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ON THE NUMBER OF SOLUTIONS FOR A CLASS OF LINEAR COMPLEMENTARITY PROBLEMS*

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In this note, we consider the linear complementarity problem $w = M z + q$, $w \geqq 0$, $z \geqq 0$, $w^T z = 0$, when all principal minors of M are negative. We show that for such a problem for any q, there are either 0, 1, 2, or 3 solutions. Also, a set of sufficiency conditions for uniqueness are stated.

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§1. Introduction. In this note we consider the following problem: Given a $n \times n$ matrix M and a n-vector q, find vectors w and z which satisfy the inequalities

$$w = M z + q \qquad (1.1)$$

$$w \geqq 0, \ z \geqq 0 \qquad (1.2)$$

$$w^T z = 0 \qquad (1.3)$$

This problem is called the linear complementarity problem, and is a unifying body of knowledge covering topics in mathematical programming, game theory, economic equilibrium theory, mechanics, etc.

In this note we will consider the case when M has all principal minors negative. Such problems were introduced in [4], and several properties were established there. Our aim is to prove that for any given q such problems have either 0, 1, 2, or 3 solutions.

In section 2, we state some properties of (1.1-3) proved in [4], and in section 3, we prove our main result. In section 4, we prove some important properties of such matrices.

§2. Some properties of LCP's with $M \in N$.

We say that a matrix M is in N if and only if all its principal minors are negative. Then, we can prove that:

Lemma 2-1: Let $M \in N$. Then, either $M < 0$ or there exists a $d > 0$ such that $M\,d > 0$.

Proof: Follows from Theorem 6.1, Corollary 6.2, and Theorem 6.3 of [4].

Lemma 2.2: Let $M \in N$, and $M < 0$. Then, for each $q \geqq 0$, (1.1-3) has a solution, and has no solution for $q \ngeq 0$. Also, it has exactly two solutions for $q > 0$.

Proof: Follows from Corollary 6.1 of [4] by noting that the non-degeneracy assumption is not required for $q > 0$.

Given a choice, $A_i \in \{u_i, -M_i\}$,$i = 1, \ldots, n$, we say $pos(A) = \{y : y = A\,x,\ x \geqq 0\}$ is a complementary cone, and F a (n-1)-face of pos(A) if it can be expressed as $F = pos(B)$, where B is nx(n-1) submatrix of A. As in [4], we say that two complementary cones on a face F are properly situated if the intersection of the two cones is F. We call such faces F proper. We can then prove:

Lemma 2.3: Let $M \in N$. Then the two complementary cones incident on the (n-1)-faces of any complementary cone other than pos(I) are properly situated.

Proof: See Lemma 6.2, [4].

§3. The main theorems:

In this section, we prove the main theorems and thus establish the required property that for $M \not< 0$, for any q, there are 1, 2, or 3 solutions, and two when $q \geqq 0$, $q_i = 0$ for some i.

Theorem 3.1:

Let $M \in N$ and $q \in R^n$. Then, if $M \not< 0$ and $q \not\leq 0$, (1.1-3) has a unique solution.

Proof: We will prove this theorem by induction. Since no such 1×1 matrix exists, the theorem is true for $p = 1$. Now, assume that the theorem is true for $p \times p$ matrices, for $p = 1, 2, \ldots, r$. We now show that the theorem is true for $p = r + 1$.

Let $q = -Md < 0$, $d > 0$ (such a d exists from Lemma 2.1.). Then (1.1-3) has a solution $w^1 = 0$, $z^1 = d$. Now, assume that there is another solution, (w^2, z^2) to (1.1-3) such that $w^1 \neq w^2$, $z^1 \neq z^2$. Let $I = \{i : z_i^2 > 0\}$ and $J = \{i : z_i^2 = 0\}$.

Now, $|I| \geqq 2$ since the contrary implies that $q_i > 0$ for some i. Now, if $|I| = p - 1$, then, since q lies in two adjacent cones on a facet of pos(-M), we have a contradiction, as all faces of pos(-M) are proper. See Lemma 2.3. For simplicity, assume $I = \{1, 2, \ldots, m\}$ and $J = \{m + 1, \ldots, p\}$.

Rewriting (1.1) as

$$\begin{bmatrix} w_1^2 \\ \hline w_2^2 \end{bmatrix} = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] \begin{bmatrix} z_1^2 \\ \hline z_2^2 \end{bmatrix} + \begin{bmatrix} q_1 \\ \hline q_2 \end{bmatrix} \tag{3.2}$$

where $z_1^2 > 0$, $z_2^2 = 0$, we note that $- M_{11} z_1^2 = q_1 < 0$. Hence, $M_{11} \not< 0$.

We will now construct a p-vector v and a number t^* such that (1.1-3) with $q + t^* v$ has two solutions (w^3, z^3) and (w^4, z^4) with $z^3 > 0$ and $\{i : z_i^4 > 0\} = \{1,2, . . ., m + 1\}$.

Since $M_{11} \not< 0$, there is a m-vector $c > 0$ such that $-M_{11} c < 0$. For $\epsilon \geqq 0$, define

$$v(\epsilon) = \begin{bmatrix} - M_{11} \\ - M_{21} \end{bmatrix} c - \epsilon u_{m+1} + \epsilon \sum_{j=m+2}^{p} u_i$$

$$= \sum_{i=1}^{m} - M_i c_i - \epsilon u_{m+1} + \epsilon \sum_{j=m+2}^{p} u_i$$

Consider the system

$$- \sum_{i=1}^{n} M_i \Delta z_i^1 = v(\epsilon)$$

For sufficiently small $\epsilon > 0$, the above system must satisfy $\Delta z_i^1 > 0$, $i = 1, . . ., m$. Also, consider the system

$$- \sum_{i=1}^{m} M_i \Delta z_i^2 + \sum_{i=m+1}^{p} u_i \Delta w_i^2 = v(\epsilon)$$

which has a unique solution:

$$\Delta z_i^2 = c_i \qquad i = 1, . . ., m$$

$$\Delta w_i^2 = \begin{cases} -\epsilon & i = m + 1 \\ \epsilon & i = m + 2, . . ., p \end{cases}$$

Define $\Delta z_i^2 = 0$, $i \in J$ and $\Delta w_i^2 = 0$, $i \in I$, and thus p vectors Δz^2 and Δw^2, and we consider the two solutions $(0, z^1 + t \Delta z^1)$, $(w^2 + t \Delta w^2, z^2 + t \Delta z^2)$ for $t \geqq 0$, when q in (1.1) is replaced by $q + t v(\epsilon_0)$ for some sufficiently small $\epsilon_0 > 0$ for which $\Delta z_i^1 > 0$, $i = 1, . . ., m$.

By the construction of $v(\varepsilon_0)$, we have that

$$z_i^1 + t \Delta z_i^1 > 0 \qquad i = 1, \ldots, m$$

$$z_i^2 + t \Delta z_i^2 > 0 \qquad i = 1, \ldots, m$$

$$w_j^2 + t \Delta w_j^2 > 0 \qquad j = m+2, \ldots, p .$$

$$\Delta w_{m+1}^2 \leq 0 .$$

Now, for some sufficiently large $t = t^*$, we will have either $z_j^1 + t^* \Delta z_j^1 = 0$ for some $j = m+1, \ldots, p$ or $\delta = w_{m+1}^2 + t^* \Delta w_{m+1}^2 = 0$. But, if any $z_j^i(t^*) = z_j^i + t^* \Delta z_j^i = 0$, $i = 1$, but $\delta > 0$; then we have found two solutions for (1.1-3) with $z_j^1(t^*) = 0$ and $z_j^2(t^*) = 0$. But this is impossible by the induction hypothesis, since then we would have found two solutions for the system

$$w_1^2 = M_{11} z_1^2 + q_1$$

extracted from (3.2) with M_{11} a $(r \times r)$ matrix. Hence, $\delta = 0$, and the two solutions for $q + t^* v(\varepsilon_0)$ in pos(-M) satisfy the required property. Now, continuing the above procedure, we will have found two solutions, one with $|I| = p - 1$, which is a contradiction. Thus, for $q = -Md$, $d > 0$ has a unique solution.

Now, for any $q \not\leq 0$, consider the line $L(\lambda) = (1 - \lambda)(-Md) + \lambda q$. For $\lambda = 0$, $L(\lambda)$ lies in the unique cone, C, say. Then, either $L(\lambda)$ lies inside C, when q has a unique solution, or the line cuts a facet of C. Since all facets of C are proper, the line leaves C and enters C′. Thus, if the remaining line lies inside C′, q has a unique solution. Otherwise, the above argument can be continued, since this line never crosses any facet of pos(I), and thus all facets encountered are proper. This, then completes the induction argument.

<u>Theorem 3.2:</u> Let $q > 0$, and $M \not< 0$. Then (1.1-3) has exactly three solutions.

<u>Proof</u>: Consider the line $L(\lambda) = (1 - \lambda)\,\hat{q} + \lambda\, q$ for some $\hat{q} \not\geq 0$. Then, from Theorem 3.1, $\hat{q}$ lies in exactly one cone C. Now, this line must encounter a facet of pos(I), and would thus add two additional solutions, giving the result.

<u>Theorem 3.3</u>: Let $0 \neq q \geqq 0$, with $q_i = 0$ for some i $M \not< 0$. Then (1.1-3) has exactly two solutions, with one solution degenerate.

<u>Proof</u>: Now, the line from $\hat{q} > 0$ to q will meet all proper facets inside pos(I), but since it ends at the boundary, exactly one solution will be lost.

§4. Further Properties of $M \in N$:

We now establish some further properties of $M \in N$, which also give an indication as to how to construct matrices $M \in N$, $M < 0$.

The first result we prove is:

Lemma 4.1: Let M be a real matrix. Then, $M \in N$ iff all proper principal minors of M^{-1} are positive, **and det $M^{-1} < 0$.**

Proof: Let M and M^{-1} be partitioned as

$$M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] , \qquad M^{-1} = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right] .$$

Then, it is easy to see that

$$\det M \,.\, \det L_{22} = \det \left(\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] \left[\begin{array}{c|c} I & L_{12} \\ \hline 0 & L_{22} \end{array}\right] \right)$$

$$= \det \left[\begin{array}{c|c} M_{11} & 0 \\ \hline M_{12} & I \end{array}\right]$$

$$= \det \; M_{11} \,.$$

and since $M \in N$, we have our result.

As a consequence of Lemma 4.1, we can establish a further structure on the several solutions to (1.1-3) guaranteed by Theorems 3.2 and 3.3.

Theorem 4.2: Let $0 \neq q \geqq 0$, and $M \not< 0$. Then, if $w_i = 0$ in some solution to (1.1-3), then $w_i > 0$ in all other solutions.

Proof: Assume we have two solutions with $w_i^1 = 0$ and $w_i^2 = 0$. Consider the following equivalent system to (1.2):

$$z = M^{-1} w + M^{-1} q \qquad (4.1)$$

and extracting the system

$$\bar{z} = \bar{M}\,\bar{w} + \bar{q} \qquad (4.2)$$

obtained by dropping the i^{th} row and the i^{th} column of M^{-1} in (4.1), we note that $\bar{M}$ has all principal minors positive (from Lemma 4.1), and the reduced complementarity problem has two solutions. This violates a well-known result relating to these systems, and we have a contradiction. (For example, see Samelson, Thrall, Wesler [5]).

We now show how to construct matrices $M \in N$ such that $M < 0$. For this, we introduce the class of matrices Z. We say P is in Z iff $P_{ij} \leqq 0$ if $i \neq j$ and $P_{ii} \geqq 0$ for all i. We also say a matrix P is in Q iff its determinant is negative but all its proper principal minors are positive. Then, we can prove:

Theorem 4.3: Let $P \in Z \cap Q$. Then, $P^{-1} \in N$ and $P^{-1} < 0$.

Proof: Since $P \in Q$, from Lemma 4.1, $M = P^{-1} \in N$.

Now, consider the complementarity problem (1.1-3) with M, and the transformation (4.1) of (1.2). Assume $M \not< 0$. Then, from Theorem 3.3, (1.1-3) has a solution for $q = (1, 0, \ldots, 0)^T$ with $z_i > 0$ for at least one i. Assume $z_1 > 0$. Then, from (4.1) we see that

$$z_1 = \sum_{j=2}^{n} P_{ij}\, w_j - P_{11} < 0$$

since $P_{11} > 0$ and $P_{ij} \leqq 0$, and we have a contradiction. Now, assume $z_i > 0$ for $i \neq 1$, and without loss of any generality, let $i = n$. Extract the system (4.2) by dropping the n^{th} equation in (1.2) and the n^{th} column of P. Since $\bar{M}$ has all principal minors positive, from the theorem of [5], (4.2) has the unique solution $\bar{z} = 0$, $\bar{w} = (1, 0, \ldots, 0)^T$.

Now, the extended solution to (1.4) also satisfies (1.1-3), we must have $z = 0$, which is a contradiction. And we thus have our theorem.

§5. Concluding Remarks:

Recently A. Mas-Collel [3] has generalized the Gale-Nikaido Theorem [1] to hypotheses which considerably weaken the positive principal minors property. He also proved this result for situations with negative principal minors. An important use of Theorem 3.1 has been made by Kojima and Saigal [2] to prove some uniqueness theorems for the nonlinear complementarity problems with the hypothesis of negative principal minors on suitable submatrices of the Jacobians.

By observing that for $M \in N$, M^{-1} has the property that all <u>proper</u> principal minors of M^{-1} are positive, as proved in Lemma 4.1, a simpler proof of Theorem 3.1 can be obtained by using the partition theorem of Samelson, Thrall, Wesler [5].

References

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