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A UNIFICATION AND GENERALIZATION OF THE EAVES AND KOJIMA FIXED POINT REPRESENTATIONS OF THE COMPLEMENTARITY PROBLEM

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Abstract. Although the mappings used in the Eaves and Kojima fixed point representations appear to be different, this paper shows that they are essentially the same -- a unification that is accomplished via a geometric programming argument in the more general setting of the "geometric complementarity problem".

Key words. Complementarity problem, fixed point problem, dual cones, geometric programming.

CONTENTS

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1. Introduction. Given a mapping f from a subset of R^n into R^n, the corresponding complementarity problem consists of finding all solutions (x, y) to the conditions

$$x \geq 0 \qquad \text{and} \qquad y \geq 0$$
$$0 = \langle x, y \rangle$$
$$y = f(x).$$

Eaves [4] characterized the solutions to this problem as essentially the fixed points for a vector variable z under a mapping defined by minimizing $\|x - z\|$ subject to $x \geq 0$. He then used the resulting fixed point representation, along with Browder's fixed point theorem [1], to establish the basic theorem of complementarity.

Subsequently, Kojima [14] characterized the same solutions as the fixed points for a vector variable z under a mapping defined by expressing z as $x - y$ for some $x \geq 0$ and some $y \geq 0$ for which $0 = \langle x, y \rangle$. He then used the resulting fixed point representation to extend the basic theorem of Eaves and unify it with other existence theorems.

The key to relating the Eaves and Kojima representations is a demontration that the Eaves minimization produces the Kojima orthogonal decomposition, and vice versa. Such a demonstration extends the well-known fact that minimizing $\|x - z\|$ over a given vector space $\mathcal{X}$ in R^n amounts to expressing z as $x - y$ for some $x \in \mathcal{X}$ and some $y \in \mathcal{Y} \triangleq \mathcal{X}^{\perp}$.

Actually, these two facts are special cases of a theory that deals with arbitrary closed convex dual cones $\mathcal{X}$ and $\mathcal{Y}$ in R^n. In addition to relating the Eaves and Kojima representations, this theory leads to fixed point representations of the geometric complementarity problem.

2. The geometric complementarity problem. Given a relation Γ on R^n (i.e., a

subset of $R^n \times R^n$), and given both a closed convex cone $\mathcal{X}$ in R^n and its "dual" $\mathcal{Y} \triangleq \{y \in R^n | 0 \le \langle x, y \rangle$ for each $x \in \mathcal{X}\}$, consider the following important problem.

Problem $\mathcal{C}$. Calculate all solutions (x, y) to the conditions

(I) $x \in \mathcal{X}$ and $y \in \mathcal{Y}$

(II) $0 = \langle x, y \rangle$

(III) $(x, y) \in \Gamma$.

We term this problem the geometric complementarity problem because of its intimate connections with (generalized) geometric programming. In particular, when $\Gamma = \{(x, y) \in R^n \times R^n | y = \nabla g(x)\}$ for some objective function g, both [23] and section 3.1.1 of [21] show that the resulting solutions (x, y) provide all "critical solutions" x to the resulting geometric programming problem $\mathcal{A}$. Moreover, when $\Gamma = \{(x, y) \in R^n \times R^n | y \in \partial g(x)\}$ for the same objective function g, both [24] and section 3.1.4 of [21] show that the resulting solutions (x, y) provide all "primal and dual optimal solution pairs" (x, y) if the resulting "geometric dual" problems $\mathcal{A}$ and $\mathcal{B}$ have no "duality gap". (It is worth noting that sections 3.3.1 and 3.3.4 of [21], as well as [23] and [24], contain the seeds for an even more general geometric complementarity problem -- one with explicit constraint functions. However, since the significance of that problem within optimization is already well documented by the preceding references, and since its relevance outside of optimization is not yet known, there is not yet any compelling reason to formalize it and further study it.)

Problems $\mathcal{C}$ for which there are no "variational principles" (i.e., for which Γ has neither the preceding gradient form that leads to an equivalent optimization problem nor the preceding subgradient form that leads to an equivalent pair of dual optimization problems) frequently arise as alternative formulations of "equilibrium problems" (as indicated, for example, in [9] and [22]). In independent work, problems $\mathcal{C}$ of that general type were first studied by Saigal [26], who had been stimulated by the work of Karamardian [10,12]. Karamardian had assumed that $\Gamma = \{(x, y) \in R^n \times R^n | y = f(x)\}$ for some mapping f, and had been motivated by the work of Habetler and Price [8], who had further assumed that $\mathcal{X}$ is both "pointed" and "solid". Habetler and Price were generalizing earlier work on the original complementarity problem -- namely, problem $\mathcal{C}$ with both $\mathcal{X}$ and $\mathcal{Y}$ being the non-negative orthant R^n_+ (a closed convex polyhedral cone that is self-dual and both pointed and solid). That earlier work and its offshoots was performed by Lemke [15], Cottle [2], Cottle and Dantzig [3], Eaves [5], Merrill [17], Saigal [25], Eaves and Saigal [6], Karamardian [11], Saigal and Simon [27], Moré [18, 19], Fisher and Gould [7], Kojima [13], and Megiddo and Kojima [16].

3. <u>The key theorem</u>. The following problem essentially generalizes the Eaves minimization problem.

<u>Problem</u> $\mathcal{M}(z)$. For the vector z from R^n calculate a corresponding vector x that minimizes $(1/2)\|x - z\|^2$ over $\mathcal{X}$.

The following problem generalizes the Kojima orthogonal decomposition problem.

<u>Problem</u> $\mathcal{D}(z)$. For the vector z from R^n calculate corresponding vectors x and y in R^n so that

(I) $\quad x \in \mathcal{X} \quad$ and $\quad y \in \mathcal{Y}$

(II) $$0 = \langle x, y \rangle$$

(III′) $$z = x - y.$$

The following theorem summarizes the requisite facts about problems $\mathcal{M}(z)$ and $\mathcal{B}(z)$.

Theorem 0. Problem $\mathcal{M}(z)$ has a unique solution x; and problem $\mathcal{B}(z)$ has a unique solution (x, y), with y determined from both z and x by relation (III′).

Proof. The closedness of $\mathcal{X}$ and the continuity and unboundedness of $\|\cdot\|$ obviously imply the existence of a solution x to problem $\mathcal{M}(z)$. Moreover, that solution x is unique because of the convexity of $\mathcal{X}$ and the strict convexity of $(1/2)\|\cdot - z\|^2$.

Now, problem $\mathcal{M}(z)$ is a geometric programming problem of the general type defined in section 2.1 of [21]. Moreover, differentiation of $(1/2)\|\cdot - z\|^2$ shows that the defining conditions for problem $\mathcal{B}(z)$ are just the appropriate optimality conditions for problem $\mathcal{M}(z)$, as described in section 3.1.1 of [21]. In fact, the convexity of both $\mathcal{X}$ and $(1/2)\|\cdot - z\|^2$ implies, via Theorem 1 in section 3.1.1 of [21], that the solutions x to problem $\mathcal{B}(z)$ are identical to the solutions x to problem $\mathcal{M}(z)$. Consequently, problem $\mathcal{B}(z)$ has a unique solution (x, y), with y determined from the unique solution x to problem $\mathcal{M}(z)$ by relation (III′). q.e.d.

It should be noted that the second assertion of Theorem 0 generalizes the work of Moreau [20], who had assumed that both $\mathcal{X}$ and $\mathcal{Y}$ are pointed and solid while using a different proof.

4. The main result. In view of Theorem 0, the given relation Γ determines another relation

$$Z \triangleq \{(z, \zeta) \in R^n \times R^n \mid (x, x - \zeta) \in \Gamma \text{ for the unique solution } x \text{ to problem } \mathcal{M}(z)\}$$

$$= \{(z, \zeta) \in R^n \times R^n \mid (x, x - \zeta) \in \Gamma \text{ for the unique solution } (x, y) \text{ to problem } \mathcal{B}(z)\}.$$

Associated with the realtion Z is the following fixed point problem.

Problem $\mathcal{F}$. Calculate all fixed points z for the relation Z; that is, calculate all solutions z to the condition

(IV) $$(z, z) \in Z.$$

Problem $\mathcal{F}$ with Z specified by the defining formula is essentially a direct generalization of the Eaves fixed point problem, while problem $\mathcal{F}$ with Z specified by the second formula is a direct generalization of the Kojima fixed point problem.

The following theorem establishes the equivalence between problems $\mathcal{C}$ and $\mathcal{F}$.

Theorem 1. Each solution (x, y) to problem $\mathcal{C}$ produces a solution $x - y$ to problem $\mathcal{F}$; and each solution z to problem $\mathcal{F}$, along with the unique solution x to problem $\mathcal{M}(z)$, produces a solution $(x, x - z)$ to problem $\mathcal{C}$.

Proof. Given a solution (x, y) to problem $\mathcal{C}$, let $z \triangleq x - y$. Then, $y = x - z$, and hence $(x, x - z) \in \Gamma$. Moreover, (x, y) solves problem $\mathcal{B}(z)$; so Theorem 0 asserts that x is the unique solution to problem $\mathcal{M}(z)$. Consequently, $(z, z) \in Z$, and thus z is a solution to problem $\mathcal{F}$.

Conversely, given a solution z to problem $\mathcal{F}$, along with the unique solution x to problem $\mathcal{M}(z)$, let $y \triangleq x - z$. Then, $(z, z) \in Z$, and hence $(x, y) \in \Gamma$. Moreover, Theorem 0 asserts that (x, y) is the unique solution to problem

$\mathcal{D}(z)$. Consequently, (x, y) is a solution to problem $\mathcal{C}$. q.e.d.

In subsequent papers, our fixed point representation(s) of the geometric complementarity problem will be used to help extend known theory about the original complementarity problem.

Finally, it is worth noting that there is at least one more fixed point representation of the geometric complementarity problem -- a representation due to Saigal [26]. However, it has no direct relation to the fixed point representations discussed here.

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