money. They can do so approximately, in the short run. However, I show that normally they must allow the marginal utility of money to drift in the long run. This is so even if money pays interest or if consumers can borrow and lend. Roughly speaking, capital markets cannot fully compensate for a lack of insurance.

This conclusion is somewhat contrary to the spirit of Milton Friedman's theory of the optimum quantity of money (Friedman, 1969). The conclusion also directly contradicts statements in my paper on the subject (1980a). Friedman claimed that consumers would hold the socially optimal quantity of money if money earned interest at a rate equal to consumers' pure rate of time preference. In my paper, I showed that in a rigorous version of Friedman's model, there would be no equilibrium if money earned such a high rate of interest. The problem is that consumers would want to hold an infinite amount of money. I also claimed that if the rate of interest were only slightly less than the pure rate of time preference, then there would exist an equilibrium. I suggested that in this equilibrium, consumers would keep their marginal utilities of money nearly constant, so that the equilibrium would be nearly Pareto optimal. This suggestion is false. Equilibria may not exist if the rate of interest is too close to the rate of time preference. When an equilibrium does exist, consumers are not necessarily able to keep their marginal utilities of money nearly constant. These two points are demonstrated by an example in section 5.

In my previous paper (1980a), I made an error. It is not true that equilibria exist whenever the interest rate is less than each consumer's rate of time preference. Equilibria necessarily exist only if the interest rate is sufficiently close to zero. The error is explained and corrected in Appendix II of this paper.

The analysis of this paper owes a great deal to Robert Lucas' paper, "Asset Prices in an Exchange Economy," (1978). He was, I believe, the first to discuss the martingale property in terms of a general equilibrium model. I adopt his way of introducing assets into such a model. The advantage of his approach is its simplicity. He introduces assets without ever introducing production.

Lucas' conclusion is somewhat different from my own. He concludes that asset prices would normally not have the martingale property. It is easy to understand intuitively why Lucas reaches this conclusion. Lucas' equilibrium is a temporary equilibrium in the sense that consumers face a different budget constraint in each period. Hence, the prices of each period require a separate normalization. He has a single consumption good in his model and he normalizes prices so that the price of this good is always one. This means that asset prices are always measured in terms of current consumption good. The martingale property has to do with comparison of asset prices in different time periods. In a temporary equilibrium model, asset prices in different time periods are measured in different units. It makes sense to compare these prices only if there is reason to expect that the units of different periods are somehow of equivalent value. One sense of value is marginal utility. In Lucas' model, the marginal utility of the consumption good should fluctuate if its supply fluctuates. As Lucas points out, his asset prices would indeed form a martingale if the marginal utility of the consumption good were constant. In conclusion, Lucas' criticism of the martingale property is based on his choice of units for prices.

It seems to me most natural to measure asset prices in terms of money. If this is done, then one can justify at least a short-run martingale property.

The plan of the paper is as follows. In the next section, I define a version of Lucas' model with complete Arrow-Debreu markets for contingent claim contracts. I prove that in such a model, asset prices have the martingale property. In section 3, I discuss Lucas' model and show that if one simply renormalizes the prices in his model, one obtains the Arrow-Debreu prices. (The link between Arrow-Debreu prices and Lucas' prices has been pointed out by LeRoy and La Civita (1980).) In section 4, I introduce money into Lucas' model. In section 5, I show by means of an example that in this monetary model, the equilibrium marginal utilities of money are not necessarily constant. They are not necessarily even asymptotically constant as the rate of interest on money approaches consumers' common pure rate of time preference. Section 5 is the only section of the paper which is analytically difficult (besides Appendix II). In section 6, I show that the monetary model may be interpreted as a model with borrowing and lending. In section 7, I point out that money and credit cannot be counted on to provide perfect insurance. This conclusion is implied by the example of section 5. In section 8, I give an argument to support a short-run martingale property.

There is a large literature on the martingale property of asset prices. An excellent survey has been written by LeRoy (1979a). LeRoy has stressed that asset prices have the martingale property if consumers are risk neutral (LeRoy, 1973, 1979b). The link with my work is clear, for in a Lucas model risk neutral consumers would have constant marginal utility for the consumption good.

2. Complete Markets Give Rise to the Martingale Property

In this section, I define a model which includes Lucas' notion of assets in an Arrow-Debreu model of pure exchange with an infinite horizon and random fluctuations. I show that the model has an equilibrium and that equilibrium asset prices have the martingale property.

The Underlying Stochastic Process

There is an underlying stochastic process $\{s_t\}_{t=-\infty}^{\infty}$ which influences utility functions and the availability of commodities. I assume that $\{s_t\}$ is a stationary Markov process on a set S. S is a Borel subset of a Euclidean space.

If $f : S \to R$ is a real-valued function, (R denotes the real numbers), then $f(s_t)$ denotes both a number and a random variable. $E\, f(s_t)$ denotes the expected value of the random variable. The same notation applies if f is a vector-valued function.

I nowhere need the assumption that $\{s_t\}$ is Markov. I make this assumption only so that I can use notation which facilitates comparison of my work with that of Lucas.

Utility

There are L goods and I consumers, and the utility function of the i^{th} consumer for consumption at any time t is $u_i : R_+^L \times S \to R$, $i = 1,\ldots,I$. (R^L denotes L - dimensional Euclidean space and R_+^L is the cone of vectors in R^L with non-negative components.) $u_i(x,s_t)$ is the utility to consumer i at time t of the bundle x if the state of the environment is s_t.

Consumers must plan their economic life over an infinite future. They choose a consumption program $(x_0,x_1,\ldots)$, where $x_t : S \to R_+^L$ is a measurable function for all t. I assume that the functions x_t are uniformly bounded.

A consumer's utility for such a program is the expected value of discounted utility. Each consumer discounts future utility at the rate ρ, where $0 < \rho < 1$. ρ is the pure rate of time preference. Therefore, the utility to consumer i at time zero for a program $(x_0, x_1, \ldots)$ is

$$U_i(x_0, x_1, \ldots) = E \sum_{t=0}^{\infty} \int (1+\rho)^{-t} u_i(x_t(s_t), s_t).$$

Assets

An asset is a right to a fluctuating stream of commodities. An asset is described by a measurable function $a : S \to R_+^L$. The bundle yielded by the asset at time t is $a(s_t)$.

There are C assets where C is a positive integer. The c^{th} asset is described the function $A_c : S \to R_+^L$. $A_c(s)$ should be thought of as a column vector. Its components are denoted by $A_{kc}(s)$. $A(s)$ denotes the $L \times C$ matrix whose kc^{th} entry is $A_{kc}(s)$.

Investment Programs

An investment program for a consumer is a sequence of uniformly bounded measurable functions $(\gamma_o, \gamma_1, \ldots)$, where $\gamma_t : S \to R_+^C$.

$\gamma_{tc}(s)$ denotes the share of the c^{th} asset owned by the consumer at the end of period t if the state of the environment is s. If the consumer holds $\gamma_t(s)$ at the end of period t, then at the beginning of period $t+1$ he receives the commodity bundle $\sum_{c=1}^{C} A_c(s_{t+1}) \gamma_{tc}(s_t) \in R_+^L$. I denote this bundle by $A(s_t) \gamma_t(s_t)$.

The Initial Investment

The model specifies each consumer's initial holdings of assets, $\gamma_{i,-1} \in R_+^C$. $\gamma_{i,-1,c}$ is the proportion of asset c held by consumer i at the beginning of period zero. I assume that $\gamma_{i,-1,c} \geqq 0$ and that

$\sum_{i=1}^{I} \gamma_{i,-1,c} = 1,$ for all c.

Allocations

An allocation for the economy is of the form $((\underset{\sim}{x}_i), (\underset{\sim}{\gamma}_i))_{i=1}^{I}$, where each $\underset{\sim}{x}_i = (x_{i0}, x_{i1}, \ldots .)$ is a consumption program and each $\underset{\sim}{\gamma}_i = (\gamma_{i0}\ \gamma_{i1}, \ldots .)$ in an investment program. The allocation is feasible if $\sum_{i=1}^{I} x_{it}(s) = \sum_{c=1}^{C} A_c(s)$, for all t and s, and if $\sum_{i=1}^{I} \gamma_{itc}(s) = 1$, for all t, s and c.

Prices

A price system for goods is of the form $\underset{\sim}{p} = (p_0, p_1, \ldots)$, where each $p_t : S \to R_+^L$ is a measurable function and $0 < E \sum_{t=0}^{\infty} p_{tk}(s_t) < \infty$, for $k=1,\ldots L$. These prices are Arrow-Debreu prices for contingent claims contracts as in Arrow (1963-4) and Debreu (1959), Chapter 7. All trading takes place at some imaginary starting point, say at time - 1. I must call the starting date imaginary, for I do not specify the state of the environment at the starting date. $p_{tk}(s)$ is the price at the starting point for a unit of good k to be delivered during period t if the state of the environment is s.

An asset price system is of the form $q = (q_0, q_1, \ldots,)$, where $q_t : S \to R_+^C$ is a measurable function and $0 < E \sum_{t=0}^{\infty} q_{tc}(s_t) < \infty$, for all c. $q_{ct}(s)$ is the price at the starting point for the delivery of an asset c in period t if s occurs.

A price system is of the form $(\underset{\sim}{p}, \underset{\sim}{q})$, where $\underset{\sim}{p}$ is a price system for goods and $\underset{\sim}{q}$ is a price system for assets.

Remark: Strictly speaking, the model does not include a complete set of Arrow-Debreu markets. In order to have all such markets, allocations and prices at time t must depend on the complete history of the environment up to time t, $(\ldots, s_{t-1}, s_t)$. However, in a full Arrow-Debreu model, consumers would not need to recall history, for utility functions and supplies at time t depend only on s_t, $\{s_t\}$ is Markov, and consumers observe s_t.

The Budget Constraint

The budget set of consumer i, given a price system $(\underset{\sim}{p}, \underset{\sim}{q})$ is

2.1) $\beta_i(\underset{\sim}{p}, \underset{\sim}{q}) = \{(\underset{\sim}{x}, \underset{\sim}{\gamma}) \mid \underset{\sim}{x}$ is a consumption program, $\underset{\sim}{\gamma}$ is an investment program and

$$\sum_{t=o}^{\infty} p_t \cdot x_t + q_0 \cdot (\gamma_0 - \gamma_{i,-1}) + \sum_{t=1}^{\infty} q_t \cdot (\gamma_t - \gamma_{t-1})$$

$$\leqq p_0 A_0 \gamma_{i,-1} + \sum_{t=1}^{\infty} p_t A_t \gamma_{t-1}\}.$$

In this formula, $p_t \cdot x_t$ denotes $E \sum_{k=1}^{L} p_{tk}(s_t) x_{tk}(s_t)$. Similarly, $q_t \cdot \gamma_t$ denotes $E \sum_{c=1}^{C} q_{tc}(s_t) \gamma_{tc}(s_t)$. A_t denotes the random variable whose value at s_t is the $L \times C$ matrix $(A_{kc}(s_t))$. Finally, $p_t A_t \gamma_{t-1}$ denotes $E \sum_{c=1}^{C} \sum_{k=1}^{L} p_{tk}(s_t) A_{kc}(s_t) \gamma_{t-1,c}(s_t)$.

Demand

The demand correspondence of consumer i is

$\xi_i(\underset{\sim}{p}, \underset{\sim}{q}) = \{(\underset{\sim}{x}, \underset{\sim}{\gamma}) \in \beta_i(\underset{\sim}{p}, \underset{\sim}{q}) \mid U_i(x) \geqq U_i(\overline{x}_i)$, for all $(\overline{\underset{\sim}{x}}, \overline{\underset{\sim}{\gamma}}) \in \beta_i\, \underset{\sim}{p}, \underset{\sim}{q})\}$.

Observe that $\xi_i(\underset{\sim}{p}, \underset{\sim}{q})$ may be empty.

Equilibrium

An Arrow-Debreu equilibrium consists of $(((\underset{\sim}{x}_i, \underset{\sim}{\gamma}_i))_{i=1}^{I}, (\underset{\sim}{p}, \underset{\sim}{q}))$, where

i) $((\underset{\sim}{x}_i, \underset{\sim}{\gamma}_i))_{i=1}^{I}$ is a feasible allocation,

ii) $(\underset{\sim}{p}, \underset{\sim}{q})$ is a price system and

iii) $(\underset{\sim}{x}_i, \underset{\sim}{\gamma}_i) \in \xi_i(\underset{\sim}{p}, \underset{\sim}{q})$, for all i.

The equilibrium is said to be stationary if $(x_{it}, \gamma_{it}) = (x_{i0}, \gamma_{i0})$ and $(p_t, q_t) = (1+\rho)^{-t}(p_0, q_0)$, for all t.

The Martingale Property

The price system (p,q) has the martingale property if

$$q_t(s_t) = E\,[q_{t+1}(s_{t+1}) + p_{t+1}(s_{t+1})\,A\,(s_{t+1}) \mid s_t] \tag{2.2}$$

almost surely, for all t. By "almost surely," I mean that the equality holds with probability one. $E\,[q_{t+1}(s_{t+1}) + p_{t+1}(s_{t+1})\,A(s_{t+1}) \mid s_t]$ denotes the conditional expectation of the random variable $q_{t+1}(s_{t+1}) + p_{t+1}(s_{t+1})\,A(s_{t+1})$ given s_t. Observe that I need not condition on s_n for $n < t$, because the process $\{s_t\}$ is Markov.

The martingale property is easier to interpret if one uses current value prices. Let $Q_t\,(s_t) = (1+\rho)^t q_t$ and let $P_t = (1+\rho)^t p_t$, for $t = 0,1,\ldots$. Then, $(\underset{\sim}{p}, \underset{\sim}{q})$ has the martingale property if and only if

$$Q_t(s_t) = (1+\rho)^{-1}\,E\,[Q_{t+1}(s_{t+1}) + P_{t+1}(s_{t+1})\,A\,(s_{t+1}) \mid s_t]$$

almost surely, for all t. In words, the current price at time t equals the discounted expected value of the sum of the price in period $t+1$ and the value of the dividend in period $t+1$.

Assumptions

I list the assumptions I use.

2.3) $\{s_t\}$ is a stationary Markov process. s_t varies over S.

2.4) $u_i : R_+^L \times S \to R$ is measurable, for all i.

By this, I mean that u_i is measurable with respect to the Borel σ-field on $R_+^L \times S$. (Recall that S is a Borel subset of a Euclidean space.)

2.5) $u_i(\cdot, s) : R_+^L \to R$ is continuous, strictly increasing and strictly concave, for all s.

By strictly increasing, I mean that $u_i(x', s) > u_i(x, s)$ if $x' \geqq x$ and $x' \neq x$.

2.6) $u_i(x,s)$ is bounded as a function of s, for all i and x.

2.7) $\sum_{i=1}^{I} \gamma_{i,-1,c} = 1$, for all c, and $\gamma_{i,-1,c} \geqq 0$, for all i and c.

2.8) $A_{kc}(s)$ is bounded as a function of s, for all c and k.

2.9) For all i, there exists $r > 0$ such that $\sum_{c=1}^{C} A_{kc}(s)\, \gamma_{i,-1,c}(s) \geqq r$, for all s and k.

Theorems

Assume that assumptions 2.3 - 2.9 apply.

2.10 Theorem There exists a stationary Arrow-Debreu equilibrium.

2.11 Theorem If $(((\bar{x}_i,\bar{\gamma}_i))_{i=1}^{I},(p,q))$ is an Arrow-Debreu equilibrium, then (p,q) has the martingale property (2.2).

A Lemma

In proving theorem 2.10, I make use of a simple lemma, which expresses the fact that if share prices have the martingale property, then consumers cannot gain by trading in assets.

2.12 Lemma Suppose that (p,q) has the martingale property (2.2). If $(x,\gamma) \in \beta_i(p,q)$, then $\sum_{t=0}^{\infty} p_t \cdot x_t \leqq \sum_{t=0}^{\infty} p_t A_t \gamma_{i,-1}$.

Proof First of all, I show that

2.13) $$q_0 \cdot \gamma_{i,-1} = \sum_{t=1}^{\infty} p_t A_t \gamma_{i,-1}.$$

By applying the martingale property (2.2) repeatedly, I obtain that $q_0(s_0) = E\,[p_1(s_1)\,A(s_1) + \dots + p_T(s_T)\,A(s_t) \mid s_0] + E\,[q_T(s_T) \mid s_0]$ almost surely, for all $T > 0$. Hence, $q_0 \cdot \gamma_{i,-1} = \sum_{t=1}^{T} p_t A_t \gamma_{i,-1} + q_T \cdot \gamma_{i,-1}$. By assumption, $\sum_{t=1}^{\infty} E\, q_{tc}(s_t) < \infty$, for all c, so that $\lim_{T\to\infty} \sum_{c=1}^{C} E\, q_{Tc}(s_t) = 0$. Since $q_T \cdot \gamma_{i-1} \leqq \sum_{c=1}^{C} E\, q_{Tc}(s_t)$, (2.13) follows.

I now prove the lemma. The budget constraint for consumer i, (2.1), may be written as

$$\sum_{t=0}^{\infty} p_t \cdot x_t \leqq \sum_{t=0}^{\infty} (q_{t+1} + p_{t+1} A_{t+1} - q_t)\cdot\gamma_t + (q_0 + p_0 A_0) \cdot \gamma_{i,-1}.$$

By the Martingale property (2.2), the infinite sum on the right hand side of this inequality is zero. Hence, $\sum_{t=0}^{\infty} p_t \cdot x_t \leqq p_0 A_0 \gamma_{i,-1} + q_0 \cdot \gamma_{i,-1}$ $\leqq \sum_{t=0}^{\infty} p_t A_t \gamma_{i,-1}$. The equality follows from (2.13).

Q.E.D.

<u>Proof of theorem 2.10</u> First of all, I define a one-period economy which corresponds to the economy of the theorem. Known theorems imply that the one-period economy has an equilibrium. I then show that a stationary equilibrium corresponds to the one-period equilibrium.

The commodity space for the one-period economy is the set of bounded measurable functions from S to R^L. The consumption set of each consumer is the set of such functions with non-negative components. Call this consumption set X. The utility function of each consumer is $V_i : X \to R$, defined by $V_i(x) = Eu_i(x(s_0), s_0)$, for i = 1,....,I. The initial endowment of each consumer is $\omega_i \in X$, defined by $\omega_{ik}(s) = \sum_{c=1}^{C} A_{kc}(s)\gamma_{i,-1,c}(s)$,

for k = 1,...,L and $s \in S$. In matrix notation, $\omega_i = A\,\gamma_{i,-1}$.

An allocation is of the form $(x_i)_{i=1}^{I}$, where $x_i \in X$, for all i. It is feasible if $\sum_{i=1}^{I} (x_i(s) - \omega_i(s)) = 0$, for all s. A price system is a measurable function $p : S \to R_+^L$ such that $0 < Ep_k(s_0) < \infty$, for k=1,...L.

The budget set of consumer i, given p, is $\hat{\beta}_i(p) = \{x \in X \mid p \cdot x \leq p \cdot \omega_i\}$, where $p \cdot x$ denotes $E\, p(s_0) \cdot x(s_0)$. The demand correspondence of consumer i is $\hat{\xi}_i(p) = \{x \in \hat{\beta}_i(p) \mid V_i(x) \geq V_i(\bar{x}),$ for all $\bar{x} \in \beta_i(p)\}$. An equilibrium is of the form $((x_i), p)$, where (x_i) is a feasible allocation, p is a price system and $x_i \in \hat{\xi}_i(p)$, for all i.

If S is a finite set, then it follows from the standard equilibrium existence theorem that the one-period economy has an equilibrium. (See Debreu (1959), p. 83.) If S is an infinite set, then one can apply results from a paper of my own (1972). By Appendix II of that paper, the utility functions V_i are continuous with respect to a certain topology called the Mackey topology. It then follows from (1972), theorems 1 and 2, that an equilibrium exists.

Let $((x_i), p)$ be an equilibrium for the one-period economy. I now define a stationary Arrow-Debreu equilibrium for the economy of the theorem. For $t = 0,1,\ldots$, let $x_{it} = x_i$ and $p_t = (1+\rho)^{-t}p$. It is easy to show that $q_0(s_0) = E[\sum_{t=0}^{\infty} (1+\rho)^{-t} p(s_t)\, A(s_t) \mid s_0]$ is well-defined and that

$E\, q_{0c}(s_0) < \infty$, for $c = 1,\ldots,C$. For $t = 1,2,\ldots$, let $q_t = (1+\rho)^{-t} q_0$. Let $\gamma_{it} = \gamma_{i,-1}$, for $t = 0,1,\ldots$. Finally, let $\underset{\sim}{p} = (p_0, p_1, \ldots)$, $\underset{\sim}{q} = (q_0, q_1, \ldots)$, $\underset{\sim}{x}_i = (x_{i0}, x_{i1}, \ldots)$ and $\underset{\sim}{\gamma}_i = (\gamma_{i0}, \gamma_{i1}, \ldots)$. I claim that $(((\underset{\sim}{x}_i, \underset{\sim}{\gamma}_i))_{i=1}^{I}, (\underset{\sim}{p}, \underset{\sim}{q}))$ is a stationary equilibrium. Clearly, it is stationary. Also, $((\underset{\sim}{x}_i, \underset{\sim}{\gamma}_i))_{i=1}^{I}$ is a feasible allocation and $(\underset{\sim}{p}, \underset{\sim}{q})$ is a price system. Hence, I need only show that $(\underset{\sim}{x}_i, \underset{\sim}{\gamma}_i) \in \xi_i(\underset{\sim}{p}, \underset{\sim}{q})$, for all i.

First of all, I show that $(\underset{\sim}{x}_i, \underset{\sim}{\gamma}_i)$ satisfies consumer i's budget constraint (2.1). Since $\underset{\sim}{\gamma}_{it} = \underset{\sim}{\gamma}_{i,-1}$, for all t, it follows that $(\underset{\sim}{x}_i, \underset{\sim}{\gamma}_i)$

satisfies the budget constraint if

$$2.14) \quad \sum_{t=0}^{\infty} p_t \cdot x_{it} \leq \sum_{t=0}^{\infty} p_t A_t \gamma_{i,-1}.$$

Observe that $p_0 \cdot x_i \leq p_0 \cdot \omega_i$, since x_i satisfies the budget constraint of the one-period economy. Recall that $\omega_i = A \gamma_{i,-1}$. Hence, $p_t \cdot x_i = (1+\rho)^{-t} p \cdot x_i \leq (1+\rho)^{-t} p \cdot \omega_i = (1+\rho)^{-t} p A \gamma_{i,-1} = p_t A_t \gamma_{i,-1}$, for all t. Thus, inequality 2.14 is valid and $(\underset{\sim}{x}_i, \underset{\sim}{\gamma}_i) \in \beta_i(\underset{\sim}{p}, \underset{\sim}{q})$.

Next, observe that $\underset{\sim}{x}_i$ solves the problem

$$\max \ \{V_i(\underset{\sim}{x}) \mid \sum_{t=0}^{\infty} p_t \cdot x_t \leq \sum_{t=0}^{\infty} p_t A_t \gamma_{i,-1}\}.$$

This follows from the facts that $U_i(\underset{\sim}{x}) = E \sum_{t=0}^{\infty} (1+\rho)^{-t} u_i(x_t(s_t))$ and that $p_t = (1+\rho)^{-t} p$, for all t.

Finally, let $(\underset{\sim}{x}, \underset{\sim}{\gamma}) \in \beta_i (\underset{\sim}{p}, \underset{\sim}{q})$. $(\underset{\sim}{p}, \underset{\sim}{q})$ clearly has the martingale property (2.2). Hence, it follows from lemma 2.12 that $\sum_{t=0}^{\infty} p_t \cdot x_t \leq \sum_{t-0}^{\infty} p_t A_t \gamma_{i,-1}$. Therefore by the previous paragraph, $U_i(\underset{\sim}{x}) \leq U_i(\underset{\sim}{x}_i)$. This proves that $\underset{\sim}{x}_i \in \xi_i(\underset{\sim}{p}, \underset{\sim}{q})$.

Q.E.D.

<u>Proof of theorem 2.11</u> The budget constraint for consumer i, (2.1), may be written as

$$2.15) \quad \sum_{t=0}^{\infty} p_t \cdot x_t \leqq \sum_{t=0}^{\infty} (q_{t+1} + p_{t+1} A_{t+1} - q_t)\gamma_t + (q_0 + p_0 A_0)\gamma_{i,-1}.$$

First of all, I show that

$$E [q_{t+1}(s_{t+1}) + p_{t+1}(s_{t+1}) A (s_{t+1}) \mid s_t] - q_t(s_t) \geqq 0 \quad \text{almost surely,}$$

for all t. Suppose that this were not the case. Then, there would exist t, c and a measurable subset B of S such that Prob $[s_t \in B] > 0$ and

$$E [q_{t+1,c}(s_{t+1}) + \sum_k p_{t+1,k}(s_{t+1}) A_{kc}(s_{t+1}) \mid s_t] - q_{tc}(s_t) < 0, \quad \text{for almost}$$

every $s_t \in B$. It follows at once from (2.15) that $\overline{\gamma}_{itc}(s_t) = 0$, for all $s_t \in B$ and for all i. Since $\sum_{i=1}^{I} \overline{\gamma}_{itc} = 1$, this is impossible.

Next, I show that

$$E [q_{t+1}(s_{t+1}) + p_{t+1}(s_{t+1}) A_{t+1}(s_{t+1}) \mid s_t] - q_t(s_t) \leqq 0 \quad \text{almost surely,}$$

for all t. If this were not so, there would exist t, c and a measurable subset B of S such that Prob $[s_t \in B] > 0$ and

$$E [q_{t+1,c}(s_{t+1}) + \sum_k p_{t+1,k}(s_{t+1}) A_{kc}(s_{t+1}) \mid s_t] - q_{tc}(s_t) > 0, \text{ for almost}$$

every $s_t \in B$. It follows that the consumer could make the right hand side of inequality (2.15) arbitrarily large simply by making $\gamma_{tc}(s_t)$ sufficiently large for all $s_t \in B$. This is impossible since each consumer i is in equilibrium at $(\overline{\underset{\sim}{x}}_i, \overline{\underset{\sim}{\gamma}}_i)$.

This proves that

$$E [q_{t+1,}(s_{t+1}) + p_{t+1}(s_{t+1}) A_{t+1}(s_{t+1}) \mid s_t] - q_t(s_t) = 0 \quad \text{almost surely,}$$

for all t, which is the martingale property (2.2).

Q.E.D.

3. The Lucas Model of Asset Prices

In this section, I relate Lucas' work to that of the previous section. I show that Lucas' asset prices may be obtained from the Arrow-Debreu prices just defined simply by renormalization.

First of all, I specialize the model of the previous section, so as to obtain a model which is essentially that of Lucas. Let there be only one consumer and one consumption good, so that $I = L = 1$. Let there continue to be C assets. Because there is one consumer, his initial holdings must be the vector $\gamma_{-1} = (1,\ldots,1) \in R_+^C$. I write this vector as $\vec{1}$. $u : [0,\infty) \to [0,\infty)$ denotes the current period utility function of the one consumer. He discounts future utility at the rate $\rho > 0$, as before. Since u is deterministic, random variation appears only in the assets, $A(s_t)$, so that I may identify s_t and $A(s_t)$. I call the resulting random variable $A_t = (A_{t1},\ldots,A_{tC})$. $\mathcal{A}$ denotes the range of variation of A_t.

Prices are normalized so that the price of the consumption good is always 1. Hence, a stationary price system is described by a function $Q : \mathcal{A} \to R_+^C$. This function is assumed to be continuous. The price of asset c at time t is $Q_c(A_t)$.

Lucas' definition of equilibrium involves the notion of a valuation function. A valuation function is a continuous function $v : R_+^C \times \mathcal{A} \to R$. $v(\gamma,A_0)$ may be interpreted as the discounted expected value of the consumer's current and future flow of utility, given that he holds the bundle of assets γ at time zero when the state of the environment is A_0.

A Lucas equilibrium consists of an asset price system Q and a valuation function v such that

3.1) $v(\gamma,A_0) = \max \{u(x) + (1+\rho)^{-1} E[v(\gamma',A_1) \mid A_0] : \gamma' \in R_+^C$

and $x + Q(A_0)\ (\gamma' - \gamma) \leqq A_0 \cdot \gamma\}$ and

3.2) $v(\vec{1},A_0) = u(\sum_{c=1}^{C} A_{0c}) + (1+\rho)^{-1} E\ [v(\vec{1},A_1) \mid A_0]$, for all $A_0 \in \mathcal{Q}$

A Lucas equilibrium is really a temporary equilibrium. If one eliminates the evaluation function from his model, one arrives at the following. The consumer chooses an infinite horizon program of the form $(\underset{\sim}{x},\underset{\sim}{\gamma}) = ((x_0(A_0), \gamma_0(A_0)), (x_1(A_0,A_1), \gamma_1\ (A_0,A_1)),\ldots.)$. Observe that consumption and investments at time t depend on $A_0,\ldots,A_t$. The consumer's budget set is $\beta_i(Q) = \{(\underset{\sim}{x},\underset{\sim}{\gamma}) \mid x_t(A_0,\ldots,A_t) + Q(A_t) \cdot (\gamma_t(A_0,\ldots,A_t)$ $- \gamma_{t-1}(A_0,\ldots,A_{t-1})) \leqq A_t \cdot \gamma_{t-1}(A_0,\ldots,A_{t-1})$, for all $A_0,\ldots,A_{t-1}$ and for all $t\}$. His demand, $\xi_i(Q)$, is the set of solutions to the problem $\max\{E[\sum_{t=o}^{\infty} (1+\rho)^{-t}\ u(x_t(A_0,\ldots,A_t)) \mid (\underset{\sim}{x},\underset{\sim}{\gamma}) \in \beta_i(Q)\}$. Equilibrium occurs if Q is such that $(\bar{\underset{\sim}{x}},\bar{\underset{\sim}{\gamma}}) \in \xi_i(Q)$, where $\bar{x}_t(A_0,\ldots,A_t) = \sum_c A_{tc}$ and $\gamma_t(A_0,\ldots,A_t) = \vec{1}$, for all $A_0,\ldots,A_t$ and for all t. This equilibrium is clearly a temporary equilibrium since the consumer faces a different budget constraint in each time period. It is not hard to see that the price system Q of a Lucas equilibrium is also the price system of a stationary temporary equilibrium. (It is sufficient to use Blackwell (1965), theorem 6, part f, in order to assert that a program which satisfies the Bellman equation, 3.1, also solves the infinite horizon optimization problem. Aloisio Araujo also has shown me a simple proof of this fact.)

By a temporary equilibrium, I mean an equilibrium in which there are no forward markets and consumers have a different budget constraint in each period.

Grandmont (1977) has written an excellent survey of temporary equilibrium theory.

I now show that the prices of a Lucas equilibrium correspond to those of an Arrow-Debreu equilibrium. First of all, I make the following assumptions.

3.3) $u: [0,\infty) \to [0,\infty)$ is continuously differentiable, bounded, strictly increasing and strictly concave.

3.4) The transition probabilities of A_t are generated by a continuous function $F : \mathcal{A} \times \mathcal{A} \to [0,1]$. $F(A',A) = \text{Prob}\,[A_1 \leqq A' \mid A_0 = A]$.

3.5) The Markov process A_t has a unique stationary distribution.

I will assume that the A_t are distributed according to the stationary distribution.

Assumptions similar to (3.3) - (3.5) were made by Lucas. I add the following assumption.

3.6) $\mathcal{A}$ is compact.

By Lucas' Propositions 1 and 3, there exists a Lucas equilibrium (Q,v). For each t and A, let $p_t(A) = (1+\rho)^{-t} \frac{du}{dx}(\sum_c A_c)$ and let $q_t(A) = p_t(A)Q(A)$. This defines a price system $(\underset{\sim}{p},\underset{\sim}{q})$. Let $(\underset{\sim}{\overline{x}}\ \underset{\sim}{\overline{\gamma}})$ be defined by $\overline{x}_t(A) = \sum_c A_c$ and $\overline{\gamma}_t(A) = \vec{1}$, for all t and A.

Proposition $((\underset{\sim}{\overline{x}},\underset{\sim}{\overline{\gamma}}),\ (\underset{\sim}{p},\underset{\sim}{q}))$ is a stationary Arrow-Debreu equilibrium.

Proof Lucas shows that Q satisfies the equation

$$\frac{du}{dx}\left(\sum_c A_{0c}\right) Q\,(A_0) = (1+\rho)^{-1}\, E\left[\frac{du}{dx}\left(\sum_c A_{1c}\right)\left(Q(A_1) + A_1\right) \mid A_0\right],$$

for all $A_0 \in \mathcal{A}$ (See Lucas (1978), equation 6, p. 1434.) When written in terms of p_t and q_t, this equation becomes

$$q_t(A_t) = E[q_{t+1}(A_{t+1}) + p_{t+1}(A_{t+1})\, A_{t+1} \mid A_t], \quad \text{for all } t \text{ and } A_t.$$

That is, $(\underset{\sim}{p},\underset{\sim}{q})$ has the martingale property. Hence by lemma 2.13, if $(\underset{\sim}{x},\underset{\sim}{\gamma})$ belongs to the Arrow-Debreu budget set of the consumer, (2.1), then,

$$E\sum_{t=0}^{\infty} p_t(A_t)\, x_t(A_t) \leq E \sum_{t=0}^{\infty} p_t\,(A_t)\, A_t\, \vec{1}.$$

(Assumption 3.6 guarantees that these infinite sums make sense.) The program $(\underset{\sim}{\bar{x}},\underset{\sim}{\bar{\gamma}})$ satisfies this constraint since $\bar{x}_t(A) = A \cdot \vec{1}$, for all t and A. Since $p_t(A) = (1+\rho)^{-t}\,\frac{du}{dx}\,(\bar{x}_t(A))$, for all t and A, it follows easily that $(\underset{\sim}{\bar{x}},\underset{\sim}{\bar{\gamma}})$ solves the problem

$$\max\,\{E\sum_{t=0}^{\infty}(1+\rho)^{-t}\, u(x_t(A_t)) \mid E\sum_{t=0}^{\infty} p_t(A_t)\, x_t(A_t) \leq E\sum_{t=0}^{\infty} p_t(A_t)A_t \cdot \vec{1}\}.$$

Therefore, $(\underset{\sim}{\bar{x}},\underset{\sim}{\bar{\gamma}})$ belongs to the Arrow-Debreu demand correspondence.

Q.E.D.

It is possible to obtain a Lucas equilibrium from an Arrow-Debreu equilibrium. Add the following assumptions.

There exists $\underline{r} > 0$ such that $A_c \geqq \underline{r}$, for all c and for all $A \in \mathcal{A}$.

The support of the stationary distribution of A_t is $\mathcal{A}$.

The second assumption means that if B is an open subset of R^C which intersects $\mathcal{A}$, then Prob $[A_t \in B] > 0$.

By Theorem 2.10, the economy has stationary Arrow-Debreu equilibrium, $((\bar{\underset{\sim}{x}}, \bar{\underset{\sim}{y}}), (p,q))$. Let $Q(A) = p_0(A)^{-1}q_0(A)$. It is easy to see that Q is well-defined and almost everywhere equal to a continuous function. Hence, I may assume that Q is continuous. By Lucas' Proposition 1, there exists a continuous bounded function $v : R_+^C \times \mathcal{A} \to [0,\infty)$ satisfying expectation 3.1, with Q as just defined. It is not hard to prove that Q and v form a Lucas equilibrium.

I am not the first to notice the link between Arrow-Debreu equilibrium and Lucas equilibrium. This link is used by LeRoy and La Civita (1980).

4. A monetary Model

I now define a temporary equilibrium model with money and assets. It seems to me that the martingale property is best discussed in terms of some such model, for one normally thinks of the martingale property as referring to prices of assets in terms of money. I show that asset prices satisfy the martingale property, provided that the marginal utility of money is constant.

In a temporary equilibrium, prices and programs must be history dependent. It is most convenient to allow them to depend on the infinite past. In order to do so, I introduce new notation. Σ denotes the set of doubly infinite paths $\{s_t\}_{t=-\infty}^{\infty}$, where $s_t \in S$, for all t. $\underset{\sim}{s} = (\ldots, s_{-1}, s_0, s_1, \ldots)$ denotes a typical element of Σ. s_t may be viewed as random variable, $s_t : \Sigma \to S$. $\mathscr{A}$ denotes the smallest σ - field on Σ such that all the random variables s_t are measurable with respect to $\mathscr{A}$. $\mathscr{A}_t$ denotes the smallest σ-field on Σ such that all the random variables s_n are measurable with respect to $\mathscr{A}_t$, for $n \leq t$. $\mathscr{A}_t$ represents the information available at time t. $E[\,\cdot \mid \mathscr{A}_t]$ denotes the conditional expectation with respect to $\mathscr{A}_t$. Finally, $\sigma : \Sigma \to \Sigma$ denotes the shift operator defined by $\sigma(\underset{\sim}{s})_t = s_{t+1}$.

The asset function A is now written as $A_0 : \Sigma \to R_+^{L \times C}$, where $R_+^{L \times C}$ denotes the set of $L \times C$ matrices with non-negative entries. $A_0(\underset{\sim}{s}) = A(s_0)$, where A is as in section 2. A_t denotes $A_0 \circ \sigma^t$. That is, $A_t(\underset{\sim}{s}) = A(s_t)$. $A_t(\underset{\sim}{s})$ is the matrix of asset bundles available in period t.

The utility function of consumer i is written as $u_{i0} : R_+^L \times \Sigma \to R$. $u_{i0}(x, \underset{\sim}{s}) = u_i(x, s_0)$, where u_i is as in section 2. u_{i0} gives the utility from consumption at time zero. The utility of consumption at time t is given by $u_{it}(x, \underset{\sim}{s}) = u_i(x, s_t)$. Consumers discount future utility at the rate ρ. ρ is the pure rate of time preference.

I permit consumers to have endowment flows which are not associated with assets. I do so because in an equilibrium all assets might eventually belong to one consumer. In this situation, the economy would really have only one consumer, unless consumers had income which came from some other source than assets. Of course, a one consumer economy is not at all interesting from the point of view of monetary theory, since money never changes hands in such an economy.

The initial endowment of consumer i is given by a function $\omega_{i0} : \Sigma \to R_+^L$. ω_{i0} is measurable and depends only on s_0. $\omega_{it} = \omega_{i0} \circ \sigma^t$ gives the endowment in period t.

Programs are now written as $\underset{\sim}{x} = (x_t)_{t=0}^{\infty}$ and $\underset{\sim}{\gamma} = (\gamma_t)_{t=0}^{\infty}$, where $x_t : \Sigma \to R_+^L$ and $\gamma_t : \Sigma \to R_+^C$ are measurable with respect to $\mathscr{A}_t$. Similarly, a price system is written as $(\underset{\sim}{P},\underset{\sim}{Q})$, where $\underset{\sim}{P} = (P_t)_{t=0}^{\infty}$ and $\underset{\sim}{Q} = (Q_t)_{t=0}^{\infty}$ $P_t : \Sigma \to R_+^L$ and $Q_t : \Sigma \to R_+^C$ are measurable with respect to $\mathscr{A}_t$.

$\gamma_{i,-1} : \Sigma \to R_+^C$ gives the initial investments of consumer i held at the end of the period -1. $\gamma_{i,-1}$ is measurable with respect to $\mathscr{A}_{-1}$.

I now introduce money. I follow Friedman (1969) and allow money to earn interest. It earns interest at a constant nominal rate $r \geqq 0$. The total nominal stock of money is one. Each consumer i pays a lump sum tax $\tau_i r$ each period. I assume that $\sum_i \tau_i = 1$, so that the nominal stock of money remains constant.

The initial money holding of consumer i is given by a function $M_{i,-1} : \Sigma \to [0,1]$, which is measurable with respect to $\mathscr{A}_{-1}$. $M_{i,-1}(\underset{\sim}{s})$ is the quantity of money held by consumer i at the end of period -1.

Let the price system $(\underset{\sim}{P},\underset{\sim}{Q})$ and the program $(\underset{\sim}{x},\underset{\sim}{\gamma})$ be given.

$M_{it}(\underset{\sim}{P},\underset{\sim}{Q}, \underset{\sim}{x},\underset{\sim}{\gamma},\cdot) : \Sigma \to [0,\infty)$ gives the money held by consumer i at the end of period t if he follows the program $(\underset{\sim}{x},\underset{\sim}{\gamma})$ when the price system is $(\underset{\sim}{P},\underset{\sim}{Q})$. M_{it} is defined by induction on t. $M_{i,-1}(\underset{\sim}{P},\underset{\sim}{Q},\underset{\sim}{x},\underset{\sim}{\gamma},\underset{\sim}{s}) = M_{i,-1}(\underset{\sim}{s})$. Suppose that $M_{i,t-1}(\underset{\sim}{P},\underset{\sim}{Q},\underset{\sim}{x},\underset{\sim}{\gamma},\underset{\sim}{s})$ has been defined. Then,

$$4.1) \quad M_{it}(\underset{\sim}{P},\underset{\sim}{Q},\underset{\sim}{x},\underset{\sim}{\gamma},\underset{\sim}{s}) = (1+r)M_{i,t-1}(\underset{\sim}{P},\underset{\sim}{Q},\underset{\sim}{x},\underset{\sim}{\gamma},\underset{\sim}{s}) - \tau_i r + P_t(\underset{\sim}{s}) \cdot (A_t(\underset{\sim}{s})\gamma_{t-1}(\underset{\sim}{s}) + \omega_{it}(\underset{\sim}{s}) - x_t(\underset{\sim}{s})) + Q_t(\underset{\sim}{s}) \cdot (\gamma_{t-1}(\underset{\sim}{s}) - \gamma_t(\underset{\sim}{s})).$$

Given $(\underset{\sim}{P},\underset{\sim}{Q})$, the <u>budget set</u> of consumer i is $\beta_i(M_{i,-1}, \gamma_{i,-1},\underset{\sim}{P},\underset{\sim}{Q})$ $= \{(\underset{\sim}{x},\underset{\sim}{\gamma}) \mid M_{i,t}(\underset{\sim}{P},\underset{\sim}{Q},\underset{\sim}{x},\underset{\sim}{\gamma},\underset{\sim}{s}) \geqq 0$ almost surely, for all $t\}$. His demand correspondence is the set of solutions to the problem

$$\max \{ E[\sum_{t=0}^{\infty} (1+\rho)^{-t} u_{it}(x_t(\underset{\sim}{s}),\underset{\sim}{s}) \mid (\underset{\sim}{x},\underset{\sim}{\gamma}) \in \beta_i(M_{i,-1},\gamma_{i,-1},\underset{\sim}{P},\underset{\sim}{Q})\}.$$

An <u>allocation</u> is of the form $((\underset{\sim}{x}_i,\underset{\sim}{\gamma}_i))_{i=1}^{I}$. It is feasible if

$$\sum_i x_{it}(\underset{\sim}{s}) = \sum_c A_{tc}(\underset{\sim}{s}) + \sum_i \omega_{it}(\underset{\sim}{s}), \quad \text{almost surely, for all } t.$$

A <u>monetary equilibrium</u>, given the initial conditions $((M_{i,-1},\gamma_{i,-1}))_{i=1}^{I}$, consists of $(((\underset{\sim}{x}_i,\underset{\sim}{\gamma}_i))_{i=1}^{I}, (\underset{\sim}{P},\underset{\sim}{Q}))$, where

i) $((\underset{\sim}{x}_i,\underset{\sim}{\gamma}_i))_{i=1}^{I}$ is a feasible allocation,

ii) $(\underset{\sim}{x}_i,\underset{\sim}{\gamma}_i) \in \xi_i(M_{i,-1}, \gamma_{i,-1},\underset{\sim}{P},\underset{\sim}{Q})$, for all i, and

iii) there exists $b > 0$ such that $b \leqq P_{tk}(s) \leqq b^{-1}$ and $b \leqq Q_{tc}(s) \leq b^{-1}$ almost surely, for all t, k and c.

The last condition excludes equilibria with prices which diverge to infinity or converge to zero. It guarantees that the long-run average real rate of interest equals the nominal rate r.

The notation for an equilibrium that I have just introduced is cumbersome to use. In subsequent sections of the paper, I eliminate assets from the model. In a model without assets, I use the more convenient notation $((\underset{\sim}{x}_i), \underset{\sim}{P}, (\underset{\sim}{M}_i))$ for a monetary equilibrium. In this expression, $(\underset{\sim}{x}_i)$ is a goods allocation, $\underset{\sim}{P}$ is a price system for goods and $\underset{\sim}{M}_i = (M_{i,-1}, M_{i0}, M_{i1},\ldots)$, where $M_{it}(s)$ is the equilibrium money balance of consumer i at the end of period t. Notice that the vector $(\underset{\sim}{M}_i)$ includes the initial condition $(M_{i,-1})_{i=1}^{I}$.

The monetary model has other equivalent formulations, which are discussed in section 6 and Appendix III. In section 6, it is pointed out that the money balances may be interpreted to be credit balances. In Appendix III, I show that r may be interpreted as a rate of deflation. I also show there that one may introduce market determined interest rates.

Given a monetary equilibrium $(((\underset{\sim}{x}_i,\underset{\sim}{\gamma}_i))_{i=1}^{I}, (\underset{\sim}{P},\underset{\sim}{Q}))$, it is possible to define associated marginal utilities of money $(\underset{\sim}{\lambda}_i)_{i=1}^{I}$, where $\underset{\sim}{\lambda}_i = (\lambda_{it})_{t=0}^{\infty}$ and each $\lambda_{it} : \Sigma \to (0,\infty]$ is measurable with respect to $\mathscr{S}_t$. $\lambda_{it}(\underset{\sim}{s})$ is the marginal utility of money of consumer i at time t. There is a technical difficulty, which is that it is not obvious that the $\lambda_{it}(\underset{\sim}{s})$ are finite. Assume that they are finite.

It is not hard to see that the marginal utilities of money satisfy the following relation.

4.2) For almost every $\underset{\sim}{s}$ and for all t,

$$\lambda_{it}(\underset{\sim}{s})\, Q_{tc}(\underset{\sim}{s}) \geqq (1+\rho)^{-1}\, E[\lambda_{i,t+1}\,(Q_{t+1,c} + P_{t+1}\, A_{t+1,c}) \mid \mathscr{S}_t](\underset{\sim}{s}),$$

for all c, with equality if $\gamma_{itc}(\underset{\sim}{s}) > 0$.

Now suppose that the marginal utilities of money are constant. Then, since $\sum_{i=1}^{I} \gamma_{itc}(s) = 1$, almost surely, for all t and c, (4.2) implies that

$$Q_t(s) = (1+\rho)^{-1}E[Q_{t+1} + P_{t+1} A_{t+1} \mid \mathscr{A}_t](s) \text{ almost surely, for all } t.$$

In summary, constancy of the marginal utilities of money imply the martingale property.

This condition guaranteeing the martingale property is the same as that of theorem 2.11. It is not hard to show that if the marginal utilities of money are constant, then the prices (P,Q) are simply current value versions of the prices of an Arrow-Debreu equilibrium.

I do not know whether monetary equilibria exist. However, the existence question is not really relevant here. I will argue that even if equilibria did exist, consumers would not necessarily be self-insuring, so that the model does not provide a justification for the martingale property.

If there are no assets, then monetary equilibria do exist, provided that r is sufficiently small. This fact is proved in Bewley (1980a). As I mentioned in the Introduction, the error in the previous paper is corrected in Appendix II of this paper.

5. An Example

The example of this section is of a monetary economy in which consumers' marginal utilities of money are not constant in any equilibrium. They do not even become constant asymptotically as the interest rate converges to consumers' pure rate of time preference. In fact, there is an interest rate $\underline{r}$ less than the rate of time preference and such that no equilibrium exists if the interest rate exceeds $\underline{r}$.

There are no assets in the example. The example may be modified so as to include a small amount of assets. I do not give the modified example, for assets complicate and obscure the argument. At the end of the section, I discuss briefly what happens if assets are included.

The Example There is one kind of consumption good and there are two consumers, indexed by 1 and 2. The random variables, s_t, are independently and identically distributed. s_t takes on two values, a and b, each with probability $1/2$. The utility function of each consumer is $u(x) = \log(x+1)$. Notice that utility is deterministic. The pure rate of time preference of each consumer is 0.1. The initial endowment of consumer i at time t is $\omega_{it}(\underset{\sim}{s}) = \omega_i(s_t)$, where ω_i is defined as follows.

$\omega_1(a) = \omega_2(b) = 1/4$. $\omega_1(b) = \omega_2(a) = 1^3/4$. Notice that the total supply of the good is constant, since $\omega_{1t}(\underset{\sim}{s}) + \omega_{2t}(\underset{\sim}{s}) = 2$, for all $\underset{\sim}{s}$ and t. There is one unit of money in the economy. The tax paid by each consumer each period is $r/2$, where r is the interest rate.

Let $((\underset{\sim}{x}_1,\underset{\sim}{x}_2),\underset{\sim}{P},(\underset{\sim}{M}_1,\underset{\sim}{M}_2))$ be a monetary equilibrium for the above example and let $(\underset{\sim}{\lambda}_1,\underset{\sim}{\lambda}_2)$ be the vector of associated marginal utilities of money.

In Appendix I, I prove that the $\underset{\sim}{\lambda}_i$ exist and are finite-valued.

In what follows, I make use of conditions (5.1) and (5.2) listed below. If either condition were violated, it would be possible to construct a program in the consumer's budget set that gave him higher expected utility. This would contradict the fact that the $\underset{\sim}{x}_i$ are equilibrium programs.

5.1) For all t, for $i = 1,2$ and for almost every $\underset{\sim}{s}$,

$$\lambda_{it}(\underset{\sim}{s})\ P_t(\underset{\sim}{s}) \geqq \frac{du_{it}(x_{it}(\underset{\sim}{s}),\underset{\sim}{s})}{dx} = (x_{it}(\underset{\sim}{s}) + 1)^{-1},$$

with equality if $x_{it}(\underset{\sim}{s}) > 0$.

5.2) For all t, for $i = 1,2$ and for almost every $\underset{\sim}{s}$,

$$\lambda_{it}(\underset{\sim}{s}) \geqq (1+r)(1.1)^{-1}\ E[\lambda_{i,t+1} \mid \mathcal{J}_t](\underset{\sim}{s}), \text{ with equality if}$$

$$M_{it}(\underset{\sim}{s}) > \underline{M}_{it}(\underset{\sim}{s}),$$

where $\underline{M}_{it}(\underset{\sim}{s})$ denotes the minimum balance of consumer i at the end of period t. $\underline{M}_{it}(\underset{\sim}{s})$ is the smallest balance consumer i can hold and be sure that with probability one he will never be obliged to hold negative balances. The $\underline{M}_{it}$ are defined in Appendix I.

Since each consumer is obliged to pay the tax $^r/2$ every period, $\underline{M}_{it}(\underset{\sim}{s})$ may be positive. In fact, the $\underline{M}_{it}(\underset{\sim}{s})$ may be made arbitrarily close to $^1/2$ by letting r be close enough to the pure rate of time preference, 0.1. As r approaches 0.1, each consumer's desire for money balances increases without bound. This demand for money forces prices down toward zero, so that the tax $r/2$ becomes infinitely burdensome relative to consumers' monetary incomes, $P_t(\underset{\sim}{s}) \cdot \omega_{it}(\underset{\sim}{s})$. It is for this reason that money does not provide perfect insurance asymptotically as r approaches the rate of time preference.

It is possible to bound $\underline{M}_{it}(\underset{\sim}{s})$ away from $r/2$ in a given equilibrium. By the definition of a monetary equilibrium, there exists $\underline{P} > 0$ such that $P_t(s) \geqq \underline{P}$ almost surely, for all t. Since $\omega_{it}(\underset{\sim}{s}) \geqq 1/4$, for 1 = 1,2 and for all $\underset{\sim}{s}$ and t, it follows that each consumer has an income of at least $(1/4)\underline{P}$ in each period. This fact implies that

5.3) $\underline{M}_{it}(\underset{\sim}{s}) \leqq \max\,(0, 1/2 - (1/4)\, r^{-1}\underline{P})$ almost surely for $i = 1,2$ and for all t.

This definition of a monetary equilibrium asserts that there is $\overline{P}$ such that $P_t(\underset{\sim}{s}) \leqq \overline{P}$ almost surely, for all t. Also since $(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ is a feasible allocation, $x_{it}(\underset{\sim}{s}) \leqq 2$ almost surely, for $i = 1,2$ and for all t. Hence (5.1) implies that

5.4) $\lambda_{it}(\underset{\sim}{s}) \geqq (3\overline{P})^{-1}$ almost surely, for $i=1,2$ and for all t.

I now show that if $0 \leqq r < 0.1$, then consumers' marginal utilities of money cannot be constant in a monetary equilibrium with interest rate r. Let $\varepsilon > 0$ be such that $\varepsilon < (0.1 - r)(2.1 + r)^{-1}$. I prove that

5.5) there does not exist $\lambda > 0$ such that Prob $\{\underset{\sim}{s} \mid |\lambda_{it}(\underset{\sim}{s}) - \lambda| \leqq \lambda\varepsilon$, for $i = 1$ or $2\} \geqq 3/4$, for all t, where Prob stands for "probability of."

Suppose that there did exist λ as in (5.5). Let $\Sigma_i = \{\underset{\sim}{s} \mid |\lambda_{j0}(\underset{\sim}{s}) - \lambda| \leq \lambda\varepsilon$, for $j = 1,2$, and $M_{i0}(\underset{\sim}{s}) > \underline{M}_{i0}(\underset{\sim}{s})\}$, where $i = 1,2$. Since $M_{10}(\underset{\sim}{s}) + M_{20}(\underset{\sim}{s}) = 1$, it follows from (5.3) that $\Sigma_1 \cup \Sigma_2 = \{\underset{\sim}{s} \mid |\lambda_{i0}(\underset{\sim}{s}) - \lambda| \leq \lambda\varepsilon$, for $i = 1$ and $2\}$. Hence, Prob $(\Sigma_1 \cup \Sigma_2) \geq {}^3/4$.

Since $M_{i0}(\underset{\sim}{s}) > \underline{M}_{i0}(\underset{\sim}{s})$ when $\underset{\sim}{s} \in \Sigma_i$, inequality 5.2 implies that for almost every $\underset{\sim}{s} \in \Sigma_i$,

$$\lambda_{i0}(\underset{\sim}{s}) = (1+r)\ (1\cdot 1)^{-1}\ E[\lambda_{i1} \mid \mathscr{I}_0](\underset{\sim}{s}) = ({}^1/2)\ (1+r)\ (1.1)^{-1}(\lambda^a_{i1}(\underset{\sim}{s}) + \lambda^b_{i1}(\underset{\sim}{s})),$$

where $\lambda^a_{i1}(\underset{\sim}{s}) = E[\lambda_{i1} \mid \mathscr{I}_0$ and $s_1 = a](\underset{\sim}{s})$ and $\lambda^b_{i1}(\underset{\sim}{s}) = E[\lambda_{i1} \mid \mathscr{I}_0$ and $s_1 = b](\underset{\sim}{s})$. Let $\underset{\sim}{s} \varepsilon \Sigma_i$. If $|\lambda^a_{i1}(\underset{\sim}{s}) - \lambda| \leq \lambda\varepsilon$ and $|\lambda^b_{i1}(\underset{\sim}{s}) - \lambda| \leq \lambda\varepsilon$, then $\lambda(1-\varepsilon) \leq \lambda_{i0}(\underset{\sim}{s}) \leq (1+r)\ (1\cdot 1)^{-1}\ \lambda(1+\varepsilon)$, which is impossible because of the choice of ε. Therefore, either $|\lambda^a_{i1}(\underset{\sim}{s}) - \lambda| > \lambda\varepsilon$ or $|\lambda^b_{i1}(\underset{\sim}{s}) - \lambda| > \lambda\varepsilon$. I have proved that Prob $\{\underset{\sim}{s} \mid |\lambda_{i1}(\underset{\sim}{s}) - \lambda| > \lambda\varepsilon$, for $i = 1$ or $2\}$ $\geq ({}^1/2)$Prob $(\Sigma_1 \cup \Sigma_2) \geq ({}^1/2)({}^3/4) = {}^3/8$. But by hypothesis, Prob $\{\underset{\sim}{s} \mid |\lambda_{i1}(\underset{\sim}{s}) - \lambda| > \lambda\varepsilon$, for $i = 1$ or $2\}$ = 1 - Prob $\{\underset{\sim}{s} \mid |\lambda_{i1}(\underset{\sim}{s}) - \lambda| \leq \lambda\varepsilon$, for $i = 1$ and $2\}$ $\leq {}^1/4$. This contradiction proves (5.5).

5.6) there exists no monetary equilibrium if r exceeds $0\cdot 1$.

Inequality $5\cdot 2$ implies that $E\lambda_{it} \leq (1\cdot 1)\ (1+r)^{-1}\ E\lambda_{i,t-1}$ $\leq \ldots \leq (1\cdot 1)^t\ (1+r)^{-t}\ E\lambda_{i0}$. Hence, $\lim_{t\to\infty} E\lambda_{it} = 0$ if $r > 0\cdot 1$. This contradicts the fact that the λ_{it} are bounded away from zero (see 5.4).

Finally, I prove that

5.7) there exists $\underline{r}$ such that $0 < \underline{r} < 0.1$ and no monetary equilibrium exists if $0.1 \geqq r \geqq \underline{r}$.

The idea of the argument is as follows. By inequality 5.4, consumers' marginal utilities of money are bounded away from zero. Some consumer must at some time have a marginal utility of money very near to the lowest level it ever reaches. If this is so and if r is close to 0.1, then it follows that the consumer's marginal utility of money must remain near its minimum level for a long time afterward. (This assertion follows from inequality 5.2.) If the consumer's marginal utility of money is nearly constant over a long period of time, then he does not protect himself against a run of bad luck by increasing his marginal utility of money and so buying less. (Bad luck occurs if the consumer's indowment is only $1/4$.) In fact, I show that the consumer will with positive probability eventually hold negative money balances. This contradicts the definition of a monetary equilibrium.

I now turn to the formal proof. Let $\underline{\lambda} = \min_i \inf_t \text{ess inf } \lambda_{it}$, where $\text{ess inf } \lambda_{it} = \sup \{c > 0 \mid \text{Prob } [\lambda_{it}(\underset{\sim}{s}) < c] = 0\}$. By 5.4, $\underline{\lambda} > 0$.

I first show that

5.8) $P_t(\underset{\sim}{s}) \leqq (2\,\underline{\lambda})^{-1}$ almost surely, for all t.

Since $x_{1t}(\underset{\sim}{s}) + x_{2t}(\underset{\sim}{s}) = 2$ almost surely, for all t, it follows that for each t and almost every $\underset{\sim}{s}$, there exists i such that $x_{it}(\underset{\sim}{s}) \geqq 1$. By inequality 5.2, for this i one has $\underline{\lambda}\, P_t(\underset{\sim}{s}) \leqq \lambda_{it}\, P_t(\underset{\sim}{s}) = (1+x_{it}(\underset{\sim}{s}))^{-1} \leqq 1/2$. This proves 5.8.

I next prove that

5.9) $M_{it}(\underset{\sim}{s}) \geqq 1/2 - (8\, r\, \underline{\lambda})^{-1}$ almost surely, for $i = 1,2$ and for all t.

If the state s_{t+1} is bad for consumer i, then his endowment is $\omega_{i,t+1}(\underset{\sim}{s}) = 1/4$, so that by inequality 5.8 he can earn at most $(8\, \underline{\lambda})^{-1}$ by selling his endowment. Hence, if s_{t+1} is bad for consumer i, then $M_{i,t+1}(\underset{\sim}{s}) \leqq (1+r)\, M_{i,t}(\underset{\sim}{s}) - r/2 + (8\, \underline{\lambda})^{-1}$. But then, $M_{i,t+1}(\underset{\sim}{s}) < M_{i,t}(\underset{\sim}{s})$, unless $M_{it}(\underset{\sim}{s}) \geqq 1/2 - (8\, r\, \underline{\lambda})^{-1}$. In fact, if $M_{it}(\underset{\sim}{s}) < 1/2 - (8\, r\, \underline{\lambda})^{-1}$, then $M_{i,t+K}(\underset{\sim}{s}) < 0$ if K is sufficiently large and if $s_{t+1},\ldots,s_{t+K}$ are all bad for consumer i. But $s_{t+1},\ldots,s_{t+K}$ may all be bad for i with positive probability. Since $M_{i,t+K}(\underset{\sim}{s})$ is almost surely non-negative, a contradiction occurs unless 5.9 is true.

I now choose an event and a time period for which the marginality utility of money of one of the consumers is very close to $\underline{\lambda}$. Choose t such that for i = 1 or 2, Prob $\{\underset{\sim}{s} \mid \lambda_{it}(\underset{\sim}{s}) < \underline{\lambda}\,(1+\varepsilon)\} > 0$, where $\varepsilon > 0$ will be determined below. Without loss of generality, I may assume that $i = 1$ and $t = 0$. Let

5.10) $\Sigma_0 = \{\underset{\sim}{s} \mid \lambda_{10}(s) < \underline{\lambda}\ (1+\varepsilon)\}.$

By assumption, Prob $\Sigma_0 > 0$.

Let δ be a small positive number. δ will be determined below. I now show that the ε of 5.10 and the $\underline{r}$ of 5.7 may be chosen so that

5.11) $\lambda_{it}(\underset{\sim}{s}) \leqq (1+\delta)\ \underline{\lambda}$ for almost every $\underset{\sim}{s} \in \Sigma_0$ if $0 \leqq t \leqq 40$.

Inequality 5.2 and 5.10 together imply that

5.12) $\underline{\lambda}(1+\varepsilon) \geqq \lambda_{1,0}(\underset{\sim}{s}) \geqq \left(\frac{1+r}{1.1}\right)^t \left[(^1/2)^t\ \lambda_{1t}(\underset{\sim}{s}) + (1-(^1/2)^t)\underline{\lambda}\right]$, for almost every $\underset{\sim}{s} \in \Sigma_0$ and for all t.

The second inequality above follows from the fact that conditional on the history $(\ldots.,s_{-1}s_0)$, there are 2^t possible values of λ_{it}, each occurring with equal probability and all at least as large as $\underline{\lambda}$.

A rearrangement of 5.12 yields $\lambda_{1t}(\underset{\sim}{s}) \leqq \underline{\lambda} + \underline{\lambda}\ 2^t \left[\left(\frac{1.1}{1+r}\right)^t (1+\varepsilon) - 1\right]$ $\leqq \underline{\lambda} + \underline{\lambda}\ 2^{40}\left[\left(\frac{1.1}{1+\underline{r}}\right)^{40} (1+\varepsilon) - 1\right]$. Clearly for given δ, $2^{40}\left[\left(\frac{1.1}{1+\underline{r}}\right)^{40}(1+\varepsilon) - 1\right]$ $< \delta$, provided that $\underline{r}$ is sufficiently close to 0.1 and ε is sufficiently small.

I now show that

5.13) $P_t(\underset{\sim}{s})\ x_{1t}(\underset{\sim}{s}) \geqq 3(8\ \underline{\lambda})^{-1}$, for $t = 0,1,\ldots,40$ and for almost every $\underset{\sim}{s} \in \Sigma_0$,

provided that δ is sufficiently small and (5.11) is true.

In order to prove (5.13), I express the equilibrium consumption of the consumers and the price of the consumption good as functions of the marginal utilities of money. I drop the variables t and $\underset{\sim}{s}$, for the moment, so that x_i is the consumption of consumer i, λ_i is his marginal utility of money and P is the price of the consumption good, all at one moment of time and in one state of the world. These variables satisfy the following relations.

$$(x_1+1)^{-1} \leqq \lambda_1 P, \text{ with equality if } x_1 > 0,$$

$$(x_2+1)^{-1} \leqq \lambda_2 P, \text{ with equality if } x_2 > 0, \text{ and}$$

$$x_1 + x_2 = 2.$$

Solving these relations, I obtain that $Px_1 = (^2/3)\lambda_1^{-1}$, if $\lambda_1 \leqq (^1/3)\lambda_2$ and $Px_1 = (4\ \lambda_1\ \lambda_2)^{-1}\ (3\ \lambda_2 - \lambda_1)$, if $(^1/3)\lambda_2 \leqq \lambda_1 \leqq 3\ \lambda_2$. Px_1 is a non-increasing function of λ_1 and a non-decreasing function of λ_2. Therefore, if $\lambda_1 \leqq (1+\delta)\ \underline{\lambda}$ and $\lambda_2 \geqq \underline{\lambda}$, it follows that

$$Px_1 \geqq (4(1+\delta)\,\underline{\lambda}^2)^{-1}\,(3\underline{\lambda} - (1+\delta)\,\underline{\lambda}) = (^1/_4)(1+\delta)^{-1}\,(2-\delta)\,\underline{\lambda}^{-1} > 3(8\underline{\lambda})^{-1},$$

provided that δ is sufficiently small. By the definition of $\underline{\lambda}$, $\lambda_{2t}(\underset{\sim}{s}) \geqq \underline{\lambda}$, for all t and almost every $\underset{\sim}{s}$. By 5.11, $\lambda_{1t}(\underset{\sim}{s}) \leqq (1+\delta)\,\underline{\lambda}$, for almost every $\underset{\sim}{s} \in \Sigma_0$ and for $t = 0,1,\ldots,40$. Therefore, 5.13 is true.

I now assume that δ is so small that 5.13 is true. Also, I assume that $\underline{r}$ is so close to 0.1 and ε is so small that 5.11 is true. I also assume that $\underline{r} > {}^1/20$. This determines $\underline{r}$ and ε. Notice that $\underline{r}$ does not depend on $\underline{\lambda}$ or on the particular equilibrium in any way. $\underline{r}$ is a true a priori bound.

I now derive a contradiction. Let $\Sigma_0' = \{\underset{\sim}{s} \in \Sigma_0 \mid s_1 = s_2 = \ldots = s_{40} = a\}$. Clearly, Prob $\Sigma_0' > 0$. I will show that $M_{1,40}(\underset{\sim}{s}) < (^1/2) - (8r\underline{\lambda})^{-1}$ if $\underset{\sim}{s} \in \Sigma_0'$. This contradicts 5.9.

By 5.9, $M_{2,0}(\underset{\sim}{s}) \geqq (^1/2) - (8r\underline{\lambda})^{-1}$ almost surely, so that $M_{1,0}(\underset{\sim}{s}) \leqq {}^1/2 + (8r\underline{\lambda})^{-1}$ almost surely. If $\underset{\sim}{s} \in \Sigma_0'$ and $1 \leqq t \leqq 40$, then $\omega_{1t}(\underset{\sim}{s}) = {}^1/4$, so that by 5.8, $P_t(\underset{\sim}{s}) \cdot \omega_{1t}(\underset{\sim}{s}) \leq (8\underline{\lambda})^{-1}$ almost surely. Also by 5.13, $P_t(\underset{\sim}{s})x_{1t}(\underset{\sim}{s}) \geqq 3(8\underline{\lambda})^{-1}$ almost surely. Hence, $M_{1,1}(\underset{\sim}{s}) = (1+r)\,M_{1,0}(\underset{\sim}{s}) - (^1/2)r - P_1(\underset{\sim}{s})x_1(\underset{\sim}{s}) + P_1(\underset{\sim}{s})\,\omega_{11}(\underset{\sim}{s}) \leqq (1+r)\,(^1/2 + (8r\underline{\lambda})^{-1}) - (^1/2)r - 3(8\underline{\lambda})^{-1} + (8\underline{\lambda})^{-1} \leqq {}^1/2 + (8r\underline{\lambda})^{-1} - (8\underline{\lambda})^{-1}$ almost surely. Continuing by induction on t, one obtains $M_{1,40}(\underset{\sim}{s}) \leqq {}^1/2 + (8r\underline{\lambda})^{-1} - 40(8\underline{\lambda})^{-1} < {}^1/2 - (8r\underline{\lambda})^{-1}$ almost surely for $\underset{\sim}{s} \in \Sigma_0'$. The second inequality here follows from the fact that $r < {}^1/20$. Since inequality 5.9 has been contradicted, there exists no equilibrium if $0.1 \geqq r \geq \underline{r}$.

The results of this section remain true if a small quantity of assets is included in the example. The proofs of 5.5 and 5.6 apply without change, so that consumers are not self-insuring if $r < 0.1$ and no equilibrium exists if $r > 0.1$. Finally, no equilibrium exists if r is close to 0.1. Asset prices are bounded above by $r^{-1} \sup_t \operatorname{ess\,sup} P_t A_t$, which is a multiple of $r^{-1} \underline{\lambda}^{-1}$. Hence, assets cannot provide a reserve against a sufficiently long run of bad luck.

6. A Credit Model

In this section, I show that the monetary model is equivalent to a model with credit. This implies that credit does not make consumers self-insuring.

I continue to consider a model with no assets. Recall that in a monetary equilibrium, $((\underset{\sim}{x}_i), \underset{\sim}{P}, (\underset{\sim}{M}_i))$, money balances evolve according to the equation

$$M_{it} = (1+r)M_{i,t-1} - r\tau_i + P_t \cdot (\omega_{it} - x_{it}),$$

where $P_t \cdot (\omega_{it} - x_{it})$ denotes the random variable $P_t(\underset{\sim}{s}) \cdot (\omega_{it}(\underset{\sim}{s}) - x_{it}(\underset{\sim}{s}))$. This equation may be rewritten as $(M_{it} - \tau_i) = (1+r)(M_{i,t-1} - \tau_i) + P_t \cdot (\omega_{it} - x_{it})$. Let $C_{it} = M_{it} - \tau_i$ and interpret this as consumer i's net credit balance at the end of period t. The above equation becomes

$$C_{it} = (1+r)\, C_{i,t-1} + P_t \cdot (\omega_{it} - x_{it}).$$

The constraint $M_{it} \geqq 0$ becomes

$$C_{it} \geqq -\tau_i.$$

τ_i may be interpreted as a legal upper limit on consumer i's debt.

This upper limit or debt may seem to be arbitrary and unnecessarily restrictive. I now replace it by the constraint

6.1) $\liminf_{t\to\infty} C_{it}(\underset{\sim}{s}) > -\infty$ almost surely.

This constraint simply says that consumers cannot engage in Ponzi schemes. It seems to be a minimal restriction on borrowers. I call the model with this constraint the unlimited credit model.

I do not know whether this model has an equilibrium. What is important is that the example of section 5 shows that even if such equilibria did exist, consumers would not be self-insuring.

Let $((\underset{\sim}{x}_i), \underset{\sim}{P}, (C_i))$ be an equilibrium for the unlimited credit model, where $C_{it}(\underset{\sim}{s})$ is the credit balance of consumer i at the end of period t. Let $(\underset{\sim}{\lambda}_1, \underset{\sim}{\lambda}_2)$ be the associated marginal utilities of credit. These may be defined as in Appendix I. They are essentially bounded and bounded away from zero. I claim that assertions 5.5, 5.6 and 5.7 all apply to this equilibrium.

Assume that $r > 0$ and observe that if $C_{it}(\underset{\sim}{s}) < -r^{-1} \sup_t \text{ess sup } P_t \cdot \omega_{it}$, then consumer i would never be able to pay even the interest on his debt, so that he would violate constraint 6.1. Hence,

6.2) $$C_{it}(\underset{\sim}{s}) \geq -r^{-1} \sup_t \text{ess sup } P_t \cdot \omega_{it} \text{ almost surely, for all } t.$$

By the definition of equilibrium, $P_t(\underset{\sim}{s})$ is essentially bounded. The $\omega_{it}(\underset{\sim}{s})$ are bounded in the example. Hence, the right hand side of 6.2 is finite. It follows that it is possible to define minimum credit balances $\underline{C}_{it}(\underset{\sim}{s})$ (see Appendix I). $\underline{C}_{it}(\underset{\sim}{s})$ is the smallest credit balance that consumer i can hold at the end of period t and be sure never to violate condition 6.4.

If $r = 0$, let $\underline{C}_{it}(\underset{\sim}{s}) = -\infty$. In any case, no equilibrium exists if $r = 0$, for inequality 5.2 becomes an equality and implies that $\lim_{t\to\infty} E\,\lambda_{it} = \infty$.

In section 5, replace M_{it} and $\underline{M}_{it}$ by C_{it} and $\underline{C}_{it}$, respectively. Also, replace the equation $M_{1t}(\underset{\sim}{s}) + M_{2t}(\underset{\sim}{s}) = 1$ by the equation $C_{1t}(\underset{\sim}{s}) + C_{2t}(\underset{\sim}{s}) = 0$. Finally, replace the restriction $M_{it}(\underset{\sim}{s}) \geqq 0$ by restriction 6.2. Then, all the results and arguments of section 5 apply with certain obvious changes. For instance, inequality 5.3 becomes $\underline{C}_{it}(\underset{\sim}{s}) \leqq -\frac{1}{4}\, r^{-1}\underline{P}$, and inequality 5.9 becomes $M_{it}(\underset{\sim}{s}) \geqq (8\, r\, \underline{\lambda})^{-1}$.

7. Money and Credit Do Not Provide Perfect Insurance

The problem brought out in the previous two sections is that in an infinite horizon model with no insurance markets, only infinite financial reserves can provide perfect insurance. This fact makes full self-insurance infeasible, even if one allows consumers to borrow.

It might seem that one could avoid the infinite reserve problem by using a model in which consumers have finite lifetimes. For instance, one might use a version of Samuelson's model with overlapping generations (1958). But if a consumer is mortal, he is sure not to be self-insuring, for near the end of his life his effective rate of time preference would exceed the interest rate.

Another objection to my argument might be that I do not allow credit institutions to take risks. In the models of the previous section, lenders take no risks at all, for I require that debtors never default. But it seems to me best to make a sharp distinction between credit and insurance for the sake of conceptual clarity. One can also make the following argument based on the problem of moral hazard. A debtor might promise to behave so as to default only rarely, but how would the creditor know when the rare events occurred? If he could know, then specific insurance contracts could be written.

My conclusion is that only complete insurance markets can provide perfect insurance. One implication of this assertion is that the martingale property of asset prices is wedded to the Arrow-Debreu model.

A series of papers have studied the extent to which money or credit make consumers self-insuring. These include Schechtman (1976), Yaari (1976), Bewley (1977) and Grossman, Levhari and Mirman (1979). These papers all studied the problem of one consumer. That is, they took a partial equilibrium point of view. The general conclusion is that consumers can insure themselves to some extent, but they cannot insure themselves perfectly, except asymptotically as the rate of interest approaches the rate of time preference from below.

I mention in passing that the example of section 5 is a counterexample to the proposition that rational expectations equilibria give the same result as do Arrow-Debreu equilibria. I have never seen this proposition stated formally, but there seems to be a tendency among many macro-economists to identify the two kinds of equilibria. For instance in a recent paper, Lucas says that "one may sometimes (though certainly not always) think of contingent-claim equilibrium as being determined via a sequence of 'spot' markets, in which current prices are set given certain expectations about future prices." (This quotation is from section 5 of Lucas (1980).) It is possible to be more precise about the relation between temporary and Arrow-Debreu equilibrium. If an economy has one consumer or many identical consumers, then indeed a temporary equilibrium with rational expectations will normally be optimal and so corresponds to an Arrow-Debreu equilibrium. However, if there are diverse consumers, then temporary equilibria may not be Pareto optimal, even if consumers have rational expectations and have access to perfect capital markets. As I have just shown, consumers may not be able to insure themselves perfectly. It is probably true that temporary equilibria in stochastic models are almost never Pareto optimal when consumers are diverse, although I have not proved this assertion. It may be true that temporary equilibria with rational expectations become Pareto optimal asymptotically as one lets the rate of time preference go to zero. One can define a form of equilibrium for an economy in which consumer's rates of time preference are zero. I called this equilibrium stationary equilibrium in another paper (1980b). A stationary equilibria is Pareto optimal. It is intermediate between temporary equilibrium and Arrow-Debreu equilibrium. A stationary equilibrium is temporary in that it involves no forward trading. A stationary equilibrium is Arrow-Debreu in that each consumer has one budget constraint for all time, instead of a different one in each period. The constraint is that long-run average expenditure per period not exceed long-run average income.

8. A Short-Run Martingale Property

It is possible to justify a short-run version of the martingale property, for money can provide nearly perfect insurance against every day short-lived fluctuations. I argued this point in a previous paper (1977). As I mentioned before, this paper used a partial equilibrium model of one consumer. In this model, money paid no interest, and prices, income and preferences fluctuated according to a stationary stochastic process. I showed that if the consumer's time horizon were sufficiently distant and if his pure rate of time preference were sufficiently small, then his optimal program would be such as to make him nearly perfectly self-insuring. These results of mine are generalizations and interpretations of results obtained by Schechtman (1976).

One could give a more convincing justification of the short-run martingale property by taking the following general equilibrium approach. One should prove that the monetary equilibrium of section 4 has a stationary equilibrium with interest rate equal to zero. A stationary equilibrium is one in which prices and all other variables fluctuate according to a stationary process. One should prove that as the rate of time preferences goes to zero, the marginal utilities of money become asymptotically constant. Letting the rate of time preference go to zero corresponds to speeding up the fluctuations in the economy so that they become short-run fluctuations.

In order to make this argument, one has to prove that monetary equilibria exist when assets are present. As I mentioned earlier, I know that monetary equilibria exist only when there are no assets. And in this case, I do not know whether equilibria exist which are stationary. It seems to me that one must deal with stationary equilibria if one is to prove that the marginal utilities of money become constant as the rate of time preference goes to zero.

APPENDIX I

The Marginal Utility of Money and Minimum Money Balances

I here prove that in a monetary equilibrum consumers have well-defined marginal utilities of money and minimum money balances. Since I use these objects only in section 5 and Appendix II, I make strong assumptions which are satisfied in these two places. I assume that there are no assets. Also, I make the following assumption which avoids measure theoretic technicalities.

A.I.1) s_t is a Markov process on a finite set S.

I am here using the notation of section 4. Recall that the utilities and initial endowments are determined by functions $u_i: R_+^\ell \times S \to R$ and $\omega_i: S \to R_+^L$. Since S is finite, ω_i is bounded and hence all feasible allocations are bounded. Finally, I assume that

A.I.2) for any $s \in S$ and for all i, $u_i(\cdot, s) = R_+^L \to R$ is continuously differentiable, increasing and concave.

Let $((\underset{\sim}{x}_i)_{i=1}^I, \underset{\sim}{P})$ be a monetary equilibrium with interest rate r.

I first define the marginal utilities of money. I proceed by defining T-period horizon marginal utilities of money, λ_{it}^T, and letting T go to infinity. The definition is by induction on T.

For $T = 0$, I let $\lambda_{it}^0(\underset{\sim}{s}) = \min \{a \mid \frac{\partial u_{it}(x_{it}(\underset{\sim}{s}), \underset{\sim}{s})}{\partial x_k} \leq aP_{tk}(\underset{\sim}{s}),$ for $k = 1, \ldots, L\}$, where $t = 0, 1, \ldots$. By assumption A.I.2, $\lambda_{it}^0(\underset{\sim}{s})$ is well-defined. Given λ_{it}^{T}, let $\lambda_{it}^{T+1}(\underset{\sim}{s}) = \max\{\lambda_{it}^{0}(\underset{\sim}{s}), (1+r)(1+\rho)^{-1}E[\lambda_{it}^{T}|\mathscr{I}_t](\underset{\sim}{s})\}$. It should be clear that $\lambda_{it}^{T+1}(\underset{\sim}{s}) \geqq \lambda_{it}^{T}(\underset{\sim}{s})$ almost surely, for all i, t and T.

I now show that the $\lambda_{it}^{T}(\underset{\sim}{s})$ are essentially bounded, uniformly in t and T. That is, there exists $b > 0$ such that $\lambda_{it}^{T}(\underset{\sim}{s}) \leqq b$ almost surely, for all t and T. By the definition of λ_{it}^{T}, it is sufficient to show that the λ_{it}^{0} are essentially bounded. To see that this is so, observe that since the u_i are continuously differentiable and the feasible allocation $(\underset{\sim}{x}_i)$ is essentially bounded, it follows that the $\frac{\partial u_{it}(x_{it}(\underset{\sim}{s}),\underset{\sim}{s})}{\partial x_k}$ are essentially bounded. Also, by the definition of a monetary equilibrium, the prices $P_{tk}(\underset{\sim}{s})$ are essentially bounded away from zero. These facts together imply that the λ_{it}^{0} are essentially bounded.

It now follows that the limits $\lambda_{it}(\underset{\sim}{s}) = \lim_{T\to\infty} \lambda_{it}^{T}(\underset{\sim}{s})$ are well-defined and finite almost surely. The λ_{it} are the desired marginal utilities of money.

I now define the minimum money balances. I define T-period horizon minimum balances and let T go to infinity. Again, the definition is by induction on T. Let $\underline{M}_{it}^{0}(\underset{\sim}{s}) = 0$. Suppose that $\underline{M}_{it}^{T}(\underset{\sim}{s})$ has been defined. Let

A.I.3) $\underline{M}_{it}^{T+1}(\underset{\sim}{s}) = (1+r)^{-1} \max\{0, \tau_i r - \underset{\mathscr{I}_t}{\text{ess inf}}\,[P_{t+1}\cdot\omega_i - \underline{M}_{i,t+1}^{T}](\underset{\sim}{s})\}$.

I must now define the symbol "ess inf". The random variable following $\underset{\mathscr{I}_t}{\text{ess inf}}$ in A.I.3 is measurable with respect to $\mathscr{I}_{t+1}$. Let $f: \Sigma \to R$ be any random variable measurable with respect to $\mathscr{I}_{t+1}$. For each $a \in S$, let $f_a(\underset{\sim}{s}) = E[f\,|\,\mathscr{I}_t$ and $s_{t+1} = a](\underset{\sim}{s})$. Let

$\operatorname{ess\,inf}_{\mathscr{J}_t} f(\bar{\underset{\sim}{s}}) = \min \{ f_a(\underset{\sim}{s}) \mid a \in S \text{ and Prob } [s_{t+1} = a \mid s_t = \bar{s}_t] > 0 \}$. Assumption A.I.1 implies that $\operatorname{ess\,inf}_{\mathscr{J}_t} f$ is well-defined. Clearly, $f(\underset{\sim}{s}) \geq (\operatorname{ess\,inf}_{\mathscr{J}_t} f)(\underset{\sim}{s})$ almost surely. Assumption A.I.1 implies that if $g: \Sigma \to R$ is $\mathscr{J}_t$- measurable and such that $f(\underset{\sim}{s}) \geq g(\underset{\sim}{s})$ almost surely, then $g(\underset{\sim}{s}) \leq \operatorname{ess\,inf}_{\mathscr{J}_t} f(\underset{\sim}{s})$ almost surely.

It should be clear that $\underline{M}^T_{it}(\underset{\sim}{s}) \leq \tau_i$ almost surely and that $\underline{M}^{T+1}_{it}(\underset{\sim}{s}) \geq \underline{M}^T_{it}(\underset{\sim}{s})$ almost surely, for all T. Hence, the limits $\underline{M}_{it}(\underset{\sim}{s}) = \lim_{T\to\infty} \underline{M}^T_{it}(\underset{\sim}{s})$ are well-defined. The $\underline{M}_{it}$ are the desired minimum balances.

APPENDIX II

A Correction of a Previous Paper

In a paper called "The Optimum Quantity of Money" (1980a), I claimed incorrectly that monetary equilibria exist for all interest rates less than the smallest pure rate of time preference of any consumer. The example of section 5 above is a counterexample to this assertion. It is true, however, that equilibria exist if the interest rate is sufficiently close to zero. The origin of my error was that I did not realize that consumers can hold no less than the minimum money balances which I define in Appendix I of this paper. The actual error is the "hence" in the fifth line of the second full paragraph of page 194 of my previous paper (1980a). I discovered this error myself, but it was also pointed out to me by Martin Hellwig.

By correcting the error in my previous paper, I strengthen its main conclusion and contradict a suggestion made there. That paper analyzes Friedman's idea of the optimum quantity of money (Friedman, [1969]). Friedman had suggested that if all consumers had the same pure rate of time preference, then money should earn interest at that rate. He called the quantity of money that would be held in that situation the optimum quantity of money. I constructed a rigorous version of Friedman's model and showed that generically the optimum quantity of money would be infinite and so could not be realized. I also went on to suggest that if the interest rate were set sufficiently close to the rate of time preference, then the allocation of the monetary equilibrium would be arbitrarily close to being Pareto optimal. In fact, I made this suggestion the thesis of the paper, expressed in the first paragraph.

The example of section 5 strengthens the conclusion that the optimum quantity of money cannot be realized, but the example contradicts the suggestion that Pareto optimality can be approached asymptotically. Recall that in the example monetary equilibria do not exist if the interest rate is too close to the rate of time preference.

I now give a correct statement of the theorem on the existence of monetary equilibria (theorem 1 in my previous paper). I must make one slight change in the model of the previous paper. In that paper, the tax paid by consumer i in each period was τ_i, where $\tau_i > 0$, $\sum_{i=1}^{I} \tau_i = r$, and r was the interest rate. I now let the tax of consumer i be $\tau_i r$, where $\sum_{i=1}^{I} \tau_i = 1$. I use the notation of my previous paper. δ_i denotes consumer i's discount factor applied to future utility. $\delta_i^{-1} - 1$ is his pure rate of time preference.

<u>Theorem</u> Suppose that assumptions 1-9 of Bewley (1980a) apply. Suppose also that $\omega_i(a) \neq 0$, for all i and a. There exists $\bar{r} > 0$, depending on $\max_i \delta_i$, and there exists $\underline{\delta} < 1$ such that a monetary equilibrium exists whenever $0 \leq r < \bar{r}$ and $\underline{\delta} \leq \delta_i < 1$, for all i.

In proving the existence theorem in my previous paper, I proceeded as follows. I truncated the economy, eliminating all periods after period N. I further changed the model by giving each consumer one unit of utility for each unit of money held at the end of period N. I then proved that the resulting finite horizon economy had a monetary equilibrium. The key steps were to prove that prices in these finite horizon equilibria were bounded and bounded away from zero, uniformly in N. I then allowed N to go to infinity and applied a Cantor diagonal argument to obtain a monetary equilibrium in the limit.

The error is in the proof that prices are bounded away from zero. This assertion is lemma 8 of the previous paper. Before stating this lemma correctly, I recall some notation from the previous paper.

The exogenous process $\{s_n\}$ is a Markov process which takes values in a finite set A. The initial money stock of each consumer is fixed at the end of period zero and is independent of the history of $\{s_n\}$ up to time zero. Therefore, all allocations, prices and so on can be written as functions of the history of the process from period 1. A typical history is written as $a_1, a_2, \ldots, a_n$.

The correct statement of lemma 8 (of the previous paper) is as follows.

A.II.1) Lemma Let $\delta_1, \ldots, \delta_I$ be fixed and such that $0 < \delta_i < 1$, for all i. There exist $\overline{r} > 0$, $\underline{p} \in R_+^L$ and $\overline{\lambda} > 0$ such that $\underline{p} \gg 0$ and the following are true. Let $(p, (x_i))$ be an N-period monetary equilibrium with interest rate r, where $0 \leqq r \leqq \overline{r}$. Let (λ_i) be the associated vector of marginal utilities of money. Then, $p_n(a_1, \ldots, a_n) \geqq \underline{p}$ and $\lambda_{in}(a_1, \ldots, a_n) \leqq \overline{\lambda}$, for all histories $a_1, \ldots, a_n$ and for all n.

Before proving this lemma, I recall some more notation and some results from the previous paper. $\alpha_{in}(a_1, \ldots, a_n)$ denotes the marginal utility of expenditure of consumer i in period n. It is defined to be the smallest number α such that $\frac{\partial u_i(x_{in}(a_1, \ldots, a_n), a_n)}{\partial x_k} \leqq \alpha\, p_{nk}(a_1, \ldots, a_n)$, for all k, where $(p, (x_i))$ is as in the lemma.

$\bar{\omega} \in R_+^L$ denotes a vector such that $\sum_{i=1}^{I} \omega_i(a) \ll \bar{\omega}$, for all $a \in A$. $\underline{q}$ are $\bar{q}$ are L-vectors such that $0 \ll \underline{q} \ll Du_i(x,a) \ll \bar{q}$, for all $a \in A$ and for all $x \in R_+^L$ such that $0 \leqq x \leqq \bar{\omega}$. The vectors $x_{in}(a_1,\ldots,a_n)$ satisfy these last inequalities, so that

A.II.2) $\lambda_{in}(a_1,\ldots,a_n)p_n(a_1,\ldots,a_n) \geqq Du_i(x_{in}(a_1,\ldots,a_n),a_n) \gg \underline{q}$, for all i, n and $a_1,\ldots,a_n$,

where (λ_i) is as in the lemma.

It is also true that

A.II.3) $p_n(a_1,\ldots,a_n) \leqq \max_i \alpha_{in}^{-1}(a_1,\ldots,a_n)\bar{q}$, for all $a_1,\ldots,a_n$ and for all n.

This statement follows from lemma 1 of the previous paper.

I use the following result, which follows from lemma 2 of the previous paper.

A.II.4) $\max_i \alpha_{in}(a_1,\ldots,a_n) < b \min_i \alpha_{in}(a_1,\ldots,a_n)$, for all n and $a_1,\ldots,a_n$,

where $b = \max_k \underline{q}_k^{-1}\bar{q}_k$

I also use the minimum money balances, defined in Appendix I of this paper. $\underline{M}_{in}(a_1,\ldots,a_n)$ denotes the minimum money balance of consumer i in the

N-period equilibrium $(p,(x_i))$. $\underline{M}_{in}(a_1,\ldots,a_n)$ is the minimum money balance at the end of period n when the history of the state of the exogenous stochastic process is $a_1,\ldots,a_n$. Observe that $\underline{M}_{in}(a_1,\ldots,a_n) = 0$, for all $a_1,\ldots,a_n$.

I use the following facts, which correspond to formulas 28-30 of the previous paper. I have simply corrected these formulas by taking account of the minimum money balances.

A.II.5) $\lambda_{iN}(a_1,\ldots,a_N) = \max\,(\alpha_{iN}(a_1,\ldots,a_N),1)$. If $n < N$, then

$$\lambda_{in}(a_1,\ldots,a_n) = \max\,\{\alpha_{in}(a_1,\ldots,a_n),$$
$$\delta_i(1+r)\,E\,[\lambda_{i,n+1}(a_1,\ldots,a_n,s_{n+1}) \mid s_n = a_n]\}\,.$$

A.II.6) $\lambda_{iN}(a_1,\ldots,a_N) > 1$ only if $M_{in}(p,x_i,a_1,\ldots,a_N) = 0$.

If $n < N$, then $\lambda_{in}(a_1,\ldots,a_n)$

$> \delta_i(1+r)\,E\,[\lambda_{i,n+1}(a_1,\ldots,a_n,s_{n+1}) \mid s_n = a_n]$

only if $M_{in}(p,x_i,a_1,\ldots,a_n) = \underline{M}_{in}(a_1,\ldots,a_n)$.

A.II.7) For all n, $\lambda_{in}(a_1,\ldots,a_n) > \alpha_{in}(a_1,\ldots,a_n)$ only if

$x_{in}(a_1,\ldots,a_n) = 0$.

<u>Proof of lemma A.II.1</u>. It is sufficient to find $\bar{\lambda}$ as in the lemma, for by A.II.2 I may let $\underline{p} = (\bar{\lambda})^{-1}\underline{q}$.

Let $\hat{r} > 0$ be such that $(1+\hat{r})^{-1} > \max_i \delta_i$. Let K be a positive integer such that

A.II.8) $\min_i (\delta_i(1+\hat{r}))^{-K} b^{-1} > 1$

Let

A.II.9) $\bar{\lambda} = b + b^2(\bar{q} \cdot \bar{\omega}) \sum_{k=1}^{K} (1+\hat{r})^{k-1} \max \tau_i^{-1}.$

By assumption 7, $\tau_i > 0$, for all i, so that $\bar{\lambda} < \infty$.

Let

A.II.10) $\epsilon = \min_{i,a} \underline{q} \cdot \omega_i(a).$

Since $\omega_i(a) \neq 0$, for all i and a and since $\underline{q} \gg 0$, it follows that $\epsilon > 0$.

Let $\bar{r}$ be such that $0 < \bar{r} \leq \hat{r}$ and so small that

A.II.11) $\bar{r} \sum_{k=1}^{K} (1+\bar{r})^{k-1} < \epsilon(b^2(\bar{q} \cdot \bar{\omega}))^{-1}$.

I claim that $\bar{r}$ and $\bar{\lambda}$ satisfy the conditions of the lemma. Let $(p,(x_i))$ and (λ_i) be as in the lemma. I must show that

A.II.12) $\lambda_{in}(a_1,\ldots,a_n) \leqq \bar{\lambda}$, for all i, for all histories $a_1,\ldots,a_n$ and for all n.

I prove A.II.12 by backwards induction on n. First of all, A.II.12 is true for $n = N$. In order to see that this is so, fix $a_1,\dots,a_N$ and let i be such that $M_{in}(p,x_i,a_1,\dots,a_N) > 0$. By A.II.5 and A.II.6, $\alpha_{iN}(a_1,\dots,a_N) \leqq \lambda_{iN}(a_1,\dots,a_N) = 1$. Hence, A.II.4 implies that $\alpha_{jN}(a_1,\dots,a_N) \leqq b$, for all j. But then by A.II.5, $\lambda_{jN}(a_1,\dots,a_N) \leqq \max(b,1) = b$, for all j. Finally, by A.II.9, $b \leqq \bar{\lambda}$. This proves A.II.12 for $n = N$.

Suppose by induction that A.II.12 is true for $n+1,\dots,N$. I now show that

A.II.13) $\underline{M}_{i,n+t}(p,x_i;a_1,\dots,a_{n+t}) \leqq \max(0, \tau_i - r^{-1}\,\varepsilon\,(\bar{\lambda})^{-1})$, for $t = 0,1,\dots,N-n$ and for all i and $a_1,\dots,a_{n+t}$.

I repeat the argument used to prove inequality 5.9 of the present paper. Since $\lambda_{i,n+t}(a_1,\dots,a_{n+t}) \leqq \bar{\lambda}$, for all i, it follows from A.II.2 that $p_{n+t}(a_1,\dots,a_{n+t}) \geqq (\bar{\lambda})^{-1}\underline{q}$. It follows that a lower bound on consumer i's income in any of the periods $n+1,\dots,N$ is $rM_i + \varepsilon(\bar{\lambda})^{-1}$, where M_i is his money balance at the end of the previous period and ε is as in A.II.10. His tax payments are $r\tau_i$. Clearly, if $rM_i + \varepsilon(\bar{\lambda})^{-1} \geqq r\tau_i$, then he can keep his money holdings positive indefinitely simply by never spending money on consumption. It follows that the smallest non-negative number M_i satisfying this inequality is an upper bound on $\underline{M}_{i,n+t}$. This number is the right hand side of inequality A.II.13.

I now prove that $\lambda_{in}(a_1,\dots,a_n) \leqq \bar{\lambda}$, for all i,n and $a_1,\dots,a_n$. Suppose that $\lambda_{in}(a_1,\dots,a_n) > \bar{\lambda}$, for some i. Without loss of generality, I may assume that $i = 1$, so that

A.II.14) $\lambda_{1n}(a_1,\ldots,a_n) > \bar{\lambda}$.

I prove that A.II.14 implies the following.

A.II.15) There exist i and a history $a_{n+1},\ldots,a_{n+T}$ following a_n such that $\lambda_{i,n+t}(a_1,\ldots,a_{n+t}) \geqq (\delta_i(1+r))^{-t}b^{-1}\bar{\lambda}$ and $M_{i,n+t}(p,x_i;a_1,\ldots,a_{n+t}) \geqq \tau_i - \bar{\lambda}^{-1}b^2(\bar{q}\cdot\bar{\omega})\ (\sum_{k=1}^{t}(1+r)^{k-1})$, for $t = 0,\ldots,T$, where $T = \min(K,N-n)$.

This statement leads to a contradiction. First of all, suppose that $T = N-n$. Then, A.II.9 and A.II.15 imply that $M_{iN}(p,x_i;\ a_1,\ldots,a_N) > 0$. But then by A.II.6, $\lambda_{iN}(a_1,\ldots,a_N) = 1$. However by A.II.9 and A.II.15, $\lambda_{iN}(a_1,\ldots,a_N) \geqq (\delta_i(1+r))^{n-N}b^{-1}\bar{\lambda} \geqq b^{-1}\bar{\lambda} > 1$, which is a contradiction.

Suppose that $T = K$. Then, A.II.8 and A.II.15 imply that $\lambda_{i,n+K}(a_1,\ldots,a_{n+K}) \geqq (\delta_i(1+r))^{-K}b^{-1}\bar{\lambda} > \bar{\lambda}$, which contradicts the induction hypothesis. This proves that A.II.15 leads to a contradiction and hence that A.II.14 is impossible. Hence, the induction step in the proof of A.II.12 will be completed once A.II.15 is proved.

I now prove A.II.15. Let i be such that $M_{i,n}(p,x_i;\ a_1,\ldots,a_n) \geqq \tau_i$, where $a_1,\ldots,a_n$ are as in A.II.14. Such an i exists by the assumption that $\sum_{i=1}^{I} M_{i0} = \sum_{i=1}^{I}\tau_i = 1$ (assumption 7).

I first show that $\lambda_{in}(a_1,\ldots,a_n) \geqq b^{-1}\bar{\lambda}$. Observe that $\bar{\lambda} < \lambda_{1n}(a_1,\ldots,a_n)$ $= \max(\alpha_{1n}(a_1,\ldots,a_n),\delta_1(1+r)E[\lambda_{1,n+1}(a_1,\ldots,a_n,s_{n+1}) \mid s_n = a_n])$ $\leqq \max(\alpha_{1n}(a_1,\ldots,a_n),\delta_1(1+r)\bar{\lambda}) = \alpha_{1n}(a_1,\ldots,a_n)$. The second inequality follows from the induction hypothesis on n (regarding A.II.12). Hence by A.II.4 and A.II.5, $\lambda_{in}(a_1,\ldots,a_n) \geq \alpha_{in}(a_1,\ldots,a_n) \geq b^{-1}\alpha_{1n}(a_1,\ldots,a_n) > b^{-1}\bar{\lambda}$. I have now proved that i exists such that the inequalities of A.II.15 are satisfied for $t = 0$.

I now prove by induction on t that $a_{n+1},\ldots,a_{n+T}$ exist as in A.II.15. Suppose that the conditions of A.II.15 are satisfied for t no larger than some non-negative integer, call it t again. I may suppose that $t < T$. Then, $M_{i,n+t}(p,x_i;a_1,\ldots,a_{n+t}) \geq \tau_i - \bar{\lambda}^{-1}b^2(\bar{q}\cdot\bar{\omega})(\sum_{k=1}^{t}(1+r)^{k-1}) > \max(0,\tau_i - r^{-1}\varepsilon(\bar{\lambda})^{-1})$. The last inequality follows from A.II.9 and A.II.11. Hence by A.II.13 and A.II.6, $\lambda_{i,n+t}(a_1,\ldots,a_{n+t}) = \delta_i(1+r)E[\lambda_{i,n+t+1}(a_1,\ldots,a_{n+t},s_{n+t+1}) \mid s_{n+t} = a_{n+t}]$, so that for some a_{n+t+1}, $\lambda_{i,n+t+1}(a_1,\ldots,a_{n+t+1}) \geq (\delta_i(1+r))^{-1}\lambda_{i,n+t}(a_1,\ldots,a_{n+t})$ $\geq (\delta_i(1+r))^{-(t+1)}b^{-1}\bar{\lambda}$. The last inequality follows from the induction hypothesis on t.

I now show that $M_{i,n+t+1}(p,x_i;a_1,\ldots,a_{n+t+1}) \geq \tau_i - \bar{\lambda}^{-1}b^2(q\cdot\bar{\omega})$ $(\sum_{k=1}^{t+1}(1+r)^{k-1})$. If $\alpha_{i,n+t+1}(a_1,\ldots,a_{n+t+1}) < \lambda_{i,n+t+1}(a_1,\ldots,a_{n+t+1})$, then by A.II.7, $x_{i,n+t+1}(a_1,\ldots,a_{n+t+1}) = 0$, so that

$$M_{i,n+t+1}(p,x_i;a_1,\ldots,a_{n+t+1}) \geq (1+r)M_{i,n+t}(p,x_i;a_1,\ldots,a_{n+t}) - r\tau_i$$

$$\geq \quad (1+r)[\tau_i - \bar{\lambda}^{-1}b^2(q\cdot\bar{\omega})(\sum_{k=1}^{t}(1+r)^{k-1})] - r\tau_i \geq \tau_i - \bar{\lambda}^{-1}b^2(q\cdot\bar{\omega})(\sum_{k=1}^{t+1}(1+r)^{k-1}).$$

The third inequality follows from the induction hypothesis on t.

Suppose now that $\alpha_{i,n+t+1}(a_1,\ldots,a_{n+t+1}) = \lambda_{i,n+t+1}(a_1,\ldots,a_{n+t+1})$. Then, by the choice of a_{n+t+1}, $\alpha_{i,n+t+1}(a_1,\ldots,a_{n+t+1}) > b^{-1}\bar{\lambda}$. It follows from A.II.4 that $\min_j \alpha_{j,n+t+1}(a_1,\ldots,a_{n+t+1}) > b^{-2}\bar{\lambda}$, so that by A.II.3, $p_{n+t+1}(a_1,\ldots,a_{n+t+1}) \leq b^2\bar{\lambda}^{-1}\bar{q}$. Hence, $p_{n+t+1}(a_1,\ldots,a_{n+t})\cdot$

$x_{i,n+t+1}(a_1,\ldots,a_{n+t+1}) \leq b^2\bar{\lambda}^{-1}(\bar{q}\cdot\bar{\omega})$. It follows that

$$M_{i,n+t+2}(p,x_i;a_1,\ldots,a_{n+t+1}) \geqq (1+r)M_{i,n+t+1}(p,x_i;a_1,\ldots,a_{n+t+1}) - r\tau_i - b^2\bar{\lambda}^{-1}(\bar{q}\cdot\bar{\omega}) \geqq (1+r)[\tau_i - \bar{\lambda}^{-1}b^2(\bar{q}\cdot\bar{\omega})\sum_{k=1}^{t}(1+r)^{k-1}] - r\tau_i - b^2\bar{\lambda}^{-1}(\bar{q}\cdot\bar{\omega}) = \tau_i - \bar{\lambda}^{-1}b^2(\bar{q}\cdot\bar{\omega})\sum_{k=1}^{t+1}(1+r)^{k-1}.$$

This completes the proof that the two inequalities of A.II.15 are satisfied for $t+1$, and so completes the induction step in the proof of A.II.15.

Q.E.D.

This completes the correction of the proof of the existence theorem in my previous paper. The other arguments and results in the paper are true, provided that certain easy adjustments are made in order to include minimum money balances. For instance, inequality 9 of that paper should read

$$\lambda_{in}(a_1,\ldots,a_n) > \delta_i(1+r)\,E[\lambda_{i,n+1}(a_1,\ldots,a_n,s_{n+1} \mid s_n = a_n] \text{ only if}$$
$$M_{in}(p,x_i;a_1,\ldots,a_n) = \underline{M}_{in}(p;a_1,\ldots,a_n),$$

where $\underline{M}_{in}(p;a_1,\ldots,a_n)$ is consumer i's minimum money balance when the price system is p.

APPENDIX III

Two More Models of Money and Credit

The monetary and credit models of sections 4 and 6 are somewhat mysterious. For one thing, the interest rate is not market determined. In order to give more insight into the models, I describe two other models, one equivalent to the monetary model and one equivalent to the model with unlimited credit. I exclude assets from the models.

Deflationary Model: Friedman (1969) suggested that the government could arrange for money to bear interest simply by steadily contracting the money supply and so causing a steady deflation. The model of section 4 is equivalent to such a deflationary model. In the new model, money bears no interest and the tax of consumer i in period t is $r\tau_i(1+r)^{-t}$, where τ_i is as in section 4. Let m_{it} be the money holdings of consumer i at the end of period t. Then, m_{it} evolves according to the equation

$$m_{it} = m_{i,t-1} - r\tau_i(1+r)^{-t} + p_t \cdot (\omega_{it} - x_{it}),$$

where p_t is the price vector at time t.

Let $((\underset{\sim}{x}_i), \underset{\sim}{P}, (\underset{\sim}{M}_i))$ be a monetary equilibrium as in section 4. Then, $((\underset{\sim}{x}_i), \underset{\sim}{p}, (\underset{\sim}{m}_i))$ is a monetary equilibrium for a deflationary model, where $p_t = (1+r)^{-t} P_t$ and $m_{it} = (1+r)^{-t} M_{it}$.

A Model with Market Determined Interest Rates: I now show how to introduce market determined interest rates. Consider the model with unlimited credit, which was introduced in section 6. Change this model by normalizing the price of the first consumption good so that it is always one. The credit balances of a consumer may now be thought of as "real" balances. Let W_{it} be the real balances of consumer i at time t. W_{it} evolves according to the equation

A.III.1) $W_{it} = (1+r_t)\, W_{i,t-1} + p_t \cdot (\omega_{it} - x_{it}),$

where p_t is the price vector at time t and r_t is the interest rate at time t. The constraint is

A.III.2) $\liminf_{t \to \infty} W_{it}(s) > -\infty$, almost surely.

I show how to pass from an equilibrium for the credit model, $((\underset{\sim}{x}_i), \underset{\sim}{P}, (\underset{\sim}{C}_i))$, to an equilibrium with market determined interest rates $((\underset{\sim}{x}_i), (\underset{\sim}{p},\underset{\sim}{r}), (\underset{\sim}{W}_i))$, where $\underset{\sim}{r} = (r_0, r_1, \ldots)$. By definition, the prices $P_{tk}(\underset{\sim}{s})$ are essentially bounded away from zero. Let $p_t(\underset{\sim}{s}) = (P_{t1}(\underset{\sim}{s}))^{-1}\, P_t(\underset{\sim}{s})$ and $W_{it}(\underset{\sim}{s}) = (P_{t1}(\underset{\sim}{s}))^{-1}\, C_{it}(\underset{\sim}{s})$, for $t \geqq 0$. Let $W_{i,-1} = C_{i,-1}$. Multiplying the equation $C_{it}(\underset{\sim}{s}) = (1+r)\, C_{i,t-1}(\underset{\sim}{s}) + P_t(\underset{\sim}{s}) \cdot (\omega_{it}(\underset{\sim}{s}) - x_{it}(\underset{\sim}{s}))$ by $(P_{t1}(\underset{\sim}{s}))^{-1}$, I obtain

A.III.3) $W_{it}(\underset{\sim}{s}) = (1+r_t(\underset{\sim}{s}))\, W_{i,t-1}(\underset{\sim}{s}) + p_t(\underset{\sim}{s}) \cdot (\omega_{it}(\underset{\sim}{s})$

$- x_{it}(\underset{\sim}{s}))$, where

$$r_t(\underset{\sim}{s}) = (P_{t1}(\underset{\sim}{s}))^{-1} P_{t-1,1}(\underset{\sim}{s}) (1+r) - 1.$$

(In order to interpret this equation when $t = 0$, let $P_{-1,1}(\underset{\sim}{s}) = 1$.) Equation A.III.3 is simply equation A.III.1. Since the prices $P_{t1}(\underset{\sim}{s})$ are bounded and the $C_{it}(\underset{\sim}{s})$ are bounded away from minus infinity, it follows that the constraint A.III.2 is satisfied.

This new concept of equilibrium is still somewhat mysterious, for one such equilibrium corresponds to each level of r in the credit model with no debt limits. Hence, the interest rates $r_t(s)$ are only partly market determined. In order to see that this is so, suppose that there are many consumers and that the random disturbances experienced by each consumer are mutually independent. Then, P_t would probably remain nearly constant, so that r_t would nearly equal r.

r is perhaps best thought of as an asymptotic real interest rate which is determined by custom and perpetuated by expectations.

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