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AGGREGATION PROCEDURE FOR CARDINAL PREFERENCES: A FORMULATION AND PROOF OF SAMUELSON'S CONJECTURE THAT ARROW'S IMPOSSIBILITY THEOREM CARRIES OVER TO CARDINAL PREFERENCES

by

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INTRODUCTION

Samuelson made the conjecture stated in the title in a 1967 paper [3]. He also formalized there the axiom of independence of irrelevant alternatives for cardinal preferences, used here. Preferences are cardinal if their representation by a numerical function is invariant under, and only under, positive linear transformations. One may think that the disregard for intensity of preferences, embedded in Arrow's treatment of profiles of ordinal rankings of alternatives, leads to the impossibility result. Samuelson's conjecture points out that this is not the way to refute the conclusions of Arrow's theorem.

There is also interest per se in aggregation of cardinal preferences. Such preferences are usually considered as von Neumann-Morgenstern utility, i.e. numerical representation of preferences over lotteries, [4]. Since uncertainty is the rule and not the exception whenever decisions are involved, it is of some importance to obtain a social N-M utility over risky outcomes. Given such a utility, the society will be able to choose a best alternative among the several feasible risky actions (i.e. lotteries).

However it is not necessary to restrict the interpretation of cardinal preferences to those induced by ordinal ranking over lotteries. One can think of cardinal preferences derived from comparisons between pairs of alternatives (like in an axionatization of a regret relation). See Alt [1] for an early work of this kind.

When working with cardinal preferences a continuity assumption is needed, in addition to unanimity and independence (see the example in the end of the next section). A standard reference for Arrow's theorem is the last chapter of his book [2].

FORMAL PRESENTATION

Let A denote a finite, nonempty set, to be referred to as the set of outcomes and let R^A denote the set of functions from A to the set of real numbers, R.

A subset X of R^A is said to be a cardinal preference relation over A if it satisfies the following three conditions: (i) X is nonempty. (ii) If x and y belong to X then there are α and β in R, $\alpha > 0$ s.t. for all a in A, $x(a) = \alpha y(a) + \beta$. (iii) If x belongs to X and α and β belong to R, with $\alpha > 0$, then y defined by, for all a in A $y(a) = \alpha x(a) + \beta$, also belongs to X.

An element x of X is referred to, sometimes, as a (cardinal) utility over A.

We denote by Ξ the set of cardinal preferences over A. For any two elements X and Y of Ξ one has; $X = Y$ or $X \cap Y = \emptyset$. Given a nonempty subset B of A, denote by Ξ_B the set of cardinal preferences over B. For X in Ξ, $X|_B$ will denote the cardinal preferences over B induced by X. This notation is justified by the fact that any element of $X|_B$ (and hence an element of R^B) is a restriction to B of some element of X (which is an element of R^A). However now it may happen that for some X and Y in Ξ; $X \neq Y$, and $X|_B = Y|_B$.

A procedure for aggregation of cardinal preferences, mentioned in the title of this note, is, by definition, a function from an n-fold cartesian product of Ξ to Ξ. In notations, f: $\Xi^n \to \Xi$ for some positive integer n (arbitrary but fixed throughout this note). Following the vast literature on Arrow's social welfare functions we may also refer to such an f as cardinal social welfare function. An element $(X_1, X_2, \dots, X_n)$ of Ξ^n will be denoted by $\underline{X}$ and will be referred to as a cardinal profile, and N will stand for the set of integers (society members) $\{1,2,\dots,n\}$.

A classical example of an aggregation procedure (cardinal social welfare function) is the "sum of utilities". Using our definitions and notations $f(\underline{X})$, in this case, is obtained as follows: for each i in N choose a representative, say x_i, of X_i which assigns utility zero to a least preferred alternative of i and if a most preferred by i alternative is strictly preferred to a least preferred alternative, it is assigned the utility of one. (We will refer to such an x_i in the sequel as a zero-one normalized representative of X_i). Now, $x = \Sigma_{i \in N} x_i \in R^A$ is a representative of $f(\underline{X})$.

The aggregation procedure f is cardinally dictatorial, by definition, if there is an j in N s.t. for all $\underline{X}$ in Ξ^n, $f(\underline{X}) = X_j$ (i.e., f is the projection on the j^{th} coordinate). It is said to satisfy cardinal independence of irrelevant alternatives, (CIIA), if for any subset B of A with three elements and for any two cardinal profiles $\underline{X}$ and $\underline{Y}$: $\underline{X}|_B = \underline{Y}|_B$ implies $f(\underline{X})|_B = f(Y)|_B$. ($\underline{X}|_B$ stands for the vector $X_1|_B, X_2|_B, \dots, X_n|_B$ etc....) An aggregation procedure f is said to satisfy cardinal unanimity, (U), if for any subset $\{a,b\}$ of A with two outcomes and for any cardinal profile $\underline{X}$: for all i in N and for all x_i in X_i, $x_i(a) > x_i(b)$ imply $x(a) > x(b)$ for some (or all) x in $f(\underline{X})$. Finally, in order to be able to use the continuity of f, we define the convergence of a sequence in Ξ by the convergence of the sequence of the corresponding zero-one normalized representatives.

CARDINAL IMPOSSIBILITY THEOREM: If $\#A \geq 4$ then a procedure for aggregation of cardinal preferences is continuous and satisfies cardinal independence of irrelevant alternatives and unanimity if and only if it is cardinally dictatorial.

It is obvious that cardinally dictatorial procedure satisfies the required independence, continuity and unanimity conditions. The rest of the paper is devoted to the proof of the converse proposition. Suppose that $f\colon \Xi^n \rightarrow \Xi$ satisfies CIIA, unanimity and continuity; we will show that f is cardinally dictatorial. The proof consists of several steps (lemmata) and will employ, of course, Arrow's impossibility theorem.

First we introduce the definitions of ordinal independence of irrelevant alternatives (OIIA) for cardinal aggregation procedures. Such a procedure satisfied OIIA if for any subset B of A with two elements and for any two cardinal profiles $\underline{X}$ and $\underline{Y}$, if $\underline{X}|_B = \underline{Y}|_B$ then $X|_B = Y|_B$ where X and Y are the values attained by the aggregation procedure, correspondingly. (Because $\#B = 2$ there is a one to one correspondence between ordinal and cardinal profiles restricted to B.)

LEMMA 1: The function f satisfies ordinal independence of irrelevant alternatives.

PROOF: Let B be a two element set, say $B = \{a,b\}$, and let $\underline{X}$ and $\underline{Y}$ be two cardinal profiles s.t. $\underline{X}|_B = \underline{Y}|_B$. By our condition there are at least two additional elements, say c and d in A. Set $C = \{a,b,c\}$ and $D = \{a,b,d\}$. We derive two cardinal profiles $\underline{X}'$ and $\underline{Y}'$ from $\underline{X}$ and $\underline{Y}$ respectively as follows: For each i in N, X_i' is obtained from X_i by moving c to a position halfway between a and b, and the same for Y_i'. Thus $\underline{X}'|_C = \underline{Y}'|_C$, $\underline{X}|_D = \underline{X}'|_D$ and $\underline{Y}|_D = \underline{Y}'|_D$. Applying CIIA to each of these equalities, we obtain: $f(\underline{X}')|_C = f(\underline{Y}')|_C$, $f(\underline{X})|_D = f(\underline{X}')|_D$ and $f(\underline{Y})|_D = f(\underline{Y}')|_D$. Since B is included in C and in D, the equality $f(\underline{X})|_B = f(\underline{Y})|_B$ follows. Q.E.D.

An ordinal profile, $\vec{P}$, is as usual, an n-vector $(P_1,P_2,\ldots,P_n)$ of preorderings of A (i.e., for each i in N, P_i is a transitive and total binary relation on A). The set of all preordering of A is denoted by Ψ, and an aggregation procedure for ordinal preferences, or Arrow's social welfare function, is a mapping from Ψ^n to Ψ.

Given cardinal preferences X, the naturally corresponding preordering in Ψ is denoted by $\Pi(X)$. (For all a and b in A: $a\Pi(X)b$ iff for all x in X, $x(a) \geqq x(b)$.) For a cardinal profile $\underline{X}$ we denote the corresponding ordinal profile by $\underline{\Pi}(\underline{X})$. It seems unnecessary to repeat here the conditions of unanimity, independence of irrelevant alternatives, (IIA), and nondictatorship. Recall however that ordinal dictator (implied by Arrow's impossibility theorem) dictats only his strict preference: Dictator's indifference between two alternatives may not be carried over by Arrow's social welfare function satisfying IIA and unanimity.

LEMMA 2: The aggregation procedure for ordinal preferences defined for all $\underline{P}$ in Ψ^n by: $F(\underline{P}) = \Pi(f(\underline{X}))$ for some $\underline{X}$ s.t. $\underline{\Pi}(\underline{X}) = \underline{P}$, is well defined, satisfied IIA and unanimity; hence by Arrow's theorem F and f are ordinally dictatorial.

PROOF: F is well defined if for any two cardinal profiles $\underline{X}$ and $\underline{Y}$ s. t. $\underline{\Pi}(\underline{X}) = \underline{\Pi}(\underline{Y})$, $\Pi(f(\underline{X})) = \Pi(f(\underline{Y}))$ holds too. But this is an immediate implication of Lemma 1. The other assertion in the lemma are obvious. Q. E. D.

In the next lemma we show, using the continuity of f that the ordinal dictator of F (and f) also imposes his indifference on the society (i.e. F is a projection).

LEMMA 3: If j in N is the ordinal dictator then for all $\underline{X}$ in Ξ^n: $\Pi(f(\underline{X})) = \Pi(\underline{X}_j)$

PROOF: Given $\underline{X}$ in Ξ^n and a and b in A, set $P_j \equiv \Pi(X_j)$ and $P \equiv \Pi(f(\underline{X}))$. If j is not indifferent between a and b, say aP_jb

and not $bP_j a$, then by Lemma 2: aPb and not bPa.

If $aP_j b$ and $bP_j a$ (i.e. for any x in X_j, $x(a) = x(b)$) define Y, s.t. that its zero-one normalized representative, y_j, satisfies: $y_j(c) = 0 < y_j(b) = y_j(a) < y_j(d) = 1$ for some c and d in A. For an appropriate $\varepsilon > 0$ and for any positive integer m define Y_j^m and Z_j^m s.t. their zero-one normalized representatives, y_j^m and z_j^m correspondingly, satisfy: For all e in A, $e \neq a$, $e \neq b$; $z_j^m(e) = y_j^m(e) = y_j(e)$ $y_j^m(a) = y_j(a) + \varepsilon/m > y_j(b) = y_j^m(b)$, $z_j^m(b) = y_j(b) + \varepsilon/m > y_j(a) = y_j^m(a)$. Now we complete the definition of the sequences of cardinal profiles $(\underline{Y}^m)_{m=1}^{\infty}$ and $(\underline{Z}^m)_{m=1}^{\infty}$ by: For each m and for each $i \neq j$ in N, $Y_i^m = Z_i^m = Y_i = X_i$. By Lemma 2, for $m = 1,2,\ldots$, for each y^m in $f(\underline{Y}^m)$, $y^m(a) > y^m(b)$ and for each z^m in $f(\underline{Z}^m)$, $z^m(a) < z^m(b)$. Since, when $m \to \infty$, $\underline{Y}^m \to \underline{Y}$ and $\underline{Z}^m \to \underline{Y}$, and since f is continuous, we have in the limit, $y(a) \geq y(b)$ and $y(b) \geq y(a)$, i.e. $a\Pi(f(\underline{Y}))b$ and $b\Pi(f(\underline{Y}))a$. However $\underline{Y}|_{\{a,b\}} = \underline{X}|_{\{a,b\}}$ so by Lemma 1, aPb and bPa. Q.E.D.

Next we will show that the social cardinal preferences depend only on the dictator's cardinal preferences.

LEMMA 4: For any two cardinal profiles $\underline{X}$ and $\underline{Y}$, if $X_j = Y_j$ then $f(\underline{X}) = f(\underline{Y})$.

PROOF: Since cardinal preferences over A are uniquely determined by their restrictions over all triplets, it suffices to show that for any subset B of A with $\#B = 3$ and for any two cardinal profiles $\underline{X}$ and $\underline{Y}$: $X_j|_B = Y_j|_B$ implies $f(\underline{X})|_B = f(\underline{Y})|_B$. Further simplification will result from an assumption that for some i in N: $k \neq i$ implies $X_k|_B = Y_k|_B$. Suppose also that for some two elements subset of B , say $\{a,b\}$ if $B=\{a,b,c\}$, $X_i|_{\{a,b\}} = Y_i|_{\{a,b\}}$. (If X_i and Y_i rank B in opposite directions, say acb and bca, then introduce Z_i on B s.t. $X_i|_{\{a,c\}} = Z_i|_{\{a,c\}}$ and $Y_i|_{\{b,c\}} = Z_i|_{\{b,c\}}$.) These assumptions are not restrictive since any profile can be obtained from any other profile by a finite number of steps satisfying these assumptions consecutively.

Choose an outcome $d \notin B$ and define new cardinal profiles $\underline{X}'$ and $\underline{Y}'$ s.t. for all i in N and an arbitrary representative x_i in X_i define a representative x_i' of X_i' by: $x_i'(d) = x_i(c)$ and $x_i'(e) = x_i(e)$ for all $e \neq d$; to define y_i' , choose a representative y_i in Y_i s.t. $y_i(a) = x_i(a)$ and $y_i(b) = x_i(b)$, and now define $y_i'(d) = x_i(c)$ and $y_i'(e) = y_i(e)$ for all $e \neq d$. Denoting by D the set $\{a,b,d\}$ we now have: $\underline{X}|_B = \underline{X}'|_B$, $\underline{Y}|_B = \underline{Y}'|_B$, $\underline{X}'|_D = \underline{Y}'|_D$. Applying CIIA we get: $f(\underline{X})|_B = f(\underline{X}')|_B$, $f(\underline{Y})|_B = f(\underline{Y}')|_B$ and $f(\underline{X}')|_D = f(\underline{Y}')|_D$. Since the dictator, j , is indiffeıent between c and d at X'_j and at Y_j' , so is the society, by Lemma 3. Hence, $f(X')|_B = f(\underline{Y}')|_B$, which in turn implies the required equality $f(\underline{X})|_B = f(\underline{Y})|_B$. Q.E.D.

Now we state the last Lemma which completes the proof of the theorem.

LEMMA 5: For any cardinal profile $\underline{X}$, $f(\underline{X}) = \underline{X}_j$.

PROOF: As mentioned in the beginning of the proof of the previous lemma, it sufficies to show for every $B \subset A$, $\#B = 3$, (and for every $\underline{X}$ in Ξ^m), that $f(\underline{X})|_B = \underline{X}_j|_B$. Given such $\underline{X}$ and B suppose that $B = \{a,b,c\}$ and for some x_j in X_j , $x_j(a) = 0$, $x_j(b) = t$, $x_j(c) = 1$ and $0 < t < 1$. We denote by X (without subindex) the corresponding aggregate cardinal preference relation $f(\underline{X})$. With this notation we have to prove that if $x(a) = 0$ and $x(c) = 1$ for some x in X then $x(t) = t$. (By Lemma 2 there is an x in X with $x(a) = 0$ and $x(c) = 1$: Lemma 3 takes care of the case when $t = 0$ or $t = 1$, or when $x_j(a) = x_j(b) = x_j(c)$ for all x_j in X_j.) The proof will be carried out in three steps: (i) $t = 1/2$, (ii) t is a binary number, (iii) any t in $[0,1]$.

Step (i): For any three outcomes a,b,c if the dictator ranks (cardinally) b halfway between a and c , so does the society.

Let X_j^o be s.t. for some x_j^o in $\underline{X}_j^o$: $x_j^o(a) = 0$, $x_j^o(b) = 1/2$ and $x_j^o(c) = 1$. Let x^o be in X^o s.t. $x^o(a) = 0$, $x^o(b) = \alpha$ and $x^o(c) = 1$. (By Lemma 2, $0 < \alpha < 1$.) We have to show that $\alpha = 1/2$.

Several cardinal preferences of j, X_j^k, $k = 1,\ldots,7$ will be used in the proof together with their corresponding social cardinal preferences X^k, $k = 1,\ldots,7$. A representative x_j^k of X_j^k will be specified by its values on $\{a,b,c,d\}$ where d is a forth outcome ($\#A \geq 4$):

$k =$	1	2	3	4	5	6	7
$x_j^k(a) =$	0	0	0	0	0	1/4	1/4
$x_j^k(b) =$	1/2	1/2	1/2	1	1/4	1/4	3/4
$x_j^k(c) =$	1	1/2	0	1/2	1/2	1/2	1/2
$x_j^k(d) =$	1	1	1	1	1	1	1

The corresponding representatives x^k of X^k, $k = 1,\ldots,8$ are obtained trivially by using CIIA and Lemma 3, except the encircled entries.

$k =$	1	2	3	4	5	6	7
$x^k(a) =$	0	0	0	0	0	α^2	α^2
$x^k(b) =$	α	α	α	1	α^2	α^2	$(1-\alpha)\alpha + \alpha$
$x^k(c) =$	1	α	0	α	α	α	α
$x^k(d) =$	1	1	1	1	1	1	1

To find $x^5(b)$ note that X_j^1 and X_j^5 agree on $\{a,b,c\}$. By CIIA x^5 restricted to $\{a,b,c\}$ should be obtained from $x^1|_{\{a,b,c\}}$ by a linear transformation which maps zero to zero and one to α. Hence it maps α to α^2. The value of $x^6(b)$ is obtained from $x^5(b)$ by CIIA and then Lemma 3 implies that $x^6(a) = x^5(b)$.

Similarily $x^7(a) = x^6(a)$ by CIIA (applied to $\{a,c,d\}$). In order to compute $x^7(b)$ note that $X_j^7|_{\{c,b,d\}} = X_j^3|_{\{c,b,d\}}$. Hence, by CIIA X^7 and X^3 agree on $\{c,b,d\}$ and the linear transformation that maps $x_j^3|_{\{c,b,d\}}$ to $x_j^7|_{\{c,b,d\}}$ yields $x^7(b) = (1-\alpha)\alpha+\alpha$. Now observe that $X_j^7|_{\{b,c,a\}} = X_j^4|_{\{b,c,a\}}$ which, again by CIIA leads to existence of linear transformation which maps 0 to α^2, α to α and 1 to $(1-\alpha)\alpha+\alpha$. The linearity implies that $\alpha=[(1-\alpha)\alpha+\alpha-\alpha^2]\alpha+\alpha^2$. This equation has three solutions for α: $0, 1/2, 1$. Since $0 < \alpha < 1$ the proof of step 1 is completed.

Step (ii): If for some x_j in X_j, $x_j(a) = 0$, $x_j(c) = 1$ and $x_j(b) = k/2^m$ with $0 < k < 2^m$, (k, m integers), then x_j also belongs to X.

The Proof is by induction on m. If $m = 1$ then we are in the case of step (i). Suppose that the conclusion holds for every positive integer smaller than m. The nontrivial case is when k is odd. Suppose (w. l.o.g. by CIIA) that $x_j(d) = (k+1)/2^m$. Since $k + 1$ is even, by the induction assumption, $x(a) = 0$, $x(c) = 1$ and $x(d) = (k+1)/2^m$ for some x in X. Define cardinal preferences Y_j where for some y_j in Y_j: $y_j(a) = 0$, $y_j(b) = k/2^m$, $y_j(c) = (k-1)/2^m$ and $y_j(d) = (k+1)/2^m$. By CIIA there is an y in Y with $y(a) = x(a) = 0$, $y(b) = x(b)$ and $y(d) = x(d) = (k+1)/2^m$. By the induction assumption, $y(c) = (k-1)/2^m$ which in turn implies also using step (i), $y(b) = k/2^m$. Hence $x(b) = k/2^m$.

The third and last step is an obvious implication of the continuity assumption. Q.E.D.

We conclude with an example of a two persons four alternatives aggregation procedure which satisfies CIIA and U, is not continuous and is not a projection.

For any profile $\underline{X}$ there is an x in $f(\underline{X})$ which attains the values; 0, 1/3, 2/3, 1: Person 1 is an ordinal dictator, whenever he is indifferent between two outcomes, person's 2 strong preferences prevail, if both are indifferent between two outcomes, their alphabetic order dictates their social order ($A = \{a,b,c,d\}$).

This mapping satisfies OIIA and U and because of the social restriction of the range it satisfies also CIIA and it is not a projection.

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