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A STUDY OF PC^1 HOMEOMORPHISMS ON SUBDIVIDED POLYHEDRONS

by

M. Kojima*

R. Saigal

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*Department of Information Sciences, Tokyo Institute of Technology, Tokyo, Japan. The work of this author was done while Visiting Northwestern University.

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ABSTRACT

In this paper we consider the problem of establishing conditions when a given piecewise continuously differentiable mapping is a homeomorphism of a closed convex polyhedral set. These conditions are a generalization of the ones used by Gale-Nikaido and are similar in spirit to those of Mas-Colell. For the special case when the mapping is piecewise linear, we give an apparently new sufficiency condition for the mapping to be a homeomorphism of R^n. The results are further extended to include the case when the Jacobians may be singular.

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§1. Introduction

Let S be a closed convex polyhedral subset of R^n, the n-dimensional Euclidian space, and let Σ be a class of closed convex polyhedral subsets of S which partition S. A function F from S into S is called piecewise continuously differentiable (PC^1 for short) on the subdivided polyhedron (S,Σ) if it is continuous, and for each piece σ in Σ, $F_\sigma \equiv F|\sigma$ (the restriction of F to σ) is a continuously differentiable mapping. The problem we consider in this paper is that of establishing conditions under which F maps S homeomorphically onto F(S); i.e., F is one to one and onto.

One of the early works establishing such a result is that of Gale and Nikaido [6], which is often used to establish the uniqueness of solutions. Their result states that if $S = \{x: a_i \leqq x_i \leqq b_i\}$ and F is a continuously differentiable mapping from S into R^n, then if the Jacobian matrix DF(x) of F has all principal minors positive, then F maps S homeomorphically onto F(S). H. Scarf [21] had conjectured that since in the nonlinear complementarity problem, such a strong requirement on the Jacobian can be considerably weakened (see, for example, Corollary 2.6, Saigal and Simon [19]) , such a weakening should be possible for the hypothesis of the Gale-Nikaido theorem.

This was verified by Mas-Colell [12]. He also further generalized the result to the case when S is a compact convex polyhedron, and showed that such a result would be false for non-convex objects. The proof of [12] involved the use of degree theoretic arguments (a possibility of which had been foreseen by H. Scarf). Later, Garcia and Zangwill [7] again verified this conjecture, using the norm-coerciveness theorem [5.3.8, 15]. Their result is on a rectangle S, but a slight weakening of the requirement on the derivatives was achieved. In this paper, we further generalize this result. In one generalization, using degree theoretic arguments similar to those of [12], we establish the result for PC^1 mappings. In the other, we find conditions under which this result holds, when the derivatives may be singular. Under a similar hypothesis involving negative determinants, we show that our approach fails for PC^1 cases. In Kojima and Saigal [11], such an hypothesis was successfully used in the context of the nonlinear complementarity problem.

In case the restriction to each piece in Σ of the mapping F is affine, we call it a piecewise linear mapping, and, for brevity, PL. Considerable attention has been paid to the study of such mappings (see for example, Eaves and Scarf [4], Fujisawa and Kuh [5], and Ohtsuki, Fujisawa and Kumagai [14]), as well as to the problem of generating PL approximations (see, for example, Charnes, Garcia and Lemke [1], Kojima [8,9], Saigal [18]). In addition, several authors have contributed to the conditions under which such mappings are onto (see for example Chien and Kuh [2], Rheinboldt and Vandergraft [15]). Also, a set of conditions under which the mapping is a homeomorphism are developed in [5] and [14]. In this paper, we present a sufficiency condition which appears weaker than that of [5], and for some examples, i.e., [Fig. 7, 5], our condition is satisfied. By an example, we show that it is not necessary, and that any condition only on all subsets of the Jacobians of the pieces cannot be necessary and sufficient.

After presenting the terminology and notation in section 2, in section 3 we calculate the local degree of certain PC^1 mappings. In section 4 we prove the extension of the Gale-Nikaido theorem for PC^1 mappings, and show by a counter example that the appropriate negative condition on certain minors of the Jacobian is not sufficient to guarantee a homeomorphism. In section 5 we prove a sufficiency condition under which a PL mapping is a homeomorphism and in section 6 we present two PL mappings which are homeomorphisms. One of these mappings is generated by the Samelson-Thrall-Wesler [20] partition theorem, and the other by the recent result of Kojima and Saigal [10] relating to the linear complementarity problem with negative principal minors. The later example is presented in the hope that it will help to generate conditions insuring homeomorphisms with the hypothesis that certain minors of the Jacobian are negative. Finally, in section 7, we show how our results can be extended to include the case when the appropriate minors of the Jacobian may be zero.

§2. Notation and Definitions

In this section we present the notation and definitions that will be needed in the subsequent sections. In particular, we establish some properties of subdivided polyhedrons and functions on them.

By a bounded polyhedron, we represent the convex combination of a finite collection of points. Also, given a set τ, we represent by H_τ the subspace spanned by τ, i.e., $H_\tau = \{y: y = \sum_{i=1}^{r} \lambda_i x_i, \sum_{i=1}^{r} \lambda_i = 1, x_i \varepsilon \tau\} - \tau$, and thus the origin is contained in H. A convex set of the form $\{x + \lambda y: \lambda \geq 0\}$ is called a half line. We will call the convex hull of a finite collection of points and half-lines in R^n a convex polyhedral set. The dimension of a set is the dimension of the subspace spanned by the set.

The interior $\mathring{\sigma}$, and the boundary $\partial\sigma$ of a set σ, are the relative interior and boundary of the set in the affine subspace $H_\sigma + \sigma$. Also, a subset τ of σ is called a face of σ if for every x and y in σ $0 < \lambda < 1$ and $(1-\lambda)x + \lambda y$ in τ imply x and y are in τ. It can be readily confirmed that the faces of convex polyhedral sets are also convex polyhedral sets. For an n-dimensional set σ, a (n-1)-dimensional face is called a facet.

Now, given a convex polyhedron S and a finite class Σ of non-empty subsets of S, we say (S,Σ) is a subdivided convex polydedron of dimension n if:

a) elements of Σ are n-dimensional convex and polyhedral, and are called pieces;

b) any two members of Σ are either disjoint, or meet on a common face;

c) the union of the pieces in Σ is S.

Let (S,Σ) be a subdivided compact polyhedron of dimension n, with S in R^n. Then, there exists an extension Σ' of Σ such that (R^n, Σ') is a subdivided polyhedron. This can be observed by defining the projection mapping:

$$\| x - P(x) \| = \min_{y \in S} \| x-y \| \qquad (2.1)$$

and noting that Σ' is generated by adding the pieces $P^{-1}(\tau)$ for τ a face of some σ in Σ to those already in Σ (see figure 2.1).

Now, let $F: S \to R^n$ be a continuous function on a subdivided polyhedron (S,Σ). We say F is PC^1, i.e., piecewise continuously differentiable, on (S,Σ) for each piece σ in Σ if there exists an open set B_σ containing σ such that $F_\sigma = F|\sigma$ can be extended to B_σ continuously differentiably. In particular, it is called piecewise linear if F_σ is affine, i.e., $F_\sigma(x) = A_\sigma x - a_\sigma$ for some nxn matrix A_σ and n vector a_σ.

Now, given a subdivided compact polyhedron (S,Σ) and a mapping $F: S \to R^n$ which is PC^1 on (S,Σ), there exists a PC^1 extension to (R^n,Σ') when the subdivision Σ is extended by the projection mapping (2.1). This mapping is $FoP: R^n \to R^n$, and as can be readily verified, it is PC^1 on Σ'.

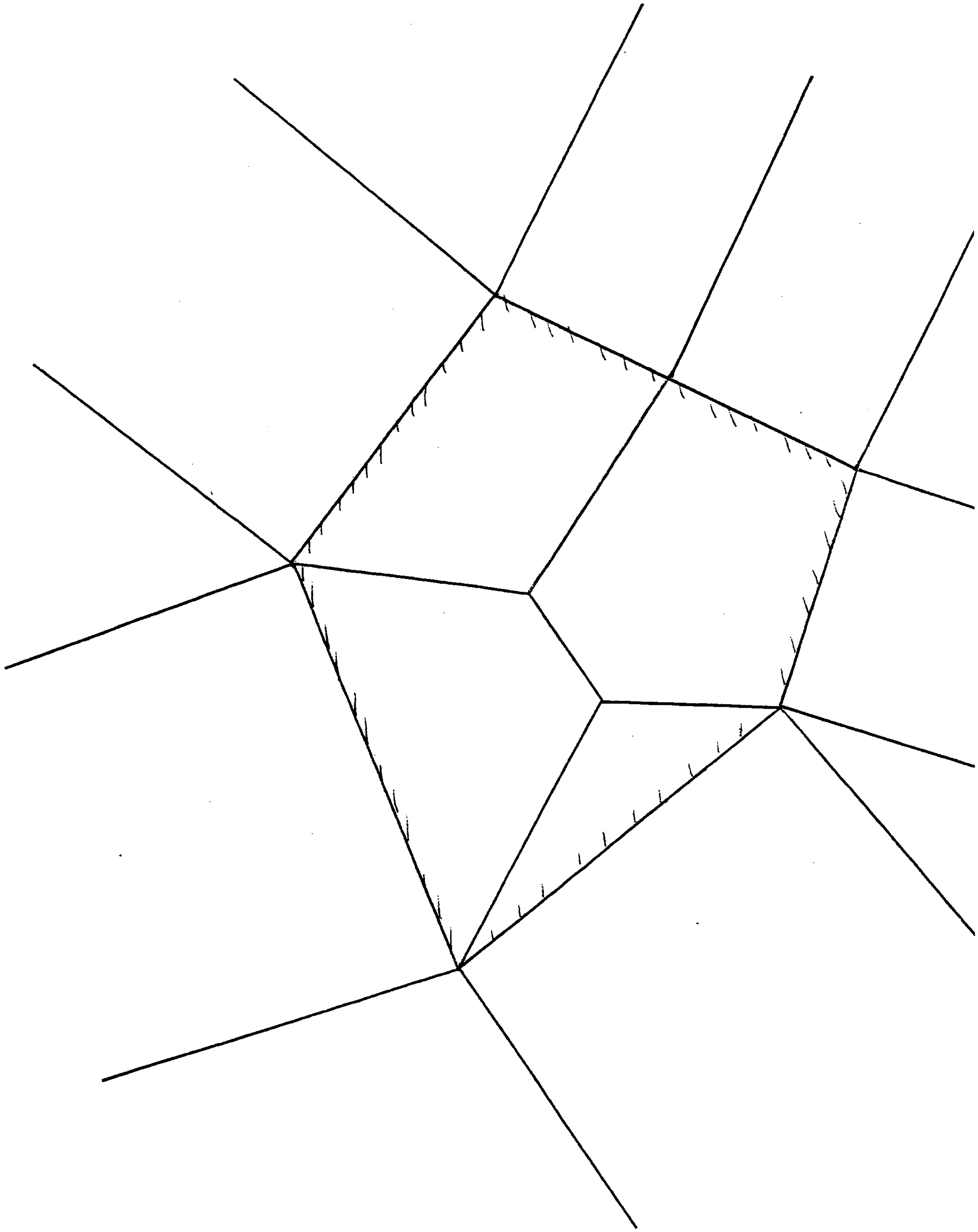


Figure 2-1

§3. Local Degree of PC^1 Mappings

In this section we consider subdivided polyhedron (R^n,Σ), and a PC^1 mapping $F: R^n \to R^n$. Our aim is to get sufficient conditions which establish the local degree of such a mapping. We now have a lemma, which will then be used to prove the main result:

Lemma 3.1 Let $\hat{x}$ be such that $DF_\sigma(\hat{x})$ is nonsingular for all σ containing $\hat{x}$. Then, there exist positive numbers α and ε such that

$$||F(x) - F(\hat{x})|| \geqq \alpha||x-\hat{x}|| \tag{3.1}$$

for all $x \varepsilon B_\varepsilon(\hat{x}) = \{x:||x-\hat{x}|| \leqq \varepsilon\}$.

Proof: Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be all the pieces of Σ which contain $\hat{x}$. Then, there is a $\delta > 0$ such that

$$B_\delta(\hat{x}) \subset \bigcup_{i=1}^{k} \sigma_i$$

Let $\sigma' \varepsilon \{\sigma_i: i=1, \ldots k\}$. For each $x\varepsilon B_\sigma(\hat{x}) \cap \sigma'$, we have

$$||F(x) - F(\hat{x})|| \geqq ||DF_{\sigma'}(\hat{x})\ (x-\hat{x})|| + o(||x-\hat{x}||)$$

and since $DF_{\sigma'}(\hat{x})$ is nonsingular, there is a $\alpha' > 0$ such that

$$||F(x) - F(\hat{x})|| \geqq 2\alpha'\ ||x-\hat{x}|| + o(||x-\hat{x}||).$$

Hence, there is a $\delta > \varepsilon' > 0$ such that

$$||F(x) - F(\hat{x})|| \geqq \alpha'||x-\hat{x}|| \text{ for all } x\varepsilon B_\varepsilon(\hat{x}) \cap \sigma'$$

and letting α be the smallest α' and ε be the smallest ε', we have our result.

Lemma 3.2 Let F be continuously differentiable, and H a subspace of R^n of dimension $\leqq$ n-1. Then F(H + v) contains no open set, for all $v\varepsilon R^n$.

Proof: Let P be the projection mapping $P:R^n \to H$, and consider FoP: $R^n \to R^n$. We note that rank (D(FoP)) is less than or equal to the dimension of H. Hence, from Sard's theorem, Milnor [13], $FoP(R^n)$ contains no open ball, and thus F(H) since $P(R^n) = H$. The result follows by considering

FoP' where P' = P + v.

We now prove our main theorem:

Theorem 3.3:

Let $\hat{x} \in R^n$, such that $\det(DF_\sigma(\hat{x}))$ is positive (negative) for every σ containing $\hat{x}$. Then, there exists $\epsilon > 0$ such that $\deg(F, B_\delta(\hat{x}), F(\hat{x})) \geq +1$ (≤ -1) for each δ in $(0, \epsilon)$.

Proof:

By Lemma 3.1, we have $\varepsilon > 0$, $\alpha > 0$ satisfying the hypothesis of the lemma. We shall now show the theorem for the case when $\det(Df_\sigma(\hat{x})) > 0$ for all σ containing $\hat{x}$.

Let $0 < \delta < \epsilon$, $B = B_\delta(\hat{x})$, $y = F(\hat{x})$ and ∂B the boundary of B. Then, from lemma 3.1,

$$||F(x) - F(\hat{x})|| \geq \alpha\delta \text{ for all } x \in \partial B.$$

Since F is continuous, there exists a $\beta > 0$ such that

$$||F(x)-q|| = \alpha\delta/2 \quad \text{for all } x \text{ in } \partial B \text{ and } q \text{ in hull } \{F(B_\beta(\hat{x}))\}$$

Let $q \in B(\hat{x})$. Now, consider the mapping $G(x) = F(x) + y - q$, and the homotopy $H: B \times [0,1] \to R^n$ defined by $H(x,t) = (1-t)G(x) + tF(x)$. Then, for (x,t) in $\partial B \times [0,1]$ we have

$$||H(x,t) - y|| = ||F(x) - (ty + (1-t)q)|| \geq \alpha/2.$$

Hence, by the homotopy invariance theorem, deg (G,B,y) = deg (F,B,y).

Since $DF_\sigma(x)$ is nonsingular for all x in $B \cap \sigma$ and σ containing $\hat{x}$, using the inverse function theorem, it can be established that $F(B_\delta(\hat{x}))$ contains an open ball U. Also, from lemma 3.2, the image $F(\tau)$ of a proper face τ of any piece σ contains no open ball.

Thus, we can choose a q in U such that $\hat{B} = \{x \varepsilon B: G(x) = q\}$ does not intersect any facet of a piece σ. Hence, as the $\hat{B}$ is a set of isolated points, and the local degree of each $x \varepsilon \hat{B}$ is +1, we have, from the fact that the degree of a mapping is the sum of local degrees (from the Poincare-Hopf theorem, Milnor [13]); that

$$\deg (G,B,y) \geqq 1$$

and we have our result.

We observe that the above result cannot be strengthened. For this, consider the piecewise linear function of Figure 3.1. The degree of this mapping at 0 is 2 and the determinant of Jacobian of the linear mapping on each piece is positive.

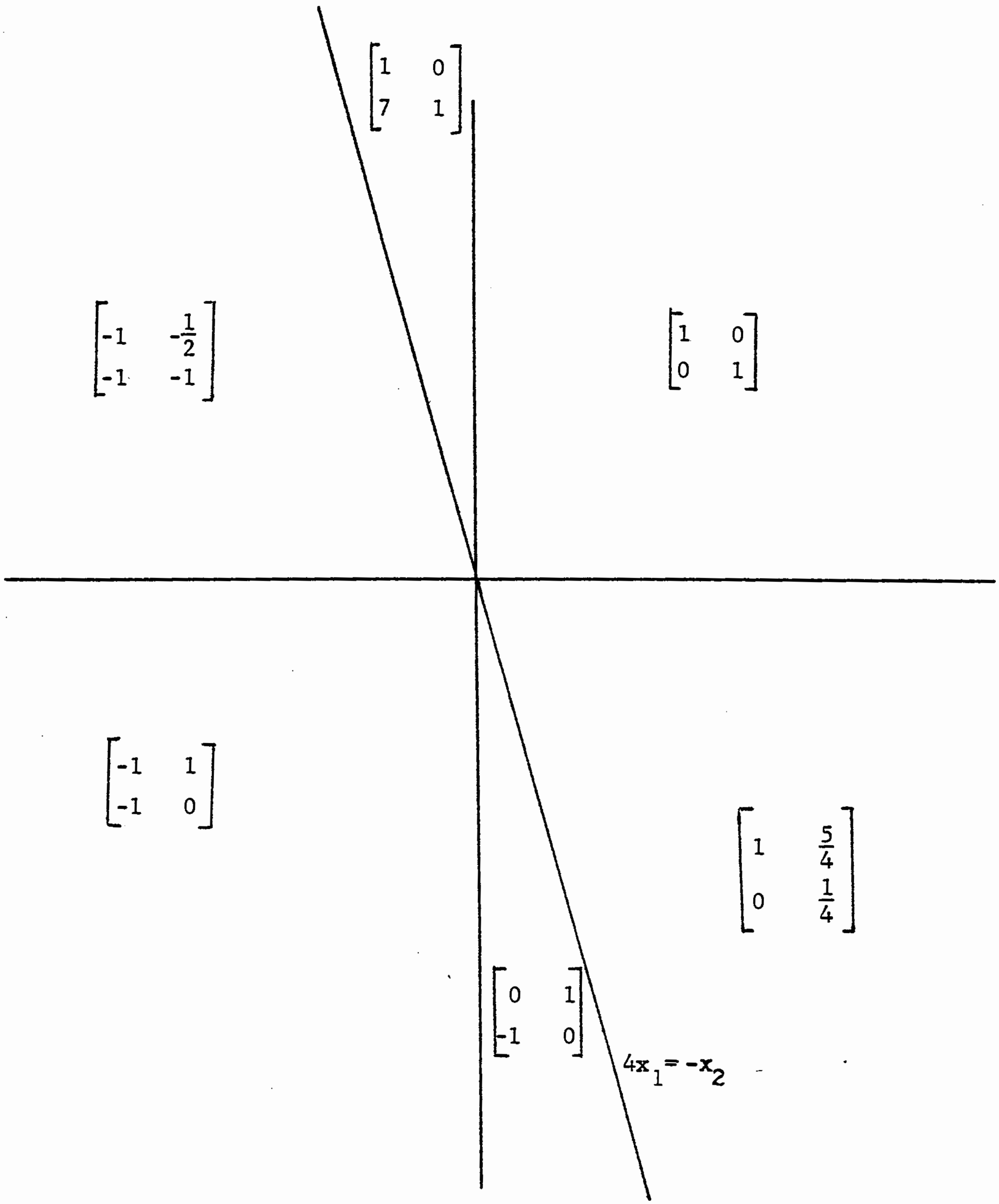


Figure 3.1

§4. PC^1 Homeomorphisms of Convex Polyhedrons

Let (S,Σ) be a subdivided compact polyhedron, and let $F: S \to R^n$ be a PC^1 mapping. In this section we consider the conditions on F and S under which F maps S homeomorphically onto F(S), i.e., F(x) = y has a unique solution for each $y \in F(S)$. The results presented in this section are in the spirit of the recent extension of the Gale-Nikaido theorem [6] by Mas-Colell [12] (see also Garcia, Zangwill [7]).

Let $P: R^n \to S$ be the projection mapping (2.1), and let $G: R^n \to R^n$ be the mapping

$$G(x) = FoP(x) + x - P(x) \tag{4.1}$$

We observe that G is a PC^1 mapping on the subdivided polyhedron (R^n, Σ').

We now state our condition, which is the same as the one used by Mas-Colell [12] (compare also with condition (ii), Corollary 2.6 of Saigal and Simon [19]).

Condition 4.1 Let x in S lie in a face T of S. Also, let x be an element of σ where σ in Σ is a piece such that $\dim \sigma \cap T = \dim T$, H_T be the subspace spanned by T and P_T the projection mapping of R^n onto H_T. Then, the linear mapping $P_T o DF_\sigma(x): H_T \to H_T$ has positive determinant.

Under this condition, the following can be proved as is done for Lemma 1 in [12] (see also [lemma 3.4, 19]).

Lemma 4.2: Let x be arbitrary, and lie in the pieces σ_i, $i = 1, \ldots, k$ in Σ'. Then $\text{Det } DG_\sigma(x) > 0$ for each $\sigma = \sigma_i$, $i = 1, \ldots, k$.

We now prove our main theorem.

Theorem 4.3: Let (S,Σ) be a subdivided compact convex polyhedron, and let $F: S \to R^n$ be a PC^1 mapping. Also, let F and S satisfy condition 4.1. Then F maps S homeomorphically onto F(S).

Proof: Extend Σ to a subdivision Σ' using the mapping (2.1), and let the mapping G of (4.1) be the corresponding PC^1 extension of F.

Now, from Theorem 2.3, since the condition 4.1 implies that the determinant of G is positive in each piece, for each x in R^n, there exists an open ball B such that

$$\deg (G,B,F(x)) \geqq 1. \tag{4.2}$$

Let A be a nxn positive definite matrix, and consider the homotopy

$$H(x,t) = (1-t)Ax + t(G(x)-y) \tag{4.3}$$

for any $y \in F(S)$. We now show that $H^{-1}(0)$ is bounded, and thus the degree of G-y is +1 since it is homotopic to a map of degree +1. But, this is true, since for sufficiently large x, $x^TAx > 0$ and

$$x^TG(x) - x^Ty = x^Tx - x^T\,(FoP(x) - P(x) - y) > 0$$

since $FoP(x) - P(x) + y$ is bounded.

Now, using the Poincare-Hopf theorem and (4.2), we conclude that, for each y in F(S), $\{x: F(x) = y\}$ is a singleton, and we are done.

Note: This theorem is false if the property of positive determinants is replaced by negative determinants. A counterexample for a PL mapping is given in Figure 4.1. This demonstrates that such an extension for C^1 mappings involving $\Sigma = \{S\}$ may also be hard, and conjecture that in this case, the result is true (see also [12]).

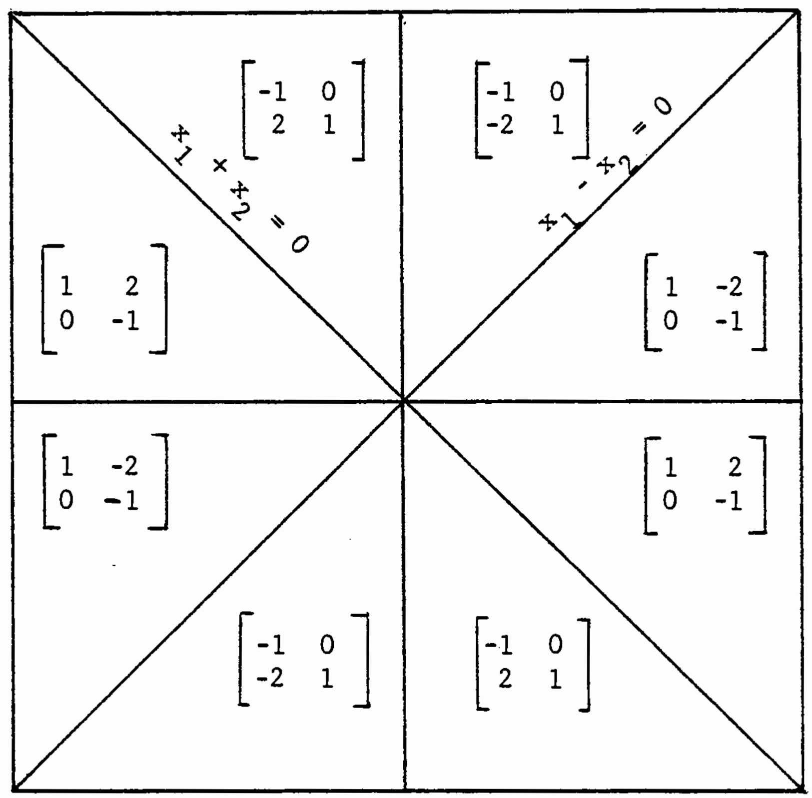


Figure 4-1

§5. On PL Homeomorphisms of R^n

In this section we give a set of sufficient conditions for a piecewise linear function in R^n to be a homeomorphism. Let (R^n,Σ) be a subdivided polyhedron, and let

$$F:R^n \to R^n$$

be piecewise linear on this subdivision, i.e., PC^1 with affine on each piece of Σ. Since Σ contains a finite number of pieces, outside some compact region, points of R^n lie in some unbounded piece in Σ. Let these unbounded pieces be numbered $\sigma_1, \sigma_2, \ldots, \sigma_k$ for some k, and let $F|\sigma_i(x) = A_i x - a_i$ for some nxn matrices A_i. Then, we can prove:

Theorem 5.1: Assume that the Jacobian matrix of each piece of linearity of F has a positive determinant. Also, let there exist a matrix B such that $(1-t)B + tA_i$ is nonsingular for each $t\varepsilon[0,1]$ and $i = 1, \ldots, k$. Then, F is a homeomorphism.

Proof: Let y be arbitrary. Then, consider the homotopy

$$H(x,t) = (1-t)Bx + t(F(x)-y);\ t\varepsilon[0,1] \tag{5.1}$$

We claim that $H^{-1}(0)$ has no unbounded component. This is true, since the contrary implies that for some σ_i, we can find a sequence $(x^p, t_p)\varepsilon H^{-1}(0), p = 1, 2, \ldots$ such that $x^p\varepsilon\sigma_i$ and $||x^p|| \to\infty$. Also, on some subsequence $x^p/||x^p|| \to x^*, t_p \to t^*, t^*\varepsilon[0,1]$ and $x^* \neq 0$. Hence, from (5.1) $(1-t_p)Bx^p + t_p(A_i x^p - a^i) - t_p y = 0$. Dividing by $||x^p||$ and taking limits, we get

$$(1-t^*)Bx^* + t^*A_i x^* = 0$$

which is a contradiction. Now, to see that it is one to one and onto, we observe that since $H^{-1}(0)$ is bounded for each y, and $\det(B) > 0$ (see Saigal [17]), from the homotopy invariance theorem, the degree of

F(x)-y is +1 for all y. The result then follows from Theorem 3.3.

The onto part of the theorem also follows from the works of several authors, including Chien and Kuh [2], Rheinboldt and Vandergraft [16]. The sufficiency condition of Theorem 5.1 is different from that of Fujisawa and Kuh [5]. In Figure 5.1 we present a homeomorphism satisfying the conditions of our theorem with B = $\begin{bmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Also, since I and -I appear as Jacobians of the pieces of linearity, no linear transform of it will satisfy the condition of [5], though there is a linear transform for which the homeomorphism of [Fig. 7,5] will satisfy the condition. Now consider the example of Figure 5.2. This is a homeomorphism which does not satisfy the condition of Theorem 5.1, and is thus a counterexample to the necessity of our condition. To see this, note that the matrices $\begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$, are Jacobians of the pieces of linearity of the non-homeomorphism of Figure 4.1, and, for these, thus, there is no matrix B satisfying the conditions of Theorem 5.1. We also observe that this example is also a counterexample to any set of necessary and sufficient conditions put on all subsets of the Jacobians of the pieces of linearity.

Theorem 5.1 is true if the property of positive determinants is replaced by negative determinants. Also, if the unbounded pieces satisfy the conditions of Theorem 5.1, it can be readily shown that $\{x : f(x) = y\}$ has an odd number of elements, if each of its elements lies interior to some piece.

A corollary to Theorem 5.1 is the following result which can also be considered as an explanation of the boundary condition 4.1. Let (S,Σ) be a subdivided compact convex polyhedron, with $F:S \to R^n$ a piecewise linear function. Then, we can prove

<u>Corollary 5.3</u>: Let (S,Σ) admit an extension (R^n,Σ') such that F can be extended to F' on R^n with $F'|\sigma'$ affine, and F' satisfying the conditions of Theorem 5.1. Then F maps S homeomorphically onto F(S).

Two applications of this corollary are given in the next section.

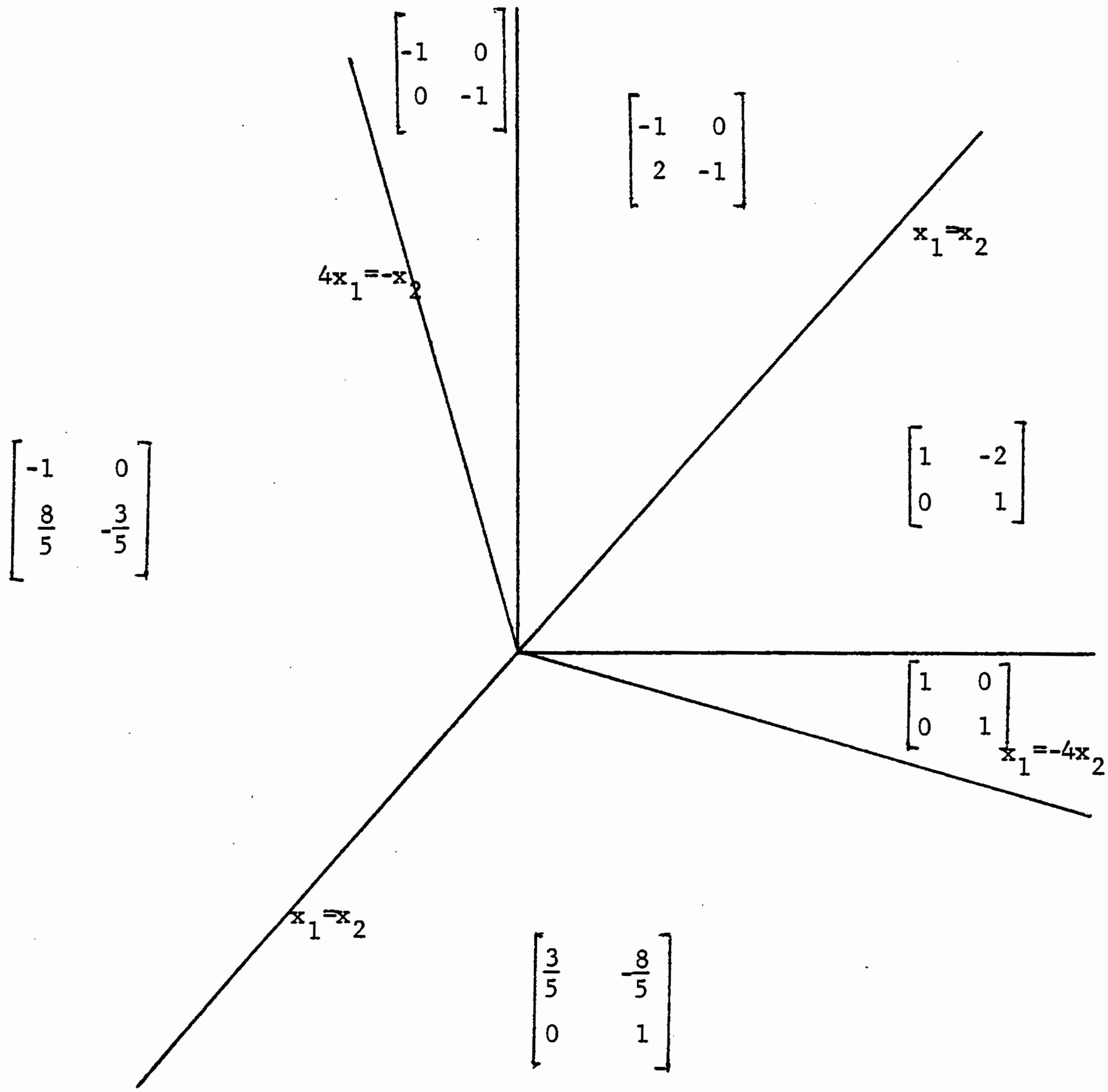
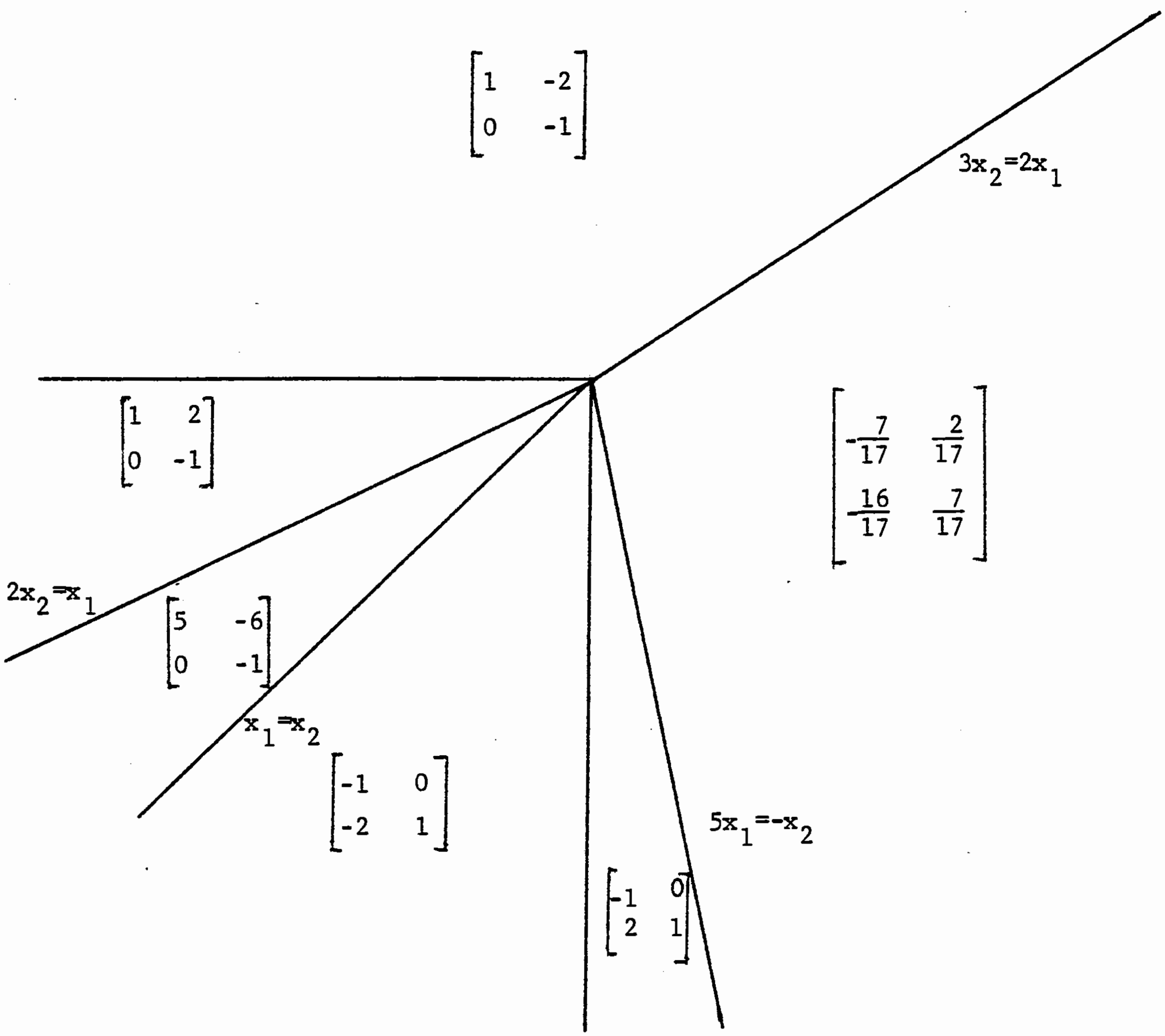


Figure 5.1



1 -2
0 -1
3x2=2x1
1 2
0 -1
-7/17 2/17
-16/17 7/17
2x2=x1
5 -6
0 -1
x1=x2
-1 0
-2 1
5x1=-x2
-1 0
2 1

Figure 5.2.

§6. Two PL Homeomorphisms

We now present two PL homeomorphisms; one satisfies the sufficiency condition of Fujisawa and Kuh [5] while the other does not. The first homeomorphism is constructed by the use of a matrix which has all principal minors positive, and thus establishes the sufficiency part of the Samelson, Thrall, Wesler [20] partition theorem. The other is constructed by using a matrix which has all principal minors negative. In the process of the construction, we will prove the main theorem of Kojima and Saigal [10], and this can be considered a degree theoretic proof of the same. We now introduce the necessary notation.

Let $I = \{1, \ldots, n\}$ and U and V be nxn nonsingular matrices. Now, for any J I, let $W_J = (W_1, \ldots, W_n)$ be the nxn matrix with

$$W_j = \begin{cases} U_j & j \in J \\ V_j & j \notin J \end{cases} \tag{6.1}$$

Also, let $\text{pos}(A) = \{y: y = Ax, s \geq 0\}$ represent the cone generated by a matrix A. For $J \subseteq I$, let $\sigma(J) = \{x: x_j \geq 0, j \in J \text{ and } x_j \leq 0, j \notin J\}$, and by $\Sigma = \{\sigma(J): J \subseteq I\}$. In this case, (R^n, Σ) is a subdivided polyhedron. Now, define the PL mapping $F: R^n \to R^n$ by

$$F(x) = \sum_{j \in J} U_j x_j + \sum_{j \notin J} V_j x_j \tag{6.2}$$

for x in $\sigma(J)$.

6.1 The First Homeomorphism

We now prove our first homeomorphism theorem:

Theorem 6.1 Let U, V, W_J, $J \subseteq I$, be defined as above, and let $\det(U) > 0$, $\det(W_J) > 0$ for each J. Then F is a PL-homeomorphism of R^n onto R^n, on the subdivided polyhedron (R^n, Σ).

Proof: On each piece of Σ, $F(x) = W_J x$ for $x \varepsilon \sigma(J)$. Also, $\det(U^{-1}W_J) = \det(U^{-1})\det(W_J) > 0$ for all J. By choosing $\bar{J} \subseteq J, \bar{J} \subsetneq I$, we can show that each principal minor of $U^{-1}W_J$ is positive. Hence, for each J, we have $\det((1-t)U + tW_J)) = \det(U)(\det((1-t)I + tU^{-1}W_J) > 0$ since $U^{-1}W_J$ has all principal minors positive (see Lemma 3.1.1, Saigal [17]). Hence, the result follows from Theorem 5.1.

As a corollary of this theorem, we prove the sufficiency part of the Samelson, Thrall, Wesler [20] partition theorem.

Corollary 6.2 Let U, V, W_J, $J \subsetneq I$ be defined as above, and $\det(U) > 0$; with $\det(W_J) = (-1)^{|J|}$ when $|J|$ is the number of elements in J. Then, the collection of cones $\Delta = \{ \text{pos}(W_J): J \subsetneq I\}$ partitions R^n.

Proof: Define

$$\bar{W}_J = \begin{cases} W_j & j \varepsilon J \\ -W_j & j \notin J \end{cases}$$

and we note that the mapping $F(x) = \bar{W}_J x$, $x \varepsilon \sigma(J)$ is a PL mapping. Also, since $\det(\bar{W}_J) > 0$, $F(x)$ is a PL-homeomorphism from theorem 6.1. This corollary follows by observing that the cones of Δ are images of the cones of Σ.

6.2 The Second Homeomorphism

In this section we consider $U = E$ (the identity matrix) and $V \nleq 0$ a matrix having all principal minors negative. Then Kojima and Saigal [10] have shown that F defined by (6.2) is not a homeomorphism of R^n on the subdivided polyhedron (R^n, Σ). In this section, we will show that there exists a PL-homeomorphism G of $Q = R^n \backslash R^n_+$ onto Q such that $F \circ G$ is the identity mapping on Q (where R^n_+ is the non-negative orthant).

Now, define $\Sigma' = \Sigma \setminus \sigma(I)$. Then (Q,Σ') is a subdivided polyhedron (which is not convex). Define $\hat{F}$ as the restriction of F as (defined by (6,2)) to Q. We now state some preliminary results.

Lemma 6.3: Let $V \nleq 0$ and have all principal minors negative. Then, there is a $d > 0$ such that $Vd > 0$.

Proof: See Lemma 2.1, [10].

Lemma 6.4: Let V have all principal minors negative. Then all proper principal minors of V^{-1} are positive.

Proof: See Lemma 4.1, [10].

Now, for $d > 0$ such that $Vd > 0$, consider the homotopy:

$$H(x,t) = (1-t)V(x+d) + t[\hat{F}(x) + Vd] \tag{6.3}$$

Lemma 6.5: $H^{-1}(0) \cap \partial Q \times [0,1] = \emptyset$.

Proof: Assume the contrary that there is a $(x,t) \in \partial Q \times [0,1]$ with $(x,t) \in H^{-1}(0)$. Then, $x \geqq 0$ with $J = \{j : x_j > 0\}$, $|J| < n$. Thus, if $J \neq \emptyset$, $(1-t)Vx + tW_J x = -Vd$, or multiplying by V^{-1}, we get

$$(1-t)x + tV^{-1}W_J x = -d. \tag{6.4}$$

Now, let A be the principal minor of V^{-1} in $V^{-1}W_J$. Then, from (6.4) we can conclude that $A\bar{x} < 0$, $\bar{x} > 0$ has a solution. But, from Lemma 6.4, A has all positive principal minors, which leads to a contradiction, [6]. Also, $J \neq \emptyset$, since the contrary implies that $x = -d$.

We are now ready to prove our main result.

Lemma 6.6: $\{x : \hat{F}(x) = -Vd\}$ is a singleton.

Proof: Assume the contrary. Then, since Lemma 6.5 implies that in $H^{-1}(0)$ no solution inside Q lies on a component intersecting ∂Q, there must be an unbounded component inside Q.

But, since $\hat{F}(x) = W_J x$ for some $J \subseteq I$, and $V^{-1} W_J$ has all positive principal minors, using arguments of theorem 5.1, we get a contradiction. Thus, the result follows.

Theorem 6.7 For any $y \varepsilon Q$, $T = \{x: F(x) = y\}$ is a singleton. Also, $T \subset Q$.

Proof: For any $y \varepsilon Q$, $y \not\leq 0$ and thus $T \cap \sigma(I) = \emptyset$. Hence, for each x in T, det DF(x) is the same as the determinant of some principal minor of V, and so det DF(x) < 0. Hence, from Theorem 3.3, deg (F,B,y) $\leqq -1$ for some neighborhood B of x.

Now consider the homotopy

$$H(x,t) = F(x) + (1-t)Vd - ty.$$

and we note that $H(x,0) = 0$ has a unique solution $x = -d$, from Lemma 6.6. Hence the degree of H(x,0) is -1. Also, $H^{-1}(0) \subset Qx[0,1]$ is bounded, and hence by the homotopy invariance theorem, H(x,1) has degree -1. Since deg $(\hat{F},B,y) = \deg(F,B,y) \leqq -1$, the result follows.

We now prove the main result of this section.

Theorem 6.8: Let $V \nleq 0$ and have all principal minors negative, U = E and F as defined by (6.2). Then there exists a PL-homeomorphism G on a subdivision of Q such that FoG is the identity on Q.

Proof: Let Δ be as in corollary 6.2 and let Δ' be the collection of polyhedrons of the type $\sigma_i = \sigma \cap \{x: x_i \leq 0\}$ $i = 1, \ldots, n$ and $\sigma \varepsilon \Delta$. Then it is readily confirmed that (Q,Δ') is a subdivided polyhedron. Define $G:Q \to Q$ by $y \to \{x: F(x) = y\}$. This is well defined by

Theorem 6.7. Also G is PL, and for y in $pos(W_J)$, $J \neq I$, $G(y) = W_J^{-1}y$, and that it is a homeomorphism of Q onto G(Q).

We now give an example of such a mapping G. Let $V = \begin{bmatrix} -1 & 2 \\ 4 & -6 \end{bmatrix}$

For this case

$$W_\emptyset = V,\ W_{\{1\}} = \begin{bmatrix} 1 & 2 \\ 0 & -6 \end{bmatrix}\quad W_{\{2\}} = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix},$$

$$W_\emptyset^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1/2 \end{bmatrix}\quad W_{\{1\}}^{-1} = \begin{bmatrix} 1 & 1/3 \\ 0 & -1/6 \end{bmatrix},\quad W_{\{2\}}^{-1} = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix}$$

The pieces of linearity of the mapping G are given in Figure 6.1

Also, as can be readily confirmed, G, in R^2, has a PL extension onto R^2 which is also a homeomorphism of R^2. For the above example, if one added $R_+^2 = \sigma(I)$ to the set Δ', and extended the mapping G to $\hat{G}$ by

$$\hat{G}(y) = \begin{cases} G(y) & y \varepsilon Q \\ Wy & y \varepsilon \sigma(I) \end{cases}$$

where $W = \begin{bmatrix} -1 & 1/3 \\ 4 & -1/6 \end{bmatrix}$ (the matrix consisting of the nontrivial columns of $W_{\{i\}}$), $\hat{G}$ maps R^2 homeomorphically onto R^2.

We conjecture that G has such an extension in n dimensional Euclidian space as well, but see no way to prove this.

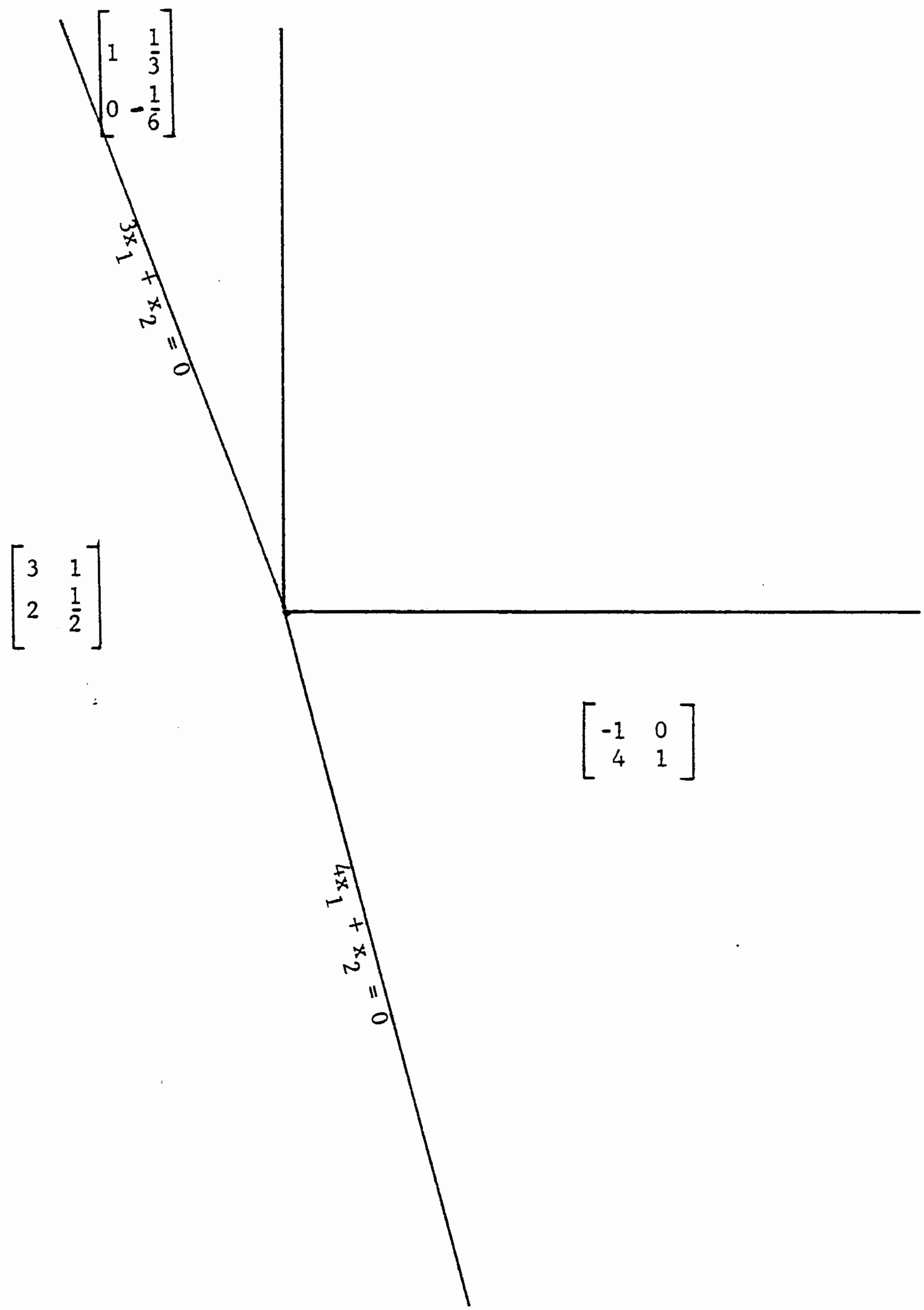

$\begin{bmatrix} 1 & \frac{1}{3} \\ 0 & -\frac{1}{6} \end{bmatrix}$
$3x_1 + x_2 = 0$
$\begin{bmatrix} 3 & 1 \\ 2 & \frac{1}{2} \end{bmatrix}$
$\begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix}$
$4x_1 + x_2 = 0$

Figure 6-1

§7. Extensions When the Jacobians May be Singular

Our aim in this section is to extend the results of Sections 3 and 4 to cases when the Jacobians of the mappings may be singular. Our main assumption is that for any y in R^n, the sets of the type $\{x : F(x) = y$ and $DF(x)$ is singular$\}$ are finite. We then show that the results of section 3 can be extended, and thus a further extension of the Gale-Nikaido theorem [6] is obtained.

We consider a subdivided polyhedron (R^n, Σ) and consider a PC^1 mapping $F: R^n \to R^n$ on it. Then, an extension of the Lemma 3.1 is the following.

Lemma 7.1: Let $\hat{x}$ be in R^n and $\sigma_1, \ldots, \sigma_k$ be the pieces in which it lies. Suppose that $\{x \varepsilon \sigma_i : F(x) = F(\hat{x})$ and $DF_{\sigma_i}(x)$ is singular$\}$ has at most a finite number of elements, for each i. Then, for each $\varepsilon_0 > 0$, there is a $0 < \varepsilon < \varepsilon_0$ such that

$$||F(x) - F(\hat{x})|| > 0 \quad \text{if } ||x - \hat{x}|| = \varepsilon.$$

Proof: Let $\varepsilon_0 > 0$, $y = F(\hat{x})$ and $X = \cup \{x \varepsilon \sigma_i : F(x) = y$ and $DF_{\sigma_i}(x)$ is singular$\}$. Since X is finite, there is a positive number $\delta < \varepsilon_0$ such that $B \equiv B_\delta(\hat{x}) \subset \cup \sigma_i$ and $\partial B \cap X = \emptyset$. Hence, for each x in ∂B, we have either $F(x) \neq y$ or $F(x) = y$ and $DF_\sigma(x)$ nonsingular. In the former case, by the continuity of F, there exists $\gamma(x) > 0$ such that $y \notin F(B_{\gamma(x)}(x))$, and in the latter case, by Lemma 3.1, a $\gamma(x) > 0$ such that $y \notin F[B_{\gamma(x)}(x) \setminus \{x\}]$. Let $V = \bigcup_{x \varepsilon \partial B} \text{int}(B_{\gamma(x)}(x))$, and, as can be readily confirmed, V is an open set in R^n with $\partial B \subset V$ and $F(x) \neq y$ for all $x \varepsilon V \setminus \partial B$.

Hence we can choose $0 < \varepsilon < \delta$ with the required property.

We now use Lemma 7.1 to compute the local degree of a mapping.

Theorem 7.2: For every piece σ in Σ y in R^n, let

(7.1) det DF (x) $\geq$ 0 for all $x\varepsilon\sigma$.

(7.2) $\{x\varepsilon\sigma$: $DF_\sigma(x)$ is singular$\}$ contains no open set.

(7.3) $\{x\varepsilon\sigma$: $F(x) = y$ and $DF_\sigma(x)$ is singular$\}$ has at most a finite number of elements.

Then, for every x in R^n and $\varepsilon_0 > 0$, there is $0 < \varepsilon < \varepsilon_0$ such that

$$\deg (F, B_\varepsilon(\hat{x}), F(\hat{x})) \geq 1.$$

Proof: Let x in R^n and $\varepsilon_0 > 0$. By Lemma 7.1, there is a positive number $\varepsilon < \varepsilon_0$ such that for $B = B_\varepsilon(\hat{x})$, $y = F(\hat{x})$ we have

$$||F(x) - y|| > 0 \text{ for all } x\varepsilon\partial B.$$

Using arguments identical to those of Theorem 3.2, we have our result.

We note that if, in (7.1) we assumed that the det $(Df_\sigma(x)) \leq 0$, then, by an identical argument, we could establish that deg $(F,y,B) \leq -1$.

We now weaken the hypothesis of Condition 4.1 so that we can obtain a further generalization of the Mas-Colell [12] generalization of the Gale-Nikaido Theorem [6].

Consider a PC^1 mapping $F:S \to R^n$ on the subdivided compact convex polyhedron (S,Σ). We now state our condition.

Condition 7.3: Let T be a face of S and σ a piece in Σ such that the dimension of $\tau = \sigma \cap T$ is the same as the dimension of T. Then

(7.4) $P_T \circ DF_\sigma(x)$: $H_T \to H_T$ has non-negative determinant for each x in τ.

(7.5) $\{x\varepsilon\tau$: $P_T . DF_\sigma(x)$ is singular$\}$ has at most a finite number of elements.

We now show that if condition 7.3 is satisfied, then the conditions of theorem 7.2 are satisfied.

Lemma 7.4: If F satisfies the condition 7.3, then the mapping G defined by (4.1) satisfies the conditions of Theorem 7.2.

Proof: Let (R^n,Σ') be the subdivision on which G, defined by (4.1), is PC^1, and let $\sigma \varepsilon \Sigma'$ be an unbounded piece. Then there exists a face F of S and $\bar{\sigma}\varepsilon\Sigma$ such that if $\tau = \bar{\sigma} \cap T$, dim τ = dim T, and $P_T(x) \varepsilon \tau$ for all $x \varepsilon \sigma$; and

$DG_\sigma(x) = P_T DF_{\bar{\sigma}}(P_T(x)) + I - P_T$ for all $x\varepsilon\sigma$. Now, by using the same argument as in the proof of Lemma 1 of [12], we have (7.1), and

$$\det DG_\sigma(x) = 0 \text{ iff } P_T DF_{\bar{\sigma}}(P_T(x)) \text{ is singular.} \quad (7.6)$$

If dim $H_T = 0$, then $DG_\sigma(x) = I$ for all $x\varepsilon\sigma$ and (7.2) holds. We now take dim $H_T \geqq 1$. Now, assume the set $\{x\varepsilon\sigma : DG_\sigma(x) \text{ is singular}\}$ contains an open set X. Then, the projection $P_T(X)$ of X into H_T is open and $P_T\ DF_{\bar{\sigma}}(P_T(x))$ is singular on $P_T(X)$. Since, dim $H_T \geqq 1$, this contradicts (7.5). Thus we have shown (7.2). It follows from (7.5) that there exists a finite number of points $x^1, x^2, \ldots, x^m$ in τ such that $P_T DF_{\bar{\sigma}}(x)$ has a positive determinant if $x\varepsilon\tau$ and $x \neq x^i$, $i = 1, \ldots, m$. Let $y\varepsilon R^n$ and $Y = \{ x\varepsilon\sigma : G(x) = y \text{ and } DG_\sigma(x) \text{ is singular}\}$. By (7.6) we obtain

$$Y \subset \bigcup_{i=1}^{m} \{x\varepsilon\sigma: P_T(x) = x^i,\ F(x^i) + x^i - x = y\}$$

$$\subset \bigcup_{i=1}^{m} \{ x\varepsilon R^n: F(x^i) + x^i - x = y\}$$

and we see that $\{x\varepsilon R^n: F(x^i) + x^i - x = y\}$ has at most one element, and thus (7.3) follows.

Thus, we obtain the following theorem.

<u>Theorem 7.5</u>: If F satisfies condition 7.3, then F maps S homeomorphically onto F(S).

<u>Proof</u>: The theorem follows directly from Theorem 7.2, Lemma 7.4, and the argument used in the proof of Theorem 4.3.

§. <u>POSTSCRIPT</u>. Recently it was brough to our attention that G. Chichilinsky, M. Hirsch and H. Scarf have also verified the extension of the Gale-Nikaido theorem as considered in [12]. In addition, Y. Kawamura has extended the homeomorphism theorem of Fujisawa and Kuh [5] to the case where the functions are Lipschitz continuous.

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