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A SIMPLE GAME OF EXCHANGE

by

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Explaining the allocation of resources in a pure exchange, barter economy is a basic problem in economic theory. Over the past two decades the methods of cooperative game theory have been widely applied to the analysis of this problem, and more recently a number of authors have approached the problem using explicit game-theoretic models of a non-cooperative nature. A central focus of both these lines of research has been the identification of relationships between various game-theoretic solutions (especially the core or the Nash equilibrium) and solutions originating in economics (especially the competitive equilibrium). For examples of these types of models and results in the non-cooperative framework see [5],[10],[12],[13], and [14]. Much of the work based on the core is presented in [6]. See also [1] and [3] for models based on the value.

In this note we study a game arising from the process of exchange. An individual's strategy in this game is an amount of each commodity to give to or receive from each other trader with whom it is physically possible for him to trade. The outcome is determined by considering the largest group of traders whose strategies are mutually consistent, carrying out their proposed trades, and leaving all other traders with their initial endowments. The principal result is that the set of strong Nash equilibria of this game coincides with the core of the underlying economy. Thus, *a fortiori*, these equilibria are Pareto optimal.

Apart from this rather straightforward result, the model that we use has three other attractive features. The first of these involves the form and features of the game we define. In particular, the strategies and payoffs in this game are very natural and uncontrived,and there is an exact identification between the feasible allocations of the economy and the possible outcomes of the game. This latter feature means that given any allocation of resources actually achieved, we can specify the strategies that would have brought this about.

The employment of the game theoretic notion of strong Nash equilibrium, instead of Nash equilibrium, is a second attractive feature. A set of strategies is a strong Nash equilibrium if no set of players can change their strategies simultaneously and make all of its members strictly better off. The chief problem with this concept is that in many games such equilibria do not exist. However when a strong Nash equilibrium does exist, as we show to be the case here for a large number of situations, it is very appealing. In particular, for exchange situations arising in economics the notion of a strong Nash equilibrium is especially appropriate since, in contrast to the usual Nash equilibrium concept, it accentuates the fact that it takes more than one person to trade.

A third aspect of interest is our use of an explicit, if simplistic, modeling of barriers to communication and trade. Typically economic models assume that all agents are physically and legally able to communicate and trade with one another. We employ a structure due to Myerson [9] within which the possibility or impossibility of trade between a particular pair of traders is made explicit. This structure appears to offer a useful framework for analysis of a number of issues.

The Game of Exchange

We consider an economy described by the characteristics of the n traders in the economy and by the physical possibilities for communication and trade between the agents. The characteristics a^i of trader i consist of his endowment ω^i of goods, his consumption set X^i and his preferences $\succsim^i$ over X^i. We assume $\succsim^i$ is continuous and increasing in each commodity and that $\omega^i \in X^i$. The possibilities for communication and trade are described by a graph g with n nodes, where the nodes are identified with the respective traders and a link $\underline{ij}$ belongs to g if i and j are able to trade. The desired interpretation is that the absence of a link $\underline{ij}$ indicates the existence of legal, institutional or physical barriers preventing communication and trade between i and j. We denote such an economy E by pair (a,g), where $a = (a^1,\ldots,a^n)$, $a^i = (\omega^i, X^i, \succsim^i)$, and g is the communications graph for this economy. Obvious special cases are those in which g is connected or even complete.

An allocation for E is an n-tuple $(x^1,\ldots,x^n)$ such that $x^i \in X^i$, $i=1,\ldots,n$ and $\sum x^i = \sum \omega^i$. Let T be a coalition, i.e. a non-empty subset of $N = \{1,\ldots,n\}$, and let $\{T^1,\ldots,T^k\}$ be the partition of T such that each T^i is g-connected and no strict superset of any T^i is g-connected.

An allocation x is feasible for T if for each $j=1,\ldots,k$, $\sum_{T^j} x^i = \sum_{T^j} \omega^i$. If an allocation is feasible for N, we simply say it is feasible. A coalition T can improve upon an allocation x if there exists an allocation y that is feasible for T such that $y^i >^i x^i$ for all $i \in T$. The set of allocations that are feasible and which cannot be improved upon by N are called Pareto optimal. The set of feasible allocations which no coalition can improve upon is called the core of E. Note that in considering the core of E it is sufficient actually to consider only the g-connected coations of E.

An n-person non-cooperative game (a game for short) G is a pair $(S,\precsim)$ where $S = S^1 \times \ldots \times S^n$, $S^i \neq \emptyset$, and $\precsim = (\precsim^1,\ldots,\precsim^n)$ are n complete pre-orders on S. The set S^i is called player i's strategy set, and $\precsim^i$ is his preference ordering over strategies. It is often natural to think of this ordering as being induced by his preferences over the outcomes arising from the strategies.

If s and t belong to S, and T is a non-empty set of players, define the strategy $(s|t^T)$ by replacing the i^{th} coordinate s^i in s by t^i for each $i \in T$. A strong Nash equilibrium (SNE) of G is a strategy s such that for no non-empty T does there exist a strategy t such that $s <^i (s|t^T)$ for all $i \in T$.

Given an arbitrary economy $E = (a,g)$, we now define a game G_E which describes the underlying non-cooperative structure of trade in this economy.

The players in the game simply correspond to the n traders in E. A strategy s^i for i consists of an (n-1)-tuple $(s^i_1,\ldots,s^i_{i-1}, s^i_{i+1},\ldots,s^i_n)$, where $s^i_j \in R^m$, $s^i_j = 0$ if $\underline{ij} \notin g$ and $\sum_j s^i_j + \omega^i \in X^i$. We interpret positive components of s^i_j as amounts of the corresponding commodities i proposes to received from j and negative components as amounts he proposes to give up in return . To define the outcomes, we make use of the concept of a consistent set of players. A non-empty subset T of N is consistent relative to the strategy s if for any $i \in T$ if $j \in T$ the $s^i_j + s^j_i = 0$ and if $j \notin T$ then $s^i_j = 0$. Since the union of consistent sets is consistent, there exists a unique maximal consistent set of players relative to any strategy. Denote this set by cons(s). Then, define the outcome corresponding to any strategy s as $p(s) = (p^1(s),\ldots,p^n(s))$, where $p^i(s) = \omega^i$, $i \notin$ cons(s) and $p^i(s) = \omega^i + \sum_j s^i_j$ for $i \in$ cons(s). The players' preferences over the outcomes are simply the individual traders' preferences over their consumption sets, and these induce preferences over strategies in the obvious way.

<u>THEOREM 1</u>: For each $s \in S$, p(s) is a feasible allocation, and for every feasible allocation x there is a strategy $s \in S$ such that p(s) = x and cons(s) = N.

Thus, the game's outcomes exactly correspond to the relevant allocations in the economy.

<u>THEOREM 2:</u> If s is a strong Nash equilibrium, then there exists $\bar{s}$ which is also a strong Nash equilibrium with $p(\bar{s}) = p(s)$ and cons(s) = N. If s is a strong Nash equilibrium

then p(s) belongs to the core of E. Further, for any core allocation x there exists a strong Nash equilibrium with p(s) = x.

Thus, by these two theorems, the set $\mathcal{N} = \{p(s) | s \text{ is a SNE}\}$ of strong Nash equilibrium payoffs coincides with Core (E). The existence of a SNE is equivalent to non-emptiness of the core of E, and limit theorems on the core also apply to $\mathcal{N}$. Thus, for example, the competitive equilibria of E are SNE payoffs, and if we have a sequence of economies E_k for which Core (E_k) shrinks to the set of competitive equilibria, the same is true of the strong Nash equilibrium payoffs.

PROOF OF THEOREM 1: That p(s) is a feasible allocation is immediate. To show the second part of Theorem 1, let x be a feasible allocation, and let $N^1,\ldots,N^k$ be the partition of N such that each N^h is g-connected and no strict superset of any N^h is connected. Note that $\sum_{N^h} (x^i - \omega^i) = 0$ for each h, so we need only consider the N^h individually. Take any N^h, say N', and select one player from N', say, without loss of generality player 1. Since N' is connected, there exists a path between 1 and any other player $j \in N'$. Select for each j a path from 1 to j involving the least possible number of links yielding a tree that is a subgraph of N'. Now define partial order relations $\leq$ and $<$ on N' by $i \leq j$ if the path from 1 to i passes through j and $i < j$ if $i \leq j$ and $i \neq j$. We define strategies for the traders in N' as follows. For every i which is an immediate follower of a j in $<$, i.e. $i < j$ and for no $t \in N'$ $i < t < j$, let $s_i^j = \sum_{t \leq i} (w^t - x^t)$ and $s_j^i = -s_i^j$. For all other $i, j \in N'$ let $s_i^j = 0$. Now repeat the same steps for all the other N^h's in the partition.

It is clear that for the resulting strategy s, $p(s) = x$, since we are working with tree structures. Further, note that $cons(s) = N$.

Note that in the case that g is connected, the construction used in this proof shows that at most (n-1) links need be active to achieve any feasible allocation. More precisely, at most $(\sum_{h=1}^{k} n^h)-k$ links need be active, where n^h is the cardinality of N^h.

PROOF OF THEOREM 2: If s is a SNE but $cons(s) \neq N$, define $\bar{s}$ by $\bar{s}^i = s^i$, $i \in cons(s)$ and $\bar{s}^i = 0$ if $i \notin cons(s)$. The first claim is now trivial to verify.

Suppose now that s is a SNE with $cons(s) = N$, but $p(s) \notin$ Core (E). Then there exists a g-connected, non-empty coalition T and an allocation x which is feasible for T such that $p^i(s) <^i x^i$ for $i \in T$. Consider the strategy $\hat{s}$ defined by identifying some member of T and then, for $i \in T$, following the procedure used in defining the strategy in the proof of Theorem 1, substituting T for N' as required. Then $p^i(s|\hat{s}^T) = x^i$, $i \in T$, and s could not be a SNE. Thus, if s is a SNE, $p(s) \in$ Core (E).

Now, take $x \in Core(E)$ and suppose $p(s) = x$, but s is not a SNE. By Theorem 1, we may take $cons(s) = N$. Since s is not a SNE, there exists some non-empty $T \subset N$ and some strategy $\hat{s}$ such that $p^i(s|\hat{s}^T) >^i p^i(s) = x^i$ for all $i \in T$. If $p(s|\hat{s}^T)$ is feasible for T, then T immediately can improve upon x, by modifying $\hat{s}$ on T to consist of trades only in T and we have a contradiction. However, it may be that $p(s|\hat{s}^T)$ is not feasible for T, since it may involve the members of T trading with members of the complement of T, although of course these trades must be those made under the original strategy s. In this case, select some individual $j \in T$, and consider the maximal connected subset C of $\mathrm{cons}(s|\hat{s}^T)$ containing j. Observe that C is not empty, since $p^j(s|\hat{s}^T) \succ^j p^j(s) \succsim \omega^j$, so trade involving j does take place at $(s|\hat{s}^T)$. Note that for $t \in C \sim T$, $p^t(s) = p^t(s|\hat{s}^T)$. To see this, note first that since $C \subset \mathrm{cons}(s|\hat{s}^T)$, $p^t(s|\hat{s}^T) = \omega^t + \sum_j (s|\hat{s}^T)^t_j$, while since $\mathrm{cons}(s) = N$, $p^t(s) = \omega^t + \sum_j s^t_j$. Since $t \notin T$, $(s|\hat{s}^T)^t = s^t$. Thus, $p^t(s) = p^t(s|\hat{s}^T)$. Meanwhile, for $t \in C \cap T \neq \emptyset$, $p^t(s) \prec^t p^t(s|\hat{s}^T)$. Thus, $p^t(s) \precsim^t p^t(s|\hat{s}^T)$ for all $t \in C$, with strict preference holding at least for $t = j$. Further, $p(s|\hat{s}^T)$ is feasible for C, since C is a maximal connected coalition. Now, given continuity and monotonicity of preferences, it is possible to find an allocation y by means of which C can improve upon x.

Some Extensions and Applications

In this section we suggest some possible extensions and applications of the basic model, and point to some open questions.

First, we should note that it is not completely standard to include the factors modeled by the graph g in the definition of the economy. However, if one identifies an exchange economy simply with the agents' characteristics, but continues to consider only the g-connected coalitions in defining the core, then our results obviously continue to hold.

An extension which is relatively easy to include is the possibility of production. One obvious approach is to use a modeling due to Hurwicz [8], which involves the introduction of fictious producing agents with whom the other agents can trade. These producers have zero endowment, flat preferences and a production set in place of the usual agent's characteristics. Naturally, in this context, the definition of the core and the strong Nash equilibrium would require only that the fictious players not be made worse off (rather than insisting on strict preference for these agents as well). A second approach is to use the model of a coalition production economy originated by Hildenbrand [7].

The introduction of externalities in consumption offers another possible line of investigation. As a first approach to this, one can take the individual preferences to be defined not just over the consumption sets but rather over the space of allocations in a non-trivial fashion. Given the well-known problem with defining an appropriate notion of the core with externalities, one probably cannot hope to obtain an analog of Theorem 2 in this context. Still, the model may be useful and interesting. For example, if g is connected, then one easily shows that any strong Nash equilibrium is Pareto optimal. This, of course, is a formalization of the celebrated Coase theorem [4] that in the absence of transactions costs externalities are compatible with optimality.

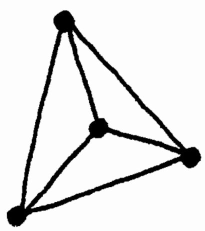
A further useful extension would be to treat transactions costs in a more explicit manner. The communications graph approach used here can be interpreted in terms of transactions costs that are either zero (if $\underline{ij} \in g$) or so large as to preclude all trade (if $\underline{ij} \notin g$). Introduction of transactions costs that vary between links and with the volume of trade flowing through a link might allow an analysis of the efficiency of various forms of market organization.

However, even the present set-up offers possibilities for research in this latter direction. For example, if g is not connected, we are effectively looking at a system of autarkies, while if g is connected but not necessarily complete one obtains models of different forms of economic systems. For example, the complete graph in part (a) of Figure 1 represents a system under which all agents are free to exchange directly with one another, while the graph in part (b) represents the existence of a middleman through which all trade flows. Figure 2 can be considered as representing two economies, where all foreign trade in the one economy must flow through an export-import agency.

A particularly interesting aspect of the question of comparing different systems involves comparative statics analysis on the equilibria as the communications graph is changed, with a view to answering such questions as precisely who gains or loses when a link is introduced or deleted from g? A specific example of this involves the role of the middleman. Suppose, for example, that initially g is complete and then all links except those with one particular agent are broken (as in Figure 1). Since the graph is still connected, the set of Pareto optima is unchanged. However, no multi-player coalitions not involving player 1 are connected in the second situation. Thus, the core and strong Nash equilibria of the second economy will include those in the first. Intuitively, one might expect that the player in the middle would do better in this situation: all trade must flow through him, which ought to improve his position. Somewhat more formally, all coalitions including this player remain as before, while those excluding him are now powerless. However, we have not yet been able to establish that he actually does gain in an appropriate sense. In fact, we have produced simple examples of games in which an arbitrarily large proportion of the points coming into the core are worse for the player in the middle than any point in the core of the game based on the complete graph. These examples, however, are not balanced, and so do not arise from markets ([2], [11]). Whether such phenomena can arise in games of exchange considered here is an open question we plan to address in future papers.

FIGURE 1

(a)



(b)

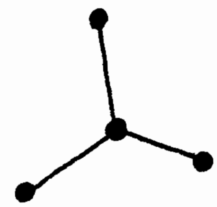
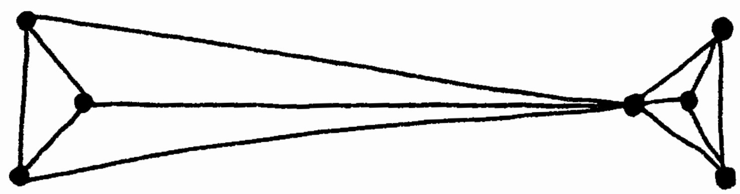


FIGURE 2



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