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A Mathematical Programming Model For Optimal Product Line Structuring

by

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A MATHEMATICAL PROGRAMMING MODEL FOR OPTIMAL PRODUCT LINE STRUCTURING *

Abstract

Firms constantly face the problem of new product introduction and retirement of existing products. Few firms have but one product, hence decisions are influenced by all products currently offered by the firm. This paper introduces a mathematical program for optimal product line structuring. Given available market research data, the model provides a method by which optimization over both marketing and production variables can be achieved. The model determines (1) the optimal number of products, (2) which products should be launched, (3) the optimal prices for these products, (4) the potential sales, and (5) the distribution of sales over market segments. Sensitivity and parametric analysis reveal information pertaining to advertising, effects of additional product introduction, effects of unexpected competition or unanticipated demand. Several intuitive characteristics of the model are shown including how the firm should differ its behavior as competition increases. The paper gives some computational results and numerous examples. It represents a theoretical framework by which market research data will be analyzed.

* A detailed version of this paper is available from the authors.

I. INTRODUCTION

Firms constantly face the problem of new product introduction and retirement of existing products. Although firms generate many product ideas, relatively few will reach the launching stage. Product lines cause additional problems. For example, some new products may cannibalize the sales of other products in the line and decrease the profitability of the entire line. Albeit a venture team's goal may be to increase sales by the introduction of a new product, these added sales should be derived primarily from competitor's product sales. Introducing new products whose sales are derived solely at the expense of other products' sales may be detrimental to the firm.

Pricing policies within product lines, therefore, become limited. Complicating matters, consumers have been shown to have strong perceived price-quality relationships. Products possessing high prices are equated with high quality [15]. Conversely, inexpensive products are sometimes associated with a shabby image. These price-product interactions often make product launching and pricing decision variables inseparable.

What management requires is a comprehensive new product planning model. Optimization must be over all the firm's existing products and products under consideration (i.e., potential products). Above all, the model inputs must be available and model outputs must be directly useful to management. The model should encompass pricing and price-product interactions. Model outputs must include: which products to launch, how to price launched products, potential sales of these products, and which market segments will be captured or left unsatisfied. Economies of scale and set-up costs should be considered. Management also requires extensive model diagnostics. Input-data accuracy must be related to optimal solution sensitivity. Effects of advertising, unexpected demand or competition, must be related to the model input. In other words, optimal solution sensitivity should be expressed in terms relevant to managerial decisions.

One additional criterion for this model is extensibility. The truly interesting problem facing management is the temporal one. Planning over time requires consideration of federal tax rates, tax laws, depreciation, market size changing over time, the time value of money, relative market segment size changing over time, pricing changing over time, preferences changing over time, and many other dynamic factors. In the real world, timing can be crucial and fundamental to the decision process.

A generalized optimization procedure is proposed here to deal with these problems. The model presented in this paper was extended to include financial, temporal, economic and other marketing elements not formally integrated in any previous model to our knowledge. The model formally considers pricing, launching, interactions, and price-product effects. Model inputs are indeed measurable. In fact, the measurement process itself has many interesting insights and is discussed in [9]. A new narrow-band telecommunications device furnishes an excellent demonstration of a model application [9], [19].

This paper assumes market research data for consumer preferences over these candidate products has been gathered. Hence, we represent a final step to a market research process where several candidate products are under consideration. Readers interested in the measurement techniques of conjoint, tradeoff analysis, direct utility assessment, and evaluative theory should refer to [6], [11], [7] and [9], respectively. We will also assume engineering technology is known and, hence, production factors relating to these candidate products are also available. We consider only the single period model although the model has been extended to the multi-period case [19]. For simplicity, all sets are assumed finite.

The managerial goal is to select the product set with the highest profit potential, although a non-profit objective could be used. Conditions include some competition. Model outputs include:

(i) Which products to launch, if any

(ii) How to price these products

(iii) The potential sales of these products

(iv) What market segments will be captured and which will be left unsatisfied.

(v) Which products to drop, if any, from the current product line

Parametric programming and sensitivity analysis reveal information pertaining to the effects of advertising, effects of additional product introduction, effects of unexpected competition or unanticipated demand.

II. TERMINOLOGY

Definition 1. Physical Product (PP)

A physical product, denoted p, is defined as the actual commodity described in terms of engineering variables (e.g., size, weight, etc.).

Definition 2. Generalized Product (GP)

A generalized product, denoted π, is defined as a physical product p at a particular price c. Symbolically, $\pi = (p,c)$ expressed as a two-tuple.

Note: A GP can be thought of as a priced physical product.

EXAMPLE: Suppose the firm wishes to launch a new perfume. Experimental evidence shows a more expensive perfume is perceived to have a more desirable scent. The high priced perfume and the low priced perfume, although the same physical product, are different generalized products.

Pricing within product lines tends to be limited [12]. Pricing policies must accommodate price-quality interactions. Further, pricing must reflect total line strategy. These practical pricing problems motivate the assumption of a finite set of potential prices and the notion of a generalized product.

Definition 3. Potential Product

The set of G_F of potential products consists of all GP now offered by the firm or currently being considered for launching.

Definition 4. Product Space

The set H of all GP consists of the set of potential products offered by the firm, G_F, and the competitors' GP, denoted G_C. Then $H = G_F \cup G_C$.

EXAMPLE: In the perfume example, H consists of competing brands, brands currently offered by the firm and the two new GP under consideration.

One objective is to select the optimal subset from GF to launch.

Definition 5. Product Line

A product line represents a group of similar products ordinarily differing by one attribute (otherwise a product space results) and marketed by one firm. New products often cannibalize (derive sales from) existing products' potential sales. For a discussion of another type of product line see [5].

Definition 6. Evoked Set [2], [18]

The evoked set for individual i, denoted D_i is the set of products which, if offered, would be considered for purchase. Note by definition $D_i \subseteq H$.

EXAMPLE: Let $h = |H|$, the cardinality (no. of elements) of H, and $d_i = |D_i|$.

Market research studies show d_i is small in comparison with h. In the deodorant market d_i averaged three compared with an h of eighteen [18].

Definition 7. Decision Function

A decision function, denoted α (.), maps some set of alternative products into a choice space. Given an alternative set of products, say A_j, the decision functions chooses a member product, $\pi_j \in A_j$. Symbolically, $\alpha(A_j) = \pi_j$.

The decision function for individual i will be denoted α_i (.). This model does not require that the decision functions on an individual level be known.

EXAMPLE: Consider four products, say A, B, C, and D.

If an individual i always chooses B over brands A, C, and D then

$$\alpha_i(\{A,B,C,D\}) = B$$

Deterministic preference is assumed here for simplicity, though stochastic preference is discussed in the appendix A.

Definition 8. Reduced Evoked Set

The reduced evoked set for individual i, denoted E_i, is defined as a set with the following three properties:

(1) $E_i \subseteq D_i$ (2) $E_i \subseteq G_F$ (3) if $E \subseteq D_i$ then $\alpha_i(E) \in E_i$

EXAMPLE: The reduced evoked set, then, represents the firm's products which are both in the evoked set of individual i and judged superior to all competitors' products in that evoked set. Any individual may have an empty reduced evoked set.

Definition 9. Ranking Sequence

The higher the rank of the product, the more preferred the product (i.e., the first ranked product is preferred to all others). The ranking sequence is defined as follows: $\{n(i)\}_{i=i}^{h}$ is a ranking sequence for $\alpha(\cdot)$ if for any $A \subseteq H$ and $\overline{\pi} = \min_j\{n(j) | \pi_{n(j)} \in A\}$ then $\alpha(A) = \pi_{\overline{n}}$.

EXAMPLE: Let the decision function defined over $H = \{\pi_1 \pi_2 \pi_3\}$ generate the ranking sequence (3,1,2). Then $\pi_{n(1)} = \pi_3$, $\pi_{n(2)} = \pi_1$ and $\pi_{n(3)} = \pi_2$ ($\pi_{n(1)}$=the 1st ranked product).

Conditions for the Existence of a Ranking Sequence

Several conditions are sufficient for the existence of a ranking sequence [14], [20]. These restrictions are not severe when considering product lines. Extremely dissimilar products vary in many attributes causing multidimensional problems leading to intransitivity and the lack of a ranking scheme. (Note: stochastic behavioral theory does not require transitivity of actual choice, but merely an underlying decision process which is transitive.)

Definition 10. The Market and CBMS

The set of individuals, denoted I, that possess non-empty reduced evoked sets will be referred to as the market. Symbolically, $I = \{ i | E_i \neq \emptyset \}$

Further, the market will be partitioned into choice based market segments (CBMS) which have both the same reduced evoked set, E_m, and the same decision function, $\alpha_m(.)$ defined over E_m.

EXAMPLE: The previous example described the α decision function generating the sequence (3,1,2), Let π_1 = Brand A, π_2 = Brand B, π_3 = Brand C, where $>$ is a preference relationship. Then C > A > B implies: $\alpha(\{A,B,C\}) = C$, $\alpha(\{A,B\}) = A$ etc.

Until recently market researchers considered only deterministic ordinal relationships. Many market researchers [7], [9],[16],[18], [19] are experimenting with stochastic choice; i.e., models predicting probability of choice in contrast to actual choice.

Using stochastic choice, an individual i may be described by one decision function with probability p and another decision function with probability (1-p).

Definition 11. CBMS Size

The expected size of segment m, denoted M_m, with reduced evoked set E_m and decision function $\alpha_m(.)$ is defined as follows: $M_m = |I| \cdot Pr[\{i| i \in m\}]$ where $Pr[\{i| i \in m\}]$ is the probability of an individual drawn at random from I being in CBMS m and $|I|$ is the number of individuals in the market. The size of CBMS represents the expected number of individuals which have a prescribed choice behavior. The precise manner in which M_m is calculated by assessing consumer utility [10], perceived attribute levels [8], [21], evoked sets [18] and availability [8] is discussed in [9], [19].

Definition 12. Delta Function

Let $e_i = |E_i|$ and suppose $\alpha_m(.)$ generates the ranking sequence (n(1), n(2),..., n(e_i)) where E_m is the reduced evoked set for CBMS m, then define $\delta(r,m)$ such that

$$\pi_{\delta(r,m)} = \pi_{n(r)} \quad \text{for } r \in E_m$$

EXAMPLE: Let $H = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ and the ranking sequence for CBMS m be (3,1,4) then $\pi_{\delta(1,m)} = \pi_3$, $\pi_{\delta(2,m)} = \pi_1$, and $\pi_{\delta(3,m)} = \pi_4$.

III. THE LINEAR PROGRAMMING MODEL

Let:

θ_π = the contribution margin for GP_π ($\theta_\pi > 0$)

S_p = the set-up cost for physical product p

$\overline{\pi}$ = the set of GP in the current solution

π^* = the set of GP in the optimal solution

M = the set of CBMS

$\mathcal{P}$ = the set of physical products offered or being considered for launching by the firm

C_p = the set of prices being considered for physical product p

E_m = the reduced evoked set for $CBMS_m$

$e_m = |E_m|$, the cardinality of E_m

M_m = the size of $CBMS_m$ ($M_m > 0$)

π = subscript for GP

c = subscript for cost

r = subscript for rank

$p = |\mathcal{P}|$, the cardinality of $\mathcal{P}$

i = subscript for an individual

m = subscript for CBMS

p = subscript for product

Integer interpretation for the decision variables are as follows:

(Production Variables) $X_p = \begin{cases} 1 & \text{if set-up costs are incurred for product P (i.e., physical product p is produced by the firm).} \\ 0 & \text{otherwise} \end{cases}$

(Marketing Variables) $Z_{\pi m} = \begin{cases} 1 & \text{if the GP } \pi \text{ captures or is directed by the firm at CBMS m (i.e., if } \pi = (p,c) \text{ then the firm prices product p at c so it will capture CBMS m).} \\ 0 & \text{otherwise} \end{cases}$

Then $X_p = 1$ if and only if $(p,c) \in \overline{\pi}$ for any c and $Z_{\pi m} = 1$ if and only if $\alpha_j(\overline{\pi} \cap E_m) = \pi$.

Note: The choice structure of all CBMS are known and incorporated into the constraints hence, the firm knows which CBMS are captured by each product set but not the optimal product set. The firm maximizes profit from sales minus set up costs. Since production occurs at only a finite number 10^t points, any concave increasing nonlinear production function can be accommodated [3], [4].

The firm has several structural constraints. First, a product can not capture a CBMS unless it is launched, i.e. set-up costs are incurred. This constraint set consists entirely of variable upper bounds which can be dealt with very efficiently [1], [13]. Second if some product, say $\delta(g,m)$, is launched, no product which is judged inferior to product $\delta(g,m)$ by CBMS can capture CBMS m. The model becomes:

(1) maximize $\sum_{m \in M} \sum_{\pi \in E_m} (\theta_\pi M_m) z_{\pi m} - \sum_{p \in \mathcal{P}} S_p x_p$

subject to

(2) $z_{\pi m} - x_p \leq 0$ for all $m \in M$, $\pi = (p,c) \in E_m$

(3) $\sum_{r=g+1}^{e_m} z_{\delta(r,m),m} + x_{p'} \leq 1$ for all $m \in M$, $g = 1,2,\ldots,e_m$

where $\delta(g,m) = (p',c')$ and $\sum_{r=a}^{b} z_{\delta(r,m),m} \equiv 0$ for $b < a$

($\equiv$ denotes defined to be)

This constraint set consists entirely of generalized upper bounds and can be dealt with efficiently [1].

At first, it may seem that the number of variables, i.e., $P \times C_p \times M + P$, is outragously large. Actually, the number of variables is $\sum_{m \in M} e_m \times M + P$ where both e_m and M are of reasonable size in practice.

The following theorems are of two types: (1) logical theorems and (2) computational theorems. The logical theorems show the model behaves intuitively when faced with special market structure. The computational theorems show special model structure allowing very efficient methods for model solution to be employed.

It may seem appropriate to add a constraint set to formally prevent more than one GP from capturing the same CBMS. This constraint set is shown redundant by theorem I.

THEOREM 1

$$\sum_{\pi \in G_F} z_{\pi m} \leq 1 \qquad \text{for all } m \in M.$$

PROOF: From (3) for g = 1,

$$(5) \qquad \sum_{r=2}^{e_m} z_{\delta(r,m),m} + x_{p'} \leq 1 \qquad \text{for all } m \in M \text{ where } (p',c') = \delta(1,m)$$

Letting $\pi = \delta(1,m)$ in (2) and combining this result with (5) yields,

$$\sum_{r=1}^{e_m} z_{\delta(r,m)} \leq 1 \qquad \text{for all } m \in M$$

and since $z_{\pi m} = 0$ for $\pi \notin E_m$, (4) follows.

LEMMA 1.

At the optimal solution, (x^*, z^*), $x_p = \underset{m,\pi}{\text{maximum}} \{z_{\pi m}\}$ for all $p \in \mathcal{P}$

PROOF: Constraint set (2) is equivalent to $\underset{m}{\text{maximum}} \{z_{\pi m}\} \leq x_p$ for $m \in M$ now let $\epsilon_p = \underset{m}{\text{maximum}} \{z_{\pi m}\} - x_p$ for all $p \in \mathcal{P}$. Substituting into (3) yields,

$$(6) \qquad \sum_{r=g+1}^{e_m} z_{\delta(r,m),m} + \underset{m'}{\text{maximum}} \{z_{\pi' m'}\} + \epsilon_{p'} \leq 1$$

$$\text{for all } m \in M \; g = 1,2,\ldots,e_m$$

It will now be shown that $\epsilon_p = 0$ for all optimal solutions. Assume at the optimal solution, say $(\hat{\pi}, \epsilon_1^*, \ldots, \epsilon_p^*, \ldots, \epsilon_p^*)$, that some $\epsilon_p^* > 0$. From (6), it is clear that $(\hat{\pi}, \epsilon_1^*, \ldots, \epsilon_{p-1}^*, 0, \epsilon_{p+1}^*, \ldots, \epsilon_p^*)$ is also feasible. Further, if $\epsilon_p = 0$, the objective function value is improved by $S_p \, \epsilon_p^*$. Hence, $(\hat{\pi}, \epsilon^*)$ could not be optimal unless $\epsilon_p^* = 0$ for all $p \in \mathcal{P}$.

COROLLARY 1.

$$x_p^* \leq 1 \quad \text{for all } p \in \mathcal{P} \text{ and } \quad z_{\pi m}^* \leq 1 \quad \text{for all } m \in M, \pi \in E_m$$

without any additional constraints.

PROOF: By theorem 1, $z_{\pi m} \leq 1$ and by lemma 1, $x_p \leq 1$.

We now shoe many of the integer requirements can be relaxed.

THEOREM 2

At any optimal solution $(x_p^*, z_{\pi m}^*)$, $z_{\pi m}^* \in \{0,1\}$ for all $m \in M$, $\pi \in E_m$ if and only if $x_p^* \in \{0,1\}$ for all $p \in P$.

PROOF: To prove the "if" part, it will be shown that if some $z_{\bar{\pi}\bar{m}} > 0$, and for all p, $x_p \in \{0,1\}$ then $z_{\bar{\pi}\bar{m}} = 1$ at any optimum. Let $\bar{\pi} = (\bar{p},\bar{c})$, from equation (2) $z_{\bar{\pi}\bar{m}} \leq x_{\bar{p}}$ and if $x_{\bar{p}} \in \{0,1\}$ then $x_{\bar{p}} = 1$. It will now be shown that $z_{\pi\bar{m}} = 0$ for all $\pi \neq \bar{\pi}$.

Let $z_{\pi\bar{m}} = z_{\delta(s,\bar{m}),\bar{m}}$ and $z_{\bar{\pi}\bar{m}} = z_{\delta(\bar{r},\bar{m}),\bar{m}}$ then there are two cases:

CASE ONE: $s < \bar{r}$ (i.e. $\alpha(\pi \cup \bar{\pi}) = \bar{\pi}$)

Let $\bar{\pi} = \delta(\bar{r},\bar{m})$ then by (3) $\sum_{h=r+1}^{e_m} z_{\delta(h,\bar{m}),\bar{m}} + x_{\bar{p}} \leq 1$ but $x_{\bar{p}} = 1$ hence

$z_{\delta(h,\bar{m}),\bar{m}} = 0$ for $h < \bar{r}$.

CASE TWO: $s > \bar{r}$ (i.e. $\alpha(\pi \cup \bar{\pi}) = \pi$)

Let $(q,c) = \delta(s,\bar{m})$ and $\bar{\pi} = \delta(\bar{r},\bar{m})$ now suppose that some $z_{qcm} > 0$. By equation (2) $z_{qc\bar{m}} < x_q$ hence $x_q = 1$. By equation (3),

$$\sum_{h=s+1}^{\bar{r}-1} z_{\delta(h,\bar{m}),\bar{m}} + z_{\delta(\bar{r},\bar{m}),\bar{m}} + \sum_{h=r+1}^{e_m} z_{\delta(h,\bar{m}),\bar{m}} + x_2 \leq 1$$

but $x_q = 1$ hence $z_{\delta(\bar{r},\bar{m}),\bar{m}} = 0$ a contradiction. Therefore, $z_{\pi\bar{m}} = 0$ for all $\pi = \bar{\pi}$. Substituting this result into equation (3) yields, $z_{\bar{\pi}\bar{m}} \leq 1$ (7)

We have shown that any value of $z_{\bar{\pi}\bar{m}}$ satisfying (7) will satisfy both constraint sets (2) and (3). Further the cost coefficient of $z_{\bar{p}\bar{m}}$ is $\theta_{\bar{p}} M_{\bar{m}} > 0$ hence at any optimal $z_{\bar{p}\bar{m}}^* = 1$.

The "only if" part of the theorem is obvious. If in the optimal solution $x_p = \underset{m,c}{\text{maximum}} \{z_{pcm}\}$ then since $z_{pcm} \in \{0,1\}$ it follows that $x_p \in \{0,1\}$.

Will a firm ever drop a perfectly profitable product? The following simplistic example may produce a counter-intuitive result.

EXAMPLE

Suppose the firm has only one product π_1, with 17,100 units of sales and the firm is considering launching a second product π_2. There are several competing products. Market research data indicates expected sales of 8000 units if product π_2 is launched (7,100 units coming from π_1 sales). Further, test market data indicates if π_2 was the only product sold by the firm it would expect sales of 17,000 units.

Segment No.	Reduced Evoked Set	Behavioral Rule	Information Provided	CBMS Size
$m = 1$	$e_1 = 2$	$\delta(1,m) = \pi_2$ $\delta(2,m) = \pi_1$	$M_1 + M_2 + M_4 = 17100$	$M_1 = 7100$
$m = 2$	$e_2 = 1$	$\delta(1,m) = \pi_1$	$M_1 + M_3 = 8000$	$M_2 = 1000$
$m = 3$	$e_3 = 1$	$\delta(1,m) = \pi_2$	$M_1 = 7100$	$M_3 = 900$
$m = 4$	$e_4 = 2$	$\delta(1,m) = \pi_1$ $\delta(2,m) = \pi_2$	$M_1 + M_3 + M_4 = 17000$	$M_4 = 9000$

Given

$$\theta_{11} = \$1/\text{unit} \quad \theta_{21} = \$2/\text{unit} \quad S_1 = 0 \text{ (already produced)} \quad S_2 = \$9000$$

The mathematical program becomes

$$\text{Maximize} \quad 7100(z_{111} + z_{211}) + 1000z_{112} + 2(900)z_{213} + 9000(z_{114} + 2z_{214}) - 0x_1 - 900x_2$$

$$\text{Subject to} \quad z_{plm} - x_p \leq 0 \quad \text{for permissible m and p}$$

$$z_{111} + x_2 \leq 1$$

$$z_{214} + x_1 \leq 1$$

SOLUTION

Solving this program as an ordinary linear program reveals an optimal integer basis. (Non-integer solutions are interpreted later). The optimal solution was $x_2 = z_{21j} = 1$ for $j = 1,3,4$ and $x_1 = z_{11j} = 0$ for $j = 1,3,4$.
Hence the firm should drop its old product π_1 and launch its new product π_2 capturing CBMS 1, 3 and 4. The firm achieves 17,000 units of sales (7100 + 900 + 9000). The objective function value is $33,100 (2 · 1700 - 900).

Sensitivity Analysis

Since the CBMS were based on estimates associated with each is a level of confidence. From the linear programming analysis, we can determine if the optimal solution will remain optimal given the accuracy of our data and unexpected competition. Here some CBMS size could vary as much as 89%.

Opportunity Costs

The imputed cost of π_1 capturing CBMS is \$7990, the imputed cost of π_1 capturing CBMS 4 is \$8000, and the imputed cost of π_1 capturing CBMS 2 is \$0. These values partially reflect lost dollar sales caused by cannibalism. Other dual variables also provide valuable diagnostics. Further, since variance in the input data can be reduced by additional market research expenditures, the magnitude of future expenditures is suggested.

Interpretation of Results

The solution indicates the firm should drop a product with a positive marginal profit because a more profitable product exists which consumers would not buy if both products were offered. A reasonable strategy when optimizing over the firm's entire product line rather than on an individual product basis. The firm must now consider competitive reaction to this strategy.

Note that the preceding example was necessarily trivial. Problems with hundreds of market segments can be efficiently solved.

Assume that if the firm can not meet consumers' first preferences, their second preference is a competitor's product. These are precisely the conditions of Theorem 3.

THEOREM 3 (strongly competitive case)

If $\delta(1,m) = m$ and $e_m = 1$ for all $m \in M$ then the product separable decision rule,

$$x_p = 1 \text{ if and only if } \theta_p M_p - S_p \geq 0$$

$$x_p = 0 \text{ otherwise; yields the optimal result.}$$

The first constraint set of the dual is $\lambda_{1m} \geq \theta_m M_m$ for all m

Now θ_m, $M_m > 0$ so $\lambda_{1m} > 0$. Hence from complementary slackness, $x_m^* = z_{nm}^*$.

Substituting, the primal becomes,

$$\text{MAX} \quad \sum_{m \in M} (\theta_m M_m - S_m) x_m \qquad \text{S.T.} \quad 0 \leq x_m \leq 1$$

yielding the optimal product separable decision rule (i.e., each product's launching decision is independent of each other product's launching decision).

LEMMA 2.

For the following linear program

$$\text{MAX} \quad \sum_{m \in M} \sum_{\pi \in G_F} (\theta_\pi M_m) z_{\pi m} - \sum_{p \in \mathcal{P}} S_p x_p$$

$$\text{S.T.} \qquad z_{\pi m} - x_p \leq 0 \quad \text{for } m \in M, \quad \pi = (p,c) \in G_F$$

$$\sum_{\pi \in G_F} z_{\pi,} \leq 1 \qquad \text{for } m \in M$$

Let $\hat{\pi}$ be that π which maximizes $T_\pi = \theta_\pi \sum_{m \in M} M_m - S_p$, then

(1) If $T_{\hat{\pi}} > 0$ then the optimal solution is

$$x_p = \begin{cases} 1 & \text{for } \pi = \hat{\pi} \\ 0 & \text{otherwise} \end{cases} \qquad z_{\pi m} = \begin{cases} 1 & \text{for } \pi = \pi \\ 0 & \text{otherwise} \end{cases}$$

(2) If $T_{\hat{\pi}} \leq 0$ then at the optimum $x_p = z_{\pi m} = 0$ for all $\pi = (p,c) \in G_F$ and $m \in M$

PROOF: The first dual constraint set is

$$\mu_{\pi m} + \lambda_m \geq \theta_\pi M_m \quad \text{for } \pi \in G_F, \ m \in M$$

where μ and λ are dual variables. Hence

$$(8) \qquad \sum_{m \in M} \mu_{m\pi} + \sum_{m \in M} \lambda_m \geq \theta_\pi \sum_{m \in M} M_m \quad \text{for } \pi \in G_F$$

noting that the second dual constraint set is

$$(9) \qquad -\sum_{m \in M} \mu_{m\pi} \geq -S_p \quad \text{for } \pi = (p,c) \in G_F$$

Then (8) and (9) imply that

$$(10) \qquad \sum_{m \in M} \lambda_m \geq \theta_\pi \sum_{m \in M} M_m - S_p \qquad \forall_p .$$

But the dual objective function is to minimize $\sum_{m\in M} \lambda_m$, hence (10) provides a lower bound for the dual objective function, that is; maximum $\{\max_{\pi}[T_{\pi}],0\}$

$$\text{maximum } \{\max_{\pi}[T_{\pi}],0\}$$

Let $\pi^* = (p^*,c^*)$ minimizing T_{π}, then this lower bound to obtained with,

$$\mu_{\pi m} = \begin{cases} (\theta_{\pi} - \theta_{\pi^*})M_m & \pi \neq \pi^* \\ S_{p^*} & \pi = \pi^* \end{cases}$$

$$\lambda_m = \theta_{\pi^*}M_m - S_{p^*}$$

(when $\sum_{m\in M} \lambda_m = 0$ the trivial solution is optimal). This value is achieved by the solution given in lemma 2.

In monopolistic situations, consumers who must buy the product will buy it regardless of the variety in the line, which leads to the conditions of Theorem 4.

THEOREM 4 (monopolistic case)

Let $g_F = |G_F|$. If $e_m = g_F$ for all $m \in M$ the firm will launch at most one product according to the following decision rule,

$$x_p = \begin{cases} 1 & \text{if both } \pi = (p,c) \\ & \text{maximizes } T_{\pi} \text{ and that } \max T_{\pi} > 0 \\ 0 & \text{otherwise} \end{cases}$$

PROOF: Note that the primal problem is equivalent to the L.P. in Lemma 2 with the additional constraint (i.e., by Theorem 1)

(11) $$\sum_{r=g+1}^{g_F} z_{\delta(r,m),m} + x_{\delta(g,m)} \leq 1, \text{ for all } m \in M;\ g = 1,\ldots,g_F$$

Hence any solution which is optimal for the L.P. in Lemma 2 which is also feasible for the L.P. in this theorem would be optimal for the L.P. in this theorem.

Now consider the optimal solution for Lemma 2. Avoiding the trivial solution (i.e., no products launched), we have two cases,

(1) if $\delta(g,m) \neq p^*$ then $x_{\delta(g,m)} = 0$ and (11) becomes

$$\sum_{r=g+1}^{g_F} z_{\delta(r,m),m} \leq 1 \qquad \text{for all } m \in M$$

and since by Lemma 2 $\sum_{r=1}^{g_F} z_{\delta(r,m),m} \leq 1$ $m \in M$ this result is clearly true.

(2) if $\delta(g,m) = p^*$ then $x_{\delta(g,m)} = 1$ and (11) becomes

$$\sum_{r=g+1}^{g_F} z_{\delta(r,m),m} \leq 0$$

but $z_{\pi m} = 0$ for $\pi \neq \pi^*$ hence this result is also true.

Theorem 4 illustrates another situation where product launching decisions follow a simple decision rule. Although these conditions may never be realized, they do show a consistency in model formulation.

COROLLARY 2.

The firm's monopolistic profit potential is no less than its competitive profit potential.

COROLLARY 3.

No more products will be launched in a monopolistic situation than a competitive one.

Suppose if the consumer's first preference is not satisfied (i.e., the product is not made available to them) they will upgrade their purchase to the next closest commodity (i.e., their second choice). By upgrade we mean, for example, buy the next higher quality product in the line despite its added cost. If the firm cannot satisfy the consumers first or second choice and competitor's product is chosen, we have the conditions of Theorem 5.

THEOREM 5

If $|E_m| = 2$ for all $m \in M$ and

$$\delta(r,m) = \begin{cases} m & \text{for } r = 1 \quad m = 1,\dots,M \\ m+1 & \text{for } r = 2 \quad m = 1,\dots,M-1 \\ 0 & \text{otherwise} \end{cases}$$

then $x_p^* \in \{0,1\}$ and $z_{\pi m}^* \in \{0,1\}$

PROOF: The dual becomes

$$\text{MIN} \sum_{m \in M} \lambda_m$$

$$\begin{array}{llll}
\text{S.T.} & \mu_{1m} & \geq \theta_m M_m & m = 1,\ldots,M \\
 & \mu_{2m} + \lambda_m & \geq \theta_{m+1} M_m & m = 1,\ldots,M-1 \\
 & -\mu_{1m} - \mu_{2,m-1} + \lambda_m & \geq S_m & m = 1,\ldots,M
\end{array}$$

where $\mu_{20} \equiv 0$

The dual constraints can now be partitioned as follows

$$\text{I} \left\{ \begin{array}{lll}
\mu_{1m} & \geq \theta_m M_m & m = 1,3,5,\ldots,M \\
\mu_{2m} + \lambda_m & \geq \theta_{m+1} M_m & m = 2,4,6,\ldots,M_1-1 \\
-\mu_{1m} - \mu_{2,m-1} + \lambda_m & \geq S_m & m = 1,3,5,\ldots,M_1
\end{array} \right.$$

$$\text{II} \left\{ \begin{array}{lll}
\mu_{1m} & \geq \theta_m M_m & m = 2,4,6,\ldots,M_2 \\
\mu_{2m} + \lambda_m & \geq \theta_{m+1} M_m & m = 1,3,5,\ldots,M_2-1 \\
-\mu_{2m-1} + \lambda_m & \geq S_m & m = 2,4,6,\ldots,M_2
\end{array} \right.$$

where $M_1 = 2[[\frac{M+1}{2}]] - 1$

$M_2 = 2[[\frac{M}{2}]]$

Note: $[[\cdot]]$ denotes "greatest integer in"

Note that

(1) No more than two nonzero elements appear in each column

(2) If a column contains two nonzero elements with the same sign, an element is in each of the subsets

(3) If a column contains two nonzero elements of opposite sign, both elements are in the same subset.

These conditions guarantee integer solutions [3,17].

If a consumer's second preference is the next lower quality product in the line, the conditions of Theorem 6 are met.

THEOREM 6

If $e_m = 2$ for all $m \in M$ and

$$\delta(r,m) = \begin{cases} m & \text{for } r = 1 \quad m = 1,\ldots,M-1 \\ m-1 & \text{for } r = 2 \quad m = 1,\ldots,M \\ 0 & \text{otherwise} \end{cases}$$

then $x_p^* \in \{0,1\}$ and $z_{\pi m}^* \in \{0,1\}$ at the optimum

PROOF: Similar to Theorem 7.

Marketing Consequence of Theorems

Theorem 1

No more than one generalized (priced) product will capture the same CBMS.

Theorem 2

Integer requirements can be relaxed implying both greater computational efficiency and greater scope in post-optimality analysis.

Theorem 3

Under highly competitive situations, interaction effects between products become less influential on the optimal solution. Each product's launching decision is independent of the others. In this case, strong competition implies a simple optimal decision-rule. Strong competition means the firm must meet the consumer's first preference.

Theorem 4

Under monopolistic situations, consumer's tend to buy regardless of the variety in the product line. In this case, the optimal policy of the firm is to follow a simple product decision rule. The firm is to launch that product which has the highest profit potential, assuming that profit is positive. This theorem has two important and intuitive corollaries:

(1) The firm's monopolistic profit is no less than its competitive profit potential.

(2) No more products will be launched in a monopolistic situation than in a competitive one.

The following theorems relate consumer preference to the likelihood of integer solutions.

Theorem 5

If when consumers do not find their first preference they will try to upgrade their purchase. And if their second preference is to a competitor, then easily interpretable solutions are guaranteed. Sensitivity analysis and dual costs are also straightforward.

Theorem 6

If consumers do not find their first preference they will try to downgrade their purchase, and if the firm can not satisfy their second preference, the consumers will go to a competitor, then integer solutions will result.

Interpretations for Non-Integer Solutions

The results provided by model applications motivate the following interpretations for the decision variables:

x_p = the probability of product "p" being launched or expected proportion of times "p" can be launched.

$z_{\pi m}$ = the probability of GP "π" capturing CBMS m or proportion of times "π" captures CBMS m.

Pricing Constraint

When it is undesirable to launch the same product at different prices we impose $\sum_{c \in C_p} z_{pcm} \leq 1$ for $p \in P$, $m \in M$

Product Subset Constraint

Policy considerations may indicate launching both π_1 and π_2 together would be undesirable. Then we must add the constraint, $\sum_{\pi \in N} z_{\pi m} \leq 1$ for all $m \in M$, where N is the subset from which only one product can be launched. Note that Pricing Constraints are a special case of Product-Subset constraints.

IV. EXTENSIONS

The Multi-period Case

The model has been extended to the multi-period case [19]. Added considerations are: financial, temporal, economic and other marketing elements. Federal tax rates, depreciation, market size changing over time, the time value of money, relative market segment sizes changing over time, pricing in a dynamic framework, and several other dynamic factors are included. The extended model is readily amenable to decomposition techniques. Efficient methods for solving the program enable immediate applications. These are left for a follow-on paper and are mentioned in the appendix B.

Solution Algorithm

The program described here can be efficiently solved through the use of several integer programming codes. Fortunately, the revised simplex method is very inexpensive and the cost of solving the program is very small relative to other considerations.

V. COMPUTATIONAL RESULTS

Two-hundred and fifty different market conditions were randomly generated. Each condition also included a randomly generated production structure generated from a uniform distribution as follows:

1-9 CBMS
1-9 Products
1-9 Price (randomly generated for each product)
1-999 Set-up cost (randomly generated for each product)
1-99 CBMS size (randomly generated for each CBMS)
1-9 Reduced evoked set size (randomly generated for each CBMS)

A random sequence of preference was generated for each CBMS

Revised simplex method was used (no integer requirements).

Results for first 50 runs:

5.28 = average number of products
5.36 = average number of CBMS
34.00 = average number of constraints
22.28 = average number of variables
13.58 = average percent non-zero variables
98.00 = percent integer solutions
17.64 = average number of iterations
1.16 = average number of products launched

Results for 250 runs:

98.40 = percent pure integer solutions

Data for non-integer cases:

7.75 = average number of products
7.00 = average number of CBMS
66.00 = average number of constraints
40.75 = average number of variables
6.72 = average percent non-zero variables
1.26 = average calculation time
32.00 = average number of iterations

Two cases had solutions (0, 1/2, 1)

Two cases had solutions (0, 1/3, 2/3, 1)

Computations were made on Northwestern University's CDC6400 computer system operating under SCOPE 3.3 and using the Multi-purpose Optimization System Version 3. This machine is considered relatively slow.

No.	No. Products	No. CBMS	No. Constraints	No. Variables	Percent Non-zero	Integer	Calc. Time	ITNO.	Products Launched
1	6	3	28	20	12.50	yes	.2160	13	1
2	6	5	34	23	9.59	yes	.3540	18	1
3	6	2	4	8	18.75	yes	.0230	4	2
4	2	3	8	6	27.08	yes	.0260	4	0
5	7	4	34	24	10.54	yes	.4460	22	2
6	1	7	14	8	18.75	yes	.0800	8	1
7	7	4	22	18	13.89	yes	.2130	14	1
8	6	2	18	15	15.93	yes	.1160	9	1
9	1	5	10	6	25.00	yes	.0500	6	1
10	7	9	64	39	6.33	yes	.9130	27	2
11	8	8	62	39	6.41	yes	.7870	24	2
12	7	3	26	20	12.88	yes	.1990	12	1
13	5	4	20	15	14.00	yes	.1460	11	2
14	8	7	50	33	7.39	yes	.6400	24	2
15	4	7	34	21	9.52	yes	.4040	21	1
16	8	1	16	16	20.31	yes	.0670	5	1
17	3	2	6	6	27.78	yes	.0200	3	0
18	2	2	6	5	33.33	yes	.0190	3	0
19	3	6	22	14	12.66	yes	.1660	13	1
20	3	9	38	22	8.49	yes	.5700	27	2
21	8	9	78	47	5.10	yes	1.9540	49	1
22	8	4	24	20	12.50	yes	.2360	17	2
23	8	9	108	62	4.72	yes	3.0280	54	1
24	1	9	18	10	15.00	yes	.0950	9	0
25	8	2	14	15	15.24	yes	.0730	7	0
26	6	5	30	21	10.79	yes	.3390	19	1
27	3	2	6	6	27.78	yes	.0190	3	0
28	6	8	50	31	6.90	yes	.7040	28	1
29	4	1	6	7	28.57	yes	.0190	3	1
30	4	5	24	16	12.24	yes	.2320	16	1
31	9	7	70	44	6.20	yes	1.3190	35	1
32	3	8	32	19	9.87	yes	.2950	17	2
33	7	6	50	32	7.69	yes	.7080	27	1
34	1	7	14	8	18.75	yes	.0770	8	1
35	7	8	84	49	5.66	yes	1.6540	38	1
36	9	7	60	39	6.71	yes	1.0620	33	1
37	7	5	42	28	9.35	yes	.4280	18	2
38	3	5	20	13	13.85	yes	.1330	11	1
39	1	7	14	8	18.75	yes	.0790	8	1
40	9	6	58	38	6.90	no*	.9480	29	-
41	4	3	18	13	15.81	yes	.1080	9	1
42	6	8	70	41	6.06	yes	1.2750	35	1
43	1	4	8	5	30.00	yes	.0350	5	1
44	4	9	44	26	7.60	yes	.6020	25	2
45	6	9	80	46	5.43	yes	1.3380	34	1
46	6	3	24	18	13.19	yes	.1850	12	1
47	3	5	16	12	14.58	yes	.0940	9	1
48	9	1	10	14	17.86	yes	.0460	5	0
49	5	6	42	26	8.61	yes	.4750	22	1
50	7	7	70	42	6.39	yes	1.1060	29	1

* : Solution included (0, 1/2, 1)

Calc. Time = calculation time (excludes input/output)

ITNO. = the number of iterations needed for convergence

VI. CONCLUSION

This paper introduced a mathematical model for optimal product line structuring. The model was formulated to analyze available market research data and product decisions relevant to managerial concerns. After formulating the model and giving an example, several theorems were presented which demonstrated some intuitive properties of the model. Other theorems produced results allowing more efficient computational procedures. Interpretations for non-integer solutions occurred only 1.6% of the time. Future research is now being directed toward proving corresponding results for the multi-period model, integrating the model with measurement techniques and extending the model to directly include such factors as advertising.

APPENDIX A:

A Note: Probability of Choice

The probability that an individual drawn at random from the population will choose product π, given measured individual characteristics S, denoted $P[\pi | S,E_i]$ equals the probability of occurance of a decision rule yielding this choice. Symbolically,

$$Pr[\pi | S,E_i] = Pr[\{\underline{\alpha}(\cdot), \underline{\alpha}_S(H) = \pi\}]$$

where $Pr[\cdot]$ is a member of a parameter family of probability distributions.

$\underline{\alpha}_S(\cdot)$ is the decision function for an individual with characteristics S. Several authors introduced assumptions which allow the calibration of $P[\pi | S,H]$, (for example, see [16]).

It follows that, if $\alpha_m(E_m^*) = \pi$ where $\alpha_m(\cdot)$ is the decision function for CBMS, m and E_m is the corresponding evoked set, the probability of an individual selecting product π can be computed as follows

$$Pr[\pi | S,H] = Pr[\{i | i \in Q\} | S]$$

where

$$\begin{cases} Q = \underset{i \in M}{\cup} Q_m \\ Q_m = \{i \mid \underline{\alpha}_i = \underline{\alpha}_m,\ E_i = E_m\} \\ M = \{m \mid \underline{\alpha}_m\ (E_m) = \pi\} \end{cases}$$

Remembering that $Q_1, \ldots, Q_{|M|}$ are mutually exclusive and the properties of $P[\cdot]$ yields

$$Pr[\pi, S, H] = \sum_{m \in M} Pr[\{i \mid i \in Q_m\} \mid S]$$

(1) The calibration of $Pr[\{i \mid i \in Q_m\} \mid S]$ is possible using a probability of choice model.

(2) $|M|$ is usually found to be small since many ranking sequences are unlikely.

APPENDIX B:

The Multi-Period Formulation

$$\text{Maximize } \left\{ \sum_{t=1}^{T} \sum_{p \in \mathcal{P}} \left[\sum_{c \in C_p} \sum_{m \in M} \theta_{pc} M_m(t) z_{pcm}(t) - \frac{S_p x_p(t)](1-r_a)(1-r_d)^{-t}}{(1-r_a)} + r_a D_{pt} x_p(t)(1+r_d)^{-t} \right\}$$

S.T.

$$\sum_{r=1}^{e_m} z_{\delta(r,m),m}(t) \leq 1 \qquad \text{for all } m,t$$

$$z_{\delta(r,m),m}(t) - y_{\delta(r,m)}(t) \leq 0 \qquad \text{for all } r,m,t$$

$$\sum_{r=g+1}^{e_m} z_{\delta(r,m),m}(t) + y_{\delta(g,m)}(t) \leq 1 \qquad g=i,\ldots,(e_m-1) \text{ for all } t,m$$

$$\sum_{t=1}^{T} x_p(t) \leq 1 \qquad \text{for all } p$$

$$\sum_{c \in C_p} y_{pc}(g) - \sum_{t=1}^{g} x_p(t) \leq 0 \qquad \text{for all } p \quad g=1,\ldots,T$$

Decision Variables: $x_p(t)$, $y_{pc}(t)$, $z_{pcm}(t) \in \{0,1\}$

p = product c = price(cost) m = market segment t = time period T = no. of periods

$$z_{\delta(r,m),m}(t) = \begin{cases} 1 & \text{if the } r^{\underline{th}} \text{ ranked product for segment "m" captures "m" in period "t"} \\ 0 & \text{otherwise} \end{cases}$$

$$y_{\delta,(r,m)}(t) = \begin{cases} 1 & \text{if the } r^{\underline{th}} \text{ ranked product for "m" is launched by "t"} \\ 0 & \text{otherwise} \end{cases}$$

$$x_p(t) = \begin{cases} 1 & \text{if product "p" is launched in time period "t"} \\ 0 & \text{otherwise} \end{cases}$$

$M_m(t)$ = the size of CBMS "m" at time "t"

S_p = the set up cost for product "p"

r_a = the tax rate $\quad$ r_d = the discount factor

D_{pt} = the present value of depreciation for product "p" launched at time "t"

The dual can be written as

$$\text{Minimize} \quad \sum_{t,m} \lambda_{1m}(t) + \sum_{p} \lambda_{4p}$$

$$\Pi(p,c,m)\ \lambda_{1m}(t) + \lambda_{2m\Delta(p,c,m)}(t) + \sum_{r=1}^{\Delta(p,c,m)-1} \lambda_{3mr}(t) \geq \Delta_{pcm}(t) \quad \forall_{p,c,m,t}$$

$$\sum_{m=1}^{M} \lambda_{2m\Delta(p,c,m)}(t) + \sum_{m=1}^{M} \lambda_{3m\Delta(p,c,m)} + \lambda_{5p}(t) \geq 0 \quad \forall_{p,c}$$

$$\lambda_{4p} - \sum_{k=t}^{T} \lambda_{5p}(k) \geq h_p(t) \quad \forall_{p,t}$$

<u>where</u>

$$\Pi(p,c,m) = \begin{cases} 1 & \text{if product "p" at price "c" is in market "m"'s evoked set} \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta(p,c,m) = \begin{cases} r & \text{the rank of product "p" at price "c" from market "m"} \\ 0 & \text{if product "p" at price "c" not in market "m"'s evoked set} \end{cases}$$

$$h_p(t) = (r_a D_{pt} - S_p + S_p r_a)(1 + r_d)^{-t}$$

Analysis of the dual variables similar to single period case leads to some interesting results. These will be explored in a follow-on paper.

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