DISCUSSION PAPER NO. 325

AN EXTENDED SINGLE PEAK CONDITION IN SOCIAL CHOICE

by

Ehud Kalai*

and

Zvi Ritz*+

May 1978

*Department of Managerial Economics and Decision Sciences
Graduate School of Management,
Northwestern University.

*+Ritz's research was supported by a grant from the
National Science Foundation, No. SOC76-20953.

An Extended Single Peak Condition in Social Choice

The purpose of this note is to show that the single peak condition (see Black [1948] and Arrow [1963]) is a special case of a simpler condition of inseparability. This inseparability condition characterizes a large family of domains that admit nondictatorial Arrow-type social welfare functions. It follows, by Maskin [1976a] or Kalai-Muller [1977], that they also admit Gibbard-Satterthwaite([1973][1975]) nonmanipulable voting procedures.

Let A denote a set of alternatives with at least two elements, and let Σ denote the set of all transitive anti-symmetric total binary relations on A. Let Ω, a non-empty subset of Σ, denote the admissible preference relations in the society. For an integer n ($n \geq 2$) an n-person social welfare function (SWF) on Ω is a function $f : \Omega^n \to \Sigma$ which satisfies the following conditions.

1. Unanimity. For every $P \in \Omega^n$, if $P = (p_1, p_2, \ldots, p_n)$, $x, y \in A$ and for $i=1,2,\ldots,n$, xp_iy then $xf(P)y$.

2. Independence of irrelevent alternatives (IIA). For $x,y \in A$ and $P,Q \in \Omega^n$ if [xp_iy if and only if xq_iy for $i=1,2,\ldots,n$] then [$xf(P)y$ if and only if $xf(Q)y$].

f is dictatorial if there exists an i, $1 \leq i \leq n$, for which $f(P) = p_i$ for every $P \in \Omega^n$. f is nondictatorial if it is not dictatorial. Ω is nondictatorial if it admits a nondictatorial n-person SWF f. (This definition is independent of n by

Maskin [1976b] and Kalai-Muller [1977]).

Ω is said to contain an inseparable ordered pair (IOP) of alternatives if there are $s,t \in A$ such that

i. For some $p,q \in \Omega$ spt and tqs, i.e. the pair (s,t) is not trivial, and

ii. For no $p \in \Omega$ and $x \in A$ spxpt, i.e. (s,t) is inseparable.

Theorem:

If Ω contains an inseparable ordered pair of alternatives then Ω is nondictatorial.

Proof:

Let (s,t) be an inseparable ordered pair in Ω. Let h be any SWF on the set $\{s,t\}$ in which person 1 is not decisive for the ordered pair (s,t). For example use majority rule, or any dictator different from 1.

Define $f:\Omega^n \to \Sigma$ as follows.

For $x,y \in A$ and $P \in \Omega^n$, if $\{x,y\} = \{s,t\}$ and sp_1t, then $xf(P)y$ if and only if $xh(P)y$. In any other case $xf(P)y$ if and only if xp_1y. That f is well defined and obeys both unanimity and IIA is very easy to show. So it remains to prove that f is transitive. Suppose that for $x,y,z \in A$ $(x \neq y, y \neq z)$ and $P \in \Omega^n$, the social outcome is $xf(P)yf(P)z$. We have to show that in all the following cases, this implies $xf(P)z$.

Case 1. $x \notin \{s,t\}$, $y \notin \{s,t\}$. Then, by definition, $xf(P)yf(P)z$ implies xp_1yp_1z, which in turn implies xp_1z and therefore $xf(P)z$.

Case 2. $s=x$.

i) $y=t$. Then $xf(P)y$ implies sp_1t and $sh(P)t$ and therefore $z \neq s$. $yf(P)z$ implies tp_1z. Hence it is sp_1z which implies $xf(P)z$.

ii) $y \neq t$. Then $xf(P)y$ implies sp_1y, which implies $z \neq t$ (otherwise it is sp_1yp_1t - contradicting the inseparability condition). Therefore it is xp_1z and $xf(P)z$.

Case 3. $x=t$, $y=s$.

i) tp_1s. Then obviously, since everything is determined by 1, $xf(P)z$.

ii) sp_1t. Then $xf(P)y$ implies $th(P)s$ which implies that $z \neq t$. Therefore $yf(P)z$ implies sp_1z which in turn implies tp_1z and therefore $xf(P)z$.

Case 4. $x=t$, $y \neq s$.

i) sp_1t. Then $yf(P)z$ implies $z \neq s$ and hence $xf(P)z$.

ii) tp_1s. Then since everything is determined by 1 obviously $xf(P)z$.

Case 5. $x \notin \{s,t\}$, $y=s$. If sp_1t then $xf(P)y$ implies both xp_1s and xp_1t. Therefore, for every z it is $xf(P)z$. On the other hand, if it is tp_1s then $z \neq t$ and xp_1sp_1z implies $xf(P)z$.

<u>Case 6</u>. $x \notin \{s,t\}$, $y=t$. If sp_1t then $xf(P)y$ implies xp_1t and by inseparability, xp_1s. Therefore, for every z it is xp_1z and therefore it is $xf(P)z$. If, on the other hand, it is tp_1s, then, since everything is determined by 1, $xf(P)z$.

Hence f is transitive and therefore f is SWF on Ω. f is non-dictatorial by the careful choice of h. Q.E.D.

The following examples show that domains containing inseparable pairs are frequent in the social choice literature.

<u>Example 1</u>. Single peaked preferences. Let $q \in \Sigma$, and define the set of single peaked preferences relative to the linear order q by $\Omega q = \{p \in \Sigma$: for every three distinct alternatives x,y,z if xqyqz then it is not the case that xpy and zpy$\}$.

If there are $x,y \in A$ such that for every $z \in A$, xqyqz, then the pair (x,y) is inseparable, i.e. there are two distinct alternatives x,y which are always top and 2nd top ranked by linear order q. Notice that this condition is met for every finite A and for some versions of infinite A.

<u>Example 2</u>. If A contains exactly 3 alternatives ($|A|=3$), then whenever Ω is obtained by deleting one allowed preference in Σ, then Ω contains an inseparable ordered pair. For example, if we delete the preference xpypz, then the pair (x,z) becomes an inseparable ordered pair.

<u>Example 3</u>. If $|A|=2$ then there is an obvious inseparable pair.

References

Arrow, K.J., (1963): Social Choice and Individual Values, J. W. Wiley, New York, 2nd ed.,(1963).

Black, D.J., (1948): "The decision of a committee using a special majority," Econometrica, 16 (1948).

Gibbard, A., (1973): "Manipulation of voting schemes: A general result," Econometrica, 41 (1973), 587-602.

Kalai, E. and E. Muller, (1977): "Characterization of Domains Admitting Nondictatorial Social Welfare Functions and Nonmanipulable Voting Procedures," Journal of Economic Theory, 16 (1977), 457-469.

Maskin, E., (1976a): "On Strategy Proofness and Social Welfare Functions when Preferences are Restricted," Harvard University and Darwin Coolege, Cambridge, 1976.

__________ (1976b): "Social Welfare Functions on Restricted Domains," Harvard University and Darwin College, Cambridge, 1976.

Satterthwaite, M.A., (1975): "Strategy-proofness and Arrow's Conditions: Existence and correspondence theorems for voting procedures and Social Welfare functions," Journal of Economic Theory, 10 (1975), 187-217.