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ON THE BOUNDEDNESS OF THE FEASIBLE SET WITHOUT CONVEXITY ASSUMPTIONS

by

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1. In the present paper we establish the boundedness of the set of attainable economic states (the feasible set) [3/]

$$A_\omega \equiv [(\prod_i X_i) \times (\prod_j Y_j)] \cap M_\omega$$

under conditions weaker than those postulated in other available results, in particular without assuming the convexity of the aggregate production possibility set Y.

The boundedness of A_ω in the presence of convexity was proved by Debreu in (2), pp. 77-78 of [1], and has been used in certain proofs of the existence

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3/ M_ω and other symbols are defined in Sec. 2 below. ω refers to the initial endowment.

of a competitive equilibrium. [4/] But boundedness of the feasible set is of significant interest in a much broader class of problems where one may not be able to assume convexity. This issue may arise in dealing with resource allocation mechanisms designed to perform satisfactorily (e.g., in the sense of convergence to a Pareto optimum) in "non-classical" environments where convexity may be absent. This is so when existence or stability of equilibria is being established through arguments involving boundedness, as in certain fixed point proofs.

In particular, the problem of the boundedness of the feasible set was encountered in proving the (probabilistic) convergence to a Pareto optimum of the so-called B-process (a decentralized stochastic adjustment process in [4]). The objective in 5.4.3 of [4] is to show that the B-process would perform satisfactorily not only in "classical" environments (in which the convexity of various sets is assumed), but also in certain "non-classical" environments, particularly those in which

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4/ In [2], however, Debreu proves the existence of the competitive equilibrium given assumptions that do not imply the boundedness of the feasible set. Example A, Sec. 3 below, satisfies Debreu's assumptions (a) - (d),pp. 259-260, [2], yet the feasible set A_ω is unbounded. (It may be noted that in Example A the total production set Y is not irreversible,i.e., it is not the case that $Y \cap (-Y) \subseteq \{0\}$; hence, since n=2 and $Y = \dot{Y}$,assumption (iii) of our Theorem 1 below is violated. On the other hand, assumptions (i) and (ii) of Theorem 1 are satisfied.) Debreu's assumptions (a) - (d), pp. 259-260, [2], do imply assumptions (i) and (ii) but not (iii) of our Theorem 1 below. Our (ii) is implied by Debreu's (a.1); that our (i) holds follows from the statement (p. 267, [2], first sentence under "(b) General Case.") that "the set of attainable states of the economy E(Y) is bounded."

convexity may be absent. Since boundedness of the feasible set is important in the proof of satisfactory performance of the B-process, one needs a result on the boundedness of the feasible set similar to Debreu's (20), pp. 77-78 of [1], but not containing convexity among its assumptions. Theorem 1 below provides such a result. Corollary 1, which follows it, is of interest because its conditions are more easily verified, even though (as shown by Example B below) it is weaker than Theorem 1. Debreu's result on boundedness in (2), p. 77 of [1] follows from Corollary 1. Both in the form of conditions used in Corollary 1 and in the techniques of proof, this note owes a great deal to methods pioneered by Debreu in [1] and [2]. (However, without convexity one does not have available the proposition that a set containing the origin contains its asymptotic cone, a proposition which plays a crucial role in proofs using methods pioneered by Debreu.)

2. Notation. Because of the close relationship to the work of Debreu [1], we use a (slightly modified) version of his notation; in particular, the feasible set (Y_F in [4]) is denoted by A_ω or A. For typographical reasons, and to avoid confusion with the symbol A used for the feasible set, the asymptotic cone of a set is usually denoted by the symbol representing the set with a dot over it; thus $\dot{X}$ for the asymptotic cone of X. At times, however, the asymptotic cone of a set is denoted by the symbol representing the set with an outsize A in front of it. (Thus, $\mathcal{A}X$ for the asymptotic cone of X.)

Specifically, the following symbols are used. [5/]

ω_i : the i-th (individual) initial endowment $i \in \{1,2,\ldots,m\}$

$\omega \equiv \sum_i \omega_i$

5/ Lower case symbols (e.g., x_i, y_j, ω_i) are elements of the (finite dimensional) commodity space R^l. The sets X_i, Y_j, etc., are subsets of the commodity space R^l.

X_i : the i-th (individual) consumption set $i \in \{1,\ldots,m\}$

$X \equiv \sum_i X_i$

Y_j : the j-th (individual) production set $j \in \{1,\ldots,n\}$

$Y \equiv \sum_j Y_j$

$\dot{X}$ or $\mathcal{A}X$: the asymptotic cone of X (Debreu's $\underset{\sim}{A}X$)

$\dot{Y}$ or $\mathcal{A}Y$: the asymptotic cone of Y (Debreu's $\underset{\sim}{A}Y$)

X^* : the convex hull of $\dot{X}$

Y^* : the convex hull of $\dot{Y}$

$A_\omega \equiv [(\prod_i X_i) \times (\prod_j Y_j)] \cap M_\omega$

where

$M_\omega \equiv \{\langle x_1,\ldots,x_m,y_1,\ldots,y_n\rangle : \sum_i x_i = \sum_j y_j + \sum_i \omega_i\}$

$\prod$ and x stand for Cartesian products. The index i runs over $\{1,\ldots,m\}$, the index j over $\{1,\ldots,n\}$.

3. Example A. The following example shows that the irreversibility assumption on the total production set [(iii) in Theorem 1 below] cannot be dispensed with in any of the results of this paper.[6/]

In this example there are two goods, and one consumer (m=1) with $X^1 = \Omega$, $\omega^1 = 0$, and preferences are described by the Cobb-Douglas function $u = x_1 x_2$ (where x_k is the amount of the k-th good taken by the consumer). Also, there are two production units (n=2), with the respective production sets

$$Y^1 = \{(y_1,y_2) : y_1 = -y_2,\ y_1 \geqq 0\}, \text{and } Y^2 = \{(y_1,y_2) : y_1 = -y_2, y_1 \leqq 0\}.$$

Verifying Debreu's assumptions (a) - (d), it is clear that $X = X^1 = \Omega$ so that (a.1) holds. The postulated utility function $u = x_1 x_2$ satisfies

6/ See Footnote (4), p. 2, concerning the relationship of this example to Debreu's [2].

(b.1), (b.2), and (b.3). (c.1) holds because $(\{\omega\} + Y) \cap X = Y \cap \Omega = \{0\} \neq 0$, Y being the negatively inclined 45° -line through the origin. To verify (c.2) we take $\ddot{Y} = Y$ which is closed and convex, and note that $D = (\text{Int}\,\Omega) \cup \{0\}$, so that, for i=1, $\{\omega_i\} + \Lambda\ddot{Y} - D = Y - D \ni 0$ and also $X^i = \Omega \ni 0$. (d.1) holds because each Y^i contains the origin. (d.2) holds because $\Lambda X = \Omega \ni 0$, $\Lambda Y = Y \ni 0$, and $Y \cap \Omega = \{0\}$. On the other hand, (i) of our Theorem 1 is satisfied because here $B_{\omega} = Y \cap (\Omega - \{0\}) = \{0\}$ is bounded. (ii) of our Theorem 1 also holds because here $\dot{X} = X = \Omega$, so that $\dot{X} \cap (-\dot{X}) = \{0\}$; also, because m=1. On the other hand (iii) is violated because n=2 and the two points (-1,1) and(1,-1) are both contained in $Y = \dot{Y}$.

Thus (iii) cannot be dispensed with in our Theorem 1. The same conclusion applies to our Corollary 1 and [for(iii")] Theorem 2, since the example also satisfies (i') of Corollary 1 as well as (i") and (ii") of Theorem 2.

It is easily seen that an analogous example involving two consumers (m=2) can be constructed to show that assumption (ii) [resp. (ii'")] cannot be dispensed with in Theorem 1, Corollary 1, or Theorem 2.

4. Theorem 1.

for every $\omega \in X-Y$, the set $A \equiv A_{\omega}$ is bounded if:

(i) for every $\omega \in X-Y$ the set $B_{\omega} \equiv Y \cap (X - \{\omega\})$ is bounded;

(ii) $\dot{X} \cap (-\dot{X}) = \{0\}$ or m=1;

and

(iii) $\dot{Y} \cap (-\dot{Y}) = \{0\}$ or n=1.

($\dot{X} \equiv \Lambda X \equiv$ the asymptotic cone of X)

Proof

Suppose A is not bounded. Since [7/] $A \equiv \{<\vec{x},\vec{y}> : \vec{x} \in \vec{X}, \vec{y} \in \vec{Y}: x = y + \omega\}$, at least one component of $<\vec{x},\vec{y}>$ must be unbounded. Suppose it is one of the components of $\vec{y}$, say y_j. I.e., there exists a sequence of points [8/] $\vec{z}^\nu \equiv <\vec{x}^\nu, \vec{y}^\nu>$, $\vec{z}^\nu \in A$, $\nu = 1,2,\ldots$, such that $|y_j^\nu| \to \infty$. But $\vec{z}^\nu \in A$ means that $y^\nu = x^\nu - \omega$, hence $y^\nu \in Y \cap (X - \{\omega\}) \equiv B_\omega$, i.e., all y^ν are elements of the bounded set B_ω.

Now this obviously cannot happen when n=1, since in this case $y^\nu \equiv y_1^\nu \in B_\omega$. We may therefore suppose that $n > 1$ and define $y^\nu_{)j(} \equiv \sum_{s \neq j} y_s^\nu = y_1^\nu + \ldots + y_{j-1}^\nu + y_{j+1}^\nu + \ldots + y_n^\nu$, $u^\nu \equiv y_j^\nu + y^1_{)j(}$, and $v^\nu \equiv y_j^1 + y^\nu_{)j(}$. (In a more explicit notation, we would write u_j^ν and v_j^ν.)

Then $u^\nu + v^\nu = y^1 + y^\nu$ and hence $v^\nu = y^1 + y^\nu - u^\nu$. Defining the (unordered) sets $U \equiv \{u^1, u^2, \ldots\}$ and $V \equiv \{v^1, v^2, \ldots\}$, we now show that [9/] $\emptyset \neq \dot{U} \cap (-\dot{V}) \neq \{0\}$.

Since we are supposing that there exists an unbounded squence y_j^ν,

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7/ We write

$$\vec{x} = <x_1,\ldots,x_m>,\ x = \Sigma x_i$$
$$\vec{y} = <y_1,\ldots,y_n>,\ y = \Sigma y_j.$$

8/ We write

$$\vec{x}^\nu = <x_1^\nu,\ldots,x_m^\nu>, x^\nu = \Sigma x_i^\nu\ ,$$
$$\vec{y}^\nu = <y_1^\nu,\ldots,y_n^\nu>, y^\nu = \Sigma y_j^\nu\ .$$

9/ In fact, $\dot{U} = -\dot{V} \neq \{0\}$, but only the inclusion $\dot{U} \subseteq -\dot{V}$ follows from the proof on the next page. ($\dot{V} \subseteq -\dot{U}$ follows from the fact that $V \subset -U + y^1 + B_\omega$, so that $\dot{V} \subseteq A(-U + y^1 + B_\omega) = A(-U)$ where the last equality is due to the boundedness of $y^1 + B_\omega$.)

the set U is unbounded and hence there must exist an element $y^* \in \dot{U}$, $y^* \neq 0$. Because $\dot{U}$ is a cone, we may without loss of generality assume that $|y^*| = 1$.

Now $y^* \in \dot{U}$ implies that there exists 10/ a subsequence of elements in U, also to be denoted by u^ν, such that $|u^\nu| \neq 0$ and $\frac{u^\nu}{|u^\nu|} \to y^*$.

We shall show that $-y^* \in \dot{V}$, i.e., that there is 10/ a sequence $t^\nu = \lambda^\nu v^\nu$, where $v^\nu \in V$, $\lambda^\nu > 0$, and $|t^\nu| = 1$, such that the t^ν converge to $-y^*$. Writing $b^\nu \equiv y^1 + y^\nu$ and $w^\nu \equiv -u^\nu$, this sequence is, naturally, given by

$$t^\nu = \frac{1}{|w^\nu + b^\nu|} \cdot (w^\nu + b^\nu),$$

since $w^\nu + b^\nu = v^\nu \in V$.

Now

$$t^\nu = \frac{w^\nu + b^\nu}{|w^\nu + b^\nu|} = \frac{w^\nu}{|w^\nu|} \cdot \frac{|w^\nu|}{|w^\nu + b^\nu|} + \frac{b^\nu}{|w^\nu + b^\nu|} .$$

But $|w^\nu| \to \infty$ because $|y_j^\nu| \to \infty$. On theother hand, $|b^\nu|$ is bounded, since $y^\nu \in B_\omega$ for all ν. Also, by hypothesis, $\frac{w^\nu}{|w^\nu|} \to -y^*$.

Also, by hypothesis,

$$\frac{w^\nu}{|w^\nu|} \to -y^* .$$

Hence,

$$\lim t^\nu = \lim \frac{w^\nu}{|w^\nu|} \cdot \lim \frac{|w^\nu|}{|w^\nu + b^\nu|} + \lim \frac{b^\nu}{|w^\nu + b^\nu|} = (-y^*)\cdot 1 + 0=(-y^*).$$

$$= (-y^*)\cdot 1 + 0 = (-y^*).$$

Thus, $|y_j^\nu| \to \infty$ implies the existence of $y^* \neq 0$ such that $-y^* \in \dot{V}$ as well as $y^* \in \dot{U}$. But $U \subseteq Y$, and also $V \subseteq Y$, by construction of u^ν and v^ν and from the definition $Y \equiv \Sigma Y_j$. (Note that no assumption of convexity is made here.)

10/ See Fenchel [3], pp. 42-43; the topology of rays is introduced on p. 2 of [3].

Hence [11/] $\dot{U} \subseteq \dot{Y}$ and $\dot{V} \subseteq \dot{Y}$. Therefore, $y^* \in \dot{U}$ implies $y^* \in \dot{Y}$. But we also established $-y^* \in \dot{V}$, hence $-y^* \in \dot{Y}$. Thus $y^* \in \dot{Y} \cap (-\dot{Y})$ and $y^* \neq 0$ for $n > 1$, contradicts (iii). It follows that one cannot have $|\vec{y}^{\infty}| \to \infty$ for $<\vec{x}^{\nu}, \vec{y}^{\nu}> \in A$. Similarly, one can show that (ii) rules out the possibility of $|\vec{x}^{\nu}| \to \infty$ for $<\vec{x}^{\nu}, \vec{y}^{\nu}> \in A$. Hence A is bounded.

5. Corollary 1.

If (i') $\dot{X} \cap \dot{Y} = \{0\}$

(ii) $\dot{X} \cap (-\dot{X}) = \{0\}$ or $m = 1$,

(iii) $\dot{Y} \cap (-\dot{Y}) = \{0\}$ or $n = 1$,

then

A is bounded.

Proof.

Let ω in X-Y be such that $B_\omega \equiv Y \cap (X - \{\omega\})$ is not bounded. Then there is an element $\tilde{x} \neq 0$ such that $\tilde{x} \in \mathcal{A} B_\omega$. But [12/]

$$\mathcal{A} B_\omega \subseteq \mathcal{A} Y \cap \mathcal{A}(X - \omega) = \mathcal{A} Y \cap \mathcal{A} X,$$

and so $\tilde{x} \neq 0$ is an element of all the above sets, hence of $\dot{X} \cap \dot{Y}$, thus violating (i').

11/ In general $S \subseteq T$ implies $\dot{S} \subseteq \dot{T}$ (Debreu [1], 1.9,o., p.22)

12/ The first inclusion is based on the proposition $\mathcal{A}(S \cap T) \subseteq \mathcal{A} S \cap \mathcal{A} T$, which follows from footnote 11.

6. Note that Corollary 1 is weaker than Theorem 1, as shown by the example below where (i) is satisfied but (i') is not.

<u>Example B.</u> (See Figure 2)

$$\omega = 0,\ X = \{(x_1, x_2):\ x_1 \geqq 0,\ i = 1,2\}$$

$$Y = \{(x_1, x_2):\ x_2 \leqq x_1^2,\ x_1 \leqq 0\}\ .$$

Here (i') is not satisfied because the positive half-axis Ox_2 belongs to both $\dot{X}$ and $\mathring{Y}$ (since both are closed by definition), but (i) does hold. (See Fig. 3)

7. <u>Relation to Debreu's Result.</u> The boundedness result in (2), p. 77 in Debreu's [1] follows from Corollary 1:

(a) if X has a lower bound for $\leqq$, condition (ii) of the Corollary holds;

(b) if $Y \cap \Omega = \{0\}$ then (i') of the Corollary holds and $0 \in Y$;

(c) if [13/] $Y \cap (-Y) = \{0\}$, or $n = 1$, then condition (iii) of the Corollary holds. Unlike (2), p. 77 in [1], Corollary 1 does not assume Y to be convex.

8. <u>Theorem 2.</u> The following theorem is weaker than Corollary 1 (and therefore weaker than Theorem 1 from which it follows). It is stated here however, because its conditions are more readily verifiable in some applications.

[13/] Although Debreu writes $Y \cap (-Y) \subset \{0\}$, the inclusion symbol (which is to be interpreted as weak inclusion) can be replaced by equality because Debreu's other assumptions imply $0 \in Y$.

Theorem 2. [14/]

A is bounded if

(i")

- (i".1) $X^{*} \cap Y^{*} = \{0\}$,
  or
- (i".2) $\dot{X} \cap Y^{*} = \{0\}$ and $m = 1$,
  or
- (i".3) $X^{*} \cap \dot{Y} = \{0\}$ and $n = 1$,
  or
- (i".4) $\dot{X} \cap \dot{Y} = \{0\}$ and $m = n = 1$;

and

(ii")

- (ii".1) $X^{*} \cap (-X^{*}) = \{0\}$,
  or
- (ii".2) $\dot{X} \cap (-\dot{X}) = \{0\}$ and $m = 2$,
  or
- (ii".3) $m = 1$;

and

(iii")

- (iii".1) $Y^{*} \cap (-Y^{*}) = \{0\}$,
  or
- (iii".2) $\dot{Y} \cap (-\dot{Y}) = \{0\}$ and $n = 2$,
  or
- (iii".3) $n = 1$.

14/ A direct proof of Theorem 2 which is very similar to Debreu's proof on pp. 77-78 of [1] can be found in [5].

Example A.

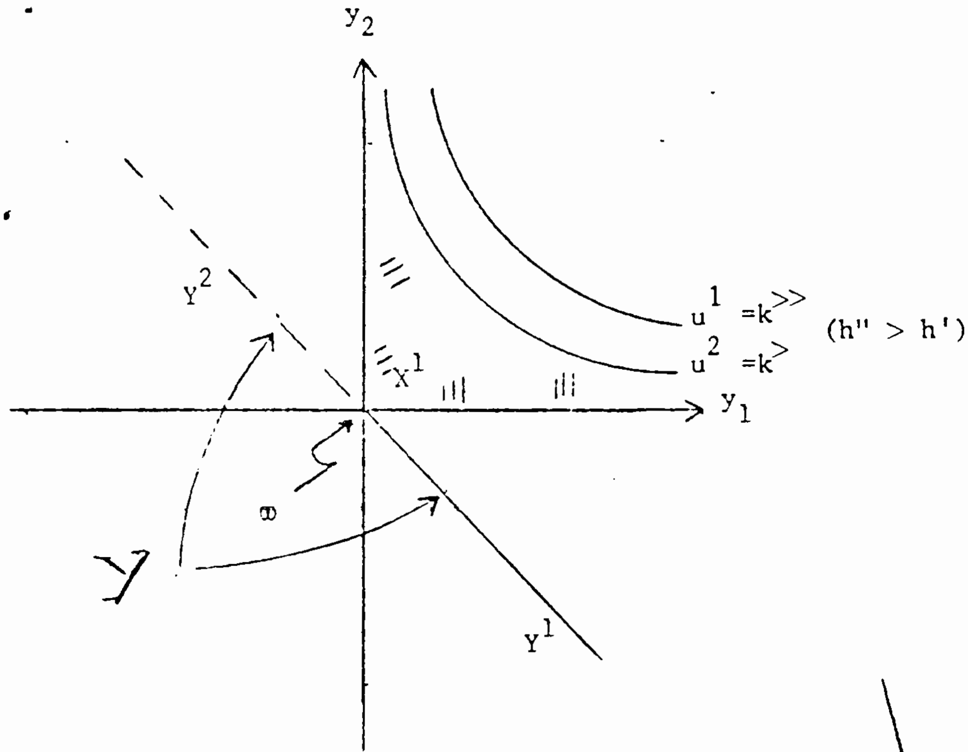


FIGURE 1

Example B.

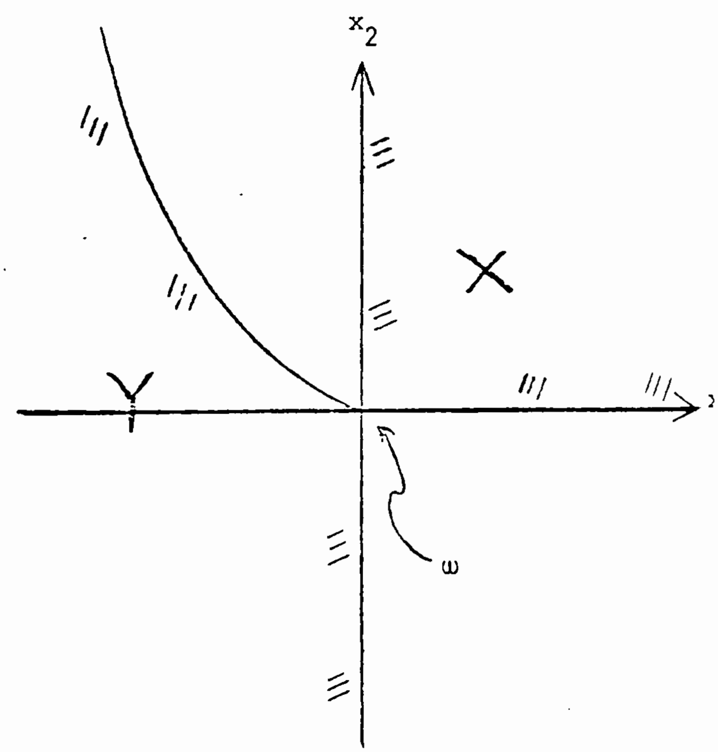
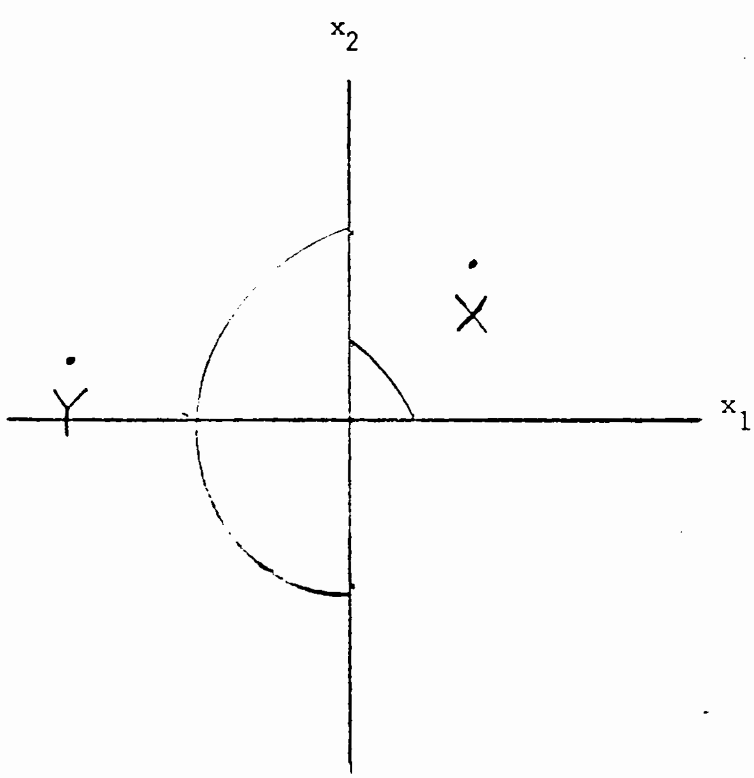


FIGURE 2

Example B.



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