DISCUSSION PAPER NO. 325

AN EXTENDED SINGLE PEAK CONDITION IN SOCIAL CHOICE

by

Ehud Kalai*

and

Zvi Ritz**

May 1978

*Department of Managerial Economics and Decision Sciences
Graduate School of Management,
Northwestern University.

**Ritz's research was supported by a grant from the National Science Foundation, No. SOC76-20053.
An Extended Single Peak Condition in Social Choice

The purpose of this note is to show that the single peak condition (see Black [1948] and Arrow [1963]) is a special case of a simpler condition of inseparability. This inseparability condition characterizes a large family of domains that admit nondictatorial Arrow-type social welfare functions. It follows, by Maskin [1976a] or Kalai-Mullor [1977], that they also admit Gibbard-Satterthwaite([1973][1975]) nonmanipulable voting procedures.

Let $A$ denote a set of alternatives with at least two elements, and let $\Sigma$ denote the set of all transitive antisymmetric total binary relations on $A$. Let $\cap$, a non-empty subset of $\Sigma$, denote the admissible preference relations in the society. For an integer $n$ ($n \geq 2$) an $n$-person social welfare function (SWF) on $\cap$ is a function $f: \cap^n \to \Sigma$ which satisfies the following conditions.

1. **Unanimity.** For every $P \in \cap^n$, if $P = (p_1, p_2, \ldots, p_n)$, $x, y \in A$ and for $i=1,2,\ldots,n$, $x_i \leq y_i$ then $x_f(y)$.

2. **Independence of irrelevant alternatives** (IIA). For $x, y \in A$ and $P, Q \in \cap^n$ if $x_P \leq y$ if and only if $x_Q \leq y$ for $i=1,2,\ldots,n$ then $x_f(P \cap Q)$ if and only if $x_f(i) \leq Q$.

$f$ is **dictatorial** if there exists an $i$, $1 \leq i \leq n$, for which $f(P) = p_i$ for every $P \in \cap^n$. $f$ is **nondictatorial** if it is not dictatorial. $\cap$ is **nondictatorial** if it admits a nondictatorial $n$-person SWF $f$. (This definition is independent of $n$ by
Maskin [1976b] and Kalai-Muller [1977]).

\( \cap \) is said to contain an inseparable ordered pair (IOP) of alternatives if there are \( s, t \in A \) such that

1. For some \( p, q \in \cap \) spt and tqs, i.e. the pair \( (s, t) \) is not trivial, and

2. For no \( p \in \cap \) and \( x \in A \) spxpt, i.e. \( (s, t) \) is inseparable.

**Theorem:**

If \( \cap \) contains an inseparable ordered pair of alternatives then \( \cap \) is nondictatorial.

**Proof:**

Let \((s, t)\) be an inseparable ordered pair in \( \cap \). Let \( h \) be any SWF on the set \([s, t]\) in which person 1 is not decisive for the ordered pair \((s, t)\). For example use majority rule, or any dictator different from 1.

Define \( f: n^N \rightarrow \Sigma \) as follows.

For \( x, y \in A \) and \( p \in n^N \), if \([x, y] = [s, t]\) and \( s p_t \), then \( x f(P)y \) if and only if \( x h(P)y \). In any other case \( x f(P)y \) if and only if \( x p_t y \). That \( f \) is well defined and obeys both unanimity and IIA is very easy to show. So it remains to prove that \( f \) is transitive. Suppose that for \( x, y, z \in A \) \((x \neq y, y \neq z)\) and \( p \in n^N \), the social outcome is \( x f(P)y f(P)z \). We have to show that in all the following cases, this implies \( x f(P)z \).
Case 1. \( x \notin \{s, t\}, y \notin \{s, t\}. \) Then, by definition, \( x f(P) y f(P) z \) implies \( x_1 y_1 P_1 z, \) which in turn implies \( x_1 z \) and therefore \( x f(P) z. \)

Case 2. \( s = x. \)

i) \( y = t. \) Then \( x f(P) y \) implies \( s P_1 t \) and \( s h(P) t \) and therefore \( z \neq s. \) \( y f(P) z \) implies \( t P_1 z. \) Hence it is \( s P_1 z \) which implies \( x f(P) z. \)

ii) \( y \neq t. \) Then \( x f(P) y \) implies \( s P_1 y, \) which implies \( z \neq t \) (otherwise it is \( s P_1 y P_1 t \) - contradicting the inseparability condition). Therefore it is \( x P_1 z \) and \( x f(P) z. \)

Case 3. \( x = t, y = s. \)

i) \( t P_1 s. \) Then obviously, since everything is determined by \( l, x f(P) z. \)

ii) \( s P_1 t. \) Then \( x f(P) y \) implies \( t h(P) s, \) which implies that \( z \neq t. \) Therefore \( y f(P) z \) implies \( s P_1 z \) which in turn implies \( t P_1 z \) and therefore \( x f(P) z. \)

Case 4. \( x = t, y \neq s. \)

i) \( s P_1 t. \) Then \( y f(P) z \) implies \( z \neq s \) and hence \( x f(P) z. \)

ii) \( t P_1 s. \) Then since everything is determined by \( l \) obviously \( x f(P) z. \)

Case 5. \( x \notin \{s, t\}, y = s. \) If \( s P_1 t \) then \( x f(P) y \) implies both \( x P_1 s \) and \( x P_1 t. \) Therefore, for every \( z \) it is \( x f(P) z. \) On the other hand, if it is \( t P_1 s \) then \( z \neq t \) and \( x P_1 s P_1 z \) implies \( x f(P) z. \)
Case 6. \( x \notin \{s, t\}, y = t \). If \( sp_t \) then \( xf(P)y \) implies \( xp_t \) and by inseparability, \( xp_t \). Therefore, for every \( z \) it is \( xp_tz \) and therefore it is \( xf(P)z \). If, on the other hand, it is \( tp_t \), then, since everything is determined by \( 1, xf(P)z \).

Hence \( f \) is transitive and therefore \( f \) is SWF on \( \Sigma \). \( f \) is non-dictatorial by the careful choice of \( h \). Q.E.D.

The following examples show that domains containing inseparable pairs are frequent in the social choice literature.

**Example 1.** Single peaked preferences, let \( q \in \Sigma \), and define the set of single peaked preferences relative to the linear order \( q \) by \( \Omega_q = \{ p \in \Sigma : \) for every three distinct alternatives \( x, y, z \) if \( xqyqz \) then it is not the case that \( xy \) and \( zpy \} \).

If there are \( x, y \in A \) such that for every \( z \in A \), \( xqyqz \), then the pair \( (x, y) \) is inseparable, i.e., there are two distinct alternatives \( x, y \) which are always top and 2nd top ranked by linear order \( q \). Notice that this condition is met for every finite \( A \) and for some versions of infinite \( A \).

**Example 2.** If \( A \) contains exactly 3 alternatives (\(|A| = 3\)), then whenever \( \emptyset \) is obtained by deleting one allowed preference in \( \Sigma \), then \( \emptyset \) contains an inseparable ordered pair. For example, if we delete the preference \( xqyz \), then the pair \( (x, z) \) becomes an inseparable ordered pair.

**Example 3.** If \(|A| = 2\) then there is an obvious inseparable pair.
References


