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GENERALIZED NETWORK PROBLEMS YIELDING TOTALLY BALANCED GAMES *

by

Ehud Kalai and Eitan Zemel

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Northwestern University

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ABSTRACT

A class of multi-person mathematical optimization problems is considered and is shown to generate cooperative games with nonempty cores. The class includes, but is not restricted to, numerous versions of Network flow problems. It is shown that for games generated by Linear Programming optimization problems, optimal dual solutions correspond to points in the core. Also a special class of network flow problems for which every point in the core corresponds to an optimal dual solution is exhibited.

INTRODUCTION

Network flow models have been applied extensively for analyzing systems in the areas of communication, transportation, distribution, integrated production and the like. In systems of this type one can usually identify certain objects, not necessarily physical, which could be described as "flowing" through the system. For instance, one may consider a fluid flowing in a pipeline network, traffic moving or goods being transformed through a transportation network, currencies being exchanged through a network of exchange dealers, telephone calls transmitted through a telephone network, physical "inputs" being transformed through a production system into "finished goods" etc. Typically, the nodes of the network correspond to the different possible "states" in which the "flow" can be found while the arcs correspond to the active elements in the system which can transform the "flow" from "state" to "state".

Given a system of this type, one is naturally interested in employing it in the most efficient way possible. For instance, given a transportation system, one likes to find a transportation pattern which maximizes the flow of traffic between two specified terminals or, if costs are present, to find a transportation pattern which supports a given intensity of flow at the lowest possible cost. Similarly, in production problems with many alternative paths of production, one would like to find a production schedule which maximizes total output (or profits) without exceeding production capacities and without disturbing the order of opertions in which production should take place.

This sentiment is amply reflected in the O.R. literature concerning network flow problems. Originating with the Pioneering work of Ford and Fulkerson, [8], a vast number of algorithms for finding optiomal flows have been proposed in the last 25 years. To date, we have available a wide choice of extremely fast algorithms for two important versions of such models namely the maximal flow problem and the mimimum cost flow problem, [1,5,7,8,10,12,15,21]. Practical applications of these models abound and problems involving many thousands of variables and constraints are being solved routinely.

A closer examination of the above mentioned literature reveals, however, one common underlying feature. Almost without exception, these models are predicated on the assumption that the network is fully controlled by one individual or by a group of individuals with identical interests. When we analyse networks in which various components are controlled by different individuals, with different, and possibly conflicting, objectives, we soon realize that the optimization problem becomes a multiperson optimization problem and game theoretic considerations arise.

Consider, for example, a maximum flow problem in a network in which arcs are owned by different individuals. We can easily find the optimal flow in this network by the classical methods. However, to sustain this optimal flow we must secure the cooperation of some critical arc owners. Their cooperation can be secured provided that they get paid "enough". However the concept of "enough" requires carefull consideration. A natural critirion suggested by the game thoery literature is that the payoff will be such that no group of owners can generate a higher payoff for themselves when acting on their own by using only their portion of the network. In game theoretic language we require the payoff distribution to be in the core of the resulting cooperative game. This seems like a natural necessary condition on the payoff distribution because a coalition of arc owners that can generate more profits for themselves than was allocated to them by the grand coalition will tend to break cooperation and act on their own. However, the existence of such a payoff distribution is not an obvious fact. One can easily generate examples of games for which the requirements of the various coalitions are inconsistent resulting in an empty core.

The most elementry problem to analyse is that of a maximum flow problem of one commodity through a network possesing a single source and a single sink. Every arc has a flow capacity constraint and is owned by some player. A unit flow from source to sink yields a unit profit. It was shown in Kalai-Zemel [14] that for such problems the resulting core is not empty (for example payoff distributions that correspond to minimum cuts in the network are always core allocations).

In this paper we study broad generalizations of the maximal flow problem which posses this property. The resulting class of problems contains various network flow problems involving, for instance, costs (profits) on the individual arcs, multiple sources and sinks, multi-commodity flows, networks with losses and gains, networks with production nodes, etc. In addition, various types of other optimization problems which are not related to networks are also covered. A similar approach of this type was taken earlier in Owen [16], Ichiishi [13] and Billera [3]. However the family of optimization problems and the resulting games are different there. Also, simultaneous and independently of our work, Dubey and Shapley [7a] have developed a different set of sufficient conditions for optimization problems to guarantee that they generate games with non-empty cores. Many of the examples of games with this property can be shown to fit into both models (the Dubey-Shapley model as well as ours). Thus it seems that both models are very general.

The organization of the paper is as follows. In section II we introduce some concepts from Game Theory, and examine the notion of totally balanced games. In section III we discuss a class of optimization problems which yield games having this property. In section IV we discuss an economic example - The Shapley - Shubik Market Games. In section V we discuss games arising from certain types of linear programming problems, which includes a large variety of network flow problems as well as other types of problems. In section VI we discuss a certain class of network flow problems for which the core of the resulting games can be fully described in terms of optimal dual prices. We conclude the paper with a few examples.

II. Concepts from Game Theory

Let $N = \{1,...n\}$ $(n > 0)$ be a set of players. A coalition consists of any non empty subset of N. We denote the set of all coalitions of N by C. An n - person cooperative game with sidepayments (a game for short) is a function V from the set of coalitions to the set of real numbers. For a coalition $S \in C$, $V(S)$ may be thought of as the value (monetary or otherwise) that the coalition S can generate for its members if it operates on its own. For example, in the maximal flow problem described earlier, $V(S)$ may stand for the maximum flow that S can sustain using only its portion of the network.

For a game V, the core of V is defined by

$$\text{CORE }(V) = \left\{ x \in R^n : \sum_{i \in N} x_i = V(N), \sum_{i \in S} x_i \geq V(S) \text{ for every } S \in C \right\}$$

A point in the core corresponds to a distribution of the total profits (or costs) to the different players of the game. The constraints imposed on CORE (V) ensure that no coalition would have an incentive to split from the grand coalition, N, and do better on its own. Thus, for an allocation of V(N) to belong to CORE (V) is a necessary condition for long term coopertion between the players. Attempting to implement profit distributions which violates some of the core constraints is likely, in the long run, to result in some coalition breaking cooperation.

In general, a game V may or may not have a non empty core. In light of the discussion of the proceding paragraph, it is apparent that establishing whether or not CORE (V) = $\emptyset$ is of considerable practical relevance. Moreover, given that CORE (V) $\neq \emptyset$ we would obviously like to find one or several points which belong to this set. Both these issues can be settled by solving the following, typically huge, linear program:

(1) $\min \sum_{j \in N} x_j$

(2) s. t. $\sum_{j \in S} x_j \geq V(S)$ for every $S \in C$

It is quite obvious that CORE (V) $\neq \emptyset$ if and only if the optimal solution of (1) and (2) is equal to V(N), in which case any optimal solution to this program lies in CORE (V). Alternatively, taking the linear programming dual of (1) - (2) we get an equivalent necessary and sufficent coalition for CORE (V)$\neq \emptyset$ which is based on the concept of balanced sets. If $B = (S_1 \ldots S_k)$ is a collection of coalitions and $\lambda = (\lambda_1 \ldots \lambda_k)$ is a set of non negative real numbers such that for every $i \in N$ $\sum_{j: i \in S_j} \lambda_j = 1$, then we call such a B a <u>balanced collection</u>. The set of coeffients λ is refered to in this case as <u>balancing weights</u>. We call a game V <u>balanced</u> if, for every balanced collection B and every corresponding set of balancing weights λ we have

$$\sum_{S_i \in B} \lambda_i V(S_i) \leq V(N).$$

Bondareva, [4], using the duality theory of linear programming, and Shapley, [18], proved the following:

<u>Theorem 1</u> A game has a non empty core if and only if it is balanced.

Let V be a game with n players, and $S \in C$ a coalition. We denote by V^S the $|S|$ players game obtained by restricting V to coalitions T S. (V^S is the restriction of the function V to 2^S). We will be concerned in this paper with the family of highly stable games for which

$$\text{CORE } (V^S) \neq \emptyset \text{ for every } S \in C.$$

Applying Theorem 1 we get that a game belong to this family, iff each of its subgames, V^S, $S \in C$ is balanced. Games with this property are called <u>totally balanced games</u>.

III. <u>Optimization Problems Resulting in Totally Balanced Games</u>

In this section we describe a general class of mathematical optimization problems with the property that the cooperative games induced by them are totally balanced.

Verbally, the games may be described as follows. Every player controls a set of variables. Every coalition has a feasible set constraining the variables of its members. Denote this feasible set by Y^S. There is a common objective function which is used by all coalitions. The value of a coalition S, V(S), will be the maximum of the objective function over the feasible set of S, Y^S.

Formally, for every player $i \in N$ we let d^i be a positive integer denoting the number of variables under i^{th} control. Let $d = \sum_{i=1}^{n} d^i$. Also, Let $D_i = \{j | \sum_{\ell=1}^{i-1} d_\ell < j \leq \sum_{\ell=1}^{i} d_\ell\}$ be the set of variables under i's control. For every coalition S let $R^S = \{y \in R^d : y^j = 0, \text{ for } j \in \bigcup_{i \in S} D_i\}$. In other words R^S is the subspace that assignes value zero to variables which are not in S's control. For a point $y \in R^d$ let y^S be its orthogonal projection on R^S. Let $Y = \{Y^S\}_{S \in C}$ be a collections of sets in R^d with the property that $Y^S \quad R^S$. We call the collection Y, <u>balanced</u> (see Billera and Bixby [2] for difinitions and references) if for every balanced collection of coalitions B = $(S_1 \ldots S_k)$ with balancing set of weights $\lambda = (\lambda_1 \ldots \lambda_k)$ and for every collection of feasible poin s $(y_1, \ldots y_k)$ with $y_i \in y^{Si}$, one has $\sum_{i=1}^{k} \lambda_i y_i \in Y^N$. The collection Y is called <u>totally balanced</u> if the same property holds for the restriction of Y to the subsets of every coalition $S \in C$.

Having defined the feasible set for an optimization problem we now define the objective function. Let f be a real valued function on R^d. f is called <u>super balanced</u> if for every balanced collection $B = (S_1 \ldots S_k)$ with balancing set of weights $\lambda = (\lambda_1 \ldots \lambda_k)$ and for every collection of feasible points $(y_1 \ldots y_k)$ with $y_i \in R^{Si}$ one has $f(\sum_{i=1}^{k} \lambda_i y_i) \geq \sum_{i=1}^{k} \lambda_i f(y_i)$. f is called <u>totally super balanced</u> if the same property holds for every restriction of the set of players. Finally, f is called <u>(totally) bounded</u> on Y if $\sup_{y \in Y^N} f(y)$ ($\sup_{y \in Y^T} f(y)$ for every $T \in C$) is finite.

Define the game asociated with pair (Y,f) by

$$V(S) = \sup_{y \in Y^S} f(y)$$

Lemma 1 If Y is totally balanced and f is totally bounded and totally super blanced on Y then the game induced by (Y,f) is totally balanced.

Proof We show that the induced game, V, is balanced. The totally balanced part is proved in the same way. Let $\mathcal{B} = (S_1, S_2, \ldots, S_k)$ be a balanced collection of coalitions with balancing weights $(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Let $\varepsilon > 0$ be arbitrary. We will show that $\sum_{i=1}^{n} \lambda_i V(S_i) - k\,\varepsilon \leq V(N)$. For every i, $1 \leq i \leq k$, there is a $y_i \in Y^{S_i}$ with $f(y_i) > V(S_i) - \varepsilon$. Consider $y = \sum_{i=1}^{k} \lambda_i y_i \in Y^n$. $f(y) \geq \sum_{i=1}^{k} \lambda_i f(y_i)$. Therefore $V(N) \geq f(y) \geq \sum_{i=1}^{k} \lambda_i [V(S_i) - \varepsilon] \geq \sum_{i=1}^{k} \lambda_i V(S_i) - k\,\varepsilon$.

Remark 1. The reader who is familiar with the theory of cooperative games without side payments will notice that this lemma can be easily formulated for such games. This can be done by considering n objective functions $u_1, u_2, \ldots u_n$ representing the utilities of the n players with properties similar to f above to induce totally balanced games without side payment (see Billera-Bixby [2]).

Remark 2. The converse of lemma 1 is also true. This was first established by Shapley-Shubik [19] (see section IV below). It also follows from the results of Kalai-Zemel [14].

IV. An Economic Example: Shapley - Shubik Market Games

The following games where discussed in shapley - Shubik [19]. Using the terminology of lemma 1, they can be described as follows.

We assume that all the d_i's of the players are identical and equal c. c may be thought of as the number of commodities that are traded in some market. With each player $i \in N$ we associate a pair (w_i, u_i). w_i is a point in the non-negative orthant of $R^{\{i\}}$ and represents the initial amounts that player i holds of the various comodities. $u_i: R^{\{i\}} \to R$ which is concave and continuous, may be thought of as the utility or the monetary gain that player i can get out of a given bundle. For every $y \in R^N$ we define $f(y) = \sum_{i=1}^{n} u_i(y^i)$. In other words the value of an allocation y is the sum of the values that the individual players have for it. For every coalition S we define Y^S by

$$Y^S = \{y=(y_1,y_2,\dots,y_n) \in R^S : \sum_{i \in S} y_i = \sum_{i \in S} w_i \text{ and } y_i \geq 0 \text{ for } i=1,2,\dots,n\}.$$

It is easy to check that (Y,f) satisfies the assumptions of lemma I and thus the induced market game is totally balanced and has non empty cores for all its subgames. Shapley and Shubik, [19], have established also the converse of this statement i.e. that every totally balanced game can be generated by a pair (Y,f) arising from a market game of this type.

IV. Linear Programming Games

We now turn our attention to a special class of optimization problems, which can be expressed as linear programs and which satisfy the conditions of lemma 1. This class includes, among other problems, all the generalizations of network flow problems reffered to in the introduction. Thus, it will be shown that the games associated with such optmization problens are totally balanced. Moreover, the linear nature of the representation of such games leads to a natural and computationally feasible procedure to identify some points which are known to be in the core.

We define the optimization problem as follows. For each player $i \in N$ let d_i denote the dimension of the space of variables controlled by this player. Let $d = \sum_{i=1}^{n} d_i$. Let also $c^i \in R^{d_i}$, be an arbitrary profit (or cost if the entries are negative) vector for player i. Let A_i and B_i be two arbitrary matrices associated with this player of dimensions $m_i \times d_i$, and $m \times d_i$ respectively, with m_i, $i \in N$, and m arbitrary nonnegative integers. Finally, for each $i \in N$, let $b_i \in R^{m_i}$ be an arbitrary vector of right hand sides. For every coalition $S \in \mathbf{C}$ we let

$$(3)\quad V(S) = \max \sum_{i \in S} c^i x^i$$

s.t

$$(4)\qquad A^i x^i \overset{\leq}{\underset{=}{\text{or}}} b^i \text{ for every } i \in S,$$

$$(5)\qquad \sum_{i \in S} B^i x^i \overset{\leq}{\underset{=}{\text{or}}} 0$$

For S = N one can think of problem (3) - (5) as a problem of decentralization with the different players serving as subsidiaries or subcontractors. Such problems are amenable to solution by various decomposition techniques (e.g. Dantzig - Wolfe, [6]) which stress the flow of information and control between the main organization and the individual sub units, i.e. the players. We note that numerous versions of network flow problem conform to this format. In the simplest versions of such problems, the constaints (4) correspond to the individual arc capacity constraints while (5) represents the requirement that flow be conserved at each intermediate note. Obviousely, the formulation (3) - (5) allows for significant genralization of such constraints. In the rest of this section we restrict our attention to optimization problems which yield $V(S) < \infty$, for $S \varepsilon C$, and $V(N) > -\infty$. It is a simple matter to establish the following:

<u>Theorem 1</u> The game V derived from the optimization problem (3) - (5) is totally balanced.

<u>Proof</u> We show that the game satisfy the conditions of Lemma 1. The objective function is a sum of linear functions and therefore totally super balanced. The assumption on the boudedness of $V(S)$, $S \varepsilon C$ imply that f is totally bounded as well. To show that the feasible regions are totally balanced, we note that showing balanceness is enough, since the problem restricted to subcoalitions of any coalition $S \varepsilon C$ has the same structure as

the original problem.

Let $(S_1,\dots S_k)$ and $(\lambda_1 \dots \lambda_k)$ be a balanced set of coalitions and the corresponding weights. For $i = 1,\dots k$, consider the optimization problem defined with respect to S_i. Let x_i be a feasible solution to this problem. We assume that the x_i's are augmented by zeros so that they all belong to R^d. Consider the vector $x = \sum_{i=1}^{n} \lambda_i \; x_i$. We have to show that x satisfies (4) - (5) with repect to $S = N$. But this is manifest: each, x_i satisfies constraints (5) by our assumption, and therefore so does x which is a linear combination of the x_i. Also each x_i satisfies (4) for all $j \in S_i$. Let $x_i^j \in R_j^d$ be the component of x_i which corresponds to player j. By the assumption of balanceness, x^j is a convex combination of the set $\{x_i^j : j \in S_i\}$ thus, x^j satisfies (4).

The proof of Theorem 1 is non constructive in nature. It asserts that CORE (V) $\neq \emptyset$ but does not indicate how a point in this set can be found. Theorem 2 below adresses itself to this issue. It establishes a connection between some points in CORE (V) and the optimal dual solutions to problem (3) - (5) defined with respect to N.

<u>Theorem 2</u> Let (u,w) be an optimal dual solution to (3) - (5), defined w.r.t. N, with $u = (u^i : i \in N)$. Let $x = (x_i : i \in N)$ be given by $x_i = u^i \; b^i$. Then $x \in$ CORE (V).

<u>Proof</u> We first note that

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} u^i \; b^i = u \; b = V(N)$$

as it should. Consider a coalition $S \in C$. Let $u^S = (u^i : i \in S)$. It can be easily verified that (u^S, w) is a feasible dual solution for (3) - (5) defined w.r.t. S. Thus, by the weak duality theorem of linear programming

$$\sum_{i \in S} x_i = \sum_{i \in S} u^i b^i \geq V(S).$$

Theorem 2 enables us to compute points in the core of the game V without having to compute first the 2^n constants $V(S)$, $S \in C$. (The first example of such efficiency was demonstrated for assignment games by Shapley-Shubik [20].) In addition, the allocations suggested in Theorem 2 lend themselves to economic interpretation consistent with traditional L.p. interpretations of shadow prices.

Is the converse of theorem 2 true? The following counter-example settles this question in the negative. However, in section V, we identify a class of network games for which every core allocation corresponds to an optimal dual vector.

Example I Consider the network of Figure 1 where each arc is labled by its index, (a letter), and its capacity.

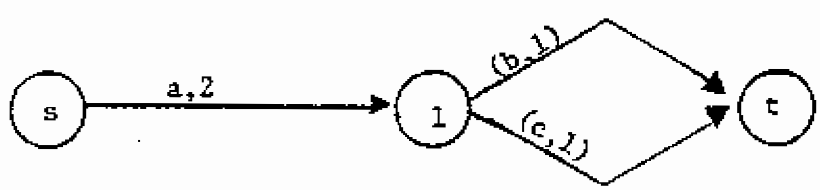


Figure 1

For each $\emptyset \neq S \quad \{a, b, c,\}$, let $V(S)$ be the maximal flow from s to t through the network containing the arcs of S only.

Obviously

$$V(i) = 0 \qquad i = a,b,c,$$

$$V(a,b) = V(a,c) = 1$$

$$V(b,c) = 0$$

$$V(a,b,c,) = 2$$

The maximal flow problem on this network has two extreme optimal dual solutions, corresponding to the two minimal cuts of the network. These yield the two core allocations

$$x_a^1 = 2, \qquad x_b^1 = 0, \qquad x_c^1 = 0$$

$$x_a^2 = 0, \qquad x_b^2 = 1, \qquad x_c^2 = 1$$

However the point

$$x_a^3 = 1 \qquad x_b^3 = 1, \qquad x_c^3 = 0$$

which is also a core allocation, does not correspond to a dual optimal solution. This is not entirely surprising since the relation between the game V and the optimization problem which yield this game is not one to one. A given game may have several linear programming representations, each possibly yielding a different optimal dual set. In the following section we present a certain standarized set of optimization problems for which this discrepency does not arise. Another class of problems with this property was given in Shapley and Shubik, [20].

V. Simple Networks

Consider a directed network $G = G(E,L)$ with one source and one sink, s and t respectively. For $j \in L$ let u_j be the capacity of this edge and c_j the associated objective value coefficient. We do not impose any condition on the sign of c_j. In practical applications one may expect some components to be negative reflecting the cost of flow, while others are positive to account for the associated revenues. Finally, for each arc $j \in L$ let $o_j \in N$ be the identity of the player which controls this arc.

The network G defines a game V in a natural way. For each $S \in C$ denote by G^S the network restricted to arcs whose owners belong to S. $V(S)$ can be defined then as the value of the optimal (i.e. maximal with respect to $c^S = (c_i: i \in S)$) s to t flow in G^S. It is a simple matter to observe that the standard arc-flow formulation of this problem, in which the variables correspond to the flows on the individual arcs, satisfies the conditions of Theorem 2.

We call the network G <u>simple</u>, if $u_j = 1$ for every, $j \in L$ and if each arc is owned by a different player, i.e. we can identify the set of arcs with the set of players. We will restrict out attention in this section to games resulting from such networks.

Let $P = (p_1 \ldots p_k)$ be the set of all simple paths from s to t on G, each regarded as a subset of edges i.e. a coalition. For each $p \in P$ let $c^p = \sum_{i \in p} c_i$. Obviously $V(p) = \max \{o, c^p\}$ for every $p \in P$ and $V(\{i\} \geq 0$ for every $i \in N$.

Thus, every $x \in$ CORE (V) must satisfy:

$$(6) \quad \sum_{i \in N} x_i = V(N)$$

$$(7) \quad \sum_{i \in p} x_i \geq c^p \qquad p \in P$$

$$(8) \qquad x_i \geq 0 \qquad i \in N.$$

The following proposition asserts that for <u>simple</u> networks the converse of this statement is also true

<u>Proposition 1</u> Let V be the game associates with a simple network G. Then

$$\text{CORE } (V) = \{x \mid x \text{ satisfies } (6), (7) \text{ and } (8)\}$$

<u>Proof</u> We have to show that $\sum_{j \in S} x_j \geq V(S)$ for every coalition $S \in C$. Let $y^S = (y_j^S : j \in S)$ be an optimal solution which yields the value V(S), and such that $y_j^S \in \{0,1\}^S$ The fact that such an integral valued solution exists follows from the fact that the underlying matrix is unimodular. It follows immediately that the non zero elements of y^S define a collection of edge-wise

disjoints pathes P' P from s to t such that the edges of these pathes belong to S. Hence,

$$V(S) = \sum_{j\in S} c_j y_j^S = \sum_{p\in P'} c^p \leq \sum_{p\in P'} \sum_{j\in p} x_j \leq \sum_{j\in S} x_j$$

Remark 3 Proposition 1 lies at the heart of the special behavior of simple networks. The reader may wish to verify that the proposition does not hold for networks which are not simple such as the one of Example 1.

Remark 4 An alternative representation of the core is as the optimal set of the linear program min $\sum_{j\in N} x_j$ subject to (7) and (8). This representation yields the following complementary slackness conditions:

(a) If there exists any optimal flow f with $f_j = 0$ for some $j \in N$ then $x_j = 0$ in any core allocation x

(b) If there exists any optimal flow , f, with $f_j = 1$ for all the arcs of a given path p, then $\sum_{j\in p} x_j = c^p$ in every core allocation x.

Proposition 1 is of little use from an algorithmic point of view since the cardinality of P is typically huge. Proposition 2 below can serve as a practical basis for deciding, for a given $x \in R^n$, whether or not $x \in$ CORE (V).

For $x \in R^n$ let G^x be the network obtained from G by replacing c with $c - x$.

Proposition 2 Let f be any optimal flow on G and let

$x \in R^n_+$ satisfy $\sum_{j \in N} x_j = V(N)$. Then $x \in$ CORE (V) if and only if f is optimal for G^x with optimal value 0.

Proof We first note that changes in the objective function leave f feasible. Its value with respect to the new objective function is

$$f(c-x) = \sum_{j \in N} f_j(c_j - x_j) = \sum_{j \in N} f_j c_j - \sum_{j \in N} f_j x_j$$
$$= V(N) - \sum_{j \in N} f_j x_j$$

assume that $x \in$ CORE (V). Then, by the complementary slackness condition (a) of Remark 2 $x_j > 0 \Rightarrow f_j = 1$. Thus

$$\sum_{j \in N} f_j x_j = \sum_{j \in N} x_j = V(N)$$

To complete the proof we note, that if f and x are such that the value of f w.r.t. network G^x is zero, f is optimal in this network iff for every path $p \in P$

$$\sum_{j \in p} (c_j - x_j) \leq 0$$

i.e. iff

$$\sum_{j \in p} x_j \geq c^p \text{ for every } p \in P$$

Using, proposition 1, we note that the last conditions holds iff $x \in$ CORE (V).

We finally come to the main theorem of this section. It states that for simple networks, every core allocation corresponds to an optimal dual solution for the corresponding optimization problem. We recall the arc-flow formulation of this problem:

(9) $\quad \text{Max} \sum_{j \in N} c_j f_j$

(10) $\quad \text{s.t.} \quad f_j \leq 1 \quad j \in N$

(11) $\quad \sum_{j \in IN_i} f_j - \sum_{j \in OUT_i} f_j = 0$ for every $i \in S$ with $s \neq i \neq t$.

(12) $\quad f_j \geq 0 \quad j \in N$

Where, for each node i, we denote by IN_i and OUT_i the set of edges coming into and going out of i respectively. The linear programming dual of (9) - (11) is

(13) $\quad \min \sum_{j \in N} u_j$

s.t.

(14) $\quad u_j + \sum_{i | j \in IN_i} w_i - \sum_{i | j \in OUT_i} w_i \geq c_j \quad j \in N$

(15) $\quad u_j \geq 0 \quad j \in N$

Theorem 3 Let $u \in$ CORE (V). Then, there exists $w = (w_i : i \in S)$ such that (u,w) is an optimal solution for (13) - (15).

Proof $u \in$ CORE (V) implies that $\sum_{j \in N} u_i = V(N)$ and $u_i \geq 0$, $i \in N$. Hence, all we have to show is that there exists w such that (14) is satisfied. Consider the network G^u. By proposition 3, the optimal value for the optimal flow problem on this network is 0. Let $(u'w')$ be any dual solution with respect to this network. Then

$$\sum_{j \in N} u'_j = 0$$

and

$$u'_j \geq 0, \quad j \in N$$

which imply that $u'_j = 0$, $j \in N$. By the dual feasibility of this solution we have that

$$\sum_{i \mid j \in IN_i} w'_i - \sum_{i \mid j \in OUT_i} w'_i \geq c_j - u_j, \quad j \in N$$

which in turn implies that

$$u_j + \sum_{i \mid j \in IN_i} w'_i - \sum_{i \mid j \in OUT_i} w'_i \geq c_j$$

i.e. that (u,w') is the required dual optimal solution.

We conjecture that Theorem 3 can serve as a practical basis for calculating the nucleolus (see Schmeidler [17a]) of the game V. If the objective function of problem (9) - (11) is to maximize flow (i.e. $c_j = 1$ $j \in OUT_s$, $c_j = 0$ otherwise) then theorem 3 simplifies to

Theorem 4 Let G be a simple network such that for each coalition S, V(S) is equal to the maximal s to t flow possible using the arcs of S only. Then the extreme points of CORE (V) are precisely the points $x = (x_j : j \in N)$ such that

$$x_j = \begin{cases} 1 & \text{if } j \in K \\ 0 & \text{otherwise} \end{cases}$$

where K is a minimal s to t cut in G.

The proof of theorem 4 follows immediately from theorem 3 and from the fact that the minimal cuts of G constitute the extreme dual solutions to the maximum flow problem on this network. For details see Ford and Fulkerson, [9] and Fulkerson [16].

VI. Examples

We conclude the paper with a few examples.

Example 2 Consider the network G_2 of Figure 2, where again each arc is labeled by its name (a letter) and its capacity. If each arc is owned by a different player, the network is simple. Consider the game defined by the max flow problem on this network. By Theorems 3 or 4, we note that the unique point in CORE (V) corresponds to the unique minimal cut in G_1. This point is given by

$$x_a = x_b = x_c = 0, \quad x_d = x_e = 1$$

Example 3 To see the complications which arise when the network is not simple, consider the network G_3 of Figure 3, which is obtained by letting one player, say a, control the three arcs a,b,c, of G_2, (or equivalently, replacing these 3 arcs by a unique arc of capacity 3). Again examine the maximal flow problem on G_3. The unique mimimum cut in G_3 consists, as previousely, of the arcs d, and e. This yield the core allocation

$$x_a = 0, \quad x_d = x_e = 1$$

However, 3 additional extreme points of CORE (V) are

$$x_a = 2 \qquad x_d = x_e = 0$$
$$x_a = 1 \qquad x_d = v, \; x_e = 0$$
$$x_a = 1 \qquad x_d = 0, \; x_e = 1$$

Cores of network games exhibit certain non monotonicities with respect to the game data. The following two examples demonstrate this behavior.

Example 4 Consider the network G_4 of Figure 4, obtained from G_2 by increasing to 2 the capacity of arc d. This increase, however, may not be in the interest of player d as the following allocation

$$x_a = x_b = x_c = 1, \qquad x_d = x_e = 0$$

belongs to the core of the new game.

Example 5 Consider network G_5 of Figure 5. Let all the per unit cost on the arcs be 0 except for C_a and C_e which are set to 2. Consider the game obtained from this network if we value each unit of flow from source to sink at one unit. (This can be achieved, say, by setting $c_a = -2$, $c_b = 0$, $c_c = 0$, $c_d = 1$, $c_e = 1-2 = -1$). The optimal flow in this network is through the path b,c,d and yields an objective function of 1. The extreme points of CORE (V) are given by

$$x_b = 1, \qquad x_j = 0 \qquad j \neq b$$
$$x_c = 1 \qquad x_j = 0 \qquad j \neq c$$
$$x_d = 1 \qquad x_j = 0 \qquad j \neq d$$

Now let us increase the value of a unit of flow from s to t to 4. A new optimal solution for the problem untilizes the paths a - d and b - c. The unique point in the core is now

$$x_b = x_d = 2, \qquad x_j = 0, \qquad j \neq b, d$$

Thus, the increase in the per unit revenue is detrimental from the point of view of player c.

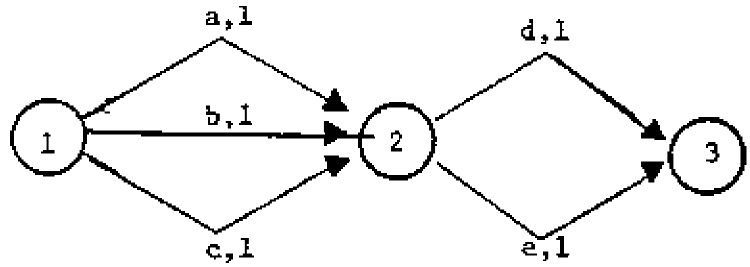


Figure 2 G_2

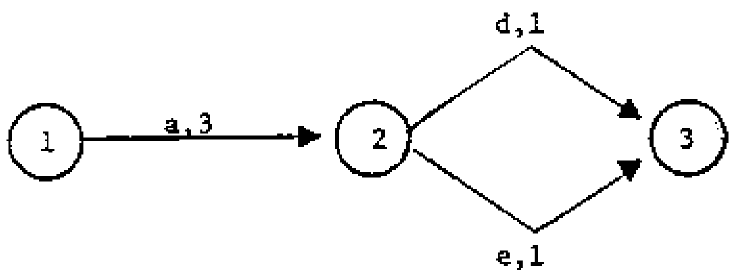


Figure 3 G_3

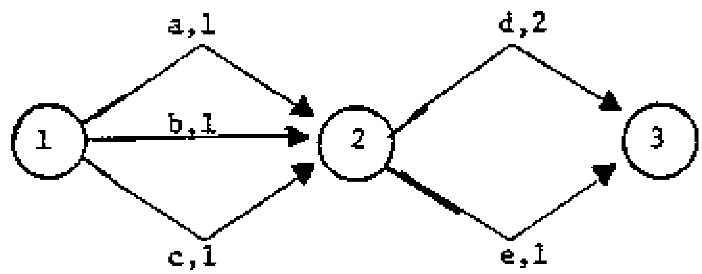
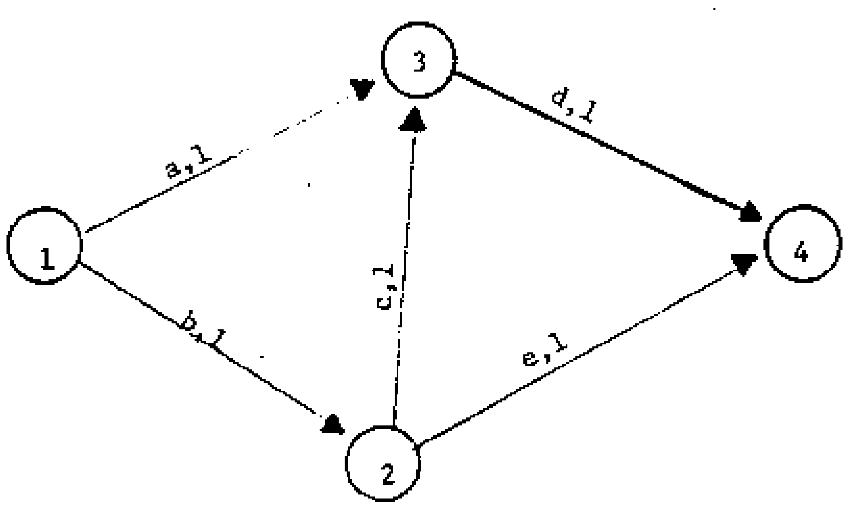


Figure 4 G_4



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