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APPORTIONMENT METHODS AND THE HOUSE OF REPRESENTATIVES *

by

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Abstract

The seemingly straightforward task of assigning seats to states according to population runs into several politically unacceptable complications. It is shown that this is a serious problem in the sense that these complications will occur for most population densities. The mathematical reasons for these complications are discussed. Finally, a simple apportionment method is suggested.

1. Introduction

A problem of integer programming is to find the family of integer s-vectors which best approximates a given family of non-integer s-vectors. The theoretical solution is trivial. Associate with each vector of the given family the closest integer vector, where the distance between vectors is determined by some given norm or metric. However this obvious approach possesses certain characteristics which can make it undesirable and/or controversial when it is used in an applied problem. This is particularly so when the stakes are high as in the apportionment of congressional seats to individual states according to population. Historically these characteristics forced, in the name of political reality, some additional side constraints on the process of apportionment. What adds interest to this story is that the method adopted and currently employed by the United States to satisfy these constraints gives rise to a new set of difficulties which can be more inequitable than the original approach!

Because of the political importance of this apportionment problem, it has received the attention of both politicians and mathematicians. Several of the names associated with this problem are well-known; for example, T. Jefferson, A. Hamilton, J. von Neumann, etc. For a witty, informative discussion of the political and mathematical history of this problem, I recommend the paper by Balinski and Young [1] (to be referred to as BY). Indeed, I learned about this problem by reading BY after hearing a lecture by H. P. Young at Northwestern, January, 1977.

Mathematically, the integer approximation problem is as follows. Assume the population of the ith state, $1 \leq i \leq s$, is

g_i and $g = \sum_{i=1}^{s} g_i$ is the total population. If there are n seats to be apportional according to population, then the problem is to find $\underline{k} = (k_1, k_2, \ldots, k_s)$, k_i a non-negative integer and $\Sigma\, k_i = n$, which best approximates $n\,\underline{p} = (np_1, \ldots, np_s)$ where $p_i = g_i/g$. That is, the seats are apportioned in a manner which approximates the ideal apportionment, usually a non-integer valued vector, which is determined by the actual population. For several years the United States used what amounted to a norm minimizing approach where of all the integer vectors satisfying $\Sigma\, k_i = n$, the solution vector was the one closest to $n\underline{p}$. The distance was determined by the sup norm $\|\underline{x}\| = \max_{1 \leq i \leq s} |x_i|$. Notice that except possibly when the fractional part of two or more components of $n\underline{p}$ are the same, the choice of $\underline{k}$ is unique.

A complication leading to the abandonment of this method is known as the Alabama paradox (BY).

Definition. Let $\underline{k}^{(n)}$ and $\underline{k}^{(n+1)}$ be integer solutions for $n\underline{p}$ and $(n+1)\underline{p}$ respectively. If some component of $\underline{k}^{(n+1)}$ is less than the corresponding component of $\underline{k}^{(n+1)}$, then we say that $\underline{p}$ has an Alabama paradox at n.

In other words, when the size of the House of Representatives is n, some state, say the ith, has k_i representatives. However, when an additional seat is added to the total house size, the ith state loses a representatives. Mathematically $\underline{k}^{(n+1)}$ would be the "best" approximation, but politically this solution would be difficult to implement, particularly if the representatives from the ith state have anything to say about it.

Since this paradox bears the name Alabama, it is reasonable to suspect it can and has occured. Not only can it occur, but in the next section it will be shown that if a norm minimizing method is used; then for "most" choices of $\underline{p}$ ("most" will be defined later) there is an n giving rise to the Alabama paradox. This is true for any norm, so the choice of distance function is not at fault! We will isolate the dynamics which causes this behavior.

Politically, this is intolerable! Therefore, a political side constraint on choice of the family of apportionment vectors is that it does not admit an Alabama paradox. Such a family is called house monotone. (Although the family depends upon $\underline{p}$, our notation will not reflect this dependency.)

Definition. Let $\{\underline{k}^{(n)}\}_{n=0}^{\infty}$ be a family of integer vectors where $\underline{k}^{(n)}$ gives the apportionment for $n\underline{p}$. That is, the components of $\underline{k}^{(n)} = (k_1^n,\ldots,k_s^n)$ are non-negative integers and they sum to n. $\{\underline{k}^{(n)}\}$ is said to be house monotone if for all $i = 1,\ldots,s$ and all $n \geq 0$, $k_i^{n+1} \geq k_i^n$.

The problem is to find a method of selecting a house monotone family which approximates the ideal family $\{n\underline{p}\}$. While the actual story is much longer and more involved, the adopted approach is essentially as described below. A norm or distance approach doesn't work, so replace it with some other measure of equal apportionment. For example, at the ideal apportionment $p_i/k_i = p_j/k_j$ for all $i,j = 1,2,\ldots,s$. Notice that g p_i/k_i is the density of population per representative, so a high density corresponds to an underrepresented state. It seems reasonable to award the state with the highest density the additional seat resulting from an increase in house size.

There remains the question as to whether function g p_i/k_i

is the best choice of a measurement. Since other choices can lead to different apportionments, this is a very real political question. We will dodge this issue by keeping only the properties of this measure.

Definition. Let $\ell: (0,\infty) \times [0,\infty) \to \mathbb{R} \cup (\infty)$ be a smooth function on $(0,\infty) \times (0,\infty)$ such that for fixed x, $\ell(x,y)$ is monotonically decreasing function in y. Such an ℓ will be called a ranking function.

A related function is defined in BY which has less restrictions on it. Yet all the ranking functions seriously considered satisfy the constraints listed here. It can be shown that if a ranking function doesn't satisfy these requirements, then pathology is introduced; and this can be equated with political controversy.

The idea is essentially as described above. When $n = 0$, $\underline{k}^{(0)} = (0,0,\ldots,0)$. When going from house size n to house size $n + 1$, the additional seat is apportioned to the state which maximizes $\ell(np_j, k_j^{(n)})$. Should there be a tie, then some sort of tie breaking scheme is introduced. Since we are not reapportioning the seats, but rather we are adding to previous apportionments, this is house monotone. Currently the United States uses $\ell(x,y) = x/(y(y + 1))^{1/2}$.

A natural question is whether this approach may introduce a different form of pathology; and the answer is yes. What happens is that this method admits the possibility that a state may receive more than its fair share of representatives. If $[\ \ ]$ is the greatest integer function, then the ith state should receive either $[n\, p_i]$ or $[n\, p_i] + 1$ representatives. If the ranking function does not satisfy a certain diagonal condition, it turns out that this method <u>must</u> eventually deviate from this quota for most choices of p. This will be shown in Section 3. In Section 4 a simple method which avoids all these difficulties will be discussed.

Alabama Paradox

Let $P_n = \{\underline{x} \in \mathbb{E}^s \mid \Sigma\, x_i = n,\ x_i \geq 0\}$. Then $\underline{p} \in P_1$ and both $n\,\underline{p}$ and $\underline{k}^{(n)} \in P_n$. The difference is that $\underline{k}^{(n)}$ must be an integer lattice point of this symplex. (Notice that these lattice points define copies of P_1 which cover P_n.) If $\{\underline{k}^{(n)}\}$ is a family corresponding to $\{n\,\underline{p}\}$ which avoids the Alabama paradox, then for each n all but one of the components of $\underline{k}^{(n+1)}$ must agree with the corresponding component of $\underline{k}^{(n)}$. Therefore $\underline{k}^{(n+1)}$ is obtained by adding some coordinate unit vector to $\underline{k}^{(n)}$. There are only s choices, and they form the defining vertices for a copy of P_1 on P_{n+1}. This copy of P_1 is also given by $(\underline{k}^{(n)} + P_1) \cap P_{n+1}$.

On the other hand, the norm minimization approach selects the lattice point of P_{n+1} which is closest to $(n + 1)\underline{p}$. This is the same as first finding the copy of P_1 on P_{n+1} which contains $(n + 1)\underline{p}$, and then selecting the nearest vertex of this symplex. Now, the direction of $\underline{p}$ may be such that the selected vertex by the norm minimizing approach is <u>not</u> one of those of $\underline{k}^{(n)} + P_1$. The following statement asserts that this is a common occurence.

Theorem. <u>Let</u> $s > 2$. <u>There exists an open dense set</u> $\mathscr{B} \subset P_1$ <u>such that if</u> $\underline{p} \in \mathscr{B}$ <u>and if</u> $\{\underline{k}^{(n)}\}$ <u>is a family of integer vectors selected by a norm minimization process to approximate</u> $\{n\,\underline{p}\}$, <u>then</u> $\{\underline{k}^{(n)}\}$ <u>will have an Alabama paradox for some</u> n.

It is a simple exercise to show that the Alabama paradox cannot occur if $s = 2$.

In the proof we need notation for the non-integer part of a number. So, let $(x) = x - [x]$.

Proof. We first prove the theorem with respect to the sup norm.

Claim: If state α satisfies the following three conditions, then it will suffer the Alabama paradox

1. If $\Sigma(np_i) = j_1 \leq s - 1$, then $(n\,p_\alpha)$ is one of the largest j_1 terms.

2. $0 < (n\,p_\alpha) < 1 - p_\alpha$

3. If $\Sigma((n + 1)p_i) = j_2 \leq s - 1$, then $((n + 1)p_\alpha)$ is <u>not</u> one of the largest j_2 terms.

Imposing the additional constraints that $(n\,p_\alpha)$ is strictly larger than the $(j_1 + 1)$st largest term in the set $\{(n\,p_i)\}$ and $((n + 1)p_\alpha)$ is strictly smaller than the j_2 largest term in the set $\{((n + 1)p_i)\}$ makes this an open condition.

With the sup norm, the allocation of representatives goes as follows. First the ith state is assigned $[n\,p_i]$ representatives. This accounts for $\Sigma[n\,p_i] = n - \Sigma(n\,p_i) = n - j_1$ representatives. The last j_1 representatives are then assigned according to the magnitude of $(n\,p_i)$. Therefore, the ordering given in condition one implies that state α receives $[n\,p_\alpha] + 1$ representatives when the house size is n. Condition 2 implies that $[(n + 1)p_\alpha] = [n\,p_\alpha]$, and Condition 3 implies that state α receives $[(n + 1)p_\alpha] = [n\,p_\alpha]$ representatives when the house size is $n + 1$, a loss of one representative with the increase in house size.

Define $\mathscr{B} = \{p \in P_1 |$ for some choice of $n > 1$, the strict inequalities given in Conditions 1, 2, and 3 are satisfied.$\}$ That $\mathscr{B}$ is open follows from the comment following the conditions. We shall show that $\mathscr{B}$ is non-empty at the same time we show that

$\mathscr{B}$ is dense.

Let $\underline{p}^* \epsilon P$ such that $p_i^* \neq p_j^*$ for $i \neq j$. For any $\varepsilon > 0$ we will show there exists $\underline{p} \epsilon \mathscr{B}$ such that $\underline{p}$ is at most distance ε from $\underline{p}^*$. Actually we shall prove much more. We will prove there exists $\underline{p} \epsilon \mathscr{B}$ and integer n so that j_1 can be chosen equal to unity and j_2 is equal to 2.

Assume without loss of generality that $p_1^* > p_2^* > \cdots > p_s^* > 0$, so $0 < p_s^* < s^{-1}$. Next choose rational numbers x_i so that $\sum_{i=1}^{s} x_i = 1$, $x_i < x_s < 2/s$ for $i = 1,2,\ldots,s-1$; $x_1 < 1-p_1^*$; $x_2 < 1-p_2^*$; $x_s < x_1 + (p_1^* - p_s^*)/2$, $x_2 + (p_2^* - p_s^*)/2$. Such x_i's can always be selected if $s > 2$. Now let n be the first multiple of 10 greater than $\max(11\varepsilon^{-1},((p_1^*-p_s^*)/10)^{-1}, ((p_2^*-p_s^*)/10)^{-1})$. Define $\underline{p}' = ([n\ p_1^*],[n\ p_2^*],\ldots,[n\ p_s^*])n^{-1}$. By adding or subtracting n^{-1} to some of the components of $\underline{p}'$, it can be adjusted so that the sum of the components equals $1 - n^{-1}$. Assume this has been done. Define $\underline{p} = \underline{p}' + (x_1,x_2,\ldots,x_s)n^{-1}$. By construction $\underline{p} \epsilon P_1$, $(n\ p_i) = x_i$, and $\underline{p} \epsilon \mathscr{B}$. Also $\underline{p}$ is within distance ε of $\underline{p}^*$.

Now assume $\underline{p}^* \epsilon P_1$ is arbitrary. If some of the components are equal, approximate $\underline{p}^*$ with some vector within distance $\varepsilon/2$ where the components all differ. The above now applies. The proof is now completed for the sup norm.

The choice of $\underline{p}$ is nothing more than an adaptation of the "irrational flow on a torus" for this symplex model, where the details are carried out for this particular sup norm. But the real dynamics which makes this proof work is the denseness of the image of $((n\ p_1),(n\ p_2),\ldots,(n\ p_s))$ if the components of $\underline{p}$ satisfy some sort of rational independence condition. Thus we can get the image to enter any open set. Consequently the proof can be modified to hold for any norm since the open unit ball for any norm is a convex open set.

Let $\| \ \|$ be any norm. The above proof is altered in the obvious fashion. Orderings of the non-integer parts of numbers is done by the norm of vectors with components $1 - (n\ p_i)$, rather than by the magnitudes of $1 - (n\ p_i)$. For example, the inequality constraints on the choice of x_i's now becomes $\|(0,\ldots,1-x_s)\| < \|(0,\ldots,1-x_i,0,\ldots,0)\|$ for $i = 1,2,\ldots,s-1$, $x_1 < 1 - p_1^*$; $x_2 < 1 - p_2^*$, and $\|(1-x_1-p_1^*,1-x_2-p_2^*,0,\ldots,0)\|$ is strictly smaller than the norm of any vector with precisely two non-zero components of form $1-x_i-p_i^*$ where one choice of i is s. The existence of the x_i's follows easily by the denseness of the rationals and standard properties of norms.

In the above proof, no attempt was made to make an economical choice of n or $\underline{p}$. Indeed, for the given choice of $\underline{p}$, an Alabama paradox may have already taken place for a much smaller value of n. However, since the three conditions given in the proof characterize the paradox, it is easy to see that the closer the populations are to each other, the larger n need be before a paradox can occur (if it will!).

In the proof, the real culprit of all the problems in apportionment methods was isolated. Namely, $(n\ \underline{p}) = ((n\ p_1),\ (n\ p_2),\ldots,(n\ p_s))$ can be dense in a certain union of symplexes. This means that should anything go wrong for a small open set, it probably goes wrong for a dense open set. The proof of this denseness fact will be given next.

Since the simplex is s-1 dimensional, to locate a point $(n\ \underline{p})$ on the simplex we need only consider $(n\ p_1),(n\ p_2),\ldots,(n\ p_{s-1})$. The value of $(n\ p_s)$ is found in the following fashion. Let $\beta = \sum_{i=1}^{s-1} (n\, p_i)$. If β is an integer, then $(n\, p_s) = 0$ since $\Sigma^s(n\, p_i)$ is an integer and $0 \leq (n\, p_s) < 1$. If β is not an integer, then the same reasoning gives us that $(n\, p_s) = [\beta] + 1 - \beta$. The vector $(n\, \underline{p})$ can enter certain copies of P_1 on $P_1, P_2, \ldots, P_{s-1}$. Namely $(n\, \underline{p})$ is restricted to $\mathscr{D}$ where $\mathscr{D}$ is obtained in the following fashion. Let $\mathscr{D}_i = \{\underline{x} \epsilon P_i | 0 \leq x_j < 1,\ j = 1,2,\ldots,s-1, \text{ and } \Sigma\, x_j > i-1\}$. Let $\mathscr{D} = (\bigcup_{i=1}^{s-1} \mathscr{D}_i) \cup \{\underline{0}\}$. Notice that in an obvious fashion, $\mathscr{D}$ can be identified with a s-1 dimensional unit cube. To see this take the s-1 dimensional unit cube $C_{s-1} = \{\underline{y} \epsilon \mathbb{R}^{s-1} | 0 \leq y_i \leq 1\}$, and divide it into the following parts:

$\underline{0}$, $C^i = \{\underline{y} \epsilon C_{s-1} | i-1 < \Sigma\, y_i \leq i\}$ for $i = 1,2,\ldots,s-1$. The homeomorphism between C^i and $\mathscr{D}_i$ is the obvious one.

Assume that $p_1, p_2, \ldots, p_{s-1}$, 1 are rationally independent positive numbers such that $\sum_{i=1}^{s-1} p_i < 1$. (That is, if $\{a_i\}_{i=1}^{s}$ are rational numbers such that $a_s + \sum_{i=1}^{s-1} a_i p_i = 0$, then all the a_i's must equal zero.) Let $\underline{p}' = (p_1, p_2, \ldots, p_{s-1}, 1 - \sum^{s-1} p_i)$.

Theorem. $\{(n\, \underline{p}')\}_{n=0}^{\infty}$ is dense in $\mathscr{D}$.

Proof. In $\mathbb{R}^s$, identify in the obvious fashion the s cubes defined by the integer lattice point. That is, consider $\mathbb{R}^s/\mathbb{Z}^s$. With this identification, the differential equation $dx_i/dt = p_i$, $i = 1,2,\ldots,s-1$, $dx_s/dt = 1$ defines a flow on the unit cube where opposite faces are identified. The rationally independent condition implies that the flow is dense. In particular, the intersection of the flow with the s-1 dimensional cube corresponding to the face $x_s = 0$, denoted by C_{s-1}, is dense. Since $dx_s/dt = 1$, each point here corresponds to an integer multiple of $(p_1,\ldots,p_{s-1},1)$ followed by the identification of each coordinate with its non-integer part. This means that the terms $\{((n\ p_1),\ldots,(n\ p_{s-1}),0)\}_{n=0}^{\infty}$ are dense in C_{s-1}. The conclusion follows from the obvious topological identification of C_{s-1} with $\mathscr{D}$.

Adding to this denseness statement is the fact that the vectors $\underline{p}\epsilon P_1$ which satisfy this rational independence condition are dense in P_1. Most of our concern is with rational entries of P_1. However, using continuity of the differential equations with respect to small perturbations of the vector field (over compact intervals of time) and the denseness of the rationals, a rational point can be selected to exhibit most of the pathology the rationally independent points do. A political corollary of this is if anything can go wrong, it probably will - and densely!

Dynamics

The easiest way to see what goes wrong with house monotone methods is to project everything onto P_1. That is, instead of comparing vectors $n\ \underline{p}$ and $\underline{k}^{(n)}$, we compare unit vectors $\underline{p}$ and $n^{-1}\ \underline{k}^{(n)}$. This means the ranking function needs to be redefined as a smooth function $r\colon (0,1) \times [0,1] \to \mathbb{E}$ such that for fixed x, $r(x,y)$ is monotonically decreasing. For additional flexibility and to make the function r correspond to the 5 methods seriously considered (see BY) we could allow r to depend upon n, but the analysis is essentially the same, so we do not.

So, a ranking method works as follows. If the house size changes from n to $n + 1$, where the apportionment at n is given by $\underline{k}^{(n)}$, then a state which maximizes $r(p_i, k_i^{(n)}/n)$ receives the additional seat. This can also be described in terms of the symplexes. Corresponding to house size n, P_1 is covered with smaller copies of P_1 with edge dimension $1/n$, and the vertices at rational points with denominator equal to n. Call this an $\frac{1}{n}$ copy of the symplex. Each vertex of a $\frac{1}{n}$ copy is in the interior of a $1/(n + 1)$ copy of the symplex, and it is the only vertex in the interior. Now, a house monotone method with vertex at $\underline{k}^{(n)}/n$ selects some vertex from the $1/(n + 1)$ copy of the symplex containing $\underline{k}^{(n)}/n$, where the dynamics is described by r.

If a state is to receive its "quota" of representatives, then for house size n, the ith state should receive either $[n\ p_i]$ or $[n\ p_i] + 1$ representatives, where the latter figure is considered <u>only</u> if $(n\ p_i) \neq 0$. An apportionment family satisfying this constraint for all n is said to respect quota. It turns out that in order to respect quota the choice of the apportionment at house size n are restricted to the numerators of some vertex of the $\frac{1}{n}$ copy of the symplex containing $\underline{p}$. This follows immediately from the fact that the quota constraint restricts attention to the vertices of the s cube where the ith component is either $[n\, p_i]/n$ or $([n\, p_i] + 1)/n$. The claim follows by taking the intersection of this cube with P_1.

Theorem. _If a ranking method respects quota for all_ $\underline{p}$ _in the interior of_ P_1, _then there exists_ C _such that_ $r(x,x) = C$ _for all_ $x \in (0,1)$.

The geometric idea behind the proof follows. Define $B(\underline{p},\underline{x}) = \max\{r(p_i,x_i)\}$. B can be viewed as the altitude of a bowl over $\underline{x}$. The dynamics are designed to choose iterates which will cause the value of B to decrease; that is, to slip down the side of the bowl to the center. However, in order for the center to correspond to $\underline{p}$, $r(p_i,p_i) = r(p_j,p_j)$ for $i \neq j$.

Proof. Assume there does not exist C such that $r(x,x) = C$ for all $x \in (0,1)$. Let $\underline{p}$ be an element in the interior of P_1 such that $r(p_1,p_1) > r(p_i,p_i)$ for $i = 2,\ldots,s$. By the continuity of r, there is an open neighborhood about $\underline{p}$ such that $B(\underline{p},\underline{x}) = r(p_1,x_1)$. According to the ranking method, for any apportionment in this neighborhood state 1 gets any additional seat due to increase in house size. At the nth stage, this is a change from $\underline{k}^{(n)}n^{-1}$ to $(\underline{k}^{(n)} + \underline{e}_1)/(n+1)$, or a step size of $1/(n+1)$ along the line connecting $\underline{k}^{(n)}n^{-1}$ and $(1,0,0,\ldots,0)$. Since the harmonic series is divergent, this means that the iterates due to the ranking method must eventually leave this neighborhood about $\underline{p}$.

On the other hand, the iterates for a quota respecting apportionment are on the vertices of the $\frac{1}{n}$ copy of the symplex containing $\underline{p}$, so they must approach $\underline{p}$ with $n \to \infty$. This contradiction completes the proof.

Corollary. For all $\underline{p}$ such that $r(p_i,p_i) \neq r(p_j,p_j)$ for some choice of i and j, the ranking method does not respect quota. Indeed, the apportionment by ranking must tend arbitrarily far away from quota.

Astonishing as it may seem, the method currently used by the United States does not satisfy this basic condition!

Can this be corrected by imposing the constraint $r(p,p) = 1$ for all $p \in (0,1)$? It can, but then other problems can still crop up resulting from the positive step size of the iterates and the fact the direction (on P_1) must be on the line connecting the starting point with one of the vertices of P_1. It turns out that it is possible to choose an open set of population densities so that the apportionment for house size n respects quota, but the iterate for $n + 1$ does not. Here the edges of $B(\underline{p},\underline{x})$ are used; that is those points $\underline{x}$ where more than one index satisfies the condition $B(\underline{p},\underline{x}) = r(p_i,x_i)$. Also, the p's are selected close enough to the boundary so that the direction vectors "tend" to be "almost" tangent to the sphere with center p and radius determined by the distance from p to the nth iterate.

Discussion

The causes of the Alabama paradox turned out to be similar to consequences of the irrational flow on the torus. So, the norm minimizing approaches were replaced with ranking methods, which turn out to be similar to placing a bowl over the symplex P_1 and define a dynamic which will cause the iterates to tend to the center, or at least not slide too high up the sides. This is acceptable if the bottom of the bowl corresponds to the population density - but it need not.

However, even if a "bowl" is selected with the bottom at the correct place, all that is done is a norm is replaced with some weaker form of measure or metric. Indeed, with the appropriate scaling and convexity assumption on $r(x,y)$, B can be expressed in terms of a metric. Also, choosing an r leads to a discussion concerning the rationale behind "this" choice for r. It should! Different choices of r can lead to different apportionments!

One approach to the problem is very simple. Go back to a quasi-norm minimizing approach which is also house-monotone. Let $\| \ \|$ be your favorite norm. (Since the sup norm leads to a "rounding off" procedure, it is a prime candidate.) If $\underline{k}^{(n)}$ is the apportionment for house size n, then $\{\underline{k}^{(n)} + e_i\}_{i=1}^{s}$ are the only possible choices for house size $n + 1$ which will lead to a house monotone apportionment, where e_i is the unit vector with unity in the ith component. A choice for $\underline{k}^{(n+1)}$ is an apportionment from this set which minimizes the distance to $(n + 1)\underline{p}$.

The advantages of this approach are as follows.

1. If there are no additional side constraints and $\underline{k}^{(0)} = \underline{0}$, then $\{\underline{k}^{(n)}\}$ respects quota. If there are initial side

constraints (such as each state must have at least one representative) then it will tend to quota, and after some m it will respect quota.

2. With the possible exception of ties, the apportionment family is unique.

3. The apportionment family is house monotone.

4. The difference between the ideal apportionment, $n\,\underline{p}$, and the actual apportionment, $\underline{k}^{(n)}$, is determined by some common concept of distance. This should eliminate any need for "philisophical" discussions concerning the "ideal" measure r or B.

The proofs of the first three statements follow by simple alterations of methods used in the previous sections. For example, using the dynamical approach of the last section, if $\underline{x}^{(n)}$ corresponds to the apportionment at house size n, then the candidates for house size $n + 1$ are the vertices of the $1/(n + 1)$ copy of the symplex which contains $\underline{x}^{(n)}$. The vertex chosen, $\underline{x}^{(n+1)}$, is the one closest to $\underline{p}$. (It is unique except for ties.) An induction argument now shows that the quota statement follows.

Assume that $\underline{x}^{(n)}$ respects quota. This means $\underline{x}^{(n)}$ is a vertex of the $\frac{1}{n}$ copy of the symplex containing $\underline{p}$. To show that quota is respected for house size $n + 1$, it suffices to show that $\underline{x}^{(n+1)}$ is a vertex of the $1/(n + 1)$ copy of the symplex containing $\underline{p}$. But this is easy to verify from the geometric fact that the union of $1/(n + 1)$ copies of symplexes containing one of the candidates as a vertex covers the union of all $\frac{1}{n}$ copies of the symplex having $\underline{x}^{(n)}$ as a vertex. Since $\underline{p}$ is in the second union, it must be in the first. Thus $\underline{p}$ is in one of these $1/(n + 1)$ copies of the symplex. $\underline{x}^{(n+1)}$ must then be the candidate vertex corresponding to this symplex.

That the system tends to quota after some additional constraint has been satisfied follows from a similar geometric argument also using the fact the harmonic series diverges.

The house monotonicity follows from the construction.

References

1. Balinski, M. L., and Young, H. P., The quota method of apportionment, Amer. Math. Monthly, 82 (1975), 701-730.