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VALUES OF GAMES WITHOUT SIDEPAYMENTS

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To model cooperative games without sidepayments, AUMANN & PELEG (1960) introduced the generalized characteristic function form. Games in this form have a natural linear structure: they can be added and multiplied by positive scalars. This paper presents some results about linear value functions for such generalized characteristic function games. We first discuss the importance of linearity for value functions, and then we present two main theorems relating to value functions which generalize the Shapley value and Pareto-optimal value functions. Finally, in a concluding section, we discuss the implications of these results for the deeper question of why the sidepayments assumption is so important and useful in game theory.

1. Definitions

Let N be a nonempty finite set, representing the set of players. Nonempty subsets of N are called coalitions. For any coalition S, let R^S be the space of all vectors of real numbers with coordinates indexed on the members of S. (Equivalently, we may think of R^S as the set of all functions from S into the real numbers.)

The characteristic function form was originally defined for games in which sidepayments or utility transfers between the players are allowed. A game in characteristic fuction form, or a c-game for short, is a function v, with the set of coalitions as its domain of definition, mapping each coalition S into a real number v(S). The number v(S) is interpreted as the wealth, in units of transferable utility, which the members of coalition S would have to divide among themselves if they were to cooperate together with each other and with no one outside S.

The <u>generalized characteristic function form</u> has been defined for games without sidepayments. A game in generalized characteristic function form, or a <u>gc-game</u> for short, is a function V, with the set of coalitions as its domain, mapping each coalition S into a set $V(S) \subseteq R^S$ which is nonempty, closed, convex, and comprehensive. (A set $E \subseteq R^S$ is comprehensive iff $x \in E$ and $y_i \leq x_i \; \forall \, i \in S$ imply that $y \in E$.) The set V(S) is interpreted as the set of all utility allocations which would be feasible for the members of S if they were to cooperate with each other and with no one outside S.

The c-games have a natural linear structure as a $(2^{|N|}-1)$-dimensional vector space:

(1) $(v+w)(S) = v(S) + w(S)$

(2) $(\lambda \cdot v)(S) = \lambda \cdot (v(S))$

The linear structure for the gc-games is a bit more complicated.

If X and Y are compact convex subsets of R^S, then we define their sum as

(3') $X+Y = \{x+y \mid x \in X, \; y \in Y\}$,

and it can be shown that X+Y is also compact and convex. However, if X and Y are merely closed and convex, then the right side of (3') may not be a closed set. So if X and Y are closed convex subsets of R^S, then we define their sum as

(3) $X+Y = \underline{cl}(\{x+y \mid x \in X, \; y \in Y\})$

where <u>cl</u>(.) denotes the closure operator.

So the sum of closed convex sets is closed and convex as well.

For any two gc-games V and W, we define V+W so that

(4) $(V+W)(S) = V(S) + W(S)$.

Observe that $V(S)+W(S)$ must be a nonempty, closed, convex, and comprehensive subset of R^S, because $V(S)$ and $W(S)$ are, and therefore V+W is a gc-game.

Given any positive real number λ and any gc-game V, we define $\lambda \cdot V$ to be the gc-game such that

(5) $(\lambda \cdot V)(S) = \{\lambda \cdot x \mid x \in V(S)\}$

(using the usual scalar multiplication in the vector space R^S). We will not define scalar multiplication for gc-games and nonpositive scalars.

A <u>convex cone</u> of gc-games is a set D such that

(6a) $V \in D$ and $W \in D$ imply $V+W \in D$, and

(6b) $V \in D$ and $\lambda > 0$ imply $\lambda \cdot V \in D$.

The set of all gc-games is itself a convex cone.

A <u>value function</u> is a mapping from gc-games (or c-games) to vectors in R^N. We want to find a value function $F(.)$ such that, for each i in N, $F_i(V)$ could be reasonably interpreted as the payoff outcome which player i should expect from the game V. We may consider value functions which are not defined for all gc-games, but we will always assume that the domain of definition is some convex cone of gc-games.

A value function F is <u>linear</u> iff

(7a) $F(V+W) = F(V) + F(W)$, and

(7b) $F(\lambda \cdot V) = \lambda \cdot F(V)$,

for any gc-games V and W in the domain of F and any $\lambda > 0$. (Values of games are added and multiplied as vectors in R^N.) Actually, we will be interested in value functions which satisfy either of two conditions which are weaker than linearity: additivity and affineness. A value function F is *additive* iff

(8) $F(V+W) = F(V) + F(W)$,

for any V and W in the domain of F.

A value function F is *affine* iff

(9) $F(\lambda \cdot V + (1-\lambda) \cdot W) = \lambda \cdot F(V) + (1-\lambda) \cdot F(W)$,

for any positive number λ less than one, and for any V and W in the domain of F.

Of course, any linear value function is also additive and affine.

2. Significance of the affine or additive property assumptions.

We have introduced both the affine and additive properties because somewhat different stories can be told to justify assuming one or the other for a value function, and either one will be sufficient to derive the results in Sections 3 and 4.

The affine property can be motivated in terms of the response to risk as ROTH (1977) has suggested. Let the payoffs for all players be measured in von Neumann-Morgenstern utility scales. Suppose that the set of feasible allocations for each coalition depends of whether a certain random variable $\tilde{X}$ takes the value 0 or 1: if $\tilde{X} = 0$, which may happen with probability λ, then each coalition S has feasible set V(S); if $\tilde{X} = 1$, which may happen with probability $1-\lambda$, then each coalition S has feasible set W(S). Thus, if F is the value function which determines the payoff outcomes for each game, then F(V) is the payoff allocation for the game which results from $\tilde{X}=0$, and F(W) is the payoff allocation for the game which results from $\tilde{X} = 1$. So, before $\tilde{X}$ is known, the expected allocation of utility is

$\lambda \cdot F(V) + (1-\lambda) \cdot F(W)$.

However, if coalition S can make conditional allocation plans before $\tilde{X}$ is known, then the set of allocations of expected utility which are feasible for S (before $\tilde{X}$ is known) is

$$\{\lambda y + (1-\lambda)z \mid y \in V(S),\ z \in W(S)\}$$

which either equals $(\lambda \cdot V + (1-\lambda) \cdot W)(S)$ or is a dense subset thereof. So $\lambda \cdot V + (1-\lambda) \cdot W$ is the game facing the players before $\tilde{X}$ is known, and $F(\lambda \cdot V + (1-\lambda) \cdot W)$ is the expected utility outcome for this game. To guarantee that the expected payoff outcomes do not depend on whether the games are played before or after $\tilde{X}$ is determined, we must assume that

$$F(\lambda \cdot V + (1-\lambda) \cdot W) = \lambda \cdot F(V) + (1-\lambda) \cdot F(W)$$

or that F is affine.

Additivity is mainly of interest because it is a mathematically simpler than affineness (scalar multiplication is not required) and because additivity was the assumption used in the original derivation of the Shapley value (SHAPLEY, 1953). We can also justify additivity if we assume that each player's utility is measured in units of some physical commodity (which is not freely transferable). If two games are being played at the same time, and if V(S) is the set of allocations feasible for coalition S in one game while W(S) is its feasible set in the other game, then $\{y+z \mid y \in V(S),\ z \in W(S)\}$ is the set of utility allocations which are feasible for S if it plans its actions in both games at once. But this set either equals $(V+W)(S)$ or is a dense subset thereof. Additivity is precisely the property needed to guarantee that $F(V+W) = F(V) + F(W)$, so that the outcome allocation does not depend on whether the two games are analyzed separately or together.

3. Extensions of the Shapley value.

The Shapley value was shown (SHAPLEY,1953) to be the most natural linear value function for c-games. In this section, we study extensions of the Shapley value to gc-games.

For a function defined on c-games to be "extended" to gc-games, we must first identify the c-games as a subset of the gc-games. For any c-game v, let G(v) be the gc-game such that

(10) $$(G(v))(S) = \left\{ x \in R^S \;\middle|\; \sum_{i \in S} x_i \leq v(S) \right\}.$$

That is, G(v) is the gc-game in which each coalition S can offer its members any allocation of utility which sums to v(S) or less, so we can identify v and G(v) as representing the same game situation. Let C^0 be the range of G:

(11) $$C^0 = \{ G(v) \mid v \text{ is a c-game} \}.$$

Thus C^0 is the set of all gc-games in which utility is essentially transferable. Observe that C^0 is a convex cone of gc-games.

We will also need another map, going from gc-games to c-games. Let C^1 be the following convex cone of gc-games:

(12) $$C^1 = \left\{ V \;\middle|\; \sup_{x \in V(S)} \left(\sum_{i \in S} x_i \right) < \infty \text{ for every coalition } S \right\}.$$

Then for any $V \in C^1$, let h(V) be the c-game such that

(13) $$(h(V))(S) = \sup_{x \in V(S)} \left(\sum_{i \in S} x_i \right).$$

Let φ be the Shapley value operator on c-games

(14) $$\varphi_i(v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|S|-1)!\,(|N \setminus S|)!}{|N|!} (v(S) - v(N \setminus S)).$$

Then a value function F for gc-games is said to <u>extend the Shapley value</u> iff C^0 is a subset of the domain of F and $F(G(v)) = \varphi(v)$ for any c-game v. Our main result is that the Shapley value has an essentially unique additive extension, and that is $\varphi(h(.))$

THEOREM 1. Let F be a value function whose domain is D, a convex cone of gc-games. Then F is additive and extends the Shapley value if and only if $D \subseteq C^1$ and $F(V) = \varphi(h(V))$ for all V in D.

PROOF. To check the "if" part of the theorem, it suffices to observe that h(.) is a linear map on the cone C^1, and that $h(G(v)) = v$ for any c-game v. Thus $\varphi(h(\cdot))$ is linear and extends the Shapley value.

We now take up the "only if" part of the theorem and assume that F is an additive extension of the Shapley value.

We show first that if $V \in D \cap C^1$ then $F(V) = \varphi(h(V))$. Let 0* be the zero c-game. ($0^*(S) = 0$ for every coalition S.) Then $V + G(0^*) = G(h(V))$, because

$$(15) \quad V(S) + (G(0^*))(S) = \underline{\text{cl}}\left(\left\{y+x \,\middle|\, y \in V(S),\ x \in R^S,\ \sum_{i \in S} x_i \leq 0\right\}\right)$$

$$= \left\{z \in R^S \,\middle|\, \sum_{i \in S} z_i \leq \sup_{y \in V(S)} \left(\sum_{i \in S} y_i\right)\right\}$$

$$= G(h(V))(S).$$

Therefore:

$$(16) \quad F(V) = F(V) + \varphi(0^*) = F(V) + F(G(0^*))$$

$$= F(V + G(0^*)) = F(G(h(V))) = \varphi(h(V)).$$

It now remains to prove that $D \subseteq C^1$. We call a gc-game V <u>improper</u> if $V(S) = R^S$ for some S. Observe that D must contain improper games if $D \not\subseteq C^1$. To see this, suppose that $V \in D$ and $\sup_{y \in V(S)} \left(\sum_{i \in S} y_i\right) = \infty$; then $V+G(0^*)$ is in D (because D is a convex cone and $G(0^*) \in C^0 \subseteq D$) and $V+G(0^*)$ is improper (because $(V+G(0^*))(S) = R^S$). It therefore suffices to show that D cannot contain any improper games.

Let u^S be the c-game such that:

(18) $$u^S(T) = \begin{cases} 1 & \text{if } T = S, \\ 0 & \text{if } T \neq S. \end{cases}$$

Now, suppose V is improper and $V(S) = R^S$.

Then $V + G(u^S) = V + G(0*)$, because $G(u^S)(T) = G(0*)(T)$

if $T \neq S$, and $V(S) + G(u^S) = R^S = V(S) + G(0*)$.

If V were in D, we would have

(19) $$F(V) + F(G(u^S)) = F(V + G(u^S)) = F(V + G(0*))$$
$$= F(V) + F(G(0*)),$$

but this is impossible since $F(G(u^S)) = \varphi(u^S) \neq \varphi(0*) = F(G(0*))$.

So improper games cannot be in the domain of F.

This completes the proof of Theorem 1.

We used additivity twice in the above proof, in equations (16) and (19). If we assume that F is affine (instead of additive) then similar arguments will show that

(16') $$\tfrac{1}{2}\cdot F(V) = \tfrac{1}{2}\cdot F(V) + \tfrac{1}{2}\cdot\varphi(0*) = \tfrac{1}{2}\cdot F(V) + \tfrac{1}{2}\cdot F(G(0*))$$
$$= F(\tfrac{1}{2}\cdot V + \tfrac{1}{2}\cdot G(0*)) = F(G(\tfrac{1}{2}\cdot h(V))$$
$$= \varphi(\tfrac{1}{2}\cdot h(V)) = \tfrac{1}{2}\cdot\varphi(h(V)), \text{ and}$$

(19') $$\tfrac{1}{2} F(V) + \tfrac{1}{2} F(G(u^S)) = F(\tfrac{1}{2} V + \tfrac{1}{2} G(u^S)) = F(\tfrac{1}{2} V + \tfrac{1}{2} G(0*))$$
$$= \tfrac{1}{2} F(V) + \tfrac{1}{2} F(G(0*)).$$

With revisions such as these, we can modify the proof to show the following theorem:

THEOREM 1'. Let F be a value function whose domain is D, a convex cone of gc-games. Then F is affine and extends the Shapley value if and only if $D \in C^1$ and $F(V) = \varphi(h(V))$ for all V in D.

4. Pareto-optimal value functions.

Unfortunately, $\varphi(h(V))$ may be an infeasible allocation for a gc-game V. That is, even if $V \in C^1$, we may have $\varphi(h(V)) \notin V(N)$. In this section we investigate value functions which select allocations which are feasible and Pareto-optimal for the universal coalition N.

For any gc-game V, we will denote the (weak) Pareto-optimal frontier for the universal coalition N by $\partial V(N)$. For any two vectors x and y in R^N, we write $y > x$ iff $y_i > x_i$ for all i in N. Thus:

(20) $\partial V(N) = \{x \mid x \in V(N)$, and $y > x$ implies $y \notin V(N)\}$.

A value function F is Pareto-optimal iff $F(V) \in \partial V(N)$ for every gc-game V in the domain of F.

Let $\Delta(N)$ be the unit simplex in R^N

(21) $\{x \in R^N \mid x_i \geq 0 \;\; \forall\, i \in N$, and $\sum_{j \in N} x_j = 1\}$.

Given a vector $z \in \Delta(N)$, a value function F is utilitarian with respect to z iff $F(V) \in V(N)$ and

$\sum_{i \in N} z_i \cdot F_i(V) = \sup_{x \in V(N)} \left(\sum_{i \in N} z_i \cdot x_i \right)$ for every gc-game V

in the domain of F. Utilitarian value functions are those which are consistent with a utilitarian social choice rule of maximizing a weighted average of the players' utilities. Such social choice rules have been discussed by HARSANYI (1955).

The main result of this section relates the Pareto-optimal and utilitarian properties.

THEOREM 2. Suppose that F is an additive value function defined on a convex cone of gc-games. Then F is Pareto-optimal if and only if there is some vector z in $\Delta(N)$ such that F is utilitarian with respect to z.

PROOF. The "if" part of the theorem is easy to check. If F is utilitarian with respect to some z, then F(V) must be on the Pareto-optimal frontier $\partial V(N)$, or else F(V) could not be maximizing $\sum_{i \in N} z_i \cdot x_i$ over $x \in V(N)$.

The "only if" part of the theorem is the harder part. Let F be a Pareto-optimal value function on a convex cone D. For any gc-game V in D, let

$$(22) \qquad P(V) = \left\{ y \in R^N \;\middle|\; \sum_{i \in N} y_i \cdot F_i(V) < \sup_{x \in V(N)} \left(\sum_{i \in N} y_i \cdot x_i \right) \right\}.$$

P(V) is always an open subset of R^N, since its complement is closed. To show that F is utilitarian, it suffices to show that there is some vector z in $\Delta(N)$ such that $z \in P(V)$ for every V in the cone D.

Suppose that, contrary to the theorem, $\bigcup_{V \in D} P(V) \supseteq \Delta(N)$.

Then the P(V) sets form an open cover of $\Delta(N)$, which is a compact set. So there exists some finite collection $\{V^1, \ldots, V^k\} \subseteq D$ such that $\bigcup_{j=1}^{k} P(V^k) \supseteq \Delta(N)$.

Let $\bar{V} = \sum_{j=1}^{k} V^j$. $\bar{V} \in D$ because D is a convex cone, and $F(\bar{V}) \in \partial \bar{V}(N)$ because F is Pareto-optimal. By the separating hyperplane theorem (ROCKAFELLAR, 1970, §11), there must be some $\bar{z} \in \Delta(N)$ such that

$$\sum_{i \in N} \bar{z}_i \cdot F_i(\bar{V}) = \sup_{x \in \bar{V}(N)} \left(\sum_{i \in N} \bar{z}_i \cdot x_i \right).$$

But by the way $V^1, \ldots, V^k$ were chosen, there must be some m such that $\sum_{i \in N} \bar{z}_i \cdot F_i(V^m) < \sup_{x \in V^m(N)} \left(\sum_{i \in N} \bar{z}_i \cdot x_i \right)$;

without loss of generality, we may assume that this is true for m = 1. So there exists some $\bar{x} \in V^1(N)$ such that

$\sum_{i \in N} \bar{z}_i \cdot \bar{x}_i > \sum_{i \in N} \bar{z}_i \cdot F_i(V^1)$. Thus:

$$(23) \qquad \sum_{i \in N} \bar{z}_i \cdot F_i(\bar{V}) = \sum_{i \in N} \bar{z}_i \cdot F_i(\sum_{j=1}^{k} V^j)$$

$$= \sum_{j=1}^{k} \sum_{i \in N} \bar{z}_i \cdot F_i(V^j)$$

$$< \sum_{i \in N} \bar{z}_i \cdot \bar{x}_i + \sum_{j=2}^{k} \bar{z}_i \cdot F_i(V^j)$$

$$= \sum_{i \in N} \bar{z}_i \cdot (\bar{x}_i + \sum_{j=2}^{k} F_i(V^j))$$

$$\leq \sup_{x \in \bar{V}(N)} (\sum_{i \in N} \bar{z}_i \cdot x_i),$$

where the last inequality holds because $\bar{x} + \sum_{j=2}^{k} F(V^j)$ is a point in $\sum_{j=1}^{k} V^j(N) = \bar{V}(N)$. But this string of inequalities adds up to a contradiction of the way $\bar{z}$ was constructed. Thus $\bigcup_{V \in D} P(V)$ cannot cover all of $\Delta(N)$. This completes the proof of the theorem.

If we assume that F is affine, instead of additive, then the proof is still essentially the same. The only place where we used additivity was in line (23). By changing the definition of $\bar{V}$ to $\bar{V} = \frac{1}{k} \cdot \sum_{j=1}^{k} V^j$ and by making corresponding adjustments in the subsequent formulas involving $\bar{V}$, we can revise the proof to cover the affine case.

THEOREM 2'. Suppose that F is an affine value function defined on a convex cone of gc-games. Then F is Pareto-optimal if and only if there is some vector z in $\Delta(N)$ such that F is utilitarian with respect to z.

5. Conclusions.

Our results in Sections 3 and 4 can help us to understand the role of the sidepayments assumption in game theory.

Theorems 1 and 1' illustrate why the theory of games with sidepayments can be important even if most problems of interest do not have sidepayments. The essential step in the proof of Theorem 1 was the observation that the sum of a game without sidepayments and a game with sidepayments (a gc-game in C^0) will be a game with sidepayments. Therefore, once we find a linear solution theory (such as the Shapley value) for games with sidepayments, the theory will have a unique linear extension to the more general case, without sidepayments. The theory for the sidepayments case determines the general theory.

Theorems 2 and 2' show why the sidepayments assumption is so convenient. For contrast, consider the opposite of the sidepayments case: consider the games for which the feasible sets are strictly convex. Given $z \in \Delta(N)$, there can be only one vector x in a strictly convex set which maximizes $\sum_{i \in N} z_i \cdot x_i$. So if F is utilitarian with respect to z, and if $V(N)$ is strictly convex and $V(N) = W(N)$, then we must have $F(V) = F(W)$. Thus $F(V)$ must depend only on $V(N)$, the feasible set of the universal coalition, and cannot depend on the feasible sets of any smaller coalitions. Such a value function must be insensitive to the power distribution among the players. So Theorems 2 and 2' tell us that a Pareto-optimal linear value function on games with strictly convex feasible sets must be a simple utilitarian social choice scheme and cannot be sensitive to the game-theoretically interesting threat structures. Sidepayments are needed to create flat regions on the Pareto-optimal frontier within which the threats of the smaller coalitions can have some influence.

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