Discussion Paper No. 238

THE QUALITATIVE IMPLICATIONS OF A SIMPLE MODEL OF OPTIMAL HUMAN CAPITAL FORMATION

By

Peter J. McCabe

August 1976

Acknowledgments: The advice and encouragement of the chairman of my dissertation committee, John O. Ledyard, is gratefully acknowledged. This research was partially supported by National Science Foundation Grant SOC 71-03784 A04 (formerly GS-31346X) (Stanley Reiter, Principal Investigator).

I. Introduction

This paper describes and analyzes a simple model of optimal human capital formation when the retirement period is included as a choice variable. The problem that the representative individual is assumed to face is the choice of a plan which maximizes the present subjective valuation of consumption, leisure and human capital. The plan consists of a composite of market commodities and an allocation of effort among three competing activities of human capital formation, work and leisure. The enquiry proceeds with a study of the comparative dynamics, i.e., the response of the maximizing plan to a change in the environment. In particular, the school leaving age and the work leaving age are the choice variables which undergo the comparative dynamics analysis.

As in earlier studies, the model is formulated as a problem of optimal control. So that the problem is well defined, four requirements are imposed upon the structure of the model. These four requirements establish the program of analysis. First, the maximization problem should not be vacuous: the conditions of the model should be sufficient for the existence of at least one maximizing plan of action. Second, those candidate plans which satisfy the necessary conditions for a maximum can be given a complete characterization. Third, those plans satisfying the necessary conditions should be the maximizing plans. Equivalently, the conditions on the structure of the model should limit the search for a maximum to the plans satisfying the necessary conditions. Fourth, the structure of the model should be sufficient to provide qualitative information on the response of the individuals' demands for human capital and retirement and the supply of services in the labor market. It is worth noting that for the model of this paper the sufficient conditions for a global maximum prove to be decisive in determining the amount of information that can be obtained from the comparative dynamics.

The inaugural paper by Ben-Porath [2] studied the human income maximization problem and his concern was primarily with the time profile of human income generated by such a hypothesis. Two independent researches completed at about the same time by H. Oniki [6a] and E. Sheshinski [8] also studied the problem of human income maximization. In a recent article in this journal, Blinder and Weiss [1] offer an extensive analysis of the plans satisfying the necessary conditions in a model of utility maximization of consumption and leisure with the possibility of engaging in schooling and on the job training in order to affect the future human income stream.

This paper takes as its point of departure the work of H. Oniki; the model described in this paper incorporates the essential structure of Oniki's model as well as Sheshinski's. The entire analysis relies heavily on the comparative dynamics results developed by Oniki [6a,6b]. Because the model of this paper incorporates Oniki's structure, reference to his work should be made by the reader in order to compare the results of the comparative dynamics in this paper to his model of income maximization.

II. Outline of the Paper

Section III of the paper describes and analyzes a model which satisfies the four demands enumerated in the introduction. It is hypothesized that the individual has two motives for accumulating human capital; the first is the standard hypothesis that addition to the stock of human capital will augment future potential income and the second hypothesis is that the agent has direct preferences on the stock of human capital. Although the model imposes severe restrictions on the extent of the coverage of the admissible economic behavior and is far less general than many of the models in the literature, the model does make the retirement decision endogenous to the problem and destroys the usual separation of the utility maximization and wealth maximization problems. Therefore, the qualitative properties of the model include income terms as well as pure substitution effects.

The main restriction imposed upon the model of Section III is that effort and the stock of human capital enter in a symmetric fashion in the utility function, the wage function and the production function for human capital. The second restriction imposed upon the structure is that for a fixed amount of human and physical capital effort enters in a linear fashion in the three activities of utility, income and human capital production. This formulation rules out the possibility of a positive demand for leisure during schooling and work and rules out the possibility of part-time work and schooling. The choice of the functional form has been dictated by the fourth demand listed in the introduction. Although the other three requirements would allow for a much richer economic coverage, and hence a less restrictive model could have been presented, the strategy is to defer until Section IV the discussion of the possible generalizations and alternative formulations, in particular, models which exclude direct preferences on the stock of human capital. For example, the model of Blinder and Weiss after some minor modifications of the constraint sets satisfies the existence theorem stated in Section III and they have exhaustively demonstrated that the second requirement of the introduction, a complete characterization of the necessary conditions, is satisfied. However, the third requirement that the paths satisfying the necessary conditions are in fact maximizing has not been shown to be satisfied and the fourth requirement, as they seem to indicate, has little chance of being met.

Section IV provides some discussion of extending the specifications of the model and the difficulties to be encountered if the model builder wishes to maintain the requirements set out in the introduction. However, positive suggestions for extending the structure of the model are included.

III. Description of the Model and the Analysis

The agent chooses a lifetime consumption plan of market commodities, $0 \leq c(t)$, leisure, $0 \leq \gamma(t)$, effort allocated to the labor market, $0 \leq \beta(t)$, effort allocated to the production of human capital, $\alpha(t)$, and a stock of human capital, $E(t)$, so as to maximize over the finite planning period, $[0, T]$, his lifetime subjective valuation, $V(T) = \int_0^T e^{-\delta\tau} \{u^1(c(\tau)) + \gamma(\tau)\, u^2(E(\tau))\} d\tau$, subject to a wealth constraint, $\dot{A} = rA(t) + \beta(t)W(E(t)) - c(t)$ and a production constraint on the accumulation of human capital, $\dot{E}(t) = \alpha(t)G(E(t) - \mu E(t)$. $A(t)$ is the value of physical wealth at date t. r is the return on physical assets. $\beta(t)W(E(t))$ is the human income obtained by offering the amount of effort β and the stock of human capital E in the labor market. $\alpha(t)G(E(t))$ is the gross addition at date t of human capital when the amount of effort, $\alpha(t)$, is combined with the stock of human capital $E(t)$. μ is the rate of depreciation of the stock of human capital and δ the subjective rate of time preference. All functions are assumed to be increasing in human capital. The agent begins the plan with an initial stock of physical wealth $A(0)$ and of human capital $E(0)$ which is assumed to be non-negative. There is no occupational choice and the features of his environment relevant to his choice of an optimal plan are known to him. All functions are continuous and differentiable functions of their arguments. A variable subscripted by a second variable represents the partial derivative of that variable with respect to the second.

The formal statement of his problem is to maximize

(1) $V(T) = \int_0^T e^{-\delta\tau} \{u^1(c) + \gamma u^2(E)\} d\tau$

subject to

(2) $\dot{A} = r A + \beta W(E) - c$,

(3) $\dot{E} = \alpha G(E) - \mu E$,

(4) $A(0) = A_o$

(5) $A(T) \geq 0$,

(6) $E(0) = E_o \geq 0$,

(7) $0 \leq \gamma(t)$, $0 \leq \beta(t)$, $0 \leq c(t)$, $0 \leq \alpha(t)$,

(8) $\gamma(t) + \alpha(t) + \beta(t) \leq 1$ and

(9) $c(t) \leq K < \infty$.

A. Existence of a Maximum

The decision maker's problem is to choose a feasible vector, $b_1 = (c(t))$, $\gamma(t)$, $\beta(t)$, $\alpha(t)$)such that $V(T)$ is a maximum. The first consideration is to establish that the set of maximizers is non-empty. This section lists conditions that insure the existence of at least one maximizing vector of controls.

It is assumed that the utility function, wage function and production function are all continuous functions of the controls and states variables for fixed t, the controls are measurable functions of time and the utility function is concave in consumption of market commodities.

The set

(10) $B_1(t,A,E) = \{b_1 \in \mathbb{R}^4 : 0 \leq b_1, \alpha + \beta + \gamma \leq 1, 0 \leq c(t) \leq K < \infty, \forall (t,A,E)\}$

is clearly compact for every (t,A,E) and upper-semi-continuous in (t,A,E) since it is constant on (t,A,E).

Let

(11) $B_2 = \{z = (z_1, z_2) \in \mathbb{R}^2 : z_1 = rA + \beta W(E) - c, z_2 = \alpha G(E) - \mu E\}$

and then

(12) $M = \{(t,A(t), E(t), b_1(t)): t \in [0,T], b_1 \in B_1, (A(t), E(t)) \in B_2\}$

is a compact set. This is easy to verify given that B_1 is compact and taking account of the initial conditions $(A(0), E(0))$ and the terminal condition $A(T) \geq 0$.

Given the assumptions of continuity of the utility function for fixed t in the state variables and the controls and the assumed continuity of the differential equations then the vector valued function

(13) $\hat{f} = (f_0, f_1, f_2)$, where

$f_0 = u^1(c) + \gamma u^2(E)$, $f_1 = rA + \beta W(E) - c$ and

$f_2 = \alpha G(E) - \mu E$,

is continuous on M.

Assumption 0 . $u^1(c)$ is concave in c and increasing in c.

Finally, with assumption 0 it is immediate that

(14) $$B_3 = \left\{\hat{z} = (z_0, z_1, z_2) \in \mathbb{R}^3 : \begin{array}{l} z_0 \leq u^1(c) + \gamma u^2(E), \\ z = (z_1, z_2) \in B_2 \end{array}\right\}$$

is a convex set.

Theorem 1 (Cesari [3]). If 10-14 hold, then there exists an absolute maximum for the problem (1) - (9).

B. Necessary Conditions for a Maximum

Having considered sufficient conditions for the existence of a maximizing control vector in the previous section, this section deals with conditions which must be satisfied by a vector which is a maximizer. New variables introduced as a consequence of the necessary conditions are given an economic interpretation.

It is not the case that only vectors of controls which are maximizers satisfy these conditions. Rather the necessary conditions only isolate potential candidates for optimality. With the exception of certain special cases, the necessary conditions are not sufficient for a maximum. In section IV two cases of sequences of activities which do satisfy the necessary conditions but are dominated by a third feasible activity are mentioned. The difficulty is that the formulation of the structure of the problem which does possess computational convenience, allowing the comparative dynamics to be carried out, is not, in general, of the convex-concave programming variety.

The characterization of the possible sequences of maximizing behavior is postponed until the discussion of sufficient conditions for a maximum is concluded.

Given a maximizing vector, $b_1^* = (a^*, \beta^*, \gamma^*, c^*)$, for the problem (1) - (9), the Maximal Principle of Pontryiagin et al [7] asserts the existence of continuous, piecewise continuously differentiable functions of time, $\lambda_1(t)$ and $\lambda_2(t)$ and piecewise continuous functions of time $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, where b_1^* maximizes the Hamiltonian:

(15) $H(A(t), E(t), \alpha(t), \beta(t), \gamma(t), c(t), \lambda_1(t), \lambda_2(t), \theta_1(t), \ldots, \theta_5(t)) =$

$$e^{-\delta t}\{u^1(c) + \gamma u^2(E) + \lambda_1 [rA + \beta W(E) - c] + \lambda_2 [\alpha G(E) - \mu E]$$

$$+ \theta_1 [c-k] + \theta_2 c + \theta_3 \gamma + \theta_4 \alpha + \theta_5 \beta\}$$

and $(\lambda_1(t), \lambda_2(t), A(t), E(t))$ are solutions to the system.

(16) $\dot{\lambda}_1(t) = - \frac{\partial H}{\partial A} = - (r-\sigma)\lambda_1,$

(17) $\dot{\lambda}_2(t) = - \frac{\partial H}{\partial E} = - \lambda_1 \beta W_E - \lambda_2 \alpha G_E - \gamma u^2_E + \lambda_2 (\delta + \mu)$.

(18) $\dot{A} = rA + BW(E) - c$ and

(19) $\dot{E} = \alpha G(E) - \mu E$.

The solutions to system (17) - (19) satisfy the transversality conditions:

(20a) $A(0) = A_o$, (20b) $\lambda_1(T) A(T) = 0$ and

(21a) $E(0) = E_o$, (21b) $\lambda_2(T) E(T) = 0$.

The piecewise continuous functions θ_i, $i=1,\ldots,5$ are continuous at points of continuity of $b_1(t)$ and satisfy the conditions:

(22a) $\theta_1 \leq 0$ (22b) $\theta_1[c-k] = 0$,

(23a) $\theta_2 \geq 0$ (23b) $\theta_2 c = 0$,

(24a) $\theta_3 \geq 0$ (24b) $\theta_3[\alpha + \beta - 1] = 0$,

(25a) $\theta_4 \geq 0$ (25b) $\theta_4 \alpha = 0$ and

(26a) $\theta_5 \geq 0$ (26b) $\theta_5 \beta = 0, \forall t \in [0,T]$.

Remark: For a maximum, the constraint $\alpha + \beta + \gamma \leq 1$ must be satisfied with equality because preferences are strictly increasing.

Assumption 1: $0 < c^*(t) < K, \forall\ t$.

Assumption 1 implies that $\theta_1 = \theta_2 = 0$ and imposes no real restriction on the analysis other than the requirement that for the choice K, there exists a positive valuation on physical capital. In effect, the transversality condition is binding in an economic sense.

By means of the methods developed by Oniki [6b], the variables, $(\lambda_1^*(0), \lambda_2^*(0), \lambda_1^*(t), \lambda_2^*(t))$, may be given an economic interpretation. The superscript indicates that these are the equilibrium values. Let $(\alpha^*, \beta^*, c^*, E^*, A^*)$ be a maximizing vector for the problem.

Define

$$(27)\quad H^*(t) = \int_0^t e^{-\delta t} \{[u^1(c^*) + u^2(E^*)\gamma^*]\, d\tau + \lambda_1^* [rA^* + \beta^* W(E^*) - c^* - \dot{A}^*] + \lambda_2^* [\alpha^* G(E^*) - \mu E^* - \dot{E}]\}\, d\tau .$$

Call H^* (t) the total utility or valuation at t . $H^*(t)$ is no longer a functional and with respect to changes in the parameters (A(0), E(0)), one obtains after integration by parts :

$$(28a)\quad \frac{\partial H^*(t)}{\partial A(0)} = \lambda_1^*(0) \quad \text{and}$$

$$(28b)\quad \frac{\partial H^*(t)}{\partial E(0)} = \lambda_2^*(0) .$$

Therefore, the initial value of the co-state variable $\lambda_1^*(t)$ is equal to the increment in total utility at date t of an additional unit of initial physical wealth holdings. Similarly, the initial value of the co-state variable, $\lambda_2^*(t)$, is equal to the increment in total utility at date t due to an additional unit of initial human capital holdings.

Let Δ_A , a real number, represent a shift in the right hand side of the differential equation $\overset{\circ}{A}$. This perturbation may be due to shifts in the wage function, changes in the rate of interest for fixed (c, β, A, E) or changes in the length of the working period, and let Δ_E , also a real number, represent an adjustment to the right hand side of $\overset{\circ}{E}$. This perturbation represents an exogenous adjustment in the production of human capital for a given (α, E). In any case, call the effects of Δ_A and Δ_E a change in the capacity to produce physical and human wealth, respectively. Let δ_A and δ_E be real numbers sufficiently close to zero and append $\Delta_A \delta_A$ and $\Delta_E \delta_E$ to the right side of $\overset{\circ}{A}$ and $\overset{\circ}{E}$. Then, assuming the conditions of Oniki's theorems are satisfied, we obtain respectively:

$$(29a) \quad \frac{\partial H^*(t)}{\partial \delta_E} = \int_o^t \lambda_1^* \Delta_A d\tau \quad \text{and}$$

$$(29b) \quad \frac{\partial H^*(t)}{\partial \delta_E} = \int_o^t \lambda_2^* \Delta_E d\tau, \text{ for fixed } (\alpha, \beta, c, A, E) .$$

Each expression, in turn, may be interpreted as the marginal total utility of a change in the capacity to produce physical wealth and human capital.

Letting $\Delta_A = \Delta_E = 1$ and differentiating with respect to t,

$$(30a) \quad \frac{\partial^2 H^*(t)}{\partial \delta_A \partial t} = \lambda_1^*(t)$$

and

$$(30b) \quad \frac{\partial^2 H^*(t)}{\partial \delta_E \partial t} = \lambda_2^*(t) .$$

The co-state variable associated with the accumulation of physical wealth is equal to the instantaneous marginal utility of a unit change in the capacity to produce physical wealth. The second expression may be given a similar interpretation in terms of human capital.

If both right hand numbers of the system $(\dot{A},\dot{E})$ are perturbed in a manner which preserves the original value of H^*, the ratio

(31) $\lambda = \lambda_2(t) / \lambda_1(t)$

may be viewed as the rate of transformation of one unit of human capital stock into one unit of physical wealth; λ is in money units. This discussion will prove useful in interpreting a number of the terms produced in the study of the comparative dynamics.

The Hamiltonian (15) is assumed to be continuously differentiable in the vector (α, β, c). Therefore, a maximizing vector (α^*, β^*, c^*) satisfies the following system of first order conditions:

(32a) $u_c^1 - \lambda_1 = 0$

(32b) $-u^2 + \lambda_1 W(E) + \theta_3 + \theta_5 = 0$

(32c) $-u^2(E) + \lambda_2 G(E) + \theta_3 + \theta_4 = 0$

Where θ_i, i = 3, 4, 5 satisfy the inequalities (22) - (26).

The necessary conditions admit seven possible maximizing paths and their combinations. The classification of the paths will become clear in the sequel.

The seven possibilities are:

A. $\gamma = 1 - \alpha - \beta = 0$

AI $\alpha = 1, \beta = 0$

AII $\alpha > 0, \beta > 0$

AIII $\alpha = 0, \beta = 1$

B. $1 > \gamma = 1 - \alpha - \beta > 0$

BI $0 < \alpha, \beta = 0$

BII $0 < \alpha, 0 < \beta$

BIII $0 = \alpha, 0 < \beta$

C. $\gamma = 1$

$\alpha = 0$

$\beta = 0$

The event A has no time allocated to consumption activities (γ=0) and a sequence of education, education and work, and all time devoted to work. The event B represents a positive consumption of leisure and the same sequence of allocation of effort between work and education. C is a period of retirement.

Under B the system of necessary conditions becomes

BI

$$u_c^1 - \lambda_1 = 0$$

$$-u^2(E) + \lambda_1 W(E) = -\theta_5 - \theta_3$$

$$-u^2(E) + \lambda_2 G(E) = -\theta_3 - \theta_4, \text{ where } \theta_3 = 0 = \theta_4$$

$$\dot{\lambda}_1 = -(r-\delta)\lambda_1$$

$$\dot{\lambda}_2 = -\lambda_2 \alpha G_E - \gamma u_E^2 + (\mu+\delta)\lambda_2$$

$$\dot{A} = rA - c$$

$$\dot{E} = G(E) - \mu E$$

BII

$$u_c^1 - \lambda_1 = 0$$

$$-u^2 + \lambda_1 W(E) = -\theta_3 - \theta_5$$

$$-u^2(E) + \lambda_2 G(E) = -\theta_3 - \theta_4 \text{ where } \theta_3 = 0 = \theta_4 = \theta_5$$

$$\dot{\lambda}_1 = -(r-\delta)\lambda_1$$

$$\dot{\lambda}_2 = -\lambda_1 \beta W_E - \lambda_2 \alpha G_E - u_E^2 \gamma + (\mu+\delta)\lambda_2$$

$$\dot{E} = \alpha G(E) - \mu E$$

$$\dot{A} = rA + \beta W(E) - c$$

BIII

$$u^1_c - \lambda_1 = 0$$

$$-u^2(E) + \lambda_1 W(E) + \theta_3 + \theta_5 = 0$$

$$-u^2(E) + \lambda_2 G(E) + \theta_3 + \theta_4 = 0 \qquad \text{where } \theta_3 = \theta_5 = 0.$$

$$\dot{\lambda}_1 = -(r-\sigma)\lambda_1$$

$$\dot{\lambda}_2 = -\lambda_1 \beta W_E - \mu^2_E \gamma + (\mu+\delta)\lambda_2$$

$$\dot{A} = rA + \beta W(E) - c$$

$$\dot{E} = -\mu E$$

The system AI, AII and AIII differs only with respect to the values of the variables θ_i which can be determined from (22)-(26).

The necessary conditions for event C are:

C

$$u^1_c - \lambda_1 = 0$$

$$-u^2(E) + \lambda_1 W(E) + \theta_5 = 0$$

$$-u^2(E) + \lambda_2 G(E) + \theta_4 = 0$$

$$\dot{\lambda}_1 = -(r-\sigma)\lambda_1$$

$$\dot{\lambda}_2 = -u^2_E + (\mu + \sigma)\lambda_2$$

$$\dot{A} = rA - c$$

$$\dot{E} = -\mu E$$

Upon combining the second and third static efficiency conditions for AI, AII, AIII and BI, BII and BIII, the following allocation rule for effort between the activities of attending school and working is:

$$(33) \quad F^1(E(t), \lambda_1(t), \lambda_2(t)) = \lambda_1(t)W(E(t)) - \lambda_2(t)G(E(t)) \lesseqgtr 0$$

As $\alpha = 1$ and $\beta = 0$, $\alpha > 0$, and $\beta > 0$, $\alpha = 0$ and $\beta = 1$ for $\gamma = 0$ and as $\alpha > 0$ and $\beta = 0$, $\alpha > 0$ and $\beta > 0$, $\alpha = 0$ and $\beta > 0$ for $\gamma > 0$.

In every case $u_c^1 = \lambda_1$ and by assumption $u_c^1 > 0$ implying $\lambda_1(t) > 0$ implying $\lambda_1(t) > 0$ for every t. Therefore, F^1 may be written

$$(33a) \quad F_\lambda^1 = W(E) - \lambda G(E) \lesseqgtr 0$$

and

$$(34) \quad \frac{d\lambda}{dt} = \dot{\lambda} = - W_E\beta - [\alpha G_E - \mu]\lambda + (r-\sigma)\lambda - \frac{\gamma u_E^2}{\lambda_1(t)}$$

The mapping F_λ^1, its time derivative and the variables (λ, E) characterize completely the allocation of time between the activities of work and education. Recalling that $\lambda = \lambda_2/\lambda_1$ has the measure of "money value per unit of the stock of human capital" and recognizing that $\lambda(t)$ is the present worth of the addition to the stock of human capital in money terms at date t, F_λ^1 may be interpreted as the difference in the marginal cost ($W(E)$ - foregone income at date t) and the marginal benefit in dollars ($\lambda G(E)$) of devoting an additional unit of effort in school.

The transversality conditions on $\lambda_1(t)$ and $\lambda_2(t)$ imply that $A(T) = 0$ and $\lambda_2(T) = 0$. Assumption 1 implies $\lambda_1(t) > 0$ for all t and by assumption $E(0) \geq 0$, hence $E(T) \geq 0$. The economic interpretation for the necessary condition that $\lambda_2(T) = 0$ is that the use of time in the production of human capital has no value at the terminal time because the rewards can be obtained only by consumption of the stock or employment of the stock and the allocation of effort to produce additions to the stock of physical wealth. Therefore, no time is allocated in the final period to production of human capital.

If, on the other hand, the activity of producing human capital were a consumption good, then the consequence of the transversality condition $\lambda_2(T) = 0$, no longer ensures that no time will be allocated to the production of human capital. In this case, the shadow price reflects only the additions to total utility of an additional unit of the stock and does not include the instantaneous benefits of the activity of producing human capital.

The transversality condition $\lambda_1(T)A(T) = 0$ and $A(T) = 0$ has a natural interpretation as well.

Recall the assumption that the optimal consumption plan for every t is assumed to satisfy $0 < C(t) < K$ and it is assumed that larger quantities are valued over smaller quantities of consumption goods and physical assets have no utility value. Hence, the utility value of physical wealth is zero at the end of the planning period but the utility value of consumption is positive as reflected in the positive marginal utility of consumption. Therefore, the consumer chooses a plan of consumption which exhausts his wealth.

The transversality condition excludes any phase (in this case AI, AII, BI, BII) having any effort devoted to education as a final or terminal phase. In each of these cases $F^1_\lambda(T) \leq 0$ which requires $W(E) \leq 0$. That is, the marginal cost must be non-positive but by assumption $W(E) > 0$. For the events AIII and BIII, $F^1_\lambda(T) \geq 0$; therefore, working is an admissible final phase. The retirement period is admissible as well.

Switching between work and retirement is characterized by:

$$(35) \quad F^2(E(t), \lambda_1(t)) = u^2(E) - \lambda_1(t)W(E(t) \gtreqless 0$$

During the work period, $F^2(.) \leq 0$ and during the retirement period $F^2(t) \geq 0$. In this case the interpretation is straightforward; the decision maker allocates his additional unit of effort to that activity which augments the instantaneous utility by a greater amount.

C. Sufficient Conditions for a Maximum and the Characterization of the Maximizing Plans

So far no reference has been made to the assumptions of concavity of the wage, production and utility functions with respect to E. The Kamien-Schwartz sufficiency theorem [4] asserts that if the Hamiltonian after a choice of controls which maximizes the Hamiltonian is concave in the state variables, for a fixed λ_1 and λ_2, then the necessary conditions for a maximum are sufficient as well. The purpose of this section is to demonstrate that if this sufficient condition is to be employed, it is necessary that the utility, wage and production functions be identical and concave in E. Except for this special condition, concavity of the Hamiltonian after insertion of the maximizing values of the controls is incompatible with a sequence of schooling followed by working for example. Because the terms involving consumption depend only upon λ_1 and physical assets enter in a linear fashion and the term describing the rate of depreciation of human capital does not involve the controls, these variables may be neglected. Further, it may be assumed that leisure is zero without any loss of generality.

Recall that the switching function $F_\lambda^1 = W(E) - \lambda G(E)$ partitions the space (λ, E) into three sets representing full time schooling, full time work and part time schooling and work as F_λ^1 is less than zero, greater than zero or equal to zero.

For example, $F_\lambda^1 = 0$ may be represented as:

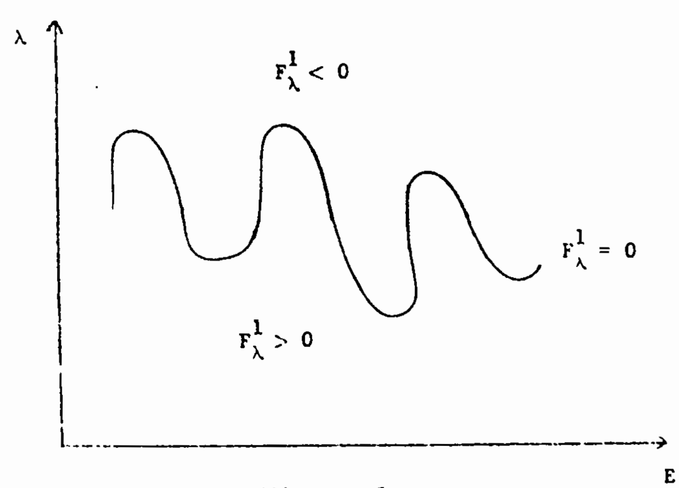

Figure 1

Again without loss of generality assume that W(E) and G(E) are concave functions of the stock of education. For example, at the moment of switching from full time schooling to full time work, $F_\lambda^1 = W(E) - \lambda G(E) = 0$ and let (E^*, λ^*) be the pair of points satisfying this relation. In order to employ the Kamien-Schwartz theorem, $H^* = W(E^*)\ (1-\alpha^*) + \alpha^* \lambda^* G(E^*)$ must be concave in E for fixed λ where α is chosen to maximize the Hamiltonian.

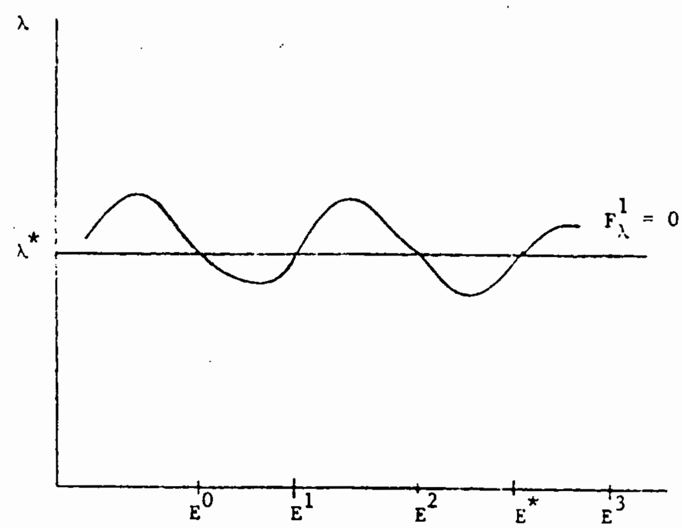

Figure 2

Therefore, the maximizing Hamiltonian for fixed λ^* as a function of E has the following scalloped shaped, i.e., the Hamiltonian is quasi-concave:

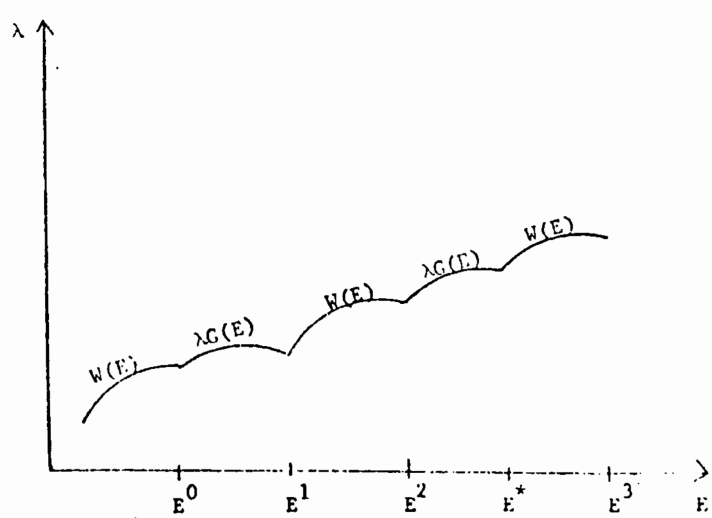

Figure 3

Notice that the slope of the curve $F^1_\lambda = 0$ is $\frac{d\lambda}{dE} = \frac{1}{G}\ \{W_E - \lambda G_E\}$; therefore, a work period preceding a school period requires $W_E \leq \lambda G_E$ and a school period followed by a work period demands $W_E \geq \lambda G_E$. In general, the maximizing Hamiltonian is not concave. The single condition which insures concavity of the maximizing Hamiltonian is that $W(E) = G(E)$ and the function be concave in E. Therefore, $F^1_\lambda = 0$ is independent on E. An example which does not satisfy the condition is when the wage function is concave in E and the production function depends only upon effort for accumulating additional gross units of human capital.

Similarly, in the case of a switch from all work to retirement, the switching function is

$F^2 = u^2(E) - \lambda_1 W(E) \lesseqgtr 0$, where $F^2 < 0$ represents the working period and $F^2 > 0$ the retirement period.

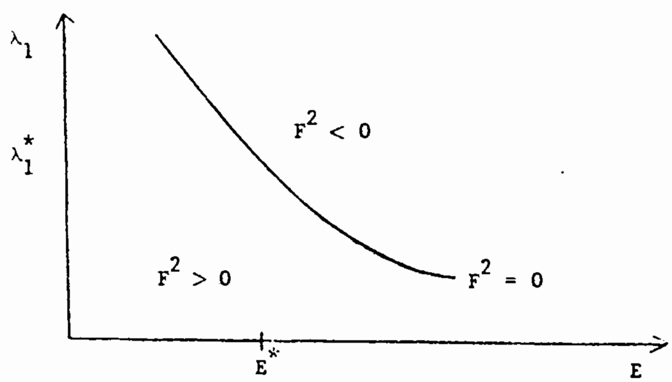

Figure 4

and the Hamiltonian has the form

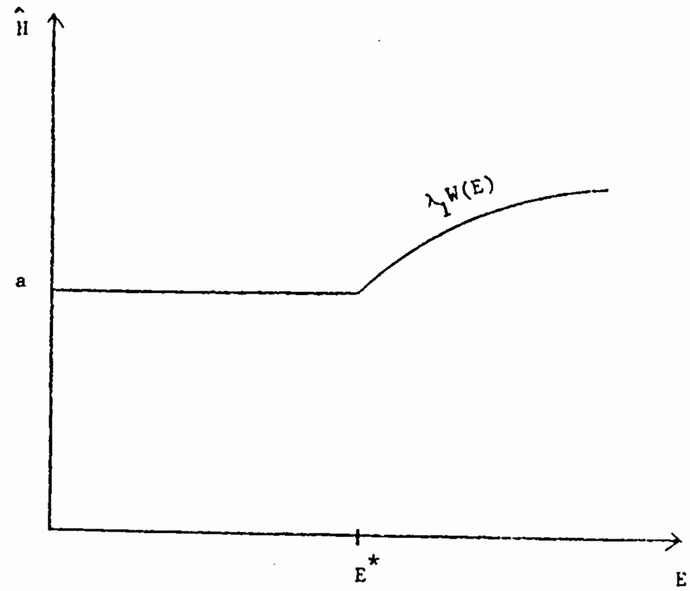

Figure 5

However, if $u^2(E) = W(E)$, then as in the case of switching from schooling to work, the Hamiltonian is concave in E. As in the case of schooling, if the utility function were independent of E, the Hamiltonian would not be concave in E. What this exercise demonstrates is the necessity of the requirement that $u^2(E) = W(E) = G(E)$ if it is to be assured that the paths satisfying the necessary conditions are in fact the maximizing ones.

Assumption 2. $u^2(E) = W(E) = G(E)$ and the functions are concave in E.

Assumption 3. $r > \delta$

Given these two assumptions, there is no part time work and schooling nor is there a positive demand for leisure during either the working or schooling periods. If there were a positive demand for leisure during the working period, it is necessary by (32b) that

$$u^2(E) - \lambda_1 W(E) = 0 \quad \text{on} \quad t \in [\bar{t},\bar{\bar{t}}] \; \bar{t} \neq \bar{\bar{t}}.$$

By assumption 2, this implies that λ_1 must be constant on this interval which is impossible given assumption 3. Similarly, once the work - leaving age is reached, the individual never returns to work at a later date.

If there is a positive demand for leisure during the schooling period, it is necessary by (32c) that

$$u^2(E) - \lambda_2 G(E) = 0 \quad \text{on} \quad t \in [\bar{t},\bar{\bar{t}}] \; \bar{t} \neq \bar{\bar{t}}.$$

By assumption 2, this implies that $\lambda_2(t)$ must be constant on this interval implying that $E(t)$ is constant. Therefore, $\lambda_2(t)$ is constant for all t and the transversality condition cannot be satisfied.

The case of part time work and part time schooling is best studied by analyzing. (33a). In this case, with assumption 2, F^1_λ becomes

$$(36) \quad F^1_\lambda = 1 - \lambda \gtreqless 0$$

For the case of part time schooling and work,

$$\dot{\lambda} = 0 \quad \text{on} \quad t \in [\bar{t},\bar{\bar{t}}] \; \bar{t} \neq \bar{\bar{t}} \quad \text{and}$$

$$(37) \quad \dot{\lambda} = - W_E \, \beta - G_E \alpha\lambda + (\mu + r)\lambda \; = \; -W_E + (\mu + r)\,\lambda = 0.$$

(37) implies that $E(t)$ is constant on this interval and hence constant over the entire planning period. Because $\lambda(t) = 1$, the transversality condition $\lambda(T) = 0$, i.e., $\lambda_2(T) = 0$, cannot be satisfied.

The next assumption proves to be sufficient to rule out cycling

Assumption 4. Let $\bar{E}(T) = \int_0^T G(E(\tau))\, d\tau + E^o$. Then for all $E(t) \leq \bar{E}(T)$, $G(E(t)) - \mu E(t) \geq 0$.

Two cases of cycling need to be investigated. Consider the possibility of work - school - work. $\dot{\lambda}$ must be positive during the working period. This implies that

$\dot{\lambda} = -W_E + (\mu + r)\lambda > 0$ and at the moment of switching $G(E)\lambda = W(E)$. Therefore, for the first instant of schooling $\dot{\lambda} = (-G_E + \mu + r)\lambda$ is positive. By assumption 4, $\dot{E}$ is non-negative during schooling and by assumption 2 $G_E(E)$ is a non-increasing function of E. Therefore, $\dot{\lambda}$ is positive for all t and a period of work is not possible and the transversality condition cannot be met. However, it is possible to have a sequence of work followed by schooling followed by retirement. In this case $\dot{\lambda}$ is positive during schooling where $\dot{\lambda} = -G_E + (\mu + r)\lambda$ but this does not imply that $\dot{\lambda} = -G_E + (\mu+\delta)\lambda_2$ is positive. If it were positive then such a sequence would not satisfy the necessary conditions.

The next proposition collects the results of subsections A through C.

Proposition 1. If assumptions 0 through 4 hold, then a solution to (1)-(9) exists and those plans satisfying the necessary conditions are maximizing plans. The sequence of activities which may satisfy the necessary conditions are: AI - AIII - C, AIII - C, AI - C, AI - AIII, AIII - AI - C, AIII, C.

D. The Results of the Comparative Dynamics

The goal of the analysis is to obtain information about the adjustment in the choice variables due to changes in the parameters of the system by employing the theorem of Oniki [6b]. Only the results of the computations are reported and because of the length of the theorem a statement of it is omitted. However, a brief heuristic discussion is offered.

A review of the necessary conditions reveals that there are two variables left to be determined; the initial values of $\lambda_1(t)$ and $\lambda_2(t)$. Each of the state and co-state variables are continuous, differentiable functions of λ_1^o, λ_2^o and the parameters, θ, of the system (1)-(9). Furthermore, the switching functions

$$(33) \quad F^1 = W(E(t^1)\lambda_1(t^1) - G(E(t^1)\lambda_2(t^1) = 0$$

and

$$(35) \quad F^2 = u^2(E(t^1)) - \lambda_1(t^1)W(E(t^1)) = 0$$

determine respectively, the school leaving age and the work leaving age [as functions of the initial values of the co-state variables and the parameters.] Therefore $F^i(E(t^i;\lambda_1^o,\lambda_2^o,\theta), \lambda_1(t^i;\lambda_2^o,\theta), \lambda_2(t^i;\lambda_1^o,\lambda_2^o,\theta)) = 0$, $i = 1,2$. What is left to be determined is whether the initial values can be written as continuous and differentiable functions of the parameters in some appropriate neighborhood of θ.

Furthermore, the analysis will require information on the adjustment in the initial values of the co-state variables when parameters and functions shift; that is, knowledge of the signs of the derivatives

$$\frac{d\lambda_1^o}{d\theta} \quad \text{and} \quad \frac{d\lambda_2^o}{d\theta}.$$

In order to obtain this information, use is made of the consequences of the transversality conditions for a maximum. In particular, the solutions of $\dot{A}(t)$ and $\dot{\lambda}_2(t)$ must satisfy at the terminal time T:

$A(T;\ \lambda_1^o,\ \lambda_2^o,\theta) = 0$ and

$\lambda_2(T;\lambda_1^o,\lambda_2^o,\theta) = 0,$ where θ denotes an m-dimensional vector of system parameters.

Both the functions $A(\cdot)$ and $\lambda_2(\cdot)$ are continuous and differentiable functions of λ_1^o and λ_2^o. If it can be established that the Jacobian of the vector valued function $(A(\cdot),\lambda_2(\cdot))$ with respect to λ_1^o and λ_2^o is different from zero, then by the implicit function theorem the initial values λ_1^o and λ_2^o are unique, continuous functions of θ having continuous first derivatives in some neighborhood of the vector θ . As in the case of the static maximization problem, the sufficiency condition employed here provides the necessary sign conditions.

If the Jacobian is different from zero, then by the chain rule theorem we have:

$$\underset{2\times 2}{\begin{bmatrix} A(T)_{\lambda_1^o} & A(T)_{\lambda_2^o} \\ \lambda_2(T)_{\lambda_1^o} & \lambda_2(T)_{\lambda_2^o} \end{bmatrix}} \underset{2\times m}{\begin{bmatrix} \frac{d\lambda_1^o}{d\theta} \\ \frac{d\lambda^o_2}{d\theta} \end{bmatrix}} + \underset{2\times m}{\begin{bmatrix} A(T)_\theta \\ \lambda^2(T)_\theta \end{bmatrix}} = \underset{2\times m}{\begin{bmatrix} 0 \end{bmatrix}}$$

where m is the dimension of the vector θ.

Upon inverting the matrix,

$$\begin{bmatrix} \frac{d\lambda_1^o}{d\theta} \\ \frac{d\lambda_2^o}{d\theta} \end{bmatrix} = \frac{\begin{bmatrix} \lambda_2(T)_{\lambda_2^o} & -A(T)_{\lambda_2^o} \\ -\lambda_2(T)_{\lambda_1^o} & A(T)_{\lambda_1^o} \end{bmatrix} \begin{bmatrix} -A(T)_\theta \\ -\lambda_2(T)_\theta \end{bmatrix}}{\begin{vmatrix} A(T)_{\lambda_1^o} & A(T)_{\lambda_2^o} \\ \lambda_2(T)_{\lambda_1^o} & \lambda_2(T)_{\lambda_1^o} \end{vmatrix}}$$

Calling the determinant in the denominator

$$(38)\quad \hat{H} = A(T)_{\lambda_1^o}\lambda_2(T)_{\lambda_2^o} - \lambda_2(T)_{\lambda_1^o}A(T)_{\lambda_1^o},$$

then with the exception of $A(T)_\theta$ and $\lambda_2(T)_\theta$, all the elements of the matrix have a known sign. The sign restrictions are obtained as a consequence of assumption 2 and from the transversality conditions.

Here we list the results of the initial computations and indicate the various sign restrictions implied by the assumptions of the model.

The switching function

$$(33)\qquad F^1 = \lambda_1 W(E) - \lambda_2 G(E)$$

is negative during the school period, zero at the moment of switching and positive during the work period. Therefore,

$$(39)\qquad \frac{dF^1}{dt} = \dot{F}^1 = G(t^1)[W_E(t^1)\lambda_1(t^1) - (\mu+\gamma)\lambda_2(t^1)] = G(t^1)\Psi(t^1) > 0.$$

The switching function

$$(35)\qquad F^2 = u^2(E) - \lambda_1 W(E)$$

is negative during the work period, zero at the moment of switching and positive during the retirement period. Therefore,

$$(40)\qquad \frac{dF^2}{dt} = \dot{F}^2 = r\,\lambda_1(t^2)W(t^2) > 0.$$

The solutions of the variational system with respect to λ_1^o and λ_2^o are:

$$(41)\quad A(T)_{\lambda_1^o} = \hat{c}(T) + \frac{G(t^1)\ e^{-(r-\sigma)t^1+rT}}{\Psi(t^1)}\left\{-e^{\mu t^1}\int_{t^1}^{t^2} W_E(\tau)e^{-(\mu+r)\tau}\,d\tau + e^{-rt^1}\right\}$$

$$+\ \frac{W(t^2)e^{rT-rt^2}}{r\lambda_1^o}\quad ,$$

$$(42) \quad E(T)_{\lambda_1^o} = -\frac{G(t^1)e^{-(r-\sigma)t^1}e^{-\mu T+\mu t^1}}{\Psi(t^1)} ,$$

$$(43) \quad \lambda_1(T)_{\lambda_1^o} = e^{-(r-\sigma)T} ,$$

$$(44) \quad \lambda_2(T)_{\lambda_1^o} = e^{(\mu+\sigma)T}\left\{\int_{t^1}^{t^2} -e^{-(r-\sigma)\tau}W_E(\tau)e^{-(\mu+\sigma)\tau}d\tau\right.$$

$$+ \frac{G\ e^{-(r-\sigma)t^1+\mu t^1}}{\Psi(t^1)}\left[\int_{t^1}^{t^2}\lambda_1(\tau)W_{EE}(\tau)e^{-(\mu+\sigma)\tau}d\tau\right.$$

$$\left.\left. + \int_{t^2}^{T}\mu^2{}_{EE}(\tau)e^{-u\tau}e^{-(\mu+\sigma)\tau}d\tau\right]\right\},$$

$$(45) \quad A(T)_{\lambda_2^o} = e^{rT}\frac{G(t^1)e^{\varphi(t^1)}}{\Psi(t^1)}\left\{e^{ut^1}\int_{t^1}^{t^2}W_E(\tau)e^{-(r+u)\tau}\,d\tau - e^{-rt^1}\right\} ;$$

$$(46) \quad E(T)_{\lambda_2^o} = \frac{G(t^1)\ e^{\varphi(t^1)-\mu(T-t^1)}}{\Psi(t^1)} ,$$

$$(47) \quad \lambda_1(T)_{\lambda_2^o} = 0 \qquad \text{and}$$

$$(48) \quad \lambda_2(T)_{\lambda_2^o} = e^{(\mu+\sigma)T}\left\{\frac{G(t^1)e^{\varphi(t^1)+\mu t^1}}{\Psi(t^1)}\left[-\int_{t^1}^{t^2}\lambda_1(\tau)\,W_{EE}(\tau)e^{-\mu\tau}e^{(\mu+\sigma)\tau}d\tau\right.\right.$$

$$\left.\left. - \int_{t^2}^{T} u^2{}_{EE}(\tau)e^{-(\mu+\sigma)\tau}e^{-\mu\tau}d\tau\right] + e^{\varphi(t^1)}e^{-(\mu+\sigma)t^1}\right\}$$

Remark 1. $\lambda_1(t^1) = \lambda_2(t^1) = e^{(\mu+\sigma)t^1}\left\{\int_{t^1}^{t^2}\lambda_1^o W_E(\tau)e^{-(r+u)\tau}d\tau\right.$

$$\left. + \int_{t^2}^{T} u_E^2(\tau)e^{-(\mu+\sigma)\tau}d\tau\right\}$$

This follows from $F^1 = 0$ and solving $\dot{\lambda}_2(t)$ on $[t_1, T]$ and taking account of $\lambda_2(T) = 0$.

Remark 2. $e^{-rt^1} = e^{\mu t^1} \int_{t^1}^{t^2} W_E(\tau) e^{-(r+u)\tau} d\tau + e^{ut^1} \int_{t^2}^{T} \frac{u_E^2}{\lambda_1^o} e^{-(\mu+\sigma)\tau} d\tau$

Remark 3. $\lambda_2(T)_{\lambda_1^o} = e^{(\mu+\sigma)T + ut^1} \int_{t^2}^{T} \frac{u_E^2}{\lambda_1^o} e^{-(\mu+\sigma)\tau} d\tau - e^{-(r-\sigma)t^1 - \varphi(t^1)} \lambda_2(T)_{\lambda_2^o}$

Remark 4. $A(T)_{\lambda_1^o} = \hat{c}(T) + \frac{W(t^2) e^{rT - rt^2}}{r\lambda_1^o} - e^{-(r-\sigma)t^1 - \varphi(t^1)} A(T)_{\lambda_2^o}$

Remark 5. $\lambda_2(T)_{\lambda_1^o} < 0$, $\lambda_2(T)_{\lambda_2^o} > 0$ by Assumption 2.

$A(T)_{\lambda_2^o} < 0$ by remark 1 and $A(T)_{\lambda_1^o} > 0$ by remark 4.

Remark 6. (38) $\hat{H} > 0$

By remarks 4 and 3

$$\text{(38a)} \qquad \hat{H} = \left[\hat{c}(T) + \frac{W(t^2) e^{rT - rt^2}}{r\lambda_1^o} \right] \lambda_2(T)_{\lambda_2^o} - \left\{ e^{(\mu+\sigma)T} \int_{t^2}^{T} \frac{u_E^2}{\lambda_1^o} e^{-(\mu+\sigma)\tau} d\tau \right\} A(T)$$

The sign is determined by Remark 5.

It is important to emphasize that the necessary condition for the application of the sufficiency theorem that $W(E) = u^2(E) = G(E)$, $G_E(E) > 0$, $G_{EE}(E) \leq 0$ proved to be crucial in determining the signs of $\lambda_2^o(T)_{\lambda_1^o}$ and $\lambda_2(T)_{\lambda_2^o}$. Aside from the concavity, the requirement that the functions be identical removed the effect of the discontinuity in $\dot{\lambda}_2(t)_{\lambda_j^o}$, $j = 1,2$ at the switching points and hence the possibility of a sign ambiguity.

The three choice variables which will undergo the comparative dynamics are $t^1(\lambda_1^o(\theta),\lambda_2^o(\theta),\theta)$, $t^2(\lambda_1^o(\theta),\lambda_2^o(\theta),\theta)$ and $C(\lambda_1^o(\theta),\theta)$. The derivatives of t^1 and t^2 with respect to a change in the parameter θ are:

$$(49)\quad \frac{dt^1}{d\theta} = t^1_{\lambda_1^o}\frac{d\lambda_1^o}{d\theta} + t^1_{\lambda_2^o}\frac{d\lambda_2^o}{d\theta} + t^1_\theta \quad \text{and}$$

$$(50)\quad \frac{dt^2}{d\theta} = t^2_{\lambda_1^o}\frac{d\lambda_1^o}{d\theta} + t^2_{\lambda_2^o}\frac{d\lambda_2^o}{d\theta} + t^2_\theta\,, \quad \text{with}$$

$$(51)\quad t^1_{\lambda_1^o} = -\frac{e^{-(r-\sigma)t^1}}{\Psi(t^1)}\,,$$

$$(52)\quad t^1_{\lambda_2^o} = \frac{e^{\varphi(t^1)}}{\Psi(t^1)}\,,$$

$$(53)\quad t^2_{\lambda_1^o} = \frac{1}{r\lambda_1^o} \quad \text{and}$$

$$(54)\quad t^2_{\lambda_2^o} = 0\,.$$

Upon making the appropriate substitutions

$$(49a)\quad \frac{dt^1}{d\theta} = \frac{e^{(\mu+\sigma)T}}{H}\left(\int_{t^2}^{T}\frac{u_E^2(\tau)e^{-(\mu+\sigma)\tau}d\tau}{\lambda_1^o}\right)\left\{A(T)_\theta\,\frac{e^{\varphi(t^1)}}{\Psi(t^1)} - t^1_\theta\,A(T)_{\lambda_2^o}\right\}$$

$$+\ \frac{\left[\lambda_2(T)_{\lambda_2^o}t^1_\theta - t^1_{\lambda_2^o}\lambda_2(T)_\theta\right]\left[\hat{c}(T) + \frac{W(t^2)e^{rT-rt^2}}{r\lambda_1^o}\right]}{\hat{H}} \quad \text{and}$$

$$(50a)\quad \frac{dt^2}{d\theta} = \frac{t^2_{\lambda_1^o}\left(-\lambda_2(T)_{\lambda_2^o}A(T)_\theta + A(T)_{\lambda_2^o}\lambda_2(T)_\theta\right)}{\hat{H}} + t^2_\theta\,.$$

The next set of terms occur often enough in the results of the computations for the comparative dynamics to justify abbreviating them in the following way. Let

$$(51) \quad \bar{W} = \int_{t^1}^{t^2} W_E(\tau)\lambda_1(\tau)(t^1 - \tau)e^{-(\mu+\sigma)\tau}d\tau < 0,$$

$$(52) \quad \bar{U} = \int_{t^2}^{T} \frac{u_E^2}{\lambda_1^o} e^{-(\mu+\sigma)\tau}d\tau > 0,$$

$$(53) \quad \bar{D} = \int_{t^o}^{t^2} W(\tau)e^{-r\tau}d\tau - \lambda_1^o \hat{c}(T) e^{-rT},$$

$$(54) \quad \bar{P} = \lambda_1^o \hat{c}(T)e^{-rT} + \frac{W(t^2)e^{-rt^2}}{r} > 0,$$

$$(55) \quad \hat{c}(T) = e^{rT} \int_o^T -c_{\lambda_1^o} e^{-r\tau}d\tau > 0 \ (c_{\lambda_1^o} < 0 \text{ by assumption 0 and (32a)},$$

$$(56) \quad \psi(t^1) = [W_E(t^1)\lambda_1(t^1) - (\mu+\sigma)\lambda^2(t^1)] > 0 \text{ by (39)},$$

$$(57) \quad \omega(t^1) = \int_o^{t^1} - G_E(\tau)d\tau + (\mu+\sigma)t^1,$$

$$(58) \quad \hat{c}(T;i) = \int_o^T \lambda_1^o e^{-r\tau}c_{\lambda_1^o}(\tau - t^i) d\tau \quad \text{where } i=1,2,$$

$$(59) \quad \bar{A} = \int_o^T e^{-r\tau}A(\tau)d\tau,$$

$$(60) \quad \hat{\Psi}(t^1) = e^{\int_o^{t^1} \eta(s)ds} \int_o^{t^1} G(E)\, e^{\int_o^{t^1} \eta(s)ds} d\tau > 0,$$

$$(61) \quad \eta(s) = G_E(E(s)) - \mu,$$

$$(62) \quad \xi = -e^{-\int_o^{t^1} \eta(s)ds + \mu t^1} \int_0^{t^1} E(\tau)\, e^{\int_o^{\tau} \eta(s)ds} d\tau \int_{t^1}^{t^2} W_E\, e^{-(u+r)\tau}d\tau$$

$$- \int_{t^1}^{t^2} W_E \int_{t^1}^{\tau} E(s)ds\, e^{-(\mu+r)\tau}d\tau < 0 \quad \text{and}$$

$$(63) \quad K = \bar{W} + \int_{t^1}^{T} \lambda_2(\tau) e^{-(\mu+\sigma)\tau} d\tau .$$

Given the earlier discussion of the interpretation of the co-state variables and their initial values several of these expressions can be described in economic terms.

$\overset{o}{_1}\hat{c}(T)$ is the total change in the present value of consumption due to a change in the total utility of an increment in the current wealth of the individual. K,(63),may be loosely interpreted as the difference in the change in total utility due to a unit change in human capital during the work period and the total change in utility over the working and retirement periods due to some change in the capacity to produce human capital.

Case 1 A change in the rate of interest

In the income maximization problem, the effect of a change in the rate of interest is due solely to a substitution effect and the sign is appropriately negative on the school leaving age. In addition, the effect of a change in the interest rate on the work leaving age is examined. In this instance the substitution effect should be negative since if the rate of interest increases and positive human income is earned the additional income can be spread over both consumption of market commodities and leisure. Finally there will be the wealth effect as well. In both cases, terms can be identified as the substitution effect and the wealth effect and with one qualification the next two propositions support the intuition.

Proposition 2a: If the present value of the sum of physical assets, $\bar{A}$, is non-positive and if $\hat{\hat{c}}(T);1)$ is non-positive, then $\frac{dt^1}{dr} < 0$.

A possible explanation for $\hat{c}(T;1) \leq 0$ is if the school leaving age occurs sufficiently early in the plan and if the adjustment in consumption is not concentrated during the school period.

Demonstration:

$$(64) \quad \frac{dt^1}{dr} = \frac{e^{\varphi(t^1)+(r+\mu+\sigma)T}}{\Psi(t^1)\ \hat{H}\ \lambda_1^o} \quad \bar{P}\ \bar{W} + \bar{U} \left\{ \bar{A} + \hat{c}(T;1) + \frac{W(t^2)e^{-rt^2}}{r\Psi(t^1)}(t^1-t^2) \right\} .$$

With the exception of $[\bar{A} + \hat{c}(T;1)]\bar{U}$ all terms are negative.

Proposition 2b: If the current worth of the sum of physical assets is non-negative $(\bar{A} \geq 0)$ and if $\hat{c}(T;2)$ is non-negative, then $\frac{dt^2}{dr} < 0$.

Demonstration:

$$(65) \quad \frac{dt^2}{dr} = \frac{1}{r\lambda_1^o\hat{H}} \left\{ -\lambda_2(T)_{\lambda_2^o} (\bar{A} + \hat{c}(T;2)) + A(T)_{\lambda_2^o} \left[e^{(\mu+\sigma)T}\ \bar{W} + \lambda_1^o\ \bar{U}(t^2-t^1) \right] \right\}$$

Case 2. A change in the initial value of physical assets, A^o.

Proposition 3: An increase in the initial value of physical assets increases the school leaving age, decreases the work leaving age and consumption increases everywhere.

Demonstration:

$$(66) \quad \frac{dt^1}{dA^o} = \frac{e^{(\mu+\delta+r)T + \varphi(T^1)}}{\hat{H}\ \Psi(t^1)} \quad \bar{U} > 0 \quad ,$$

$$(67)\quad \frac{dt^2}{dA^o} = -\frac{e^{rT}}{r\lambda_1^o \hat{H}}\, \lambda_2(T)_{\lambda_2^o} < 0 \;, \quad \text{and}$$

$$(68)\quad \frac{dc}{dA^o} = c_{\lambda_1^o} \frac{d\lambda_1^o}{dA^o} > 0, \text{ where } \frac{d\lambda_1^o}{dA^o} = \frac{-\lambda_2(T)_{\lambda_2^o} A(T)_{A^o}}{\hat{H}} .$$

Case 3. A change in the terminal time T.

Proposition 4a: The effect of an increase in the school leaving age of an increase in the terminal time is indeterminate. The first term of (69) is negative and the second term is positive.

Demonstration:

$$(69)\quad \frac{dt^1}{dT} = \frac{-e^{(\mu+\sigma T+\varphi(t^1))}}{\hat{H}\,\psi(t^1)}\, \bar{U}\, c(T) + \frac{e^{\varphi(t^1)}}{\hat{H}\,\psi(t^1)}\, \frac{u_E^2(T)}{\lambda_1^o}\, \bar{P} \,,$$

where $c(T)$ is consumption at date T and $u_E^2(T)$ is the marginal utility of human capital at date T.

Proposition 4b: An increase in the terminal time, increases the work leaving age.

Demonstration:

$$(70)\quad \frac{dt^2}{dT} = \frac{1}{r\lambda_1^o \hat{H}} \left(\lambda_2(T)_{\lambda_2^o}\, c(T) - A(T)_{\lambda_2^o}\, u_E^2(T) \right) > 0.$$

Case 4. A change in the productivity of effort in the labor market.

The case describing the adjustment in the wage function is due to Oniki [6a].

The adjustment in the wage function, $W(E)$, is represented in general form by the following parametric shift function:

$$(71) \quad W^{*}(E) = W(E) + \xi\, \delta W(E).$$

In order to preserve the conditions of the sufficiency theorem, the special case chosen is:

$$(72) \quad \delta W(E) = W(E) \qquad (73) \quad \delta W_E = W_E(E)$$

and

$$(74) \quad W^{*}(E) = (1+\xi) W(E).$$

The analysis is carried out for small adjustment in ξ for $\xi = 0$. The interpretation of the effect of ξ is to increase or decrease the productivity of effort during work relative to effort expended in schooling or leisure activities.

The switching functions are:

$$(75) \quad F^1 = \lambda_1 [W(E) + \xi\, \delta W(E)] \big|_{\xi=0} - \lambda_2 G(E) = 0$$

and

$$(76) \quad F^2 = u^2(E) - \lambda_1 [W(E) + \xi\, \delta W(E)] \big|_{\xi=0} = 0.$$

Let

$$(77) \quad x(t)_{\delta W} = \frac{\partial x}{\partial \xi} \Big|_{\xi = 0} .$$

Proposition 5: If (53) $\bar{D} > 0$, then $\frac{dt^1}{d\delta W} > 0$ and $\frac{dt^2}{d\delta W} < 0$.

If (53) $\bar{D} < 0$, then $\frac{dt^1}{d\delta W} < 0$ and $\frac{dt^2}{d\delta W} > 0$.

Demonstration:

$$(78)\quad \frac{dt^1}{d\delta W} = \frac{e^{\varphi(t^1)+rT+(\mu+\sigma)T}}{\psi(t^1)\ \hat{H}}\ \bar{U}\ \bar{D}$$

and

$$(79)\quad \frac{dt^2}{d\delta W} = \frac{e^{rT}}{r\hat{H}\lambda_1^o}\ [-\lambda_2(T)_{\lambda_2^o}\ \bar{D}]\ .$$

Therefore, the effect of increasing the productivity of work effort is to increase the period of schooling and increase the period of retirement unless the response in consumption to a change in the present value of wealth is greater than the present value of human income due to the change in the productivity of effort.

Case 5. A change in the efficiency of effort in consumption activities.

The productivity of effort in consumption was perturbed in the same manner as in the case of the wage function.

Proposition 6: An increase in the productivity of effort in consumption activities increases the period of schooling and lengthens the retirement period.

Demonstration:

$$(80)\quad \frac{dt^1}{d\delta u_2} = \frac{e^{(\mu+\sigma)T+\varphi(t^1)}}{\hat{H}\ \psi(t^1)}\ \bar{U}(\lambda_1^o\hat{c}(T)) > 0$$

and

$$(81)\quad \frac{dt^2}{d\delta u_2} = \frac{1}{r\hat{H}}\left\{-\ \lambda_2(T)_{\lambda_2^o}\ \hat{c}\ (T)\right\} < 0.$$

Case 6. A change in the initial value of physical capital, E^o.

In this case, given the symmetric and identical nature in which human capital enters the problem, there is no effect on the school leaving age and work leaving age of a change in this parameter.

Case 7. A change in the rate of time preference, δ.

Proposition 7: The effect of a change in the rate of time preference is indeterminate on the school leaving and work leaving ages.

Demonstration:

$$(82)\quad \frac{dt^1}{d\delta} = \frac{e^{(\mu+\sigma)T+\varphi(t^1)}}{\hat{H}\,\Psi(t^1)}\;\bar{U}\left\{-\hat{\hat{c}}(T;t^1) + \frac{W(t^2)e^{rT-rt^2}}{r}\,(t^2-t^1)\right\}$$

$$- \frac{e^{(\mu+\sigma)T+\varphi(t^1)}}{\hat{H}\,\Psi(t^1)\lambda_1^o}\;\bar{P}\left\{\bar{W} + \int_{t^1}^{T}\lambda_2(\tau)e^{-(\mu+\sigma)\tau}d\tau\right\}$$

and

$$(83)\quad \frac{dt^2}{d\delta} = \frac{1}{r\hat{H}\lambda_1^o}\left\{\lambda_2(T)_{\lambda_2^o}\,\hat{\hat{c}}(T;t^2)\right\} + A(T)_{\lambda_2^o}\,e^{(\mu+\sigma)T}$$

$$\left\{\bar{W} + \int_{t^1}^{T}\lambda_2(\tau)e^{-(\mu+\sigma)\tau}d\tau\right\}.$$

Therefore, if $-\hat{\hat{c}}(T;t^1)$ and $\hat{\hat{c}}(T;t^2)$ are positive and if $\bar{W} + \int_{t^1}^{T}\lambda_2(\tau)e^{-(\mu+\sigma)\tau}d\tau$ is negative then the school leaving age and work leaving age increase when the rate of time preference increases.

Case 8. An adjustment in the productivity of effort in the production of human capital.

A similar adjustment was introduced here as in the case of the wage function. However, the results are indeterminate under assumption 2. Nevertheless, the next stronger assumption does provide some information.

Assumption 5: $u^2_{EE} = W_{EE} = G_{EE} = 0.$

Proposition 8a: Given assumption 5, when the productivity of human capital increases, the school leaving age increases.

Demonstration:

$$(84)\quad \frac{dt^1}{d\delta G} = \frac{e^{(\mu+\sigma)T}}{\hat{H}\,\Psi(t^1)}\ \bar{U}\ e^{rT+\varphi(t^1)}\ \hat{\Psi}(t^1)e^{ut^1}\int_{t^1}^{t^2} W_E\ e^{-(\mu+r)\tau}d\tau + \frac{\bar{P}\ e^{\varphi(t^1)+(\mu+\sigma)(T-t^1)}}{\hat{H}}\ \frac{\lambda_2(t^1)}{\Psi(t^1)}$$

Proposition 8b: If assumption 5 holds, then $\frac{dt^2}{d\delta G}$ is indeterminate.

Demonstration:

$$(85)\quad \frac{dt^2}{d\delta G} = \frac{t^2\lambda_1^o}{\hat{H}}\left(e^{\varphi(t^1)+rT+(\mu+\sigma)(T-t^1)}\left(-\ \hat{\Psi}(t^1)e^{ut^1}\int_{t^1}^{t^2} W_E\ e^{-(u+r)\tau}d\tau\right) - \frac{A(T)\ \frac{\lambda_2(t^1)}{\lambda_2^o}}{\Psi(t^1)}\right)$$

Case 9. A change in the rate of depreciation, μ.

As in case 3, no results were obtained unless Assumption 5 is invoked.

Proposition 9: If assumption 5 holds, then $\frac{dt^1}{d\mu} < 0$ and $\frac{dt^2}{d\mu} > 0$.

Demonstration:

$$(86)\quad \frac{dt^1}{d\mu} = \frac{e^{\varpi(t^1)+(\mu+\sigma)T}}{\Psi(t^1)\ \hat{H}}\ \bar{U}\ \xi + \frac{e^{\varpi(t^1)}}{\Psi(t^1)}\ \frac{\bar{P}}{\hat{H}}\left[-\int_{t^1}^{T} \lambda_2(\tau)e^{-(\mu+\sigma)\tau}d\tau\right]$$

and

$$(87)\quad \frac{dt^2}{d\mu} = \frac{t^2\lambda_1^o}{\hat{H}}\left(-\lambda_2(T)_{\lambda_2^o}\ \xi - A(T)_{\lambda_2^o}\ e^{(\mu+\sigma)T}\int_{t^1}^{T} \lambda_2(\tau)e^{(\mu+\sigma)\tau}d\tau\right)$$

IV. Comments on Possible Extensions and Alternative Formulations

Among all of the simplications and compromises that are exhibited in this paper, perhaps the least satisfactory is the formulation of the utility function with the inclusion of the stock of human capital. It is the least satisfactory because one would prefer to concentrate on the human capital formation problem as a motive for affecting the future income stream only and attempt to study the qualitative properties of such a model when retirement is included as a choice variable. This section includes a report of some attempts that were made to study this sort of model. The analysis may be found in McCabe [5a].

Before turning to this question, permit a brief mention regarding the problem of the existence of a maximizing course of action. It can be established for a much broader class of life cycle models and less severe restrictions than imposed in this paper that a maximizing plan of action exists. However, real restrictions still need to be imposed if one wishes to confirm the existence of a maximizer. McCabe [5b] contains such a demonstration and techniques for checking whether the conditions of the various existence theorems are satisfied for wider classes of the life-cycle model.

As an indication of some of the difficulties to be encountered when attempting to satisfy the other requirements outlined in the introduction consider the next model which is closer in spirit to the Blinder-Weiss structure.

$$J(T) = \text{Max} \int_{o}^{T} u^1(c) + u^2(\gamma) d t$$

subject to

$$A = rA + \beta W(E) - c \quad \text{and}$$

$$E = \alpha C(E).$$

Possible candidate paths which satisfy the necessary conditions of Pontryagin are part-time work and schooling and cycling, that is sequences of work-school-work. However, it can be shown (see McCabe [5a] for the demonstration with a positive demand for leisure and Oniki [6b] for the income maximization case) that there exist other possible courses of action (not necessarily satisfying the necessary conditions) which would dominate in the sense of a greater total subjective life-time valuation. This is shown without reference to conditions other than that the functions are increasing in their arguments. Assumptions such as concavity of $W(E)$ and $G(E)$ are of no help in any case. This example is included because it is not uncommon for the analysis of what paths are to be ruled out stops with those which do not satisfy the necessary conditions.

Secondly, attempts to obtain qualitative information for this model in the case of a positive demand for leisure throughout the planning period proved not to be successful. The first difficulty encountered was that the solutions of the variational system of the state and co-state variables. i.e, $X(T)\backslash o_i$, $i = 1,2$ could not be computed. Although the variational system is linear it did not decompose allowing a set of solutions to be computed as it does in the problem presented in this paper.

A second less general model was tried with the production function defined in terms of the length of schooling, $G(E) = 1$. This formulation allowed the variational system's solutions to be computed but the conditions on the model were not sufficient to insure definite sign information of what would be the analogue to $\hat{H}$ of this paper. Without such sign information the comparative dynamics analysis is stymied. Of course, both versions of the model did not satisfy any known sufficient conditions for a maximum.

However, this suggests the next possible formulation. Let the wage function be $W(\beta,E)$ and $G(\alpha,E) = \alpha$ and assume the utility function $u^1(c) + u^2(\gamma)$ is concave in its arguments and $W(\beta,E)$ concave in its arguments. Although, in this formulation, it does not seem possible to study the case of part-time work and schooling, it may be possible to obtain information on a sequence of schooling, work and retirement with a positive demand for leisure throughout the plan. A second suggestion is to specify parametric forms for all remaining functions and this may achieve two ends; one, it may be possible to solve the system of variational equastions when there is a period of part-time work and schooling and two, perhaps restrictions on the magnitudes of the parameters may provide whatever additional sign information is needed (this may be possible after additional local tests are employed some of which will be discussed in the next paragraph).

Stimulated by a technique of Oniki [6a] which proved successful in the income maximization case, a static maximization problem was posed for the case where leisure enters the utility function in a linear fashion. The undetermined variables in the control problem are λ_1^o and λ_2^o. Once the controls and state variables have been chosen to satisfy the Maximal Principle, $J(T)$ it may be viewed as a function of λ_1^o and λ_2^o. Assuming that

λ_1^o and λ_2^o have been chosen to maximize $J(T)$, they must satisfy first and second order conditions for a constrained local maximum. The constrained maximization problem was formulated taking account of the conditions that $A(T) = 0$ and $\lambda_2(T) = 0$. The first and second order conditions for this extremum problem were used in an attempt to determine what if any restrictions were to be placed on the signs of the solutions to the variational system and in particular, the sign of $\hat{H}$. In general, no information was provided but if it were assumed that $W(E) = G(E) = E$ then various sign restrictions were imposed if λ_1^o and λ_2^o were to be local maximizers of the problem. In order to compare the consequences of this model with the one presented in this paper, two experiments changing the initial value of physical assets and the second a change in the terminal time are reported. The full discussion may be found in McCabe [5a].

In the case of an increase in the initial value of wealth, the school leaving and work leaving ages both decrease. Hence, the retirement period increases and the results seem to support intuition since human capital is accumulated only to enhance human income potential. However, consumption falls at each instant with an increase in the initial wealth holdings. What happens is that the increase in initial physical wealth is offset by a decline in the present worth of human income. In this model, the length of the working period does not change. As a further check, the effect of a change in physical wealth on the total valuation of the plan was studied. The total valuation of the plan increases when initial wealth increases. The increase in utility from lengthening the retirement period offsets the loss in utility from the decrease in consumption.

For the case of an increase in the terminal time both the school leaving age and the work leaving age increase and consumption increases for all $t \neq T$. It should be emphasized that these results were obtained by imposing conditions that would be satisfied if λ_1^o and λ_2^o were local maximizers for a given plan of school-work- and retirement.

V. Summary

The results of this paper indicate that if the hypothesis of maximizing behavior is invoked and if it is to be robust there is at least one formulation of the life cycle model which satisfies the four requiremrnts listed in the introduction. Secondly, it offers some indication of how rich such models will be in their qualitative implications and which parameters will yield definite qualitative information. In particular, perturbing the parameters which define the budget equation did yield information. For example, this model suggests that studying the effects of various tax policies may not be a barren issue. Furthermore, it may be possible to obtain results when imperfect capital markets are assumed [for a very interesting first attempt in a model with credit rationing see Oniki [6a]]. However, as in the case of neo-classical comparative statics, not much in the way of definite results are to be obtained by inquiring into the effects of changes in functional form and there seems to be little mileage to be gained by perturbing the parameters which defined the rate of human capital formation.