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A GAME OF BARTER WITH BARRIERS TO TRADE

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Explaining the allocation of resources under voluntary exchange is a fundamental problem for economic theory. We here analyze a simple but natural model of such exchange in which agents trade directly with one another rather than with an impersonal market. Our analysis of this model is game-theoretic. In particular, treating the process of exchange as a game in strategic form, we show that the allocations corresponding to strong Nash equilibria of this game coincide with the core of the underlying economy.

In contrast with most models of exchange, we do not automatically assume that all agents are freely able to trade with one another. Instead, we introduce a structure on the set of traders and specify that two agents can trade only if they are linked in this structure. The absence of such a link indicates the presence of legal, institutional or physical barriers which prevent direct communication between the agents in question or the flow of commodities between them. The introduction of such a structure, which was first suggested by Myerson [6], provides rich opportunities for modeling and analysis of alternative forms of market organization. As an illustration of these possibilities, we examine the question of whether there is an advantage to being a middleman. To this end, we compare the core (strong Nash equilibrium) allocations when all pairs of agents can trade directly with one another with those in the same economy when there is one agent who can trade with everyone but no bilateral exchanges not involving this middleman are possible. Using the theory of market games, we are able to show that for the case of three players, the middleman must gain, although this is not true in general three person games. We also present an example of a five person market game in which being a middleman is disadvantageous.

There have recently been a number of applications of games in strategic form to the analysis of exchange. Among the most prominent of these are [8], [9] and [10], in which the Nash equilibrium is the solution concept studied. In our model, the Nash equilibria are not particularly interesting because they are too numerous. On the other hand, the strong Nash equilibria are of interest both because of their relationship with the core and because they accentuate the essentially cooperative nature of exchange.

Description of the Game

We consider an economy described by the characteristics of the n traders in the economy and by the physical possibilities for communication and trade between the agents. The characteristics a^i of trader i consist of his endowment ω^i of goods, his consumption set X^i and his preferences $\succsim^i$ over $X^i \subset R^m$, where m is the number of commodities. We assume $\omega^i \in X^i$, all i. The possibilities for communication and trade are described by a trading structure, which is a graph g with n nodes.[1] The nodes are identified with the respective traders and a link <u>ij</u> belongs to g if i and j are able to communicate and to exchange commodities. The desired interpretation is that the absence of a link <u>ij</u> indicates the existence of legal, institutional or physical barriers preventing communication and trade between i and j. We denote such an economy E by the pair (a,g), where $a = (a^1,\ldots,a^n)$, $a^i = (\omega^i, X^i, \succsim^i)$, and g is the communications graph for this economy. Obvious special cases are those in which g is connected or even complete.

An allocation for E is an n-tuple $(x^1,\ldots,x^n)$ such that $x^i \in X^i$, $i=1,\ldots,n$, and $\sum x^i = \sum \omega^i$. Let T be a coalition, i.e. a non-empty subset of $N = \{1,\ldots,n\}$, and let $\{T^1,\ldots,T^k\}$ be the partition of T such that each T^i is g-connected and no strict superset of any T^i is g-connected. An allocation x is feasible for T if, for each $j=1,\ldots,k$, $\sum_{T^j} x^i = \sum_{T^j} \omega^i$. If an allocation is feasible for N, we simply say it is feasible. A coalition T can improve upon an allocation x if there exists an allocation y that is feasible for T such that $y^i \succ^i x^i$ for all $i \in T$. The set of allocations that are feasible and which cannot be improved upon by N are called Pareto optimal. The set of feasible allocations which no coalition can improve upon is called the core of E. Note that in considering the core of E it is sufficient actually to consider only the g-connected coalitions of E.

An n-person non-cooperative game (a game for short) G is a pair $(S,\precsim)$ where $S = S^1 \times \ldots \times S^n$, $S^i \neq \emptyset$, and $\precsim = (\precsim^1,\ldots,\precsim^n)$ are n complete pre-orders on S. The set S^i is called player i's strategy set, and $\precsim^i$ is his preference ordering over strategies. It is often natural to think of this ordering as being induced by his preferences over the outcomes arising from the strategies.

If s and t belong to S, and T is a non-empty set of players, define the strategy $(s|t^T)$ by replacing the i^{th} coordinate s^i in s by t^i for each $i \in T$. A strong Nash equilibrium (SNE) of G is a strategy s such that for no non-empty T does there exist a strategy t such that $s <^i (s|t^T)$ for all $i \in T$.

Given an arbitrary economy $E = (a,g)$, we now define a game G_E which describes the underlying structure of trade in this economy.

The players in the game simply correspond to the n traders in E. A strategy s^i for i consists of an (n-1)-tuple $(s^i_1,\ldots,s^i_{i-1}, s^i_{i+1},\ldots,s^i_n)$, where $s^i_j \in R^m$, $s^i_j = 0$ if $\underline{ij} \notin g$ and $\sum_j s^i_j + \omega^i \in X^i$. We interpret positive components of s^i_j as amounts of the corresponding commodities i proposes to received from j and negative components as amounts he proposes to give up in return. Given a strategy n-tuple $s = (s^1,\ldots,s^n)$, a non-empty subset T of N is said to be consistent relative to s if, for any $i \in T$, if $j \in T$ then $s^i_j + s^j_i = 0$

and if $j \notin T$ then $s_j^i = 0$. Since the union of consistent sets is consistent, there exists a unique maximal consistent set of players relative to any strategy. Denote this set by cons(s). Then, define the outcome corresponding to any strategy s as $p(s) = (p^1(s),\ldots,p^n(s))$ where $p^i(s) = \omega^i$ for $i \notin$ cons(s) and $p^i(s) = \omega^i + \sum_j s_j^i$ for $i \in$ cons(s). The players' preferences over the outcomes are simply the individual traders' preferences over their consumption sets, and these induce preferences over strategies in the obvious way.

<u>THEOREM 1</u>: For each $s \in S$, p(s) is a feasible allocation, and for every feasible allocation x there is a strategy $s \in S$ such that p(s) = x and cons(s) = N. Further, if s is a strong Nash equilibrium, there exists $\bar{s} \in S$ with $p(\bar{s}) = p(s)$ such that $\bar{s}$ is a strong Nash equilibrium and $\text{cons}(\bar{s}) = N$.

Thus, the game's outcomes exactly correspond to the relevant allocations in the economy. Also if one restricts attention to consistent strategies the same strategic feasible structure would resul

<u>THEOREM 2</u>: If s is a strong Nash equilibrium then p(s) belongs to the core of E. Further, if preferences are continuous and monotone, for any core allocation x there exists a strong Nash equilibrium with p(s) = x.

Thus, by these two theorems, the set $\mathcal{N} = \{p(s) \mid s \text{ is a SNE}\}$ of strong Nash equilibrium payoffs coincides with Core (E). The existence of a SNE is equivalent to non-emptiness of the core of E, and limit theorems on the core also apply to $\mathcal{N}$. Thus, for example, the competitive equilibria of E are SNE payoffs, and if we have a sequence of economies E_k for which Core (E_k) shrinks to the set of competitive equilibria, the same is true of the strong Nash equilibrium payoffs.

PROOF OF THEOREM 1: That p(s) is a feasible allocation is immediate. To show the second part of Theorem 1, let x be a feasible allocation, and let $N^1,\ldots,N^k$ be the partition of N such that each N^h is g-connected and no strict superset of any N^h is connected. Note that $\sum_{i\in N^h}(x^i-\omega^i) = 0$ for each h, so we need only consider the N^h individually. Take any N^h, say N', and select one player from N', say, without loss of generality, player 1. Since N' is connected, there exists a path between 1 and any other player $j\in N'$. Select for each j a path from 1 to j involving the least possible number of links and yielding a tree that is a subgraph of N'. Now define partial order relations $\leq$ and $<$ on N' by $i \leq j$ if the path from 1 to i passes through j and $i < j$ if $i \leq j$ and $i \neq j$. We define strategies for the players in N' as follows. For each $j\in N'$, if $i\in N'$ is such that $i < j$ and there is no $t\in N'$ with $i < t < j$, set $s_i^j = \sum_{t\leq i}(\omega^t-x^t)$ and $s_j^i=-s_i^j$. Otherwise, set $s_i^j = 0$. Now repeat the same steps for all the other N^h's in the partition. It is clear that for the resulting strategy s, $p(s) = x$, since we are working with tree structures. Further, note that $\text{cons}(s) = N$. For the final part of the Theorem, suppose that s is a SNE but $\text{cons}(s) \neq N$. Then, define $\bar{s}$ by $\bar{s}^i = s^i$, $i\in \text{cons}(s)$ and $\bar{s}^i = 0$ if $i\notin \text{cons}(s)$. The claim is now trivial to verify.

Note that in the case that $\bar{g}$ is connected, the construction used in this proof shows that at most (n-1) links need be active to achieve any feasible allocation. More precisely, at most $(\sum_{h=1}^{k} n^h)-k$ links need be active, where n^h is the cardinality of N^h.

PROOF OF THEOREM 2: Suppose that s is a SNE. By Theorem 1, we may take cons(s) = N. If $p(s) \notin$ Core (E), there exists a g-connected, non-empty coalition T and an allocation x which is feasible for T such that $p^i(s) <^i x^i$ for $i \in T$. Consider the strategy $\hat{s}$ defined by identifying some member of T and then, for $i \in T$, following the procedure used in defining the strategy in the proof of Theorem 1, substituting T for N′ as required. Then $p^i(s|\hat{s}^T) = x^i$, $i \in T$, and s could not be a SNE. Thus, if s is a SNE, $p(s) \in$ Core (E).

Now, take $x \in$ Core(E) and suppose p(s) = x, but s is not a SNE. By Theorem 1, we may again take cons(s) = N. Since s is not a SNE, there exists some non-empty $T \subset N$ and some strategy $\hat{s}$ such that $p^i(s|\hat{s}^T) >^i p^i(s) = x^i$ for all $i \in T$. If $p(s|\hat{s}^T)$ is feasible for T, then T immediately can improve upon x by modifying $\hat{s}$ on T to consist of trades only in T, and we have a contradition. However, it may be that $p(s|\hat{s}^T)$ is not feasible for T, since it may involve the members of T trading with members of the complement of T, although of course these trades must be those made under the original strategy s. In this case, select some individual $j \in T$, and consider the maximal connected subset C of $\text{cons}(s|\hat{s}^T)$ containing j. Observe that C is not empty, since $p^j(s|\hat{s}^T) >^j p^j(s) \gtrsim \omega^j$, so trade involving j does take place at $(s|\hat{s}^T)$. Note that for $t \in C \sim T$, $p^t(s) = p^t(s|\hat{s}^T)$. To see this, note first that since $C \subset \text{cons}(s|\hat{s}^T)$, $p^t(s|\hat{s}^T) = \omega^t + \sum_j (s|\hat{s}^T)^t_j$, while since cons(s) = N, $p^t(s) = \omega^t + \sum_j s^t_j$.

Since $t \notin T$, $(s|\hat{s}^T)^t = s^t$. Thus, $p^t(s) = p^t(s|\hat{s}^T)$. Meanwhile, for $t \in C \cap T \neq \emptyset$, $p^t(s) \prec^t p^t(s|\hat{s}^T)$. Thus, $p^t(s) \precsim^t p^t(s|\hat{s}^T)$ for all $t \in C$, with strict preference holding at least for $t = j$. Further, $p(s|\hat{s}^T)$ is feasible for C, since C is a maximal connected coalition. Now, given continuity and monotonicity of preferences, it is possible to find an allocation y by means of which C can improve upon x.

We should note that it is not completely standard to include the factors modeled by the graph g in the definition of the economy. However, if one identifies an exchange economy simply with the agents' characteristics, but continues to consider only the g-connected coalitions in defining the core, then our results obviously continue to hold.

The Advantage of Being a Middleman

The use of the graph structure on the set of traders permits the modeling of differing forms of market organization. For example, if g is not connected, we are effectively looking at a system of autarkies, while if g is connected but not necessarily complete one obtains models of different forms of economic systems. For example, the complete graph in part (a) of Figure 1 represents a system under which all agents are free to exchange directly with one another, while the graph in part (b) represents the existence of a middleman through which all trade flows. Figure 2 can be considered as representing two economies, where all foreign trade in the one economy must flow through an export-import agency.

PLACE FIGURES 1 and 2 here

A particularly interesting aspect of the question of comparing different systems involves comparative statics analysis on the equilibria as the communications graph is changed, with a view to answering such questions as precisely who gains or loses when a link is introduced or deleted from g? A specific example of this involves the role of the middleman. Suppose, for example, that initially g is complete and then all links except those with one particular agent are broken (as in Figure 1). Since the graph is still connected, the set of Pareto optima is unchanged. However, no multi-player coalitions not involving player 1 are connected in the second situation. Thus, the core and strong Nash equilibria of the second economy will include those in the first. Intuitively, one might expect that the player in the middle would do better in this situation: all trade must flow through him, which ought to improve his position. Somewhat more formally, all coalitions including this player remain as before, while those excluding him are now powerless. The question is then whether he does in fact gain from having this special position. In analyzing this question we assume that each trader's preferences can be represented by a continuous quasi-concave utility function which is strongly monotonic in each commodity. We also assume that the consumption set of every trader consists of the non-negative orthant of the commodity space.

Let $g_c = \{\underline{ij} : i,j \in N\}$ be the complete graph with n traders and let $g_m = \{\underline{1i} : i \in N, i \neq 1\}$. Then g_c represents free trade and g_m represents trading through a middleman (trader 1). Let $E_c = (a, g_c)$ and $E_m = (a, g_m)$ be the economies corresponding to these two structures. We are interested in comparing 1's payoffs at core or, equivalently, SNE allocations in E_c and E_m.

<u>Theorem 3</u>: Let $n \equiv 3$. For every $x \in$ Core(E_m) there exists a $y \in$ Core(E_c) with $x^1 \succsim y^1$.

Thus, for every solution allocation x which arises when he is a middleman there is a solution allocation y in the economy with unrestricted trading which is no better than x: any point becoming a solution when he becomes a middleman is better than some solution under free trade.

To prove this theorem we use the methods of the theory of balanced cooperative games.

An n-person cooperative game is defined to be a collection $V = \{V_S\}_{\emptyset \neq S \subseteq N}$ where, for every non-empty $S \subseteq N$, V_S is a non-empty, strict subset of R^n which is closed and comprehensive (i.e. if $x \in V_S$, $y \in R^n$ and $y^i \leq x^i$ for every $i \in S$ then $y \in V_S$). We let $O_S = \{x \in V_S :$ for some $y \in V_S$ $y^i > x^i$ for every $i \in S\}$. The Core (V) $= V_N \bigcap_{S \neq \emptyset} O_S^c$ where the superscript c denotes the complement of a set.

With an economy E we can associate an n-person cooperative game V(E) as follows. We let u^i be a utility function for trader i normalized so that $u^i(w^i) = 0$. For $\emptyset \neq S \subseteq N$, define

$$V_S = \{v \in R^n \mid \text{there exists some S-feasible allocation } x \text{ with } v^i \leq u^i(x^i) \text{ for every } i \in S\}$$

A collection of coalitions $\{S_r\}_{r \in R}$ is called balanced (see Scarf [7])if there is a collection of non-negative real numbers $\{\delta_r\}_{r \in R}$ such that for every $i \in N$ $\sum_{r: i \in S_r} \delta_r = 1$.

<u>Lemma 1</u>: Let $E_c = (a, g_c)$ be an economy and let V be an n-person game associated with it.

1. For every non-empty coalition S and for every $x, y \in V_S$, if $0 \leq y^j \leq x^j$ for every $j \in S$ and $y \neq x$, then $y \in O_S$. Thus the pareto surface of V_S contains no segments which are parallel to the axes of the players in S.
2. For every $x \in R^n$, $x \in$ Core (V) if and only if for some $y \in$ Core (E_c) $x = (u^1(y^1), u^2(y^2), \ldots, u^n(y^n))$.
3. For $x \in R^n$ let $T_x = \{S \subsetneq N : x \in V_S\}$. If T_x is a balanced collection of sets with weights $\{\delta_S\}_{S \in T_x}$, and if for some $S \in T_x$, $x \in O_S$ and $\delta_S > 0$,

then there exists $y \in V_N$ with $y^i > x^i$ for every $i \in N$.

Proof of Lemma 1: Part 1 follows immediately from the monotonicity of the utility functions. Part 2 follows from the definitions of the cores and from part 1.

To prove part 3 we use a method due to Scarf [7]. Assume x satisfies the hypotheses of the lemma. For every $S \in T_x$ there exists an S-feasible allocation z_S with $x^i \leq u^i(z_S^i)$ for every $i \in S$. Also for some $S' \in T_x$ $z_{S'}$ can be chosen so that $x^i < u^i(z_{S'}^i)$ for every $i \in S'$. Consider the allocation z defined by $z^i = \sum \delta_S z_S^i$, where the summation is over those $S \in T_x$ with $i \in S$. Observe that z^i is a convex combination of the z_S^i's and therefore (by the quasi-concavity of the utility functions) $u^i(z^i) \geq x^i$ for every $i \in N$. Also, $u^i(z^i) > x^i$ for every $i \in S$. Let $y^i = u^i(z^i), i \in N$. Since

$$\sum_{i\in N} z^i = \sum_{i \in N} \sum_{S \in T_x, i \in S} \delta_S z_S^i = \sum_{S \in T_x} \delta_S \sum_{i \in S} z_S^i \leq \sum_{S \in T_x} \delta_S \sum_{i \in S} \omega^i = \sum_{i \in N} \omega^i ,$$

z is feasible and thus $y = (y^1, \ldots, y^n) \in V_N$.

Proof of Theorem 3. Let V be the game associated with E_c, and V' be that associated with E_m. We wish to show that for any $z \in$ core V' there exists $w \in$ core V with $z^1 \geq w^1$. To this end, suppose $z^1 \leq v^1$ for all $v \in B \equiv V_{\{123\}} \cap O^c_{\{12\}} \cap O^c_{\{13\}} \cap O^c_{\{1\}} \cap O^c_{\{2\}} \cap^c_{\{3\}}$, $z \in B$ and $z \notin O_{\{123\}}$. (Note that the core of V' is the set of Pareto optima in B.) If we can show that $z \notin O_{\{23\}}$, then z must also be in the core of V. Then any point in core V' is at least as good for 1 as some point in core V, namely z, and we are done.

Thus, to obtain a contradiction, we will suppose that $z \in O_{\{23\}}$. Let $T = \{S \subsetneq N : S \neq \emptyset,\ S \neq \{23\}, \text{ and } z \in V_S\}$. Then $1 \in \bigcup_{S \in T} S$, since otherwise the comprehensiveness of the V_S sets would imply that z does not minimize z^1. If $\{1\} \in T$ then the proof is completed because Lemma 1.3 would contradict the Pareto optimality of z ($\{1\},\{23\}$ is a balanced collection). Then suppose $\{1\} \notin T$. We claim $i \in \bigcup_T S$, $i=1,2$. For example if $2 \notin \bigcup_T S$, i.e. $z \notin V_{\{2\}} \cup V_{\{12\}}$, then for some small enough $\epsilon > 0$ $z' \equiv z - (0,\epsilon,0)$ also minimizes z^1, while $z' \in B$, $z' \in O_{\{123\}}$ and $z' \in V_S$ for any $S \neq N$ with $2 \notin S$. Therefore for some small enough $\delta > 0$, $z'' = z' + (0,0,\delta)$ we have $z'' \in O_{\{123\}}$ and $z'' \in V_S$ for $\emptyset \neq S \neq N$ and $S \neq \{2,3\}$. Thus z^1 is minimized at an interior point of B, which is impossible. So we assume without loss of generality that $\{12\} \in T$ and either $\{3\} \in T$ or $\{13\} \in T$. If $\{13\} \in T$ then we obtain a contradiction by Lemma 1.3 ($\{12\},\{13\},\{23\}$ is a balanced collection). So we are left with the cases $T = \{\{12\},\{3\}\}$ or $T = \{\{12\},\{3\},\{2\}\}$. In either of these two cases it is possible to transfer a small amount of utility from 1 to 2 while still staying in B, which again yields a contradiction. Thus, $z \notin O_{\{23\}}$, so z is unblocked in V', and thus all points in the core of V' are at least as good for the middleman as some point in the core of V.

This result seems intuitive, and it is surprising that the proof is so involved. However, the following examples show that the analogue of this theorem is not true for general three-person cooperative games, or for market games with a larger number of players.

We first show an example of a 3-person non-market, cooperative game in which the middleman may lose. Let ϵ be a small non-negative real number and define the game V (=V(ϵ)) as follows:

$$V_{\{i\}} = \{v \in R^3 : v^i \leq 0\} \quad \text{for } i=1,2,3;$$

$$V_{\{123\}} = \{v \in R^3 : \text{for some } x \in \text{ convex hull } \{A,B,C,F\},\ v \leqq x\};$$

$$V_{\{12\}} = \{v \in R^3 : v^1 + v^2 \leqq .01\};$$

$$V_{\{13\}} = \{v \in R^3 : v^1 + v^3 \leqq 1\};$$

$$V_{\{23\}} = \{v \in R^3 : \text{ for some } x \in \text{ convex hull } \{C,D,E\},\ v \leqq x\}.$$

where

$$A = (1+\epsilon,0,0)\ ,\ B = (1,1,0)\ ,\ C = (0,0,1)\ ,$$

$$D = (0,1,0)\ ,\ E = (0,\tfrac{1}{2},\tfrac{3}{4})\ ,\ F = (0,2,0)\ ,$$

$$G = (1,0,0)\ .$$

<u>Place Figure 3 here</u>

The core of V(ϵ) is B = {(1,1,0)}. When the coalition {1,2} is no longer allowed to block the core becomes all of the triangle ABC with the exception of a small set of points near C. Thus the middleman is worse off at most of the points of the new core. Moreover, by letting $\epsilon \to 0$ we can make the proportion of the points where the middleman loses go to one. When $\epsilon=0$ the middleman is never better off and he is worse off at almost all the new points. Note, however, that when $\epsilon=0$ V_N contains segments which are parallel to the v^2 axis. Note too that these games are not balanced in the sense of Billera and Bixby [2], and so they could not come from markets.

Thus, the condition that the game be balanced is crucial to the middleman gaining in 3 person games. One might still hypothesize that Theorem 3 would continue to hold for market games with any finite number of players. However, the following example indicates that this is not the case.

Consider the economy with 5 goods and 5 people, A, B, C, D and E, whose utility functions and endowments are as follows:

$$U^A(x_1,x_2,x_3,x_4,x_5) = x_1\, x_2 \qquad \omega^A = (1,0,0,0,0)$$

$$U^B(x_1,x_2,x_3,x_4,x_5) = x_1\, x_2 \qquad \omega^B = (0,1,0,0,0)$$

$$U^C(x_1,x_2,x_3,x_4,x_5) = (x_1+x_5)\, x_3 \qquad \omega^C = (0,0,1,0,0)$$

$$U^D(x_1,x_2,x_3,x_4,x_5) = (x_2+x_5)\, x_4 \qquad \omega^D = (0,0,0,1,0)$$

$$U^E(x_1,x_2,x_3,x_4,x_5) = (x_3+x_4)(x_5+8) \qquad \omega^E = (0,0,0,0,1)$$

With all coalitions allowed, the coalition {A B} can block any allocation unless $U^A + U^B \geq 1$. There are no feasible allocations for which this is possible except those in which A and B end up with all of the goods x_1 and x_2. Hence core allocations will be those in which the goods x_1 and x_2 are efficiently distributed among A and B and the goods x_3, x_4, and x_5 are divided among C, D and E in such a way that no coalition can block the allocation.

Consider now the coalition {C E}. If C gets the bundle $(0,0,\alpha,0,1)$ and E gets $(0,0,1-\alpha,0,0)$, $U^C = \alpha$ and $U^E = 8(1-\alpha)$. If $\alpha \geq 1/9$ the marginal rate of substitution for C of good 5 for good 3 is $1/\alpha \leq 9$. For E the marginal rate of substitution between goods 5 and 3 is $8/1-\alpha \geq 9$, and hence this allocation of the goods of C and E is efficient with respect to them. Thus among the allocations which {C,E} can block are any which give $U^C = \alpha \leq 1/9$ and $U^E < 8(1-\alpha)$. Similarly $\{D,E\}$ can block any allocation which gives D, $U^D = \alpha \leq 1/9$ and $U^E < 8(1-\alpha)$.

Now consider the coalition $\{C\,D\,E\}$. If C gets $(0,0,2\alpha,0,1/2)$ and D gets $(0,0,0,2\alpha,1/2)$, E gets $(0,0,2-4\alpha,0)$. The utilities are $U^C = U^D = \alpha$ and $U^E = 8(2-4\alpha)$. If $\alpha \geq 1/18$, $MRS^C_{5,3} = (1/2)/2\alpha \leq 9/2$ while $MRS^E_{5,3} = 8/(2-4\alpha) > 9/2$. Thus the allocation is efficient with respect to C and E. The same argument shows that it is efficient with respect to D and E. As well, it is trivially efficient with respect to C and D, and no reallocation among the three agents can make all better off. Thus this allocation between C, D, and E is efficient. Thus any allocation which gives utilities $U^C = U^D = \alpha \geq 1/18$ and $U^E = 8(2-4\alpha)$ cannot be blocked by $\{C\,D\,E\}$. However, if $\alpha > 1/3$ the coalition $\{C\,E\}$ (or $\{D\,E\}$) can block since the utility which E can be guaranteed while C gets $U^C = \alpha$ is $U^E = 8(1-\alpha) > 8(2-4\alpha)$.

We now claim that at any core allocation, we must have $U^E \geq 16/3$. To see this, note that if $U^E = \beta$, then $\{C\,D\,E\}$ can achieve $U^C = U^D = (16-\beta)/32 = \alpha$ via the allocation given above, with $\alpha = 16-\beta/32$. But if $\beta < 16/3$, this allocation is blocked by either $\{C\,E\}$ or $\{D\,E\}$. Since if either E can get $8(1-\alpha) = 8-8(16-\beta)/32 = 4 + \beta/4 > \beta$ while the other member gets α, and it is then possible to redistribute so as to make both members strictly better off.

Now suppose we restrict blocking to only those coalitions containing E. Consider the allocation and associated utilities below:

$$x^A = (0,0,0,0,0) \qquad U^A = 0$$
$$x^B = (0,0,0,0,0) \qquad U^B = 0$$
$$x^C = (1,0,2/3+\epsilon,0,1/2) \qquad U^C = 1 + 3/2\,\epsilon$$
$$x^D = (0,1,0,2/3+\epsilon,1/2) \qquad U^D = 1 + 3/2\,\epsilon$$
$$x^E = (0,0,1/3-\epsilon,1/3-\epsilon,0) \qquad U^E = 16/3 - 16\epsilon$$

We will show that for suitably small ϵ, this allocation is in the core as defined above of the corresponding game V'.

The coalitions {A E}, {B E} and {A B E} clearly cannot block this allocation. The coalitions {C E} and {D E} cannot block since these coalitions cannot give C or D utility greater than 1. The coalition {C D E} similarly cannot block since they cannot simultaneously guarantee C and D utility greater than 1/2. We will now examine the marginal rates of substitution of C, D, and E to show that this distribution is efficient among them. $MRS^C_{5,3}$ is $(3/2)/(2/3+\epsilon) < 9/4$ while $MRS^E_{5,3} = 8/(2/3 - 2\epsilon) > 12$. Thus this allocation is efficient with respect to C and E. Similarly it is efficient with respect to D and E and trivially so with respect to C and D. Further, no reallocation between the three agents dominates this allocation. Adding A and B to {C D E} cannot yield a blocking coalition either. The only coalitions left to examine are those which contain C or D (but not both), E, and A and/or B.

Consider the coalition {A C E}. Can this coalition block? It will be enough to show that with their combined resources C and E cannot both improve upon their utilities in the proposed allocation. Consider the distribution $x^C = (1,0,\alpha/2,0,1)$, $x^E = (0,0,1-\alpha/2,0,0)$.

This gives $U^C = \alpha$ and $U^E = 8(1 - \alpha/2)$. If $\alpha > 2/5$, $MRS^C_{5,3} = 2/(\alpha/2) = 1/\alpha < 5/2$ and $MRS^E_{5,3} = 8(1 - \alpha/2) > 5/2$, hence the distribution is efficient between them. In the proposed allocation we have $U^E = 16/3 - 16\varepsilon > 8[1-(1+3/2\varepsilon)/2] = 8(1/2 - 3/4\varepsilon)$ for $0 < \varepsilon < 2/15$. Hence even with A's endowment, C and E cannot both improve upon the utilities in the proposed allocation. Clearly adding B to the coalition {A,C,E} will not change this. Also it is clear that the same argument shows that the coalitions {B,D,E} and {A,B,D,E} cannot block the proposed allocation.

Thus the proposed allocation is in the core if only coalitions containing E are allowed to block. Yet this allocation is clearly worse for E than any previous core allocation (when all coalitions were allowed to block). In fact, his minimal utility over core allocations has gone down from 16/3 to 16/3 - 16(4/30) = 3.2.

The preferences in this example are not strictly monotonic, as was assumed in Theorem 3. However, it is possible to modify the example so that this condition holds and yet there are still core allocations when E is a middleman which are worse for him than any point in the core of the original game. Specifically, consider modifying the above example by adding a term $r(\Sigma x_j)$ to each player's utility function, which then becomes strictly monotone for all $r > 0$. We will show that for r sufficiently small, there are still core points when E is a middleman yielding $U^E < 16/3$.

Consider the allocation

$$x^A = (\delta, \delta, 0, 0, 0)$$

$$x^B = (\delta, \delta, 0, 0, 0)$$

$$x^C = (1-2\delta, 0, \tfrac{21}{30}, 0, \tfrac{1}{2})$$

$$x^D = (0, 1-2\delta, 0, \tfrac{21}{30}, 0, \tfrac{1}{2})$$

$$x^E = (0, 0, \tfrac{9}{30}, \tfrac{9}{30}, 0),$$

This corresponds to $\epsilon = 1/30$ in the example above except that a small amount of x_1 and x_2 has been taken from C and D and given to A and B. Arguments similar to those used above show that for $0 < \delta < 1/84$, this allocation yields a utility vector that not only is in the core of the game V' with E as a middleman (and the original preference orderings), but also has the property that, for all coalitions $S \neq N$, it lies outside V'_S. Note that $U^E \leqq 16/3$ at any such point.

Now, if we consider the games $V(r)$ and $V'(r)$ obtained by adding $r\Sigma x_j$ to each player's utility function, it is clear that for any point u^r in $V_S(r)$ (resp., $V'_S(r)$) there is some point $u \in V_S$ (resp., V'_S) with $u_i^r \leqq u_i + r|S|$, $i \in S$. Now consider sequences $\{V(r_k)\}$ and $\{V'(r_k)\}$, and let $u_k \in$ core $V(r_k)$, $u_k \to u$. Then if is simple to show that $u \in$ core V, so that for any $\epsilon > 0$ there exists K such that core $V(r_k)$ is contained in an ϵ neighborhood of core V for $k \geqq K$. This in turn means that for any ϵ, the minimum of U^E on the core of $V(r_k)$ is at least $16/3 - \epsilon$ for all k large enough. If we can now show that for large enough k there are points in the core of $V'(r_k)$ which are arbitrarily close to the utility image of the allocation given above, we are done. To show this, take the utility image $\bar{u}$ of this allocation, and note that for k large enough (r_k small enough), $\bar{u}$ is not blocked by any $S \neq N$, since $V'_S(r_k)$ is within an ε-ball of V'_S. Now consider any Pareto optimal point u_k in $V'(r_k)$ with $u \leqq u_k$. (Such a point must exist since $U^i(x) \leqq U^i(x) + r\Sigma x_j$ for any $x = (x_1, \ldots, x_5)$, with strict inequality if $x \neq 0$). Since $V'_N(r_k)$ is within an ε-ball of V'_N, this sequence converges to u, and, since u_k is unblocked in $V'(r_k)$, we are done.

It is worth noting, however, that although core $V'(r_k)$ contains points strictly worse for E than any in core $V(r_k)$, in contrast to the situation with V and V′ it also contains points which are better for him than any in core $V(r_k)$. This is, in fact, generally true with strict monotonicity.

<u>THEOREM 4</u>: Consider an economy where, for some trader i with strictly monotonic preferences, the initial endowment is not Pareto optimal for $S = N \sim \{i\}$. Then restriction of blocking to coalitions containing i must result in the addition of points to the core which i strictly prefers to any point in the core with unrestricted coalition formation.

<u>Proof</u>: Consider a core allocation in the unrestricted game which gives the central trader utility $\bar{u}^i$ which is at least as high as any other core allocation. At least one trader $j \neq i$ must have utility strictly higher than his initial endowment yields because ω is not Pareto optimal for $N \sim \{i\}$. Now consider a Pareto efficient allocation which gives i the maximal utility possible

subject to the non-central traders' utilities being at least as big as their initial endowment utilities. Compactness of the feasible set and continuity of the utility functions guarantee that there is such an allocation and strict monotonicity implies that it yields $u^i > \bar{u}^i$. By construction, no coalition containing i can block this allocation; hence it is in the core when i is a middleman.

Conclusions and Possible Extensions

The model we have presented here seems a reasonable one for analysis of direct trading between individuals. The strategies are rather natural, and there is a bare minimum of institutional rules involved in defining the outcomes. Yet the resulting allocations are of interest, and the model offers interesting possibilities for applications of the type suggested in the preceding section. Moreover, several extensions would seem feasible and interesting.

One extension which is relatively easy to include is the possibility of production. An obvious approach to this is to use a modeling due to Hurwicz [5], which involves the introduction of fictitious agents with whom the other agents can trade. These agents have zero endowment, flat preferences and a production set in place of the usual agent's characteristics. Naturally, in this context, the definition of the core and the strong Nash equilibrium would require only that the fictitious players not be made worse off (rather than insisting on strict preference for these agents as well). A second approach is to use the model of a coalition production economy originated by Hildenbrand [4]. We hypothesize that our principal results would continue to hold with production.

The introduction of externalities in consumption offers another possible line of investigation. As a first approach to this, one can take the individual preferences to be defined not just over the consumption sets but rather over the space of allocations in a non-trivial fashion. Given the well-known problem with defining an appropriate notion of the core with externalities, one probably cannot hope to obtain an analog of Theorem 2 in the context. Still, the model may be useful and interesting. For example, if g is connected, then one easily shows that any strong Nash equilibrium is Pareto optimal. This, of course, is a formalization of the celebrated Coase theorem [3] that in the absence of transactions costs free bargaining will lead to optimality, regardless of externalities and the assignment of property rights.

A further useful extension would be to treat transactions costs in a more explicit manner. The communications graph approach used here can be interpreted in terms of transactions costs that are either zero (if $\underline{ij} \in g$) or so large as to preclude all trade (if $\underline{ij} \notin g$). Introduction of transactions costs that vary between links and with the volume of trade flowing through a link should provide even richer possibilities for analysis of the efficiency of various forms of market organization. Again, we expect that versions of our results could be obtained in such a model.

Finally, the examples in the preceding section with a disadvantageous middleman are reminiscent of the examples of disadvantageous monopolies (see, e.g., [1]), although the phenomena do differ somewhat. Exploring the nature of the relationship between these two classes of examples and, more generally, the monotonicity properties of the core would seem to be a difficult open problem.

Footnotes

1. Let $N = \{1,2,\ldots,n\}$. A graph on the set of nodes N is a collection of links $\underline{ij}$ where i and j are distinct elements of N and $\underline{ij} = \underline{ji}$. A graph is complete if it contains all the possible links. A path from node i to node j is a collection of nodes $i_1, i_2, \ldots, i_k$ with $i_1 = i$, $i_k = j$ and with $\underline{i_r\ i_{r+1}}$ being a link in the graph for $r=1,2,\ldots,k-1$. A graph is connected if there is a path from any one node to any other node. A graph is a tree if, given any two nodes i and j, there is a unique path between them.

Figure 1

(a) (b)

Figure 2

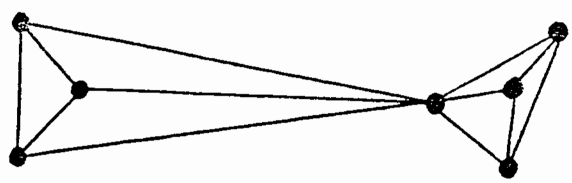
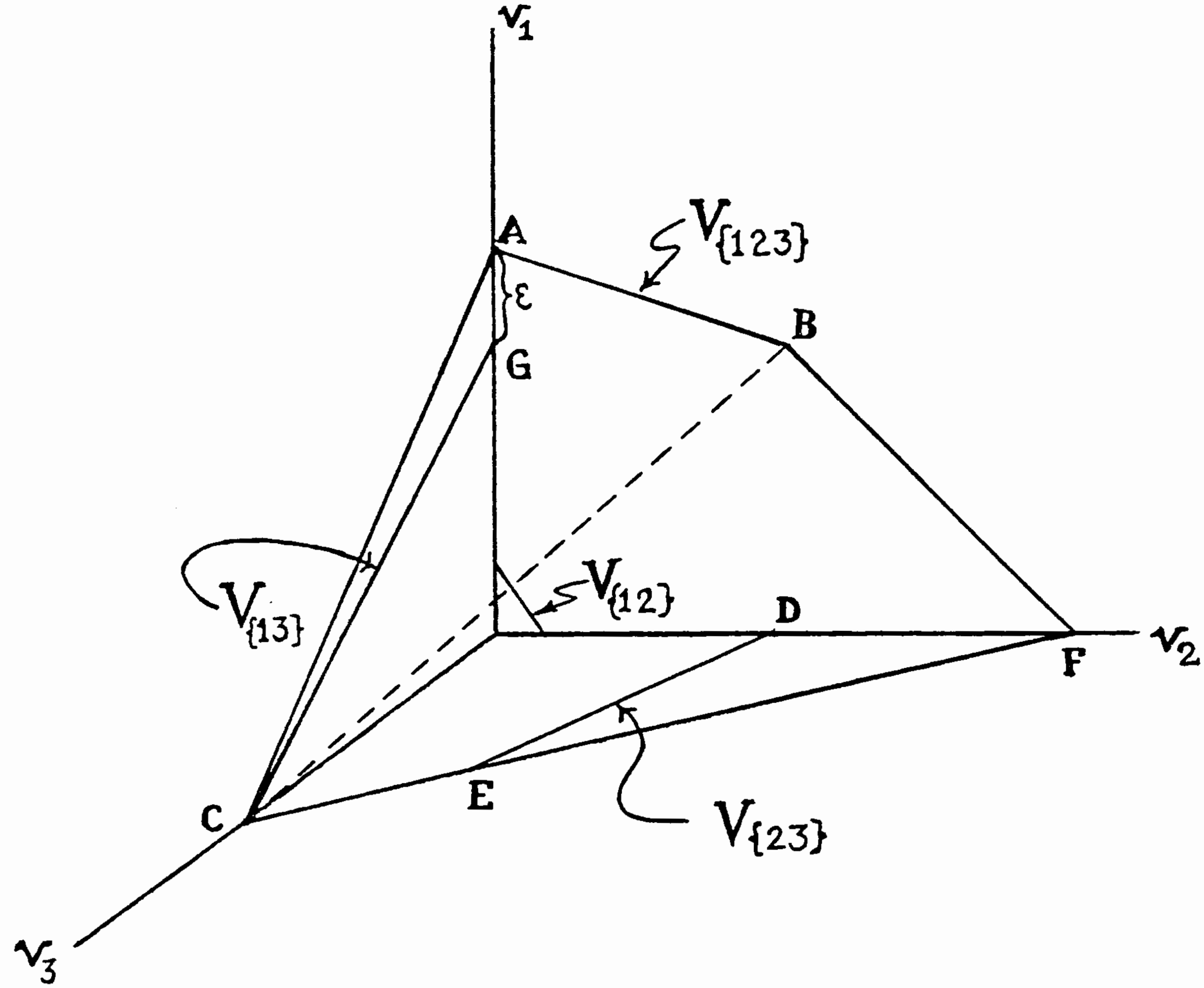


Figure 3



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