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The Allocation of Public Goods with Sealed-Bid Auctions: Some Preliminary Evaluations

by

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0. INTRODUCTION

It is now well known that it is impossible to design a mechanism, for making collective allocation decisions, which is informationally decentralized, non-manipulable, and Pareto-optimal. This impossibility has been adequately demonstrated in the work of Gibbard [4] and Satterthwaite [22] in the context of social choice theory, in the work of Hurwicz [11] in the context of resource allocation with private goods, and in the work of Green and Laffont [6], Hurwicz [12], Roberts [21] and Walker [27] in the context of resource allocation with public goods. Thus, if collective decisions are to be made, it is necessary to consider 'next best' mechanisms which either are manipulable or not Pareto-optimal (or both), and many such mechanisms have been discovered.

Mechanisms which preserve non-manipulability at the cost of optimality can be found in the work of Vickery [25] in the context of private goods (the second price auction is one) and in the work of Clarke [2] and Groves [8] in the context of public goods (<u>if the income elasticity of demand for public goods is identically zero</u>). Green and Laffont [5] have shown that these are the only non-manipulable mechanisms in the public goods context.

There are many mechanisms which preserve Pareto-optimality at the cost of non-manipulability (such as the competitive mechanism in the context of private goods); however, interest has centered on those which preserve 'some degree' of non-manipulability. In particular, a wide variety of mechanisms which have the property that Nash equilibria are Pareto-optimal have been discovered. These can be found in the work of Hurwicz and Schmeidler [17], and Maskin [18] in the context of social choice theory, and in the work of Hurwicz [13] and Schmeidler [23] in the context of resource allocation with private goods, and in the work of Groves and Ledyard [10], Hurwicz [13], and Walker [26] in the context of public goods. There are undoubtedly many others. There does not seem to be any obvious characterization of the class of mechanisms which have Pareto-efficient Nash equilibria although Hurwicz [15] has shown, under some convexity and continuity assumptions, that if one adds the requirement that the mechanism leave no one worse off than they are at their initial endowment then any such mechanism must produce the same outcomes as the Lindahl mechanism in the context of a public goods model.

The incredible variety of 'next best' mechanisms leads one to consider properties, other than optimality of Nash equilibria, in an effort to distinguish among them. In this paper we will concentrate on the analysis of collective choice or resource allocation mechanisms with respect to only one of the many possible additional dimensions of performance: performance under incomplete information. This property was chosen because of its importance with respect to the practical issue of whether or not a particular mechanism is implementable. If a mechanism is non-manipulable, then all individuals have a dominant strategy; that is, they can select their optimal response solely on the basis of their information about their own characteristic (preferences, endowments, etc.) and independently of the responses of all others. Thus, if the mechanism is also Pareto-optimal, then an optimal decision can be generated in one iteration whether or not agents have information about the responses or characteristics of others. It is for this reason that non-manipulability is so desirable and, also, unattainable in the presence of the requirement of Pareto-optimality. On the other hand, if mechanisms whose Nash equilibria are Pareto-optimal are implemented then, assuming all agents would like to be at the Nash equilibrium, some process[1/] for its discovery through information transfer must also be proposed. Of course, once this process is formalized agents may not act in the manner hypothesized and may cause the process to arrive at something other than a Nash equilibrium. Thus, if one wishes outcomes to be Pareto-optimal, consideration of the dynamic process in addition to the mechanism is important.[2/]

[1/] I use the word 'process' to indicate the full procedure used to arrive at the collective decision, while the term 'mechanism' is used to denote solely the outcome rules.

[2/] Perhaps the most striking example of the facts, that (1) implementation requires specification of an adjustment process (or institution for information transfer) and (2) outcomes may be different from those predicted as Nash-equilibria, can be found in Smith [24] whose auction mechanism seems to arrive rather consistently in experiements at Lindahl equilibria in spite of the fact that there are a multiplicity of Nash equilibria, many of which are not optimal.

One extreme specification of a dynamic process is to consider what would happen if only one iteration were allowed. That is, all agents must select their strategy (message) once and may not alter it on the basis of the observed responses of others. An example of this type of one-iteration process is the sealed-bid auction. Since implementation of mechanisms may be feasible only with a few iterations (as, for example, the Soviet planning systems), if a mechanism performs well as a one-iteration process, it may be of more interest than one which doesn't. Further, consideration of one-iteration mechanisms highlights the impact of incomplete information on the performance of these mechanisms since even a tatonnement process is a one-iteration process at its last iteration. Thus, dynamic processes can be viewed simply as a sequence of one-iteration processes.

In this paper, I analyze case by case the performance as sealed-bid auctions of twelve specific mechanisms designed to allocate public goods. In order to be able to solve explicitly for the optimal bidding strategies of consumers and, thereby, the specific outcomes generated by each mechanism, I have been forced to consider an especially limited range of utility functions. In particular, I deal only with risk-neutral consumers and utility functions with no income effects where all agents have linear marginal willingness to pay functions with identical slopes--only the intercepts vary. Although this severely limits the conclusions one can draw about the performance of these mechanisms in general, it does provide some indications and is a basis for future work.

A summary of results is presented in Section 7 .

1. PRELIMINARIES

We consider a model with n consumers (indexed by $i = 1, \ldots, n$), one private good and one public good. We let y be the amount of public good produced and consumed and let x^i be the amount of the private good consumed by i. An allocation is $z = (x^1, \ldots, x^n, y)$. Each consumer's utility function will be of the special form $u^i(x^i,y) = x^i + v^i y - \frac{K}{2} y^2$. Further, each consumer is endowed with an initial quantity, w^i, of the private good. Finally production is assumed to be linear: to produce one unit of the public good requires q units of the private good.

K, q, and the functional form of u^i are assumed to be fixed and known. Thus, each consumer is completely characterized by the pair $(v^i, w^i) = e^i$ and an economy is completely characterized by the n-truple $= (e^1, \ldots, e^n)$.

Of interest in this paper are Pareto-optimal allocations and the (social) loss which occurs if some other allocation is chosen. For the special model above, it is very easy to characterize Pareto-optimal allocations since there is a unique Pareto-optimal level of the public good, y^0, for each economy e.

Lemma 1: In the economy $e = (v^1, w^1, \ldots, v^n, w^n)$, the allocation $\overset{0}{z}(e) = (\overset{0}{x}{}^1, \ldots, \overset{0}{x}{}^n, \overset{0}{y})$ is Pareto-optimal if and only if[3/]

3/ All sums are taken from $i = 1$ to n unless otherwise specified.

$$(1.1) \qquad \overset{0}{y} = \frac{\Sigma v^i - q}{nK}$$

and

$$(1.2) \qquad \Sigma \overset{0}{x}{}^i + q \overset{0}{y} = \Sigma w^i .$$

As we will see below, almost all mechanisms, if operated as sealed-bid auctions, will not in general select a Pareto-optimal allocation. A natural issue then is the extent of the loss which arises. Since our special model involves only utility functions which are linear in the private good, a natural measure of social welfare is $W(z) = \Sigma u^i(x^i,y)$. Thus, a natural measure of the loss which occurs when, in the economy e , z is chosen instead of $\overset{0}{z}(e)$ is $L(\overset{0}{z}(e),z) = \Sigma u^i(\overset{0}{x}{}^i,\overset{0}{y}) - u^i(x^i,y)$.

<u>Lemma 2</u>: In the economy e , the welfare loss due to the allocation z is

$$(1.3) \qquad L(z;e) = [\, \Sigma (\overset{0}{x}{}^i - x^i)] + q(\overset{0}{y} - y) + \frac{nK}{2} (\overset{0}{y} - y)^2 .$$

If we let $S(z)$ be the amount of the private good which is collected from consumers for the production of the public good but not used then we can rewrite (1.3) as

$$(1.4) \qquad L(z;e) = S(z) + \frac{nK}{2} (\overset{0}{y} - y)^2 ,$$

where

$$\Sigma x^i + qy + S(z) = \Sigma w^i .$$

2. MECHANISMS AND AUCTIONS

An allocation mechanism is modeled simply as a language (set of messages), M , and an outcome rule, h , associating messages with allocations; that is, $h(m^1, \ldots, m^n) = z$. For public goods economies, allocation mechanisms have usually been formalized as consisting of two parts: (1) an allocation rule, $y(m^1, \ldots, m^n) = y$ and (2) a tax rule, $T^i(m^1, \ldots, m^n) = T^i$. In this form $h(m^1, \ldots, m^n) = [w^1 - T^1(m), \ldots, w^n - T^n(m), y(m)]$. A crucial property a mechanism must possess, if it is to be operable, is <u>weak feasibility</u>. That is, it must be true that

$$(2.1) \qquad \Sigma T^i(m) \geq qy(m) \ .$$

Since enough private good must be collected from consumers to pay for the public good, we will consider only mechanisms which satisfy (2.1).

Although there are a variety of methods for implementing any given mechanism, we are particularly interested in this paper in the performance of various mechanisms as sealed-bid auctions. That is, we wish to consider what allocations and losses may occur under a particular allocation mechanism if all consumers, knowing only their own characteristic $e^i = (w^i, v^i)$, submit a sealed-bid (message) m^i and then the mechanism is used to determine the allocation $z = h(m^1, \ldots, m^n)$. We let $\delta^i : E^i \to M$ be consumer i's bidding strategy where i bids $\delta^i(e^i) = m^i$ if his characteristic is e^i . A mechanism h will then produce the outcome z(e) in the economy $e = (e^1, \ldots, e^n)$ where $z(e) = h[\delta^1(e^1), \ldots, \delta^n(e^n)]$. It is thus very important to determine how consumers select their bidding strategies.

In this paper we accept and adopt the theory of bidding behavior which has become standard in the analysis of auctions of private goods. Examples can be found in the papers of Vickery, Wilson, and Milgrom. For our purposes we begin by assuming that all consumers believe that each consumer's characteristic is an independently and identically distributed random variable whose distribution is known to all.

Informational Postulate: Each consumer is imperfectly informed about the true environment e. Each i knows his own characteristic e^i but believes that the e^j for $j \neq i$, $j = 1, \ldots, n$ are independently and identically distributed random variables with density function $f(e^j)$. Thus, all i have the same beliefs.

Each consumer will select a bidding strategy which is best given the strategies of others. Thus, given δ^j for all $j \neq i$, consumer i will choose for each e^i the message $m^i = \delta^i(e^i)$ which maximizes[4/]

$$\mathcal{E}[u^i(y(\delta(e)/m^i), x^i(\delta(e)/m^i); e^i)]$$

where $(\delta(e)/m^i) = [\delta^1(e^1), \ldots, \delta^{i-1}(e^{i-1}), m^i, \delta^{i+1}(e^{i+1}), \ldots, \delta^n(e^n)]$.

A vector of bidding strategies $(\delta^1, \ldots, \delta^n)$ such that each δ^i is best against the other δ^j is a Bayes-equilibrium. Since we have assumed that all consumers are identical *ex ante*, the *symmetric Bayes*

[4/] $\mathcal{E}$ is the expected value operator taken with respect to the known probability distribution on $(e^1, \ldots, e^{i-1}, e^{i+1}, \ldots, e^n)$.

equilibrium in which all bidding strategies are identical, $\delta^i = \delta$ for $i = 1, \ldots, n$, is of particular interest.

Behavioral Postulate: Given the mechanism $h : M^1 \times \ldots \times M^n \rightarrow Z$ each consumer uses the symmetric Bayes equilibrium bidding strategy $\delta_h : E^i \rightarrow M^i$ where $\delta_h(e^i)$ maximizes with respect to m^i,

$$\mathcal{E}[u^i[h(\delta_h(e^1), \ldots, \delta_h(e^{i-1}), m^i, \delta_h(e^{i+1}), \ldots, \delta_h(e^n)), e^i]] .$$

With this behavioral postulate one can compute the outcome which will occur in the economy e under the mechanism h to be $z(e,h) = h[\delta_h(e^1), \ldots, \delta_h(e^n)]$.

Since the rest of the paper contains many illustrations of the postulate and its implications we turn now to the analysis of several mechanisms as sealed bid auctions.

(m)]y(m)

3. A MECHANISM WITH DOMINANT STRATEGIES

The first mechanism we propose to investigate has the property that each agent has a best bid (message) which is independent of the others' bidding strategies. This mechanism was proposed by Clarke, Groves, and Vickery (for private goods). In their more general frameworks, messages were demand functions; however, for the limited framework in this paper we can let messages be real numbers. That is, if agent i sends m^i, he is treated as if he sends the demand (marginal willingness to pay) function $m^i - Ky$. For the general mechanism, if $\varphi^i(y)$ is i's message, the outcome rule is:

$$(3.1\text{ a})\qquad y(\varphi^1, \ldots, \varphi^n) \text{ maximizes } \Sigma\varphi^i(y) - qy$$

and

$$(3.1\text{ b})\qquad x^i(\varphi^1, \ldots, \varphi^n) = w^i - T^i(\varphi^1, \ldots, \varphi^n)$$

where $$T^i(\varphi^1, \ldots, \varphi^n) = \frac{1}{n}\, qy(\varphi^1, \ldots, \varphi^n) + \max_y[\Sigma_{j\neq i}(\varphi^j(y) - \frac{1}{n}\, qy)] - \Sigma_{j\neq i}[\varphi^j(y(\varphi^1, \ldots, \varphi^n)) - \frac{1}{n} qy(\varphi^1, \ldots, \varphi^n)] .$$

For our simpler framework, since $\varphi^i(y) = m^i y - \frac{K}{2}y^2$, these rules reduce to

$$(3.2\text{ a})\qquad y(m) = \frac{(\Sigma m^j) - q}{nK}$$

and

(3.2 b) $$x^i(m) = w^i - T^i(m) \quad \text{where}$$

$$T^i(m) = \frac{1}{n} qy(m) + [(\Sigma_{j\neq i} m^j - \frac{1}{n}q) - \frac{n-1}{2} Ky^i(m)]y^i(m) - [(\Sigma_{j\neq i} m^i - \frac{1}{n} q) - \frac{n-1}{2} Ky^i(m)]y^i(m)$$

where $$y^i(m) = \frac{\Sigma_{j\neq i}(m^i - \frac{1}{n} q)}{(n-1)K} .$$

It takes a bit of manipulation but (3.2 b) can be simplified to

(3.2 b') $$T^i(m) = \frac{1}{n} qy(m) + \frac{(m^i - \overline{m})^2}{2(n-1)K}$$

where $$\overline{m} = \frac{\Sigma m^i}{n} .$$

From previous work it is known that $m^i = V^i$ is a dominant strategy for agent i ; that is, no matter what messages others send, i will be best off, ex post, if i 's message is v^i . From this it is a simple step to

Proposition 3.1: For the demand revealing mechanism (3.2),

(a) the symmetric Bayes-equilibrium bidding strategy is $\delta(v^i) = v^i$,

(b) $y^0(v) - y(v) = 0$ for all v

(c) $S(v) = \Sigma T^i(\delta(v)) - qy(v) = \frac{1}{2(n-1)K} \Sigma(\overline{v} - v^i)^2$

(d) $L(z(e),e) = L(y^0(v), x^0(v), y(\delta(v)), x(\delta(v)) = \frac{1}{2(n-1)K} \Sigma(\overline{v} - v^i)^2$.

Conclusion (d) is the result of the well-known property of the Demand revealing mechanism that a surplus in taxes must be collected in order to preserve the dominant strategy property. Thus, even though y(v) is 'optimal' by (b), the allocation z(v) is not necessarily Pareto-optimal.

Since we do not know ahead of time what $(v^1, \ldots, v^n)$ will be, in order to evaluate the loss L(v) we consider the fact that the v^i are independently and identically distributed. Thus, the loss, L(v) , is a random variable.

Proposition 3.2: If v^i is normally distributed with mean μ and variance σ^2 then for the Demand revealing mechanism (3.2)

(a) $\frac{2(n-1)K}{\sigma^2} L$ is Chi-squared distributed with $n-1$ degrees of freedom.

(b) $\mathcal{E}(L) = \sigma^2/_{2K}$

(c) $\mathrm{Var}(L) = \sigma^4/_{2(n-1)K^2}$.

For large n if v^i is distributed with mean μ and variance σ^2, then (a) - (c) hold assymptotically as $n \to \infty$.

Proof: (a) - (c) follow from the fact that $2K \cdot L$ is simply the sample variance of a random sample from a (normal) distribution with mean μ and variance σ^2 .

Before we turn to a discussion of the implications of proposition 3.2 , let us first analyse some other mechanisms so that comparisons can be made.

4. MECHANISMS WHOSE NASH EQUILIBRIA ARE PARETO-OPTIMAL

In this section we investigate a class of mechanisms which, while not generating dominant strategies, do have the property that the allocation generated at a Nash-equilibrium is Pareto-optimal. When these mechanisms are operated as sealed-bid auctions; however, Pareto-optimality is lost. In addition, as we will see below, the extent of the loss varies greatly across seemingly similar mechanisms.

A. The Basic Quadratic

One of the first mechanisms discovered which has the property that its Nash-equilibrium allocations are Pareto-optimal appears in the work of Groves and Ledyard [10] . In the world of one private and one public good, messages are real numbers. The allocation rules are

(4.1 a) $$y(m) = \Sigma m^i$$

(4.1 b) $$T^i(m) = \frac{1}{n} qy(m) + \frac{\gamma}{2}\left[\frac{n-1}{n}(m^i - \mu^i)^2 - \sigma_i^2\right]$$

where $\mu^i = \dfrac{\Sigma_{h \neq i} m^h}{n-1}$, $\sigma_i^2 = \dfrac{1}{n-2}\Sigma_{h \neq i}(m^h - \mu^i)^2$, and $\gamma > 0$ is an arbitrary constant.[5/]

[5/] An alternative form of these tax rules is $T^i(m) = \frac{1}{n} qy(m) + \delta\left[(m^i - \bar{m})^2 - \frac{1}{n}\Sigma\left[(m^h - \bar{m})^2\right]\right]$ where $\bar{m} = \frac{1}{n}\Sigma m^i$ and $\delta = \frac{n}{2(n-2)}\gamma$. See section 5.c below. Also compare to (3.2 b') above.

At this point the reader should notice a deficiency in the Basic Quadratic Rules which is also shared by many of the mechanisms we discuss below. In particular, no allowance is made for the fact that $\Sigma m^i < 0$ is possible. Thus (4.1 a) may lead to negative amounts of the public good being 'produced'. This is not a problem if one assumes that a Nash-equilibrium occurs. It is a problem if one is to use this mechanism as a sealed-bid auction since, as we will see, there will be a positive probability that $\Sigma m^i < 0$. The obvious solution to this problem is to change (4.1 a) to

$$y(m) = \begin{cases} \Sigma m^i & \text{if} \quad \Sigma m^i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

This alteration in the rules, however, leads to a significant alteration in Bayes-equilibrium bidding strategies. Indeed, in order to solve precisely for those strategies, one needs to solve a non-linear integral equation.[6/] Since I have been unable to solve that equation, I will not consider alternatives in $y(m)$. Instead I will allow negative y, act as if that made sense and indicate where possible the extent to which this might bias the reported results.

With this in mind I turn to

Proposition 4.1: For the Basic Quadratic Mechanism (4.1),

(a) the symmetric Bayes-equilibrium bidding strategy is

[6/] This equation is given for the Vacuous Mechanism, section 4 B. See the remark following Proposition 4.4.

$$\mathcal{S}(v^i) = A(v^i - \frac{1}{n} q) + B(\bar{v} - \frac{1}{n} q) \quad \text{where} \quad A = \frac{n}{nK + \gamma(n-1)} \quad ,$$

$$B = \frac{(n-1)(\gamma - nK)}{nK(nK + \gamma(n-1))} \ , \quad \text{and} \quad \bar{v} = \mathcal{E}(v) = \int vf(v)dv,$$

(b) $y^0(v) - y(v) = nB(\frac{\Sigma v^i}{n} - \bar{v})$,

(c) $S(v) = 0$

(d) $L(z(e),e) = \frac{n^2 B^2 K}{2} (\frac{\Sigma v^i}{n} - \bar{v})^2$.

Proof: Under the mechanism (4.1), each i chosses $\mathcal{S}^i(v^i)$ to maximize

$$\mathcal{E}[v^i y(m) - \frac{K}{2} y(m)^2 - T^i(m)] .$$

Thus, first order conditions are

$$0 = \mathcal{E}[v^i - Km^i - K(n-1)\mu^i - \frac{1}{n} q - \frac{\gamma(n-1)}{n} m^i + \frac{\gamma(n-1)}{n} \mu^i] .$$

Let $\bar{\mu} = \mathcal{E}(\mu^i) = \mathcal{E}(\frac{\Sigma_{h \neq i} m^h}{n-1}) = \mathcal{E}(m^h)$ where the last equality holds since the m^h are independently and identically distributed as $m(v^h)$.

Therefore,

$$(\ *\ ) \qquad m^i[K + \frac{\gamma(n-1)}{n}] = (v^i - \frac{1}{n} q) + \frac{n-1}{n} [\gamma - nK] \bar{\mu} \quad .$$

Since this equation must be true for all i and since the v^i are independently and identically distributed by taking the expected value of each side of the last equation and manipulating it follows that

$$\bar{\mu} = \frac{(\bar{v} - \frac{1}{n} q)}{nK} .$$

(a) follows from this and (*) . (b) - (d) can now be easily derived. Q.E.D.

Conclusion (d) is the vital fact. Although the Basic Quadratic Mechanism generates Pareto-optimal Nash-equilibrium allocations, it does not generate Pareto-optimal allocations when used as a sealed-bid auction. As opposed to the Demand revealing mechanism, the losses here are due entirely to the selection of non-optimal levels of the public good and not to an unutilized tax collection.[7]

As in section 3, $L(v)$ is a random variable since the v^i are.

Proposition 4.2: If v^i is normally distributed with mean μ and variance σ^2 then for the Basic Quadratic Mechanism (4.1)

(a) $\frac{1}{2\,n\,B^2 K\sigma^2} L$ is chi-squared distributed with 1 degree of freedom.

(b) $\mathcal{E}(L) = 2\,n\,B^2 K\sigma^2$

(c) $\mathrm{Var}(L) = 2\,n^2 B^4 K^2 \sigma^4$.

(a) - (c) hold assymtotically as $n \to \infty$ for any distribution on v^i with

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[7] To indicate the rather special nature of the class of economies we are considering, one can note that since K and n are known a priori, the parameter γ could be set so that $\gamma = nK$. If this were done then $B = 0$, each i has a dominant strategy, and $L(z(e),e) = 0$ for all e . We do not emphasize this fact, since it depends crucially on the special class of environments. If K were not known or if $K^i \neq K^j$ for some i and j , then one could not choose γ in this fashion.

mean μ and variance σ^2.

Proof: Follows immediately from the fact that $\left(\frac{\Sigma v^i}{n} - \bar{v}\right)^2$ is simply the variance of the sample mean of a random sample of size n from a (normal) distribution with mean $\mu(=\bar{v})$ and variance σ^2.

Of some later interest is the case[8/] in which $\gamma_0 = \gamma n^2$. For this choice of γ we have

Corollary 4.1.1: $$L(z(e),e) = \frac{n}{2K}\left[\frac{\gamma^0 - \frac{K}{n}}{\gamma^0 + \frac{K}{n(n-1)}}\right]^2 \left(\frac{\Sigma v^i}{n} - \bar{v}\right)^2$$

and

Corollary 4.2.1: (a) $\frac{2R}{K\sigma^2} L$ is chi-squared distributed with one degree of freedom.

(b) $\mathcal{E}(L) = \frac{R\sigma^2}{2K}$

(c) $\mathrm{Var}(L) = \frac{R^2\sigma^4}{2K^2}$

where $$R = \left[\frac{\gamma^0 - \frac{K}{n}}{\gamma^0 + \frac{K}{n(n-1)}}\right]^2 .$$

8/ Muench and Walker [20] identify this choice of γ as leading in the case of large numbers to good incentive properties but bad stability properties (under Cournot behavior). If the Basic Quadratic is used as a sealed bid auction mechanism, stability is a moot issue.

B. The Vacuous Mechanism

A particularly simple mechanism whose Nash equilibria are Pareto-optimal is the one which taxes everyone identically and proportionately for the public good. For this mechanism messages are real numbers,

(4.2a) $$y(m) = \Sigma m^i$$

(4.2b) $$T^i(m) = \frac{1}{n} q\, y(m) .$$

Unfortunately Nash equilibria rarely exist (for the special class of economies in this paper, existence occurs if and only if $v^i = v^j$ for all i, j). However, if this mechanism is used as a sealed-bid auction then symmetric Bayes equilibria do exist. Thus, it is of some interest to analyze its performance as an auction.

Proposition 4.3: For the Vacuous Mechanism (4.2),

(a) the symmetric Bayes-Equilibrium bidding strategy is $\delta(v^i) =$

$$\frac{1}{K}\left[(v^i - \frac{q}{n}) - \frac{(n-1)}{n}(\bar{v} - \frac{q}{n})\right] \text{ where } \bar{v} = \mathscr{E}(v^i) ,$$

(b) $$y^0(v) - y(v) = \frac{n-1}{K}\left(\bar{v} - \frac{\Sigma v^i}{n}\right)$$

(c) $$S(v) = 0$$

(d) $$L(z(e), e) = \frac{nK}{2}\left(\frac{n-1}{K}\right)^2 \left(\frac{\Sigma v^i}{n} - \bar{v}\right)^2 .$$

<u>Proof</u>: Under the mechanism (4.2) i wishes to maximize

$$\mathcal{E}[v^i y(m) - \frac{k}{2} y(m)^2 - \frac{1}{n} qy(m)].$$

First order conditions give

$$0 = \mathcal{E}(v^i - K(m^i + (n-1)\mu^i) - \frac{1}{n} q)$$

where $\mu^i = \sum_{h \neq i} m^h / (n-1)$.

Thus

$$0 = v^i - Km^i - \frac{1}{n} q - K(n-1)\mathcal{E}(\mu^i).$$

Taking expected values, one gets

$$0 = \bar{v} - K\bar{m} - \frac{1}{n} q - K(n-1)\bar{m}$$

or

$$\mathcal{E}(\mu^i) = \bar{m} = \frac{\bar{v} - \frac{q}{n}}{nK} .$$

Therefore, $v^i - Km^i - \frac{1}{n} q - \frac{(n-1)}{n} (\bar{v} - \frac{q}{n}) = 0.$ The rest follows easily.

Q.E.D.

<u>Remark</u>: As I indicated in Section 4.a, $y(m)$ may be negative.

For the Vacuous Mechanism $y(m)$

$$y(m) = [(\frac{\sum v^i}{n} - \bar{v}) + \frac{1}{n}(\bar{v} - \frac{q}{n})] \frac{n}{k}, \quad \mathcal{E}(y) = \frac{\bar{v} - \frac{q}{n}}{K} = y^o,$$

and $\mathrm{Var}(y) = (\frac{n-1}{K})^2 \frac{\sigma^2}{n}$. Therefore since $\mathrm{Var}(y) \to \infty$ as $n \to \infty$ the probability that $y(m) \leq 0$ approaches $\frac{1}{2}$. If we adjust y to

$$y(m) = \begin{cases} \sum m^i & \text{if } \sum m^i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

then first order conditions give

(*) $$0 = w - Km - (n-1)K\mathcal{E}[\mu \mid \mu \geq \frac{m}{m-1}] \quad .$$

The distribution of μ depends on the choice of the function $m(w)$ which must satisfy (*) for all w, Thus (*) is a non-linear integral equation for which the solution appears to be unknown. Analysis of the revised rules remains, therefore, an open question.

Returning to the original mechanism we can prove

Proposition 4.4: If v^i is normally distributed with mean μ and variance σ^2 then for the Vacuous Mechanism (4.2)

(a) $\frac{2K}{(n-1)^2\sigma^2} L$ is chi-squared distributed with 1 degree of freedom

(b) $\mathcal{E}(L) = \frac{(n-1)^2}{2K}\sigma^2$

(c) $\text{Var}(L) = \frac{(n-1)^4}{2K^2}\sigma^4$

(a) - (c) hold assymptotically as $n \to \infty$ for any distribution on v^i with variance σ^2.

C. The Paired-Difference Mechanism

One group of mechanisms, whose Nash equilibrium allocations are Pareto-optimal, have the additional property that their Nash allocations are also Lindahl equilibrium allocations. Among other things this means that no one is worse off than at his initial endowment. To see how these mechanisms perform as auctions, let us first consider one proposed by Walker [26], called the Paried-difference mechanism by Groves [9]. Messages are, again, real numbers. The allocation rules are

(4.3a) $$y(m) = \Sigma m^i$$

(4.3b) $$T^i(m) = [\frac{1}{n}q + m^{i+1} - m^{i-1}]y(m)$$

where $n+1 = 1$ and $1-1 = n$.

Let us look at the consumer's choice of bidding strategy. Consumer i wishes to maximize

$$\mathcal{E}[v^i y(m) - \frac{K}{2} y(m)^2 - (\frac{1}{n} q + m^{i+1} - m^{i-1}) y(m)].$$

First order conditions imply

$$0 = v^i - K(m^i + (n-1)\mathcal{E}(\mu^i)) - (\frac{1}{n} q + \mathcal{E}(m^{i+1}) - \mathcal{E}(m^{i-1})),$$

where $\mu^i = \Sigma_{h \neq i} m^h/(n-1)$. But $\mathcal{E}(\mu^i) = \mathcal{E}(m^{i+1}) = \mathcal{E}(m^{i-1}) = \bar{m}$. Thus $0 = v^i - Km^i - \frac{1}{n} q - K(n-1)\bar{m}$. This is identical to the first order conditions for the Vacuous mechanism! We have thus established

<u>Proposition 4.5:</u> Propositions 4.3 and 4.4 remain valid if we replace the Vacuous Mechanism (4.2) with the Paired-difference mechanism (4.3). That is, their performance as sealed-bid auctions is identical in the special class of environments considered. [9/]

[9/] This is because i cares only about $\mathcal{E}(m^{i+1}) - \mathcal{E}(m^{i-1})$. If i's utility function were not linear in x^i this would no longer be true. Thus, for example, if i were risk averse the Vacuous Mechanism and Paired-difference mechanism would perform differently as sealed-bid auctions. This is another observation which emphasizes the special nature of class of economies considered in this paper.

Remark: Proposition 4.5 states that a mechanism whose Nash equilibria rarely exist and a mechanism whose Nash equilibria allocations are Lindahl allocations are "isomorphic" as sealed-bid auctions. This fact illustrates why one can not infer performance of auctions directly from the properties of Nash equilibria.

D. The Shared-Cost Mechanism

The identical performance of the Vacuous and paired-difference mechanisms seems to lie in the fact that if other bidders follow identical strategies then the expected value of the (marginal) tax is identical, equal to $\frac{q}{n}$, in each. To see that there may be a deeper reason let us turn to another mechanism whose Nash allocations are Lindahl allocations. This mechanism was proposed [10/] by Hurwicz [13], and named the Shared-cost mechanism by Groves [9]. In contrast to all previous mechanisms we have considered messages are now 2-tuples; that is, $m^i = (y^i, a^i)$ The outcome rules are

(4.4a) $\quad y(m) = \Sigma y^i$

(4.4b) $\quad T^i(m) = (q - \Sigma_{j \neq i} a^j) y(m) + (\Sigma a^j - q)^2.$

It should be obvious that the expected marginal tax rate of this mechanism, $\mathcal{E}[q - \Sigma_{j \neq i} a^j]$, need not equal that of the vacuous mechanism, $\frac{q}{n}$, even when all other agents use identical bidding strategies. However, performance of the Shared-cost mechanism is identical to that of the Vacuous Mechanism.

10/ I have slightly modified the original mechanism, along lines similar to Groves [9], to make the outcome rules anonymous.

Proposition 4.6: Propositions 4.3 and 4.4 remain valid if we replace the Vacuous Mechanism (4.2) with the Shared-cost mechanism (4.4).

Proof: Consumer i wishes to maximize $\mathcal{E}[v^i y - \frac{K}{2} y^2 - (q - \Sigma_{j \neq i} a^i)y - (\Sigma a^j - q)^2$ by choosing y^i and a^i. The first order conditions give

(4.5) $$0 = -2\,[a^i + (n-1)\mathcal{E}(a^j) - q]$$

and

(4.6) $$0 = v^i - q + (n-1)\mathcal{E}(a^j) - K(m^i + (n-1)\mathcal{E}(m^j)).$$

From (4.5), by taking expected values and letting $\bar{a} = \mathcal{E}(a^j) = \mathcal{E}(a^i)$ we have that $n\bar{a} = q$ or $a^i = \frac{1}{n}q$ for all i and all v^i. Thus substituting into (4.6) and taking expected values gives $0 = v^i - \frac{1}{n}q\, Km^i (n-1)K\bar{m}$ where $\bar{m} = \mathcal{E}(m^j)$. This is identical to the first order conditions for the Vacuous Mechanism. Thus the symmetric Bayes-equilibrium building strategy for the shared cost mechanism is $\delta^i(v^i) = [\delta^*(v^i), \frac{1}{n}q]$ where $\delta^*(v^i)$ is the symmetric Bayes-equilibrium bidding strategy for the Vacuous government.

Q.E.D.

The identical performance of the Vacuous, Paired-difference, and Shared-cost mechanisms seems to occur because the latter two select Lindahl allocations and, therefore, assign Lindahl prices as marginal tax rates. Since all consumers are <u>ex ante</u> identical their expected marginal tax rates, at a symmetric Bayes-equilibrium, will be identical and therefore equal to q/n, the marginal rate of the Vacuous Mechanism. Thus one might conjecture that if utility functions are of the form [11/] $x^i + \varphi^i(y)$, then any mechanism whose Nash allocations are Lindahl allocations will perform identically to the

11/ That is, no income effects and risk-neutral behavior.

Vacuous Mechanism when used as a sealed-bid auction. 12/ Such a conjecture is, however, false as we will see by considering one more mechanism.

E. A Balanced Mechanism for $n \geq 2$

Although we have not emphasized the fact, all the mechanisms we have considered either require that $n \geq 3$ or have the property that $\Sigma T^i(m) \neq qy(m)$ for some messages. Hurwicz [16] has discovered a mechanism for which $\Sigma T^i(m) = qy(m)$ and Nash allocations are Lindahl allocations whenever $n \geq 2$. Messages are 2-tuples where $m^i = (y^i, p^i)$. The allocation rules are

(4.7a) $$y(m) = \begin{cases} \Sigma y^i & \text{if } \Sigma p^i = q \\ \Sigma p^i - q & \text{if } \Sigma p^i \neq q \end{cases}$$

(4.7b) $$T^i(m) = -p^i y(m).$$

One should notice immediately that these rules are not continuous. This has a significant effect on the performance of this mechanism as a sealed-bid auction. In particular, for a bidding strategy $\delta(v^i) = [\delta_y(v^i), \delta_p(v^i)]$, the allocation $y(v) = \Sigma_i \delta_y(v^i)$ if $\Sigma \delta_p(v^i) = q$ but $y(v) = \Sigma i\, \delta_p(v^i) - q$ when

12/ This is reinforced by observing that another mechanism proposed by Hurwicz [12] is consistent with the conjecture. That mechanism has $m^i = (y^i, p^i), y(m) = \Sigma y^i$, and $T^i(m) = -R_{in}y(m) - P_i(y_i - y_{i+1})^2 + P_{i+1}(y_{i+1} - y_{i+2})^2$ where $R_{in} = \frac{q}{n} + p_{i+1} - p_{i+2} + \ldots - p_j$, and $j = 1+n-1$ if n is odd, $j = 1+n-2$ if j is even. For this mechanism, Nash allocations are Lindahl allocations and performance as an auction is identical to the Vacuous Mechanism since $\delta(v^i) = [\delta^*(v^i), 0]$.

$\Sigma\delta_p(v^i) - q \neq 0$. For bidding strategies which are one to one functions, the probability that $\Sigma\delta_p(v^i) = q$ is zero. Thus, consumers can safely act as if $y(v) = \Sigma_i\delta_p(v^i) - q$ for all v. That mechanism does not have Nash allocations which are Lindahl and, therefore, we should expect different performance as an auction. This observation is formalized in

<u>Proposition 4.7:</u> For the Hurwicz Mechanism (4.7)

(a) The symetric Bayes-equilibrium bidding strategy is $\delta(v^i) = (\delta_y(v^i), \delta_p(v^i))$ where δ_y is any arbitrary function and

$$\delta_p(v^i) = \frac{v^i}{K+1} + \frac{(K+1)q}{n(K+1)+1} - \frac{(n-1)(K+1)\bar{v}}{(K+2)[n(K+1)+1]}$$

(b) $S(v) = 0$ and $L(z(e),e) =$

$$\frac{n}{2K}\left[\frac{K+2-nK}{(K+2)}\left(\frac{\Sigma v^i}{n} - \bar{v}\right) + \frac{(n-1)\frac{q}{n} + (n+1)\bar{v}}{n(K+1)+1}\right]^2$$

(c) If v^i is distributed with mean μ and variance σ^2 then

$$\mathcal{E}(L) = \left(\frac{(n-1)K-2}{K+2}\right)^2 \frac{\sigma^2}{2K} + \frac{n}{2K}\left[\frac{(n-1)\frac{q}{n} + (n+1)\mu}{n(K+1)+1}\right]^2$$

(d) If v^i is normally distributed with mean μ and variance σ^2 then

$$\mathrm{Var}(L) = \left(\frac{(n-1)K-2}{K+2}\right)^4 \left(\frac{\sigma^2}{2K^2}\right)\left(\sigma^2 + 2n\left[\frac{(n-1)\frac{q}{n} + (n+1)\mu}{n(K+1)+1}\right]^2\right).$$ Also

(d) holds assymptotically as $n \to \infty$ if v^i is not normally distributed.

<u>Proof</u>: The first order condition for consumer i is

$$0 = v^i - K(a^i + (n-1)\mathcal{E}(a^j)) + a^i - (a^i + (n-1)\mathcal{E}(a^j) - q).$$

Thus, $\bar{a} = \mathcal{E}(a^j) = \frac{\bar{v} + (K+1)q}{n(K+1)+1}$. The rest follows easily except for, maybe, (d).

To compute (d) we note that $L = a(bz + c)^2 = ab^2z^2 + 2abcz + ac^2$ where a, b, c are constants and $z = \frac{\Sigma v^i}{n} - \mu$. Therefore, $\mathrm{Var}(L) = a^2[b^4\mathrm{Var}(z^2) + 4b^2c^2\mathrm{Var}(z) + \mathrm{Cov}\ (b^2z^2, 2bcz)]$. But $\mathrm{Var}\ (z^2) = \frac{\sigma^2}{2n}$, $\mathrm{Var}(z) = \frac{\sigma^2}{n}$ and $\mathrm{Cov}(b^2z^2, 2bcz) = \mathcal{E}(2b^3cz^3) = 0$ since z is normally distributed with zero mean. The rest follows.

Q.E.D.

It might seem reasonable at this point to compare the performance of the six mechanisms we have considered. Such a comparison is, however, more illuminating if we first look at several other mechanism which have neither dominant strategies nor, in general, Nash equilibria which are optimal.

5. OTHER MECHANISMS

To provide a basis for comparison we turn to an investigation of some mechanisms which are usually dismissed from consideration on the grounds that their equilibria are not optimal. As we have seen above, however, the equilibirum properties of a mechanism may not be a good predictor of its performance as a sealed-bid auction.

A. The Naive Mechanism

One mechanism whose Nash equilibria exist but which yield non-optimal allocations arises from assuming the public good is to be provided by a "private" market or by voluntary contributions. For this mechanism, messages are real numbers. The allocation rules are: 13/

13/ This is the market version. The voluntary contributions model is $y(m) = \frac{\Sigma m^i}{q}$ and $T^i(m) = m^i$.

(5.1a) $y(m) = \Sigma m^i$

(5.2b) $T^i(m) = qm^i$.

Proposition 5.1: For the Naive Mechanism (5.1)

(a) The symmetric Bayes-equilibrium bidding strategy is

$\delta(v^i) = \frac{1}{K}(v^i - q) - \frac{(n-1)}{nK}[\mu - q]$, where $\mu = \mathcal{E}(v^j)$

(b) $S(v) = 0$, $L(v) = \frac{n(n-1)^2}{2K}[(\mu - \frac{\Sigma v^i}{n}) + \frac{q}{n}]^2$ where $\mu = \mathcal{E}(v^i)$

(c) $\mathcal{E}(L) = (n-1)^2 [\frac{\sigma^2}{2K} + \frac{n}{2K}(\frac{q}{n})^2]$ where $\sigma^2 = Var(v^i)$,

(d) If v^i is normally distributed then $Var(L) = \frac{(n-1)^4\sigma^2}{2K^2}[\sigma^2 - 2n(\frac{q}{n})^2]$.

Also (d) holds assymptotically as $n \to \infty$ if v^i is not normally distributed.

Proof: The first order condition for i is

$$0 = v^i - K(m^i + (n-1)\mathcal{E}(m^j)) - q.$$

Thus $\mathcal{E}(m^j) = \frac{1}{nK}(\bar{v} - q)$. The rest follows.

Q.E.D.

B. An Auctioneerless Lindahl Mechanism

One mechanism which seems to be at the heart of early conjectures about free riders involves consumers reporting marginal willingness to pay schedules. The public good is produced at a level which equates the sum of the reported functions to the price. Then each consumer is charged at their reported marginal rate. For the economies we are considering, we can let messages, m^i, be real numbers and interpret m^i as if the consumer reported the marginal willingness to pay function, $m^i - Ky$. The allocation rules are:

(5.2a) $$y(m) = \frac{\Sigma m^i - q}{nK}$$

(5.2b) $$T^i(m) = (m^i - Ky(m))y(m).$$

Proposition 5.2: For the Auctioneerless Lindahl process (5.2)

(a) The symmetric Bayes equilibria strategy is $\delta(v^i) = \frac{n}{2n-1} v^i - (\frac{n-1}{2n-1})[\frac{n-1}{n}\mu - \frac{(2n-1)}{n^2} q]$ where $\mu = \mathcal{E}(v^i)$

(b) $S(v) = 0$, $L(V) = \frac{(n-1)^2}{2nK}[\frac{n}{2n-1}(\frac{\Sigma v^i}{n} - \mu) + (\mu - \frac{q}{n})]^2$,

(c) $\mathcal{E}(L) = \frac{(n-1)^2}{(2n-1)^2} \frac{\sigma^2}{2K} + \frac{(n-1)^2}{2nK}(\mu - \frac{q}{n})^2$ where $\sigma^2 = \text{Var}(v^i)$,

(d) If v^i is normally distributed,

$$\text{Var}(L) = \frac{(n-1)^4\sigma^2}{2(2n-1)^4K^2}[\sigma^2 + \frac{2(2n-1)^2}{n}(\mu - \frac{q}{n})^2].$$

Also (d) holds asymptotically as $n \to \infty$ if v^i is not normally distributed.

C. Demand Revealing Mechanism with Surplus Distributed

One mechanism which has been analyzed as an auction mechanism (Green, Kohlberg and Laffont [7])[14/] and which in general does not generate Nash equilibrium allocations which are Pareto-optimal is the Demand-revealing mechanism with the surplus distributed. Referring back to Section 3, one can

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[14/] Their analysis differs from ours in that each agent assumed the others' bidding strategies generated messages which were independently and identically normally distributed. If the v^i are normally distributed, their analysis would be equivalent to the Bayes-equilibrium hypothesis if and only if the Bayes equilibrium were a linear function of v^i; that is, $\delta(v^i) = av^i + b$ for all i. This is true for our special class of environments (see Proposition 5.3(a) below) but is not true in general.

remember that, for the Demand-revealing mechanism, $S(v) = \frac{1}{2(n-1)K} \Sigma (\bar{v} - v^i)^2$ is collected and not returned to the economy. Suppose, however, each agent were to receive $\frac{1}{n} S(m^1,\ldots,m^n)$ where $S(m)$ is the surplus generated if messages are m. This new mechanism [15] is

$$(5.3a) \qquad y(m) = \frac{\Sigma m^i - q}{nK}$$

$$(5.3b) \qquad T^i(m) = \frac{1}{n} qy(m) + \frac{(m^i - \bar{m})^2}{2(n-1)K} - \frac{1}{n} \Sigma \frac{(m^h - \bar{m})^2}{2(n-1)K} .$$

<u>Proposition 5.3:</u> For the mechanism (5.3),

(a) The symmetric Bayes equilibrium bidding strategy is

$$\delta(v^i) = \frac{n}{n-1} [(v^i - \frac{1}{n} q) - \frac{1}{n} (\mu - \frac{1}{n} q)]$$

where $\mu = \mathcal{E}(v^j)$,

(b) $S(v) = 0$, $L(v) = \frac{n}{2(n-1)^2 K} (\frac{2v^i}{n} - \mu)^2$

(c) $\frac{2(n-1)^2 K}{\sigma^2} L$ is Chi-squared distributed with 1 degree of freedom, where $\sigma^2 = Var(v^j)$, and

(d) $\mathcal{E}(L) = \frac{1}{(n-1)^2} \frac{\sigma^2}{2K}$, $Var(L) = \frac{\sigma^4}{2(n-1)^4 K^2}$.

It is important to notice that although the Nash equilibrium allocations of the Demand revealing mechanism with the surplus distributed are not in general Pareto-optimal, the mechanism (5.3) does generate optimal allocations. The reason is that in the limited class of economies considered in this paper, the mechanism (5.3) is the Basic Quadratic Mechanism (4.1) discussed in Section 4A. To see this we note first that if we let $\gamma = \frac{2(n-2)}{n} \delta$, then

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15/ Again we have replaced the general rules with those appropriate for our simple economies.

(4.1b) can be written as

$$(4.1b') \quad T^i(m) = \frac{1}{n}\, qy(m) + \delta\left[(m^i - \bar{m})^2 - \frac{1}{n}\, \Sigma (m^h - \bar{m})^2 \right] \quad \text{where} \quad \bar{m} = \frac{1}{n}\, \Sigma\, m^i \; .$$

Next, let $s^i = \dfrac{m^i - q_n}{n K}$ and consider s^i as a translation of the messages used in (5.3) . We then write (5.3) as

$$(5.3a') \qquad y(s) = \Sigma s^i$$

$$(5.3b') \quad T^i(s) = \frac{1}{n}\, q\, y(s) + \frac{n^2 K}{2(n-1)}\left[(s^i - \bar{s})^2 - \frac{1}{n}\Sigma (s^h - \bar{s})^2 \right] \quad \text{where} \quad \bar{s} = \frac{1}{n}\, \Sigma s^i \; .$$

Therefore, if we let $\gamma = \dfrac{n(n-2)K}{(n-1)}$ in equation (4.1b) and translate messages then (5.3) can be derived from(4.1) . To see that the performance of the Basic Quadratic (when $\gamma = \dfrac{n(n-2)K}{(n-1)}$) and the mechanism (5.3) are identical, notice that under the Basic Quadratic the loss is

$$L(v) = \frac{n^3 B^2 K}{2}\left(\frac{\Sigma v^i}{n} - \mu\right)^2 = \frac{n}{2(n-1)^2 K}\left(\frac{\Sigma v^i}{n} - \mu\right)^2$$ which is the same as in Proposition 5.3(b) .

If we had considered a broader class of economies, the two mechanisms, (4.1) and (5.3), would not perform identically. This again indicates the limited scope of our inquiry, and we again caution the reader not to infer too much from our results.

6. Mechanisms which use the information in $f(\cdot)$

A final group of mechanisms worth considering, for purposes of comparison with those above, involve the use of more information. In particular, none of the mechanisms considered to this point utilized any information other than the messages of the agents.[16/] However, in computing the statistical properties of losses and in formalizing the behavior of consumers it was assumed that the v^i were independently and identically distributed according to a distribution known to everyone. One question then is what happens if we allow the mechanism to also utilize this information.

A. Central Planning

We first consider what is in effect no mechanism. That is, we let a central agent choose y without consulting the consumers. In particular we assume the allocation is chosen as follows:

(6.1a) $y(f)$ maximizes $\mathcal{E}[\Sigma v^i y - \frac{nK}{2} y^2]$

(6.2b) $T^i(f) = \frac{1}{n} qy(f)$.

16/ In some cases we relied on the knowledge that $u^i = x^i + v^i y - \frac{K}{2} y^2$ for some v^i , but that was really not necessary.

The actual taxing rules (6.2b) are unimportant as long as $\Sigma T^i(f) = qy(f)$. We have chosen (6.2b) because of their simplicity and symmetry.

Proposition 6.1: For the Central Planning Mechanism (6.1),

(a) $y(f) = \frac{\mu - \frac{q}{h}}{K}$

where $\mu = \mathcal{E}(v^i)$;

(b) $L(z(e),e) = \frac{n}{2K} \left(\frac{\Sigma v^i}{n} - \mu\right)^2$

(c) $\frac{2K}{\sigma^2}$ L is chi-squared

distributed with one degree of freedom, $\mathcal{E}(L) = \frac{\sigma^2}{2K^2}$ if v^i is normally distributed with variance T^2 . (c) holds asymptotically as $n \to \infty$ if v^i is not normally distributed.

Under the assumption that $f(v^i)$ is known, this mechanism is a standard against which to compare all others since no communication is necessary; that is, an auction and its related paperwork need not be held. If an auction were held one should expect to be able to do better. Such an expectation is fulfilled by the last mechanism we consider.

B. The d'AGVA Mechanism

It turns out that for an even larger class of environments than we have considered, it is possible to design a mechanism which uses the information in f( ) and for which the loss L(z(e),e) = 0 for all environments if the v^i are independently and identically distributed. This mechanism was discovered by d'Aspremont and Gerard-Varet [3] and, independently, by Arrow [1]. In general, their mechanism is based on the Demand Revealing Mechanism and uses $\varphi^i : Y \to R^1$ as messages, interpreted as willingness to pay functions. For our economies $\varphi^i = v^i y - \frac{K}{2} y^2$ and, as before, we will let messages be m^i where sending m^i is equivalent to sending $m^i y - \frac{K}{2} y^2$. The allocation rules then are:

$$(6.2a) \quad y(m) = \frac{\Sigma m^i - q}{nK}$$

$$(6.2b) \quad T^i(m) = \frac{1}{n} qy(m) g^i(m^i) + \frac{1}{n-1} \Sigma_{j \neq i} g^j(m^j)$$

where $g^j(a^j) = \mathcal{E}[\Sigma_{h \neq j}(m^h \cdot y(m) - \frac{K}{2} y(m)^2 - \frac{1}{n} qy(m))]$ and the expectation is taken with respect to the probability measure on m given m^j.

<u>Proposition 6.2</u>: For the d'AGVA Mechanism (6.2),

(a) the symmetric Bayes-equilibrium bidding strategy is $\delta(v^i) = v^i$ and this is a dominant bidding strategy (i.e., best for any strategy of the others),

(b) $S(v) \equiv 0$ and $L(v) \equiv 0$.

<u>Proof</u>: The first order condition of consumer i is

$$\mathcal{E}[v^i - Ky(m) - \frac{1}{n} q + \Sigma_{j \neq i} m^j - (n-1)Ky(m) - \frac{n-1}{n} q] = 0 .$$

This reduces to $v^i - m^i = 0$. The rest follows easily. Q.E.D.

It is now time to see whether any comparative judgements can be made about the variety of mechanisms we have considered.

7. Comparisons of Mechanisms as Auctions.

For the limited class of economies we have considered, each consumer cares only about the expected utility of his outcome. Since all consumers are ex ante identical, one measure of the performance of a mechanism as an auction would be $\mathcal{E}(L)$, the expected aggregate utility loss.[17] If we were to broaden the class of economies to allow, for example, risk aversion, then Var(L) becomes important. Thus, we consider both these statistics in our comparison of mechanisms. Table 1 lists $\mathcal{E}(L)$ and Var(L) for all the mechanisms considered under the assumption that the v^i are independently, identically, and normally[18] distributed. For ease of comparison I have listed the mechanisms in order, approximately, of increasing expected loss.

One must be careful not to conclude too much, due to the special nature of the economies we have considered. However, some of these results are perhaps surprising. For example, other than the d'AGVA and Basic Quadratic ($\gamma = nK$) mechanisms, each of which has special problems[19], the best performer seems to be the Demand Revealing mechanisms with the surplus distributed[20]. Next in line for large enough n (in particular,

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[17] One could use $\frac{1}{n}\mathcal{E}(L)$ but the rankings would be unchanged.

[18] These would be the asymptotic values of $\mathcal{E}(L)$ and Var (L) as $n \to \infty$ for other distributions.

[19] The d'AGVA uses information in f( ) and relies heavily on the independence of the distributions of the V^i. The Basic Quadratic (γ=nK) is only possible when $u = X^i + V^i y - \frac{K^i}{2} y^2$ and $K^i = K$ for all i.

[20] Remember, however, that this is simply the Basic Quadratic with $\gamma = \frac{n(n-2)K}{n-1}$.

TABLE 1

mechanism	$\mathcal{E}(L) = A\frac{\sigma^2}{2K} + B$		Var (L)
	A	B	
AGVA (6.2)	0	0	0
iadratic (γ=nK) (4.1)	0	0	0
emand Revealing + $\frac{1}{n}S$ (5.3)	$\frac{1}{(n-1)^2}$	0	$A^2 \frac{\sigma^4}{2K^2}$
iadratic ($\gamma=\gamma^{3}n^{2}$) (4.1)	$\left(\frac{\gamma_o - K/n}{\gamma_o + K/n(n-1)}\right)$	0	$A^2 \sigma^4/_{2K^2}$
ax$\mathcal{E}$(n) (6.1)	1	0	$\frac{\sigma^4}{2K^2}$
emand Revealing (3.2)	1	0	$\frac{\sigma^4}{2(n-1)K^2}$
indahl (5.2)	$(\frac{n-1}{2n-1})^2$	$\frac{(n-1)^2}{2nK}(\mu - {}^q/_n)^2$	$A^2 \frac{\sigma^4}{2K^2} + C_1$
cuuous (4.2)	$(n-1)^2$	0	$(n-1)^4 \frac{\sigma^4}{2K^2}$
aired difference (4.3)	$(n-1)^2$	0	"
hared cost (4.4)	$(n-1)^2$	0	"
urwicz ( )	$(n-1)^2$	0	"
urwicz (4.7)	$\left(\frac{(n-1)K-2}{K+2}\right)^2$	$\frac{n}{2K}\left[\frac{n({}^q/_n+\mu)+(\mu-{}^q/_n)}{n(K+1)+1}\right]^2$	$A^2 \frac{\sigma^4}{2K^2} + C_2$
aive (5.1)	$(n-1)^2$	$\frac{n(n-1)^2}{2K}({}^q/_n)^2$	$A^2 \frac{\sigma^4}{2K^2} + C_3$

$\frac{2n(n-1)}{n-2}\gamma_o > K$), is the Basic Quadratic ($\gamma = \gamma_o n^2$) since for those n, $0 \leq A < 1$. Perhaps most surprising is that the Demand Revealing Mechanism is equivalent in expected loss to central planning (maximize $\mathcal{E}(u)$), and worse than those previously mentioned mechanisms. However, one should note that the variance of the loss under the Demand Revealing Mechanism is less than either central planning or the Basic Quadratic ($\gamma = \gamma_o n^2$). Therefore, a clear ranking of these alternatives is not possible.

Perhaps the most surprising result is that none of the mechanisms designed to produce Lindahl allocations at a Nash equilibrium performed well as sealed-bid auctions. In fact, the Lindahl mechanism (5.2) whose Nash equilibria are not Pareto-optimal performs better.

It is probably pushing special cases too far, but for completeness we have listed in Table 2 the values of the limits as $n \to \infty$ of $\mathcal{E}(L)$, $\mathrm{Var}(L)$, $\mathcal{E}(\frac{1}{n}L)$, $\mathrm{Var}(\frac{1}{n}L)$. What these values indicate is that in very large economies if one is interested only in per-capita losses then there are five[21/] mechanisms which differ insignificantly from the ideal in that $\mathcal{E}(\frac{1}{n}L) = \mathrm{Var}(\frac{1}{n}L) = 0$ in the limit. Obviously if one is to distinguish between these it is either on the basis of their performance in small economies or in large economies with a much broader possible range of utility functions. In particular, it is important to know what happens if there

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[21/] Six, if $\mu = \frac{q}{n}$ for all n.

are income effects (i.e., $\frac{\partial}{\partial x^i}\left(\frac{\partial U/\partial x}{\partial U/\partial x}\right) \neq 0$) or if consumers display some degree of risk aversion. This is left for future work.

TABLE 2

	$\mathcal{E}(L)$	Var(L)	$\mathcal{E}(\frac{1}{n}L)$	Var $(\frac{1}{n}L)$
'AGVA	0	0	0	0
ıadratic $(\gamma=\gamma_o n^2)$	$\frac{\sigma^2}{2K}$	$\frac{\sigma^4}{2K^2}$	0	0
ɜmand Revealing $+\frac{1}{n}S$	0	0	0	0
ax $\mathcal{E}(U)$	$\frac{\sigma^2}{2K}$	$\frac{\sigma^4}{2K^2}$	0	0
ɜmand Revealing	$\frac{\sigma^2}{2K}$	0	0	0
indahl $(\mu={}^q/_n)$	$\frac{\sigma^2}{8K}$	$\frac{\sigma^2}{32K}$	0	0
acuous	∞	∞	∞	∞
ırwicz (4.7)	∞	∞	∞	∞
aive	∞	∞	∞	∞
indahl $(\mu={}^q/_n)$	∞	∞	$\frac{1}{2K}(\mu-\frac{q}{n})^2$	0

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