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INCONSISTENCIES OF WEIGHTED SUMMATION VOTING SYSTEMS

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ABSTRACT

Let α_N be any ordinal ordering of $N \geq 3$ alternatives. Select some one alternative and let α_{N-1} be some ordering of the remaining $(N-1)$ alternatives where α_{N-1} need not have any relationship to α_N. It is shown for a voting system coming from a large class of weighted summation voting systems that there exist examples of voter profiles such that if the voters vote on N alternatives the aggregated result is α_N, but if they vote on $(N-1)$ alternatives, the aggregated result is α_{N-1}. This result holds even if the voting system changes with the number of alternatives.

1. INTRODUCTION

When a group of individuals is required to rank order $N > 2$ alternatives often they adopt some sort of weighted voting procedure. In practice, the resulting ordering is interpreted, or at least it is used as though it were the group's aggregate linear ordering for the N alternatives. But is it? If it were then when some subset of the alternatives is considered separately this subset should inherit the ordering given by the original ranking of all N alternatives. However, if the subsets have cardinality two, then this subset requirement is equivalent to the condition of binary relevancy, so it follows from Arrow's theorem that this condition can be violated.

It might be argued, or at least hoped, that this "linear ordering" interpretation for a weighted vote could be partially salvaged. After all, even if the rankings of some two alternatives are transposed, there still might remain a useful relationship between the original ranking of the N alternatives and the ranking of a subset of these alternatives -- a relationship which could be exploited. The main purpose of this paper is to show that in general this is false -- there need not be any relationship whatsoever between the different rankings. This is a consequence of the stronger result proved here that *if* α_N *is a specified ordering of* N *alternatives while* α_{N-1} *is an arbitrary ordering of some subset of* $(N-1)$ *alternatives, then there exist profiles of voters so that the group outcome is* α_N *when* N *alternatives are considered but* α_{N-1} *when the specified subset of* $(N-1)$ *alternatives is considered*.

Fishburn [1] reported an example which is in the spirit of this conclusion. In his example seven voters use a Borda count to rank four alternatives as $a > b > c > d$. However, when alternative d is discarded, the voters rerank the remaining alternatives as $c > b > a$. This new ordering is the exact reversal of the ordering previously enjoyed by these three alternatives!

The Borda Count for N alternatives is where a voter casts $(N-i+1)$ points for his ith place alternative. The choice of these assigned weights is not the culprit which explains this example as our conclusion holds for most weighted voting systems commonly used. Indeed, we shall show that our conclusion depends more upon the geometry of a simplex in N dimensional Euclidean space than upon the choice of the weights. In fact the effects of the geometry are so strong that it is impossible to avoid our conclusion by devising a different weighted voting scheme for the subset of alternatives! In a future paper this geometry will be exploited to extend our conclusion to general selection processes by obtaining a statement which is in the spirit of Arrow's theorem (Saari [2]).

In order to simplify the exposition used to isolate the central *· of the proof, first we prove the theorem for a class of weighted ∆ voting systems which share properties similar to those of the Borda This will be done in Sections 2 and 3. Part of the proof (Section 3) centers around determining properties of the Condorcet triplet so common designing counter-examples for various "expected" properties of voting systems. Here we extend the basic structure of this triplet to N alternatives and determine the basic "geometric" properties which make them so important. A more detailed discussion will follow at a later date.

In Section 4 the restrictions on the voting systems are removed so that they now include most weighted voting systems. Indeed, the final theorem, Theorem 3 is general enough so it even includes systems of little practical interest since they may have no monotonicity or pareto type properties. In fact, just about the only interest in these examples is to illustrate that the conclusion holds even for weighted voting systems not satisfying Arrow's assumptions, which in turn highlights the above statement which asserts it is the geometry of the simplex which makes the result "work."

The paper ends with a brief discussion concerning how to determine restrictions on preferences or on the fraction of voters with certain profiles which ensure that the ordering could be viewed as a linear ordering. This is in terms of feasible sets from a linear programming problem.

2. THE GEOMETRY AND THE MAIN RESULT

We start by giving a geometric interpretation for cardinal and ordinal rankings of N alternatives. Let vector $\underline{x} = (x_1,\ldots,x_N)$ in the positive orthant of $\mathbb{R}^N$ ($\mathbb{R}^N_+$) denote a cardinal (complete, transitive) ordering of the N alternatives where the magnitude of component x_i measures the intensity of preferences for the ith alternative. Thus, an individual ranks alternative i ordinally over alternative j if and only if $x_i > x_j$. This means that the $\binom{N}{2}$ "indifference" hyperplanes $\{\underline{x} \mid x_i = x_j,\ i \neq j\}$ divide $\mathbb{R}^N_+$ into cones, where each cone corresponds to an ordinal ranking of the alternatives. Namely, each ordinal ranking corresponds to a unique cone which is an equivalence class of cardinal rankings of alternatives. Denote these classes or ordinal rankings by P_N. Notice that degenerate cones are admitted, e.g., the ranking of complete indifference among alternatives corresponds to the ray $t(1,1,\ldots,1), t > 0$.

Since ordinal rankings do not reflect intensity of preferences, the description can be simplified by normalizing vectors so that the sum of their components equals unity. This is equivalent to intersecting plane $P(N) = \{\underline{y} \in \mathbb{R}^N \mid y_i \geq 0,\ \sum_{i=1}^{N} y_i = 1\}$ with the cones. The resulting object is a simplex which is divided into the ordinal equivalence classes where these classes give a lower dimensional representation for P_N. For example, the complete indifference class now corresponds to the point $N^{-1}(1,1,\ldots,1)$. Figure 1 illustrates P_3. P_4 is given by the baricentric division of an equilateral tetrahedron, P_N is given by the division of an equilateral N-gon lying in $(N-1)$ dimensional space.

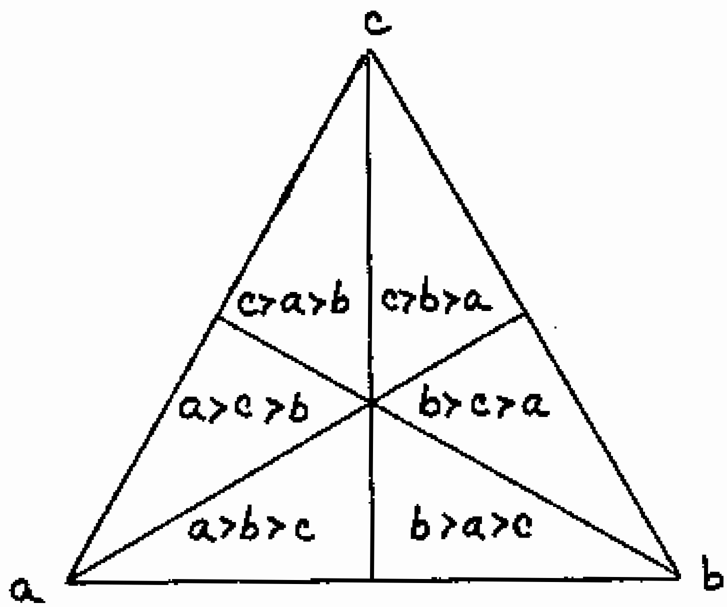


FIGURE 1. This simplex illustrates P_3, the set of all transitive ordinal rankings of the three alternatives, a,b,c. Each open triangular region represents the strict ordering given by the label. The lines correspond to indifference relations.

Let $\tilde{P}_N$ denote those orderings which only admit strict preferences, i.e., $\tilde{P}_N$ corresponds to the union of the open regions in P_N. Each subset in $P_N - \tilde{P}_N$ is contained in at least one indifference hyperplane, so it corresponds to an ordinal ranking which admits an indifference between at least two alternatives.

In the Borda count, a vector is assigned to each ordinal equivalence class. Each voter selects one vector, and the Borda count conclusion is the ordinal equivalence class containing the sum of the selected vectors. We generalize this by calling any weighted voting procedure which satisfies these conditions a Borda method. More precisely:

Definition 1: A Borda method is a summation voting method which satisfies the following.

1. Vectors are assigned to each ordinal equivalence class of P_N (or of $\tilde{P}_N$).

2. The assigned vectors lie in the closure of the corresponding equivalence class.

3. From the set of assigned vectors, each voter can select one of them, and the group ordering is given by the ordinal class containing the sum of selected vectors.

With this definition Borda methods include the usual Borda count [the weight vector lies in the interior of the $\tilde{P}_N$ classes]; plurality voting where the weight vector assigns one point for the voter's first choice alternative and zero for all others [the weight vectors are on the boundaries of the P_N classes]; cummulative voting where the voter casts non-negative integer points for each alternative as long as the point total equals a specified value k, e.g., if $k = 2$, then the voter may cast either one point each for his top two choices or two points for his top choice [more than one weight vector in each P_N class]; or voting cardinal preferences where voter casts his cardinal preference vector provided the sum of the weights either equals, or doesn't exceed some specific value [an uncountable number of vectors with different Euclidean or L_1, lengths.] This definition does not admit inverted voting systems where smaller weights correspond to more favored alternatives; these are discussed in Section 4.

Definition 2. A Borda method is symmetrical if the following hold.

a) Vectors are assigned only to classes in $\tilde{P}_N$, and each class has assigned to it only one vector.

b) Any weight vector can be obtained from any other weight vector by a permutation of the indices.

c) The weight vectors are not all the same.

The effect of (a) is to limit the number of vectors, and for some methods it carries with it the tacit assumption that the voters are not indifferent between alternatives; or, if they are, they must vote as if they had strict preferences. Condition (b) is a neutrality or symmetry condition among the alternatives which ensures that no one alternative is given a preference in the assignment of the weights and that all voters have the same selection of weight vectors. Technically this will allow us to normalize the weight vectors so that they all lie on

P(N). Although this condition can be greatly relaxed, it is a reasonable condition and the symmetry properties significantly simplify the exposition. Condition (c) is imposed to ensure that the weight vectors are not scalar multipliers of $(1,1,\ldots,1)$. Such a vector means that each voter <u>must</u> cast a vote reflecting complete indifference among the alternatives independent of the voter's actual preferences.

Let β_N correspond to a symmetrical Borda Method over the N alternatives, and let β_{N-1} correspond to a symmetrical Borda method over $N-1$ alternatives. We now state our main result, a result stated without proof in [2].

Theorem 1. <u>Let</u> $N \geq 3$. <u>Let</u> $\alpha_N \in \tilde{P}_N$ <u>be some ordering of the</u> N <u>alternatives</u>. <u>Choose some one alternative and let</u> $\alpha_{N-1} \in \tilde{P}_{N-1}$ <u>be some ordering of the remaining</u> $(N-1)$ <u>alternatives</u>. <u>Let</u> β_N <u>be a symmetrical Borda Method for the</u> N <u>alternatives and let</u> β_{N-1} <u>be a symmetrical Borda Method for the</u> $(N-1)$ <u>alternatives</u>. <u>Then there exist choices of voters' profiles so that the</u> β_N <u>Borda ranking of the</u> N <u>alternatives is</u> α_N, <u>while the</u> β_{N-1} <u>ranking of the selected</u> $(N-1)$ <u>alternatives is</u> α_{N-1}.

We illustrate this theorem by selecting for β_N the usual Borda count, for β_{N-1} a plurality election, and for α_{N-1} a reversal of the ordering.

Corollary 1. Let $N \geq 3$. <u>There exist examples where if the N alternatives are ordered by the Borda Count, and if the second from last place alternative is discarded, then the last place alternatives wins a plurality election among the remaining</u> $(N-1)$ <u>alternatives</u>.

It will be clear from the proof that the conclusion of this theorem holds for any subset of the N alternatives, not only subsets with cardinality $N-1$. This supports our assertion that the ordering resulting from a weighted Borda method need not admit the interpretation of a linear ordering.

3. PROOF OF THE THEOREM

Before proving the theorem we highlight the basic idea by proving the special case where β_3 is a plurality election and where β_2 is the Borda Count assigning $(2-i+1)$ points for the voter's ith alternative. Assume that the three alternatives are $\{a,b,c\}$ and that c is the discarded alternative.

Changing the magnitude of a vector does not affect the ordinal equivalence class in which it lies. Therefore, for convenience of exposition, we assume that all vectors lie in $P(N)$ where N is either 2 or 3. This means that the β_2 weight vectors are $(\frac{2}{3},\frac{1}{3})$ for $(a > b)$ and $(\frac{1}{3},\frac{2}{3})$ for $(b > a)$. If m voters prefer a to b while n voters prefer b to a, then the (normalized) β_2 outcome is $(\frac{m}{m+n})(\frac{2}{3},\frac{1}{3}) + \frac{n}{(m+n)}(\frac{1}{3},\frac{2}{3}) = \frac{1}{3(m+n)}(2m+n,m+2n)$, an outcome which is equivalent to majority rule.

The m voters preferring $(a > b)$ come from three $\vec{P}_3$ equivalence classes, namely $(a > b > c),(a > c > b)$, and $(c > a > b)$. The β_3 weight vectors assigned to these three classes are, respectively, $(1,0,0),(1,0,0)$, and $(0,0,1)$. Assume that, respectively, these classes have m_1,m_2,m_3 voters where $m_1 + m_2 + m_3 = m$. When these voters cast their β_3 ballots, the P_3 outcome is $\underline{V}_m = m^{-1}(m_1+ m_2,0,m_3)$. This is a point on the convex hull of the weight vectors; that is, some point on the line from vertex a to vertex c (see Figure 2). Call this set C_a. Indeed, for any point $p \in C_a$ and for any $\epsilon > 0$, a value of m and appropriately selected values of m_1,m_2,m_3, can be made so that $\underline{V}_m$ is within ϵ distance of $\underline{p}$.

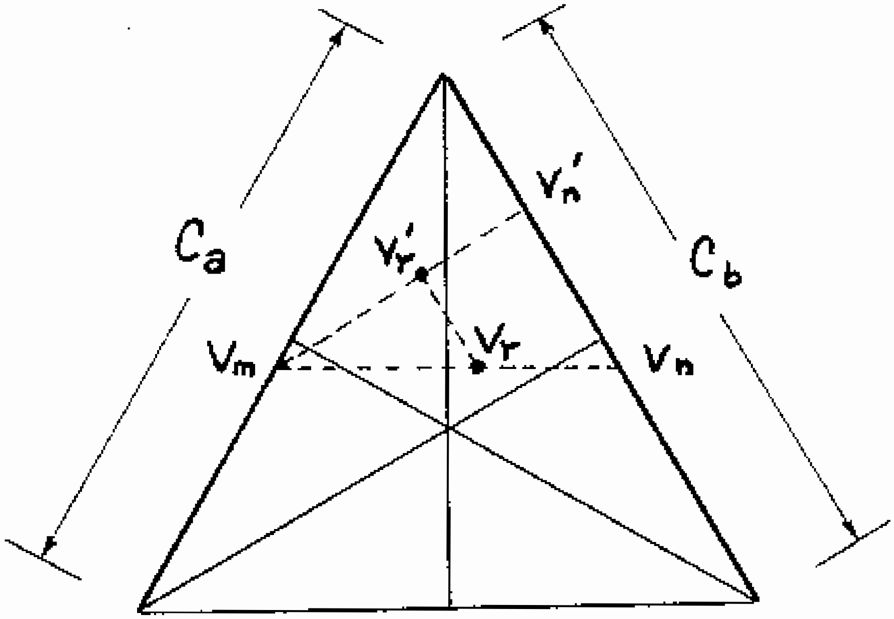


FIGURE 2. Point $\underline{V}_\gamma$ is the aggregated outcome given by a fixed convex combination of $\underline{V}_m$ and $\underline{V}_n$ where γ is determined by the percentage of voter preferences between (a,b). As $\underline{V}_n$ changes in set C_b to $\underline{V}_n'$, the aggregated outcome can change (a,b) classes. By adjusting $\underline{V}_m$, $\underline{V}_n'$ and γ, it can be seen that examples exist where almost all voters prefer b to a, yet the P_3 outcome has $a > b$.

A similar description holds for the n voters preferring b to a. Their β_3 value is vector $\underline{V}_n$ which is in the convex hull of the weight vectors $(0,1,0)$, $(0,1,0)$, and $(0,0,1)$. Denote this set by C_b. The β_3 outcome of all voters is $\lambda\underline{V}_m + (1-\lambda)\underline{V}_n$ where $\lambda = m/(m+n)$, a vector which lies in the convex hull of the convex hulls.

Now, without loss of generality assume that α_{N-1} is $(b>a)$; that is $n > m$ or $\lambda < \frac{1}{2}$. It is easy to show that if α_N also ranks $b > a$, then there are examples where the β_N outcome is α_N and the β_{N-1} outcome is $(b > a)$. So, assume α_N ranks $a > b$. If the line connecting $\underline{V}_m$ and $\underline{V}_n$ is parallel to the line connecting vectors a and b, then for $0 < \gamma < \frac{1}{2}$, $\underline{V}_\gamma = \gamma\underline{V}_m + (1-\gamma)\underline{V}_n$ lies in the region $b > a$. However, if $\underline{V}_n$ lies above this parallel line, say at point $\underline{V}'_n$ (see Figure 2), then there are some values of $\gamma < \frac{1}{2}$ so that $\underline{V}'_\gamma = \gamma\underline{V}_m + (1-\gamma)\underline{V}'_n$ lies in the region $a > b$. This is because the points $\underline{V}_m, \underline{V}_n, V'_n$ and $\underline{V}_m, \underline{V}_\gamma, \underline{V}'_\gamma$ define similar triangles. But, the line segment $\underline{V}_n, \underline{V}'_n$ is not parallel to the indifference line $a \sim b$, therefore, neither will the line $\underline{V}_\gamma, \underline{V}'_\gamma$ be parallel to this indifference line. Consequently, if γ is sufficiently close to $\frac{1}{2}$ ($\underline{V}_\gamma$ is sufficiently close to the indifference line) then $\underline{V}'_\gamma$ will lie in the region $a > b$. By an appropriate selection of the height of $\underline{V}_m$ and $\underline{V}'_n$, $\underline{V}'_\gamma$ can be selected to lie in any of the $\tilde{P}_3$ classes. Thus by an appropriate selection of m_i, n_i, the β_3 outcome $\lambda\underline{V}_m + (1-\lambda)\underline{V}_n$ can be made to lie in any of the three classes having $a > b$. This completes the proof for the special case.

The general theorem is proved in much the same way. The next several lemmas illustrate that any symmetrical Borda Method admits a similar geometric description

Lemma 1. Consider any subset, S, of preferences in $\tilde{P}_N$. Let $\tilde{S} \subset P(N)$ be the convex hull of the corresponding weight vectors. Then for voters whose preferences lie in S, their P_N outcome lies in $\tilde{S}$. Furthermore, for any $\epsilon > 0$ and any $p \in \tilde{S}$ there exists some distribution of voters so that their P_N outcome is within distance ϵ of p.

The proof is obvious.

If the weight vectors lie in the interior of the classes of $\tilde{P}_N$, then $\tilde{S}$ must meet the interior of all classes in S. However, if the weight vectors lie on the boundary of the $\tilde{P}_N$ classes, then, as we have seen with plurality voting, it is possible that $\tilde{S}$ does not meet the interior of any set of S and it may meet some sets in only one point. The following lemma asserts that this is not true for all choices of S.

Definition: Assume that the N alternatives are $a_1, a_2, \ldots, a_N$. Let $\alpha_N = (a_{\pi(1)} > a_{\pi(2)} > \ldots > a_{\pi(N)})$ be a given ordering where $\Pi = (\Pi(1), \ldots, \Pi(N))$ is some permutation of the N indices. Call the set of N linear orderings $\{\alpha_N, (a_{N(2)} > \ldots > a_{\pi(N)} > \alpha_{\pi(1)}), (a_{\pi(3)} > \ldots > a_{\pi(N)} > a_{\pi(1)} > a_{\pi(2)}), \ldots, (a_{\pi(N)}, > \alpha_{\pi(1)} > \ldots > a_{\pi(N-1)})\}$ the Condorcet N-tuple generated by α_N.

If $N = 2$, then the Condorcet 2-tuple must be $\{(a_1 > a_2), (a_2 > a_1)\}$. Permutation Π can be viewed as being a labeling of the vertices of an equilateral N-gon ordered in a counter-clockwise direction. The Condorcet N-tuple are the linear orderings corresponding to permutations obtained by rotating this N-gon. If ρ is a rotation which generates this rotation group, let $\rho(\alpha_N)$ denote the corresponding element in the Condorcet N-tuple. Then this N-tuple is given by $\{\rho^k(\alpha_N) \mid k = 1,2,\ldots,N\}$ where $\rho^k(\alpha_N) = \rho(\rho^{k-1}(\alpha_N))$ and where $\rho^N = I$ is the identity element of the rotation group.

Lemma 2. <u>Let S be any Condorcet N-tuple. Then $\tilde{S}$ has a non-empty interior which includes the complete indifference point</u> $(N^{-1},N^{-1},\ldots,N^{-1})$ <u>and which meets the interior of any set of</u> $\tilde{P}_N$. (By interior, we mean topological interior with respect to the usual metric topology on P(N)).

Proof: Without loss of generality, assume that the Condorcet N-tuple is generated by $a_1 > \ldots > a_N$ and let the corresponding weight vector be $\underline{w}_1 = (w_1,\ldots,w_N)$. According to Definition 2a, $w_1 \geq w_2 \geq \cdots \geq w_N$ where (by Definition 2b) $w_1 > w_N$ and where the weight vectors for the other classes of S are given by the appropriate permutation of coordinates w_i. It is easy to see that if the Condorcet N-tuple is given by rotating the indices of α_N in one direction with respect to order $(>)$, the weight vectors are obtained by applying the inverse rotation to the indices of w_i's, e.g., if $\underline{w}_2$ is the weight vector for ordering $(a_2 > a_3 > \ldots > a_N > a_1)$ then $\underline{w}_2 = (w_N,w_1,\ldots,w_{N-1})$.

First, we claim that $N^{-1}(\Sigma\underline{w}_i) = (N^{-1},\ldots,N^{-1})$. The rotation of the components for the weight vectors implies that each component of $\Sigma\underline{w}_i$ is the sum of the weights $w_1,\ldots,w_N$; so the vector sum is a constant multiple of $(1,1,\ldots,1)$. This multiple must be unity because the vectors $\underline{w}_i$ were normalized to lie on P(N). Thus the complete indifference point lies in $\tilde{S}$.

If $\tilde{S}$ has a non-empty interior, then trivially this point will lie in the topolotical interior. As this point lies in the closure of each set of $\tilde{P}_N$, the conclusion follows once we show $\tilde{S}$ has a non-empty interior.

$\tilde{S}$ has a non-empty interior if $\{\underline{w}_i\}_{i=1}^N$ forms a basis for $\mathbf{R}^N$. Since the vectors are normalized, this is equivalent to showing that each $\underline{w}_i$ is a vertex of $\tilde{S}$. If one of these vectors, say $\underline{w}_j$, were not a vertex, then the convex hull defined by the remaining vectors agrees with $\tilde{S}$. In particular, this means there exists

$\lambda_i \in [0,1]$, $\sum_{\substack{i=1 \\ i\neq j}}^N \lambda_i = 1$ such that

1) $\sum_{\substack{i=1 \\ i\neq j}}^N \lambda_i \underline{w}_i = \underline{w}_j$. In turn, this gives N coordinate scalar equations. By symmetry, each scalar equation contains each w_i weight once, each weight is on the right hand side in exactly one scalar equation, and each weight is multiplied by λ_i in exactly one scalar equation. Thus, by rotating N times the order of these equations by taking the first equations and making it the last, we have N sets of equations. From this it follows that each $\underline{w}_k$ satisfies a vector equation of the type (1) where the λ_i scalars retain the same value but change index. Consequently, either none of the $\underline{w}_i$'s are vertices of $\tilde{S}$, or they are equal. The first conclusion cannot occur as $\tilde{S}$ is the convex hull of these points. The second can't occur as it would imply $w_1 = w_N$, which is a contradiction (Definition 2c). Thus $\tilde{S}$ has a non-empty interior and these weight vectors are linearly independent.

Lemma 3. *If* $S_1 \subset S_2$, *then* $\tilde{S}_1 \subset \tilde{S}_2$. *In particular, if* $S_2 = \tilde{P}_N$, *then* $\tilde{S}_2$ *satisfies the stated properties of* $\tilde{S}$ *in Lemma 2.*

There is a natural projection, Π, from $\tilde{P}_N$ to $\tilde{P}_{N-1}$, where the image of this projection preserves the ordinal ordering of subsets of a P_N ordering. To prove the theorem we need to show that Π does not commute with summation of vectors; i.e., if $N \geq 3$, the geometry of $P(N)$ prevents the process of aggregation and any natural projection from commuting. In order to show this, we will need the inverse image of this projection which we call a "lift."

Definition 4. *Let* α_{N-1} *be a given ordering of* $(N - 1)$ *of the* N *alternatives. Corresponding to this ordering are* N *classes in* $\tilde{P}_N$ *which are obtained by the* N *ways the remaining alternative can be placed within the ordering* α_{N-1}. *Call this set of* N *classes the lift of* α_{N-1}.

In the special case, to prove that summation and Π do not commute we used the fact that C_a and C_b are not lines parallel to the indifference line $a = b$. Essentially, the role this geometry plays is to ensure that set $\{\frac{1}{2}\underline{\alpha} + \frac{1}{2}\underline{\beta} \mid \underline{\alpha} \in C_a, \underline{\beta} \in C_b\}$ contains the complete indifference point as an *interior point*. Notice that C_a and C_b are determined by the lifts of a Condorcet two tuple. The following lemma generalizes this to higher dimensions.

Lemma 4. *If* β *is a given ordering of* $(N-1)$ *alternatives, let* S_β *be the corresponding lift. Let* $\alpha_{N-1} \in \tilde{P}_{N-1}$, *and let* S_1 *be the* corresponding Condorcet $(N-1)$ tuple. Then the set

$$C = \{(N-1)^{-1}(\sum_{\beta\in S_1} \underset{\sim}{\alpha}_\beta) \mid \underset{\sim}{\alpha}_\beta \in \tilde{S}_\beta\}$$

has a non-empty interior which contains the complete indifferent point and which meets the interior of each set in $\tilde{P}_N$.

Proof. First we claim that $S = \bigcup_{\beta\in S_1} S_\beta$ is the disjoint union of $(N-1)$ Condorcet N-tuples; namely the Condorcet N-tuples generated by the N entries in the lift of α_{N-1}. Let α be an element of this lift. It is easy to see that the Condorcet N-tuple generated by α lies in $\bigcup_{\beta\in S_1} S_\beta$. This is because if the permutation corresponding to α is first rotated and then projected, we obtain a rotation of α_{N-1}. Thus the Condorcet N-tuples generated by elements in the lift of α_{N-1} is contained in S.

Next, we claim that any two Condorcet N-tuples are either disjoint or they are the same. To show this, we use the interpretation that a Condorcet N-tuple can be viewed as an orbit under the rotation group action; $\{\rho^k(\alpha) \mid k = 1,2,\ldots,N,\ \rho$ generates the rotation group of an equilateral N-gon $\}$. But it is well-known that such orbits have the above stated property. [If S_1 and S_2 are two Condorcet N-tuples which share $\tilde{\beta}$, then any element γ in S_1 and any element γ' in S_2 can be obtained through a rotation of $\tilde{\beta}$; e.g., $\gamma = \rho\tilde{\beta}$, $\gamma' = \rho'\tilde{\beta}$. In particular, if γ' is the generator of the Condorcet N-tuple, then $\gamma = \rho\tilde{\beta} = \rho\circ(\rho')^{-1}\gamma'$. But since $\rho\circ(\rho')^{-1}$ is a rotation, γ is in the Condorcet N-tuple generated by γ'. Thus $S_1 \subset S_2$. A cardinality argument shows that both sets must agree.]

In the lift of α_{N-1}, the N elements are obtained by the N possible ways the missing alternatives, say a_N, can be placed within the ordering α_{N-1}. Only the orderings in a lift where a_N is the most preferred and where it is the least preferred alternative are related through a rotation. Thus the N elements in the lift of α_{N-1} provide generators for $N-1$ distinct Condorcet N-tuples. The union of these N-tuples gives rise to $(N-1)\,N$ distinct orderings contained in S. But since this is the cardinality of S, S consists of the disjoint union of the Condorcet N-tuples generated by elements in the lift of α_{N-1}. This is the proof of the claim in the first sentence.

We now show that set C has a non-empty interior. For $\beta \in S_1, \tilde{S}_\beta = \sum_{i=1}^N \lambda_i^\beta \underline{w}_i^\beta$ where i ranges over the N classes in S_β, where $\underline{w}_i^\beta$ is the corresponding weight vector in S_β and where the λ's are the convex scalar weights, $\underline{\lambda}^\beta = (\lambda_1^\beta, \ldots, \lambda_N^\beta) \in P(N)$. Set C is given by the sum $\sum_{\beta \in S_1} (N-1)^{-1}\tilde{S}_\beta = \sum_{\beta \in S_1} \sum_{i=1}^N \lambda_i^\beta (N-1)^{-1}\underline{w}_i^\beta)$. This last sum can be viewed as a linear mapping $g(\lambda):(P(N))^{N-1} \to P(N)$. Since it is linear, it is an open mapping should its Jacobian have maximal rank. But the $(N-1)(N)$ rows of its Jacobian are given by the weight vectors $\underline{w}_i^\beta$. Since S contains $N-1$ Condorcet N-tuples, among these rows are the weight vectors corresponding to a Condorcet N-tuple. But the main point of the proof of Lemma 2 was to show that these weight vectors are linearly independent; thus Dg has maximal rank.

The proof of the lemma is completed once it is demonstrated that there is an interior point of $(P(N))^{N-1}$ which gets mapped via g to the indifference point. But for any Condorcet N-tuple in S, N^{-1} times the sum of the corresponding weight vectors is the complete indifference point (Lemma 2). Therefore, summing over the (N-1) Condorcet N-tuples in S and multiplying by the scalar $(N-1)^{-1}$ again yields the complete indifference point. This sum is equivalent to letting $\lambda_i^\beta = N^{-1}$ for all β and all i. Since this is an interior point in the domain of g, this completes the proof of the lemma.

We now can prove the theorem. Let α_{N-1}, α_N be the specified orderings of N-1 and N alternatives and let β_{N-1}, β_N be the specified voting procedures. Let S_1 be the Condorcet (N-1)-tuple generated by α_{N-1}. For $\underline{\gamma} = (\gamma_1, \ldots, \gamma_{N-1}) \in P^{N-1}$ let $f(\underline{\gamma}, \underline{\lambda}): P(N-1) \times (P(N))^{N-1} \to P(N)$ be $\sum_{k=1}^{N-1} \gamma_k (\sum_{i=1}^{N} \lambda_i^k \underline{w}_i^k)$ where $\underline{w}_i^k$, i = 1,...,N are the N β_N weight vectors in the lift of $\rho^k(\alpha_{N-1})$, k = 1,2,...,N-1.

The same argument we used to show that Dg has maximal rank (Proof of Lemma 4) shows that Df has maximal rank. Indeed, g is f restricted to $\underline{\gamma}^* = ((N-1)^{-1}, \ldots, (N-1)^{-1})$. Thus, according to Lemma 4, there exists point p in the interior of the $\tilde{P}_N$ class corresponding to α_N and $\underline{\lambda}^*$ in the interior of $(P(N))^{N-1}$ such that $f(\underline{\gamma}^*, \underline{\lambda}^*) = p$.

According to Lemma 2, if $\underline{\gamma}^*$ corresponds to the convex scalar weights for the β_{N-1} weight vectors of the Condorcet (N-1)-tuple generated by α_{N-1}, then the outcome is the P(N-1) complete indifference point. Also according to Lemma 2, a $\underline{\gamma} \in P(N-1)$ can be selected so that the convex sum of these β_{N-1} weight vectors lies in the interior of the $\tilde{P}_{N-1}$ class corresponding to α_{N-1}. Indeed, since the complete indifference point lies on the boundary of this $\tilde{P}_{N-1}$ class, it follows from the continuity of the convex sum and from the continuity of f that $\underline{\gamma}$ can be selected so that both $f(\underline{\gamma},\underline{\lambda}^*)$ and $f(\underline{\gamma}^*,\underline{\lambda}^*)$ lie in the interior of the same $\tilde{P}_N$ class. Since the rationals are dense within the reals, it follows from continuity that $\underline{\gamma}$ and $\underline{\lambda}^*$ can be assumed to have rational components.

Now, it is a simple arithmetic exercise to find an appropriate rational equivalent for $\underline{\gamma}$ and $\underline{\lambda}^* = (\underline{\lambda}^1,\underline{\lambda}^2,\ldots,\underline{\lambda}^{N-1})$ so that

i) $\gamma_1,\ldots,\gamma_{N-1}$ have a common denominator

and

ii) the numerator of γ_k serves as the common denominator for the components of $\underline{\lambda}^k = (\lambda_1^k,\ldots,\lambda_N^k)$.

Then, the common denominator for $\gamma_1,\ldots,\gamma_{N-1}$ is the total number of voters. The numerator for γ_k is the number of voters with $\tilde{P}_{N-1}$ ordering $\rho^k(\alpha_{N-1})$. The numerator of λ_j^k is the number of voters in the jth class of the lift of $\rho^k(\alpha_{N-1})$. By construction, with this number of voters in the designated classes, the β_{N-1} outcome is α_{N-1} while the β_N outcome is α_N. This completes the proof.

4. COMMENTS AND EXTENSIONS

The statement of the theorem requires the preselected orderings α_N and α_{N-1} to be strict orderings -- indifference is not admitted. If we had allowed the preselected orderings to contain an indifference relation, the proof would have been the same up to the point where $f(\underline{\gamma},\underline{\lambda}^*)$ lies in the appropriate class of P_N. However, the inverse image of a class admitting an indifference relation does not contain an open set. Therefore, the crucial next step of assuming that $\underline{\gamma}$ and $\underline{\lambda}^*$ have rational components may not hold, especially if the weights are rationally independent. Thus, the extension is restricted by a number theoretic reason which, for practical problems, would most likely not occur. The next theorem should cover most practical voting methods.

Corollary 2. Let β_N and β_{N-1} be symmetrical Borda methods where the weight vectors are all rational numbers. Then the conclusion of Theorem 1 holds with $\alpha_N \in P_N$ and $\alpha_{N-1} \in P_{N-1}$.

The proof of the theorem did not depend upon the direct correlation between larger weights being assigned to more favored alternatives, but rather upon the continuity of f and the topological properties of the barycentric division of $P(N)$.

Definition 5. Let G_N and $\tilde{G}_N$ be geometric divisions of $P(N)$ defined, respectively, by the geometric representations of P_N and $\tilde{P}_N$. A weighted summation voting system is said to be symmetric if the following hold for the weight vectors.

i) <u>Each class of $\tilde{P}_N$ is assigned in a one-to-one fashion to a class in $\tilde{G}_N$. The weight vector corresponding to an ordering in $\tilde{P}_N$ must lie in the closure of the assigned class of $\tilde{G}_N$.</u>

ii) <u>The weight vectors differ only by a permutation of indices.</u>

iii) <u>The weight vectors are not all the same.</u>

The group outcome is the P_N class associated with the G_N class containing the (normalized) sum of the selected vectors. Borda symmetrical systems are symmetrical. A non-Borda system which is symmetrical is one where smaller weights indicate higher preferences. Because there is no assumption of monotonicity on the mapping from $\tilde{P}_N$ to $\tilde{G}_N$, this definition includes systems which are a mixture of the above two types. Indeed, the classes in $\tilde{P}_N$ can be assigned even in a random fashion to the classes of $\tilde{G}_N$. For example, the following system was determined by using a random number table -- a normalization of the weight vectors yields a system satisfying the definition.

Preferences	Weight Vectors
$a > b > c$	(3,2,1)
$a > c > b$	(2,1,3)
$b > a > c$	(1,2,3)
$b > c > a$	(3,1,2)
$c > a > b$	(1,3,2)
$c > b > a$	(2,3,1)

Theorem 2. Let β_N, β_{N-1} be symmetric weighted summation voting systems for N alternatives and (N-1) alternatives respectively. Then the conclusion of Theorem 1 holds for these voting systems. If all the weight vectors for both systems have rational components, then the conclusion of Corollary 2 holds.

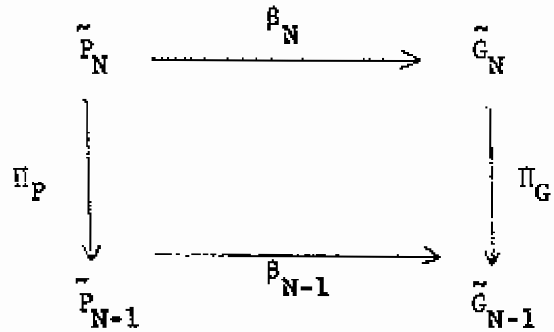
This theorem permits a mixture of voting systems; e.g., if β_N assigns larger weights for more preferred alternatives while β_{N-1} reverses this procedure.

Corollary 3. Let β_N be the Borda Count on N alternatives where (N - i+1) points are assigned for the ith place alternative and let β_{N-1} be a system on (N - 1) alternatives which assigns i points for the ith place alternative. Let $\alpha_N \in P_N$. Discard some one alternative and for the remaining set of alternatives let $\alpha_{N-1} \in P_{N-1}$. Then, there exist profiles of voters such that when they use the β_N voting system the outcome is α_N, but when they consider the subset of (N - 1) alternatives and use a β_{N-1} count, the outcome is α_{N-1}.

This theorem asserts that even nonsensical voting systems may yield "reasonable" results.

Corollary 4. Let β_3 be the voting system defined just prior to the statement of Theorem 2 and let β_2 be majority voting on the two alternatives a,b. There exist profiles of voters such that their β_3 outcome is $a > b > c$ while their β_2 outcome is $a > b$.

Proof of Theorem 2. A summation voting system β_N is characterized by the assignment of vote vectors to the ordinal preference classes such as specified in Definitions 2 and 5. For a Borda method the following diagram commutes where Π is the natural projection.



In general, this diagram will not commute unless β_N, β_{N-1} have the same monotonicity characteristics. For example, the diagram does not commute for the systems defined in Corollaries 3 and 4.

So, the proof of Theorem 2 involves modifying that of Theorem 1 in a fashion which overcomes this obstruction. In Theorem 1 the Condorcet N-tuple was used because it formed a sparse system where the convex hull of its weight vectors included the complete indifference point in its interior. According to Lemma 3, larger collections of preferences could have been admitted with the conclusion remaining the same. Thus, if we define f over all classes of $\tilde{G}_{N-1}$, the relevant properties of f will remain the same.

Let $f: P((N-1)!) \times (P(N))^{(N-1)!} \to P(N)$ be defined as $\sum_{k=1}^{(N-1)!} \gamma_k (\sum_{i=1}^{N} \lambda_i^k \underline{w}_i^k)$ where k serves as an index for the $(N-1)!$ classes of $\tilde{G}_{(N-1)}$ and where $\underline{w}_i^k$, $i = 1,\ldots,N$, are the

$N\,\beta_N$ weights in the $N\,\tilde{G}_N$ classes corresponding lift of the kth class in $\tilde{G}_{(N-1)}$. Notice, <u>this</u> lift is the inverse image of the projection Π_G. Let $\underline{\gamma}^* = (((N-1)!)^{-1},\ldots,((N-1)!^{-1}) \in P((N-1)!)$. Then, by Lemma 3 and the arguments of Theorem 1, there exists $\underline{\lambda}^*$ in the interior of $(P(N))^{(N-1)!}$ such that $f(\underline{\gamma}^*,\underline{\lambda}^*) = N^{-1}(1,\ldots,1)$. Furthermore, there exists $\underline{\lambda}$ in the interior of $(P(N))^{(N-1)!}$ such that $f(\underline{\gamma}^*,\underline{\lambda})$ is in the interior of the $\tilde{G}_N$ class assigned to α_N. An argument similar to the one used in Theorem 1 shows that a $\underline{\gamma}$ can be selected close to $\underline{\gamma}^*$ so that (i) $f(\underline{\gamma},\underline{\lambda})$ is in the interior of the same $\tilde{G}_N$ class and (ii) the convex combination of all the β_{N-1} weight vectors lies in the interior of the $\tilde{G}_{N-1}$ class corresponding to α_{N-1}. Also, the vectors $\underline{\lambda}$ and $\underline{\gamma}$ can be selected to have rational components.

The normalization of the vectors $\underline{\lambda}$ and $\underline{\gamma}$ differs from that of Theorem 1. Find a rational equivalent of $\underline{\lambda}$ and $\underline{\gamma}$ so that the following are satisfied.

(i) $\gamma_1,\ldots,\gamma_{(N-1)!}$ have a common denominator

(ii) Let γ_k correspond to the kth class of $\tilde{G}_N$. Let $\alpha(k)$ be the preference ordering in $\tilde{P}_{N-1}$ corresponding to this class, and let $S_{\alpha(k)}$ be the lift (with respect to Π_P) to $\tilde{P}_N$ of $\alpha(k)$. Corresponding to the N classes in $S_{\alpha(k)}$ are N classes in $\tilde{G}_N$ and N components of $\underline{\lambda}$. The numerator of γ_k serves as the common denominator for all N of these λ_i^j's.

The number of voters in the example is given by the common denominator of the γ_k's. The numerator of each γ_k corresponds to the number of voters with the kth $\tilde{P}_{N-1}$ preference rating. The numerator of the λ_i^j's given by the above construction corresponds to the number of voters in each $\tilde{P}_N$ class. By construction, the conclusion of the theorem holds.

The proofs of the preceding theorem used subsets of preferences large enough so that the conclusions can be obtained. Larger subsets of weight vectors lead to the same conclusion. So, another immediate extension would be to extend the weight vectors from $\tilde{P}_N$ to P_N and to allow the number of weight vectors in each class to increase. This would then include most systems commonly used as approval voting, cummulative voting, cardinal preference voting etc.

Theorem 3. Let β_N, β_{N-1} be summation weighed voting systems where the weight vectors include subsets satisfying Definition 5. Then for these voting systems the conclusions of Theorem 2 hold.

It turns out that even Definition 5 can be extended to allow classes in P_N to be assigned to classes of G_N, where $\tilde{P}_N$ need not be mapped to $\tilde{G}_N$. However, this extension doesn't seem to shed any new light about the structure nor does it seem to include voting systems of any practical interest, so it is not discussed here.

The symmetry imposed upon the components of the weight vectors isn't required, but it does simplify the exposition. All that is required for mapping f defined in the proof of Theorem 2 is that (i) there is a $\underline{\gamma}^*$ in the interior of $P((N-1)!)$ such that if its coefficients are the scalar convex weights for the β_{N-1} weight vectors then the outcome is the $\tilde{P}_{N-1}$ indifference point, and (ii) $f(\underline{\gamma}^*, \underline{\lambda})$ is an open set in G_N which includes the complete indifference point as an interior point. The symmetry of the weight vectors and the rotational properties of the Condorcet N-tuples simplified the demonstration that f has these properties. However, if weight vectors are assigned to the interior of sets of $\bar{G}_N$, even with no assumption of symmetry, it can be shown that similar conclusions always hold except now the missing alternative may need to be specified. See Figure 3 to see why the geometry forces this in P(3). To avoid f having the above properties, the convex hull of the lifts must be lines perpendicular to the base lines; but it is impossible to choose weights for this to be true with respect to all three bases of the simplex P(3).

In the context of this paper, Black's single-peakedness condition [3] can be viewed as a restriction of the preferences to classes of preferred orderings so that, with these classes, a $\underline{\gamma}^*$ cannot be found with the above property. Clearly, this restriction must not include any Condorcet N-tuple. An obvious extension of this is to allow voters to select any preference orderings but to restrict the percentage of voters in certain classes. This changes the domain of f from $P((N-1)!) \times (P(N))^{(N-1)!}$ to $\mathcal{D}$, a closed subset of it. The actual restrictions are imposed so that f will not have the above properties of $f(\gamma^*,\lambda)$ meeting the complete indifference point. This is a linear programming problem. For purposes of asserting the result is a linear ordering, these restrictions need to be imposed with respect to all binary classes.

Finally, a natural question is to determine bounds on the number of voters required for various examples to occur, or for them not to occur [2].

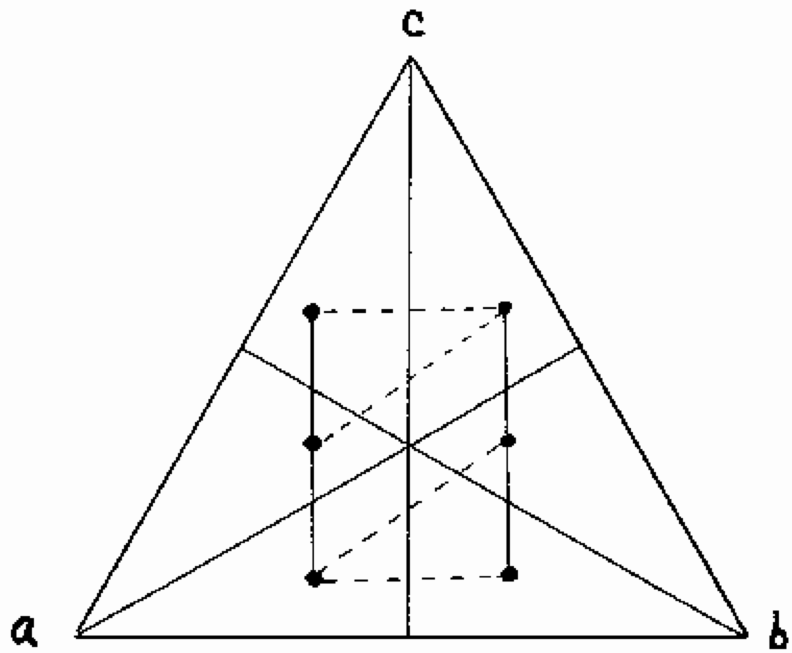


FIGURE 3. Paradoxes cannot be avoided. To ensure that the (a,b) outcome is always consistent with the P_3 outcome, the P_3 weight vectors (heavy dots) must be selected so that the convex hulls of the (a,b) lifts are lines parallel to the indifference line a = b. But then the convex hulls of the (b,c) lifts define open sets (the two triangles determined by the dotted lines); consequently the (b,c) outcome need not correspond to the P_3 outcome.

For an example to occur is to require $f(\underline{\gamma},\underline{\lambda})$ to lie in a specified set of G_N, which is described by linear constraints, while $\Sigma\, \gamma_k \underline{w}_k^{(N-1)}$ lies in a specified set of G_{N-1}. This is a linear programming problem where the minimum number of voters is determined by the smallest common denominator for $\underline{\gamma}$ in the feasible set after the normalization process. This same argument holds for determining restrictions which ensure that the group ordering is a linear ordering. Also, the probability that these examples will occur for large numbers of voters, M, can be computed in the same way. It is clear that when classes of $P_N - \tilde{P}_N$ are excluded then the conclusions of Theorem 1 and of Theorem 3 occur with positive probability as $M \to \infty$. This is because the feasible set for these examples is an open set and the examples correspond to all rational points in this set [2].

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