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CONVERSION OF SEMIMARKOV PROCESSES TO CHUNG PROCESSES

by

ERHAN ÇINLAR

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Abstract

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Structure of semimarkov processes in the sense of [4] will be clarified by relating them to Chung processes. Start with a semimarkov process. For each attractive instantaneous state whose occupation time is zero, dilate its constancy set so that the occupation time becomes positive; this is achieved by a random time change. Then, mark each sojourn interval of an unstable holding state i by (i,k) if its length is between 1/k and 1/(k-1); this is "splitting" the unstable state i to infinitely many stable states (i,k). Finally, replace each sojourn interval (of the original stable states i plus the new stable states (i,k)) by an interval of exponentially distributed length. The result is a Chung process modulo some standardization and modification.

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Keywords: Semimarkov process; Chung process; strong Markov property; random time changes; regenerative systems; sample paths; random sets.

FOOTNOTES

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2. I am grateful to Professor Chung for this information.

3. [T] is the graph of T; $[T] \subset K_i$ means that $T(\omega) \in K_i(\omega)$ for almost every $\omega \in \{T < \infty\}$.

4. Memory aid: π for projection, λ for lower limit (for a sojourn length).

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1. INTRODUCTION

Our object is to give a rectification of LÉVY's assertion in [7] concerning the qualitative structure of semimarkov processes compared with that of Chung processes. The assertion was that the two are the same since there exists a continuous strictly increasing random time change which converts the given semimarkov process into a Markov process. The details of this conversion were carried out by YACKEL [12] under certain restrictions on the transition laws involved.

Evidently[2] Lévy's original objective in introducing semimarkov processes was to replace the exponential laws governing the sojourns of a Markov process at stable states by arbitrary laws, and thus to achieve a class of processes which includes both continuous time and discrete-time Markov processes as special cases. From that point of view, the assertion mentioned serves admirably well in characterizing the sample paths in question.

However, LÉVY [7] introduced a much more fundamental definition of semimarkov processes by requiring only that they enjoy (what is now called) the strong Markov property at their times of discontinuity. Sach processes possess significant structural differences from Markov processes, especially by admitting "unstable holding states" which have no parallels in the theory of Markov processes. Then, the conversion from a semimarkov to a

Markov process requires, in addition to a random time change, splitting each unstable holding state into infinitely many stable holding states.

Our object is to give a careful account of this conversion process. For the definition and basic properties of what we call a semimarkov process we refer to ÇINLAR [4]. The results here supplement [4] and clarify the structure of semimarkov processes vis â vis Chung processes. The remainder of this section is a brief description of the most relevant aspects of such processes and the results to be obtained here.

Let X be a semimarkov process in the sense of ÇINLAR [4] with a discrete state space E. Let K_i be the constancy set for i, that is, $K_i = \{t: X_t = i\}$. A point $i \in E$ is said to be a holding state if K_i is a countable union of intervals almost surely. A holding state i is <u>stable</u> if and only if $K_i \cap [0,t]$ has only finitely many component intervals for every t almost surely; otherwise, i is called <u>unstable</u> and between any two component intervals of K_i there is a third. If i is not a holding state, it is called <u>instantaneous</u>, and the interior of K_i is almost surely empty. An instantaneous state i is called <u>repellent</u> if K_i is discrete (every point isolated), and <u>attractive</u> if K_i is perfect (no isolated points) — one or the other holds a.s. Finally, an attractive state is <u>heavy</u> or <u>light</u> according as the Lebesgue measure of K_i is a.s. positive or a.s. zero.

A Chung process is, according to [4], a Markov process with discrete state space, standard transition function, and right lower semicontinuous sample paths (see CHUNG [1] for these terms). Every Chung process is a semimarkov process. If the semimarkov process X is a Chung process, then every holding state is stable and every instantaneous state is heavy attractive. Unstable holding states never appear in the theory of Markov processes. But light attractive and repellent states do appear in the theory of Markov processes on general state spaces, and in the boundary theory of Chung processes, where a sticky boundary atom (see CHUNG [2] for the term) behaves like a light attractive state and a non-sticky boundary atom like a repellent state.

The following are the main results of this note. More precise versions of these will be stated and proved in the remaining sections.

Let D be the set of all unstable states, put $\hat{D} = D \times \{1,2,3,\ldots\}$, and let $\hat{E} = (E \setminus D) \cup \hat{D}$. Define a "projection" mapping $\pi\colon \hat{E} \cup \{\phi\} \to E \cup \{\phi\}$ by

$$(1.1) \qquad \pi(i) = \begin{cases} j & \text{if } i = (j,k) \in \hat{D}, \\ i & \text{otherwise.} \end{cases}$$

(1.2) MAIN RESULT. There exist a strictly increasing continuous process A and a strong Markov process $\hat{X}$ with state space $\hat{E}$ such that, for every $t > 0$,

$$(1.3) \qquad X_t = \pi(\hat{X}_{A_t}) \quad \text{a.s.}$$

Moreover, $\hat{X}$ is a semimarkov process; and, considered as such, it has no unstable and no light attractive states.

The time transformation (A_t) does not alter the qualitative structure of $\hat{X}$ because A is continuous and strictly increasing: The succession of states being visited remains the same, and for each state, the properties of being stable, attractive, or repellent remain invariant. So, the only qualitative structural difference between $\hat{X}$ and X is due to the space transformation effected by π: each class $\underset{\sim}{j} = \{(j,k)\colon k = 1,2,\ldots\} \subset \hat{D}$ is lumped into one state $j \in D$. Each state $(j,k) \in \hat{D}$ is a stable holding state for $\hat{X}$, and the lumped state $j \in D$ is an unstable holding state for X.

On the quantitative side, differences between $\hat{X}$ and X are due to the time change: the constancy sets K_i of attractive instantaneous states i of $\hat{X}$ lose their Lebesgue measure and become light for X, and the stable states of $\hat{X}$ lose the exponential laws which govern the sojourns in them.

If X has no repellent, light attractive, or unstable states, then the above theorem reduces to that proved by YACKEL [12], who insured this state of affairs by requiring that the transition semi-group of (X,S) be "strong." In fact, if there are no repellent and no unstable states, our first step in Section 3 leads to a semimarkov process to which YACKEL's theorem applies. Otherwise, especially in the interesting case where there are unstable states, our step in Section 4 leads to a process which is (roughly) semimarkov in the second sense of JACOD [6], which is in general not semimarkov in the strict sense employed by YACKEL [12].

Returning to the process $\hat{X}$, suppose its state space $\hat{E}$ is given the discrete topology and let ϕ be the point at infinity if $\hat{E}$ is infinite (which is the case if X has any attractive or unstable states). The process $\hat{X}$ is very close to a Chung process: the sample paths of $\hat{X}$ behave exactly as those of a Chung process except at instants t where they are in a repellent state. Thus it is easy to modify $\hat{X}$ to obtain a Chung process.

(1.4) RESULT. If X has no repellent states, then $\hat{X}$ is a Chung process with an appropriate state space $E^0 \subset \hat{E}$. If X has repellent states, then putting

$$(1.5) \qquad X'_t(\omega) = \begin{cases} \hat{X}_t(\omega) & \text{if } \hat{X}_t(\omega) \text{ is not repellent,} \\ \phi & \text{if } \hat{X}_t(\omega) \text{ is repellent,} \end{cases}$$

for every $t > 0$ and ω, yields a Chung process X'.

This proposition, while useful in simplifying the arithmetic involved in ancient quests concerning the transition functions and generators, is nevertheless a step in the wrong direction from the point of view of one watching the sample paths. For, each repellent state is indeed a non-sticky atom of the entrance boundary of the process X', and by expunging such a state i the strong Markov property is lost over the time-set K_i. The following achieves the computational desirability of a Chung process without sacrificing the repellent states.

(1.6) RESULT. There is a Chung process $\tilde{X}$ and a strictly increasing left continuous process B such that

$$X_t = \pi(\tilde{X}_{B_t}) \quad \text{a.s.} \tag{1.7}$$

If i is attractive for X then it is instantaneous for $\tilde{X}$; if i is repellent or stable for X then it is stable for $\tilde{X}$; if i is unstable for X then there are infinitely many stable states j of $\tilde{X}$ such that $\pi(j) = i$.

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The results described were put at the end of the original version of [4] with brief sketches for proofs. I am grateful to the referees of that paper for pointing out errors and technical difficulties which had escaped me, and for their insisting that careful proofs be provided.

2. INITIAL PROCESS

In this section we describe the process which we take to be given. Our notations and terminology will follow [4] very closely. For general terminology we follow DELLACHERIE [5]. The following are a few special conventions.

Let $(\Omega,\underline{\underline{M}},P)$ be a complete probability space. By a <u>history</u> $\underline{\underline{H}}$ on $(\Omega,\underline{\underline{M}})$ we mean an increasing family $\underline{\underline{H}} = (\underline{\underline{H}}_t)_{t\in\mathbb{R}_0}$ (where $\mathbb{R}_0 = (0,\infty)$) of sub-$\sigma$-algebras $\underline{\underline{H}}_t$ of $\underline{\underline{M}}$ on Ω. A history $\underline{\underline{H}}$ is said to be <u>complete</u> if $\underline{\underline{H}}_\infty = \vee_t \underline{\underline{H}}_t$ is complete and each $\underline{\underline{H}}_t$ contains all the negligible sets of $\underline{\underline{H}}_\infty$. A history $\underline{\underline{H}}$ is said to be <u>right continuous</u> if $\underline{\underline{H}}_t = \bigcap_{s>t} \underline{\underline{H}}_s$ for every t. If $X = (X_t)$ is a stochastic process defined on $(\Omega,\underline{\underline{M}})$, by the history generated by X we mean the history $\underline{\underline{H}}$ where $\underline{\underline{H}}_t = \sigma(X_s,\ s \leq t)$ and $\underline{\underline{H}}_\infty = \sigma(X_s,\ s \geq 0)$. By the right continuous complete history generated by the history $\underline{\underline{H}}$ we mean the history $\underline{\underline{G}}$ where $\underline{\underline{G}}_\infty$ is the completion of $\underline{\underline{H}}_\infty$ and, for each t, $\underline{\underline{G}}_t$ is the σ-algebra generated by $\underline{\underline{H}}_{t+} = \bigcap_{s>t} \underline{\underline{H}}_s$ and all the negligible sets of $\underline{\underline{G}}_\infty$.

If $\underline{\underline{H}}$ is a history and T is a stopping time of $\underline{\underline{H}}$ we write $T \in st\underline{\underline{H}}$, that is, $st\underline{\underline{H}}$ is the set of all stopping times of $\underline{\underline{H}}$. If $\underline{\underline{F}}$ is a σ-algebra, we write $p\underline{\underline{F}}$ (resp. $b\underline{\underline{F}}$) for the set of all positive (resp. bounded) $\underline{\underline{F}}$-measurable functions.

We take the following as given. A complete probability space $(\Omega,\underline{\underline{M}},P)$; a family $\theta = (\theta_t)_{t\in\mathbb{R}_+}$ of "shift" operators $\theta_t\colon \Omega \to \Omega$; a random variable $S_0\colon \Omega \to \mathbb{R}_+$; and a stochastic process $X = (X_t)_{t\in\mathbb{R}_0}$, $X\colon \mathbb{R}_0 \times \Omega \to \bar{E}$ where E is a countable set with the discrete topology, and $\bar{E} = E$ if E is finite and $\bar{E} = E \cup \{\phi\}$ is the one point compactification of E if E is not finite.

For each $\omega \in \Omega$, let $M(\omega)$ be the set of all $t \in \mathbb{R}_0$ such that the path $X(\omega)$ is either <u>not</u> continuous at t, or else is continuous at t and is equal to ϕ. The set $M(\omega)$ is called the discontinuity set of $X(\omega)$. It is closed in $\mathbb{R}_0$; therefore, its complement is a countable union of open intervals, which intervals are said to be contiguous to $M(\omega)$. For each $\omega \in \Omega$ and $t \in \mathbb{R}_0$ we let $S_t(\omega)$ be the time since the last discontinuity of $\mathbf{X}(\omega)$ before t; more precisely,

$$(2.1) \qquad S_t(\omega) = \begin{cases} S_0(\omega) + t & \text{if } (0,t] \cap M(\omega) = \emptyset, \\ t - \sup(0,t] \cap M(\omega) & \text{otherwise.} \end{cases}$$

Let $\underline{\underline{F}}_0^\circ = \sigma(S_0)$ and $\underline{\underline{F}}_t^\circ = \sigma(S_0, X_u, u \leq t)$ for $t > 0$; and define $\underline{\underline{F}} = (\underline{\underline{F}}_t)_{t \in \mathbb{R}_+}$ to be the right continuous complete history generated by $\underline{\underline{F}}^\circ$.

Finally, we take it as given that the objects $(\Omega, \underline{\underline{M}}, P)$, $\underline{\underline{F}} = (\underline{\underline{F}}_t)_{t \in \mathbb{R}_+}$, $\theta = (\theta_t)_{t \in \mathbb{R}_+}$, $(X,S) = (X_t, S_t)_{t \in \mathbb{R}_0}$ satisfy the following

(2.2) CONDITIONS. a) Regularity axiom (2.2) of [4] holds, and the regularization (2.14) of [4] is already achieved.

b) For all $t \in \mathbb{R}_+$ and $u \in \mathbb{R}_0$, $S_0 \circ \theta_t = S_t$ and $X_u \circ \theta_t = X_{t+u}$.

c) For every $t \in \mathbb{R}_0$, $P\{X_t = \phi\} = 0$.

d) X is progressively measurable with respect to $\underline{\underline{F}}$.

e) (X,S) enjoys the strong Markov property at all $T \in \mathrm{st}\underline{\underline{F}}$ such that $X_T \in E$ a.s. on $\{T < \infty\}$.

Then X is a strict semimarkov process in the sense of [4]. The net effect of the regularity axioms (2.2) and (2.14) of [4] is the regularization of the constancy sets

$$(2.3) \qquad K_i(\omega) = \{t \in \mathbb{R}_0 : X_t(\omega) = i\}, \quad i \in E,$$

so that each one is right closed for almost every $\omega \in \Omega$, and that a state is either instantaneous or holding or absorbing.

3. DILATION OF LIGHT STATES TO HEAVY

The object of this section is to transform X into a new semimarkov process having almost the same sample paths as X but without any light attractive states. This will be achieved through a random time change using a clock (C_t) which is continuous and strictly increasing. Therefore the qualitative structure of the paths $X(\omega)$ will not be altered; but on the quantitative side, Lebesgue measure of $K_i(\omega)$ will go from zero to something positive for each light attractive state i.

Let $i \in E$ be a light attractive state. Then its set of constancy K_i is almost surely right closed and perfect, and is progressive relative to the history $\underline{\underline{F}}$ (see [4] for details). For any $T \in st\underline{\underline{F}}$ such that[3] $[T] \subset K_i$, we have $X_T = i$ and $S_T = 0$ a.s. on $\{T < \infty\}$. The strong Markov property applied at such times implies that $(\Omega, \underline{\underline{F}}, \theta_t, K_i, P)$ is a regenerative set in the sense of ÇINLAR [3]. The following lemma is merely Theorem (4.35) of [3].

(3.1) LEMMA. There is an increasing continuous perfectly additive process $(C_t^i)_{t \in \mathbb{R}_+}$ adapted to $\underline{\underline{F}}$ such that the set of right-increase of the path $C^i(\omega)$ is equal to the $K_i(\omega)$ for almost every ω. Moreover, $C_t^i \leq e^t$ for all t.

(3.2) REMARK. By "perfectly additive" we mean that

$$(3.3) \qquad C_{t+u}^i(\omega) = C_t^i(\omega) + C_u^i(\theta_t \omega)$$

for all t and u for every $\omega \in \Omega \setminus \Omega_0$ where Ω_0 is a negligible set (independent of t and u). The regenerative sets of [3] are exactly the same as those of MAISONNEUVE [8] except that the latter are defined on a canonical space of sawtooth shaped functions. The lemma above also follows from the results of MAISONNEUVE [9], but the transformation needed from our Ω into his canonical space of right continuous paths introduces difficulties.

Let A be the set of all light attractive states (in E). Let $(p_i)_{i\in A}$ be a family of strictly positive numbers p_i with $\sum p_i = 1$. Define

(3.4) $$C_t(\omega) = t + \sum_{i\in A} p_i C_t^i(\omega),$$

(3.5) $$\tau_t(\omega) = \inf\{s: C_s(\omega) > t\},$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$. The following lemma summarizes the facts we will need on (C_t) and (τ_t).

(3.6) LEMMA. For a.e. ω, the path $C(\omega)$ is strictly increasing, is continuous, and satisfies

(3.7) $$C_{t+u}(\omega) = C_t(\omega) + C_u(\theta_t\omega)$$

for all t and u. Each τ_t is a stopping time of $\underline{\underline{F}}$. For a.e. ω, the path $\tau(\omega)$ is strictly increasing, is continuous, and satisfies

(3.8) $$\tau_{t+u}(\omega) = \tau_t(\omega) + \tau_u(\theta_{\tau_t(\omega)}\omega), \quad t,u \in \mathbb{R}_+.$$

PROOF. The convergence of the series on the right side of (3.4) is uniform over intervals $[0,t]$. So, the almost sure continuity of C follows from that of C^i for each i; see Lemma (3.1). Similarly, the perfect additivity (3.7) follows from that of C^i. It is obvious that C is strictly increasing for all ω.

Each C^i and therefore C is adapted to $\underline{\underline{F}}$. So, C is a continuous additive functional of (X,S). Thus, each τ_t is a stopping time of $\underline{\underline{F}}$. The remaining properties of the paths $\tau(\omega)$ follow from Lebesgue's theorem on time changes (see DELLACHERIE [5, p. 91]), and the properties listed for $C(\omega)$.

Let Ω_0 be the exceptional set for the first statement of the preceding lemma. Redefine $C(\omega)$ for $\omega \in \Omega_0$ by putting $C_t(\omega) = t$ identically. Then, the statement concerning the paths $\tau(\omega)$ holds for every $\omega \in \Omega$.

For each $t \in \mathbb{R}_0$, define

$$(3.9) \qquad \bar{X}_t = X_{\tau_t}, \quad \bar{S}_t = S_{\tau_t}, \quad \bar{\underline{\underline{F}}}_t = \underline{\underline{F}}_{\tau_t}, \quad \bar{\theta}_t = \theta_{\tau_t},$$

and set $\bar{\underline{\underline{F}}}_0 = \underline{\underline{F}}_0$, $\bar{\theta}_0 = \theta_0$. The following is the main result of this section.

(3.10) THEOREM. The objects $(\Omega,\underline{\underline{M}},P)$, $\bar{\underline{\underline{F}}}$, $\bar{\theta}$, $(\bar{X},\bar{S})$ satisfy Conditions (2.2) except possibly for (2.2c) which now holds for (Lebesgue) almost every t. Each $i \in E\setminus A$ has the same classification with respect to $\bar{X}$ as it does with respect to X. Each $i \in A$ is heavy attractive for $\bar{X}$.

PROOF. Condition (2.2a) for $\bar{X}$ follows from that for X, since $\tau(\omega)$ is strictly increasing and continuous for all ω. Condition (2.2b) holds for $\bar{X}$ because it holds for X and because τ has the additivity property (3.8). Condition (2.2d) follows from Lemma (3.6) and Theorem T57 in MEYER [10, p. 73]. By the same Theorem T57, for any $T \in st\bar{\underline{\underline{F}}}$ such that $\bar{X}_T \in E$ a.s. on $\{T < \infty\}$, $U = \tau_T \in st\underline{\underline{F}}$ and $X_U = \bar{X}_T \in E$ a.s. on $\{U < \infty\} = \{T < \infty\}$. Since $\bar{\underline{\underline{F}}}_\infty = \underline{\underline{F}}_\infty$, this implies the strong Markov property (2.2e) for $(\bar{X},\bar{S})$ via that for (X,S).

There remains to show the replacement for Condition (2.2c), and the remaining assertions on the classification of states. First note that the constancy set $\bar{K}_i(\omega)$ of $i \in \bar{E}$ for $\bar{X}(\omega)$ is equal to the set of all $C_t(\omega)$ such that $t \in K_i(\omega)$. Therefore, by theorems on the Lebesgue measure of the range of an increasing function, for every ω, (we write $\bar{K}_i(\omega) = \{t: \bar{X}_t(\omega) = i\}$),

$$(3.11) \qquad \ell eb\ \bar{K}_i(\omega) = \int_{K_i(\omega)} dC_t(\omega)$$

$$= \ell eb\ K_i(\omega) + \sum_{j \in A} p_j \int_{K_i(\omega)} dC_t^j(\omega),$$

where "ℓeb" denotes "Lebesgue measure."

For $i = \phi$, for almost every ω, Condition (2.2c) on X implies that $\ell eb\ K_\phi(\omega) = 0$, and by Lemma (3.1) the measure $dC_t^j(\omega)$ does not charge $K_\phi(\omega)$. Hence, $\ell eb\ \bar{K}_\phi(\omega) = 0$ for a.e. ω, and by FUBINI's theorem, $P\{\bar{X}_t = \phi\} = 0$ for (ℓeb) a.e. t.

Similarly, if $i \in E \setminus A$, $\ell eb(\bar{K}_i(\omega) \cap B) = \ell eb(K_i(\omega) \cap B)$ by (3.11) for every $B = [0,b]$ for a.e. ω since $dC_t^j(\omega)$ does not charge $K_i(\omega)$ for any $j \in A$. This and the fact that $\tau(\omega)$ is strictly increasing and continuous imply that each $i \in E \setminus A$ has the same classification for $\bar{X}$ as it does for X. Finally, if $i \in A$, the preceding reasoning shows that i is still attractive, but by (3.11)

$$\ell eb(\bar{K}_i(\omega) \cap [0,C_t(\omega)]) = p_i C_t^i(\omega),$$

which shows that i is <u>heavy</u> for $\bar{X}$.

4. SPLITTING THE UNSTABLE STATES

Throughout this section we are working with $(\Omega,\underline{\underline{M}},P)$, $\bar{\underline{\underline{F}}}$, $\bar{\theta}$, $\bar{X}$, $\bar{S}$ obtained in the preceding section, but we will omit the bars over $\bar{\underline{\underline{F}}}$, $\bar{\theta}$, $\bar{X}$, etc. altogether. Although X satisfies only a weaker version of Condition (2.2c), all the results of [4] still hold. Also, this section will not use the fact that X has no light attractive states, and hence, all the results below hold for all semimarkov processes X.

Roughly speaking, these results show that the unstable states can be dispensed with, but at the cost of replacing each by infinitely many stable states and losing some strength in the strong Markov property. The situation is similar to that of JACOD [6], who showed that semimarkov processes in his first sense are also so in his second sense, but that the converse is not true in general. Incidentally, our processes are much less restricted than JACOD's, and his results do not carry over. In fact even his methods do not work due to the weakness of Condition (2.2e) here, which requires the strong Markov property at a stopping time T to hold only on the set $\{X_T \in E, T < \infty\}$. This leaves the set $\{X_T = \phi, T < \infty\}$ free for "pathologies," and since $\{X_t = \phi\}$ may have positive probability for some fixed t, we do not even have the simple Markov property.

Let D be the set of all unstable states. Define

$$(4.1) \qquad \hat{D} = \left\{(j, \tfrac{1}{k}) : j \in D,\ k \in \{1,2,\ldots\}\right\},\ \hat{E} = (E \setminus D) \cup \hat{D},$$

and for each $i \in \hat{D}$ put[4]

$$(4.2) \qquad \pi i = j, \quad \lambda i = \frac{1}{k}, \quad \mu i = \frac{1}{k} - \frac{1}{k-1} \quad \text{if } i = (j, \tfrac{1}{k}).$$

tr.

For every $t \in \mathbb{R}_0 = (0,\infty)$ define

$$(4.3) \qquad R = \inf M, \quad R_t = R \circ \theta_t, \quad D_t = t + R_t;$$

and

$$(4.4) \qquad Y_t = \begin{cases} X_t & \text{on } \{X_t \notin D\}, \\ (X_t, \frac{1}{k}) & \text{on } \{X_t \in D,\ S_t + R_t \in (\frac{1}{k}, \frac{1}{k-1}]\}. \end{cases}$$

Set $\underline{\underline{G}}_0 = \underline{\underline{F}}_0$ and, for $t \in \mathbb{R}_0$,

$$(4.5) \qquad \underline{\underline{G}}_t = \underline{\underline{F}}_t \vee \sigma(Y_t).$$

The process Y has the state space $\hat{E} \cup \{\phi\}$. For each i, let K_i be the constancy set of i with respect to Y, that is, $K_i = \{t > 0: Y_t = i\}$. If $i \notin \hat{D}$ then $K_i = \{t > 0: X_t = i\}$ as before. If $i \in \hat{D}$ then K_i is the union of those component intervals of the constancy set $K_{\pi i} = \{t: X_t = \pi i\}$ whose lengths exceed λi but not $\lambda i + \mu i$. Conversely, if $j \in D$, then $K_j = \{t: X_t = j\}$ is the union of all the K_i with $\pi i = j$, $i \in \hat{D}$. In this sense, j is "split" into states (j,1), (j,1/2),..., each of which looks stable since it can be visited only finitely often during a finite interval.

If $i \notin \hat{D}$ then K_i is progressive relative to $\underline{\underline{F}} \subset \underline{\underline{G}}$; if $i \in \hat{D}$ then K_i is a countable union of stochastic intervals $[T_n, S_n)$ where $T_n, S_n \in \text{st}\underline{\underline{G}}$, and therefore is again progressive relative to $\underline{\underline{G}}$. Hence, the process Y is progressively measurable with respect to $\underline{\underline{G}}$. Finally, note that $Y_u \circ \theta_t = Y_{t+u}$ for all $t \in \mathbb{R}_+$ and $u \in \mathbb{R}_0$ because of the similar homogeneity of X, S, R.

In the propositions below the probabilities $P_{(i,a)}$ are the measures corresponding to the entrance laws $\{P_t((i,a),\cdot);\ t \in \mathbb{R}_0\}$ of the <u>initial</u> semimarkov process of Section 2, where (P_t) is the transition semi-group of the Markov process (X ,S) of Section 2.

(4.6) PROPOSITION. Let $T \in st(\underline{\underline{G}}_{t+})$ and suppose that $[T] \subset M \cap K_i$ for fixed $i \in \hat{D}$. Then, for any $Z \in b\underline{\underline{G}}_{\infty}$,

$$(4.7) \qquad E[Z \circ \theta_R \circ \theta_T | \underline{\underline{G}}_{T+}] = E_i[Z \circ \theta_R | R \leq \mu i]$$

almost surely on $\{T < \infty\}$.

(4.8) REMARK. We say that i is in the minimal state space of Y if and only if $P_i\{R \leq \mu i\} > 0$, and then the second member of (4.7) is well defined in the usual elementary sense. Otherwise, if $i \in \hat{D}$ is not in the minimal state space, then the hypothesis $[T] \subset M \cap K_i$ implies that $T = +\infty$ a.s. and then the conditional expectation in (4.7) is not being questioned.

PROOF. Suppose i is in the minimal state space. Let $Z \in b\underline{\underline{G}}_{\infty}$, and note that $\underline{\underline{G}}_{\infty} = \underline{\underline{F}}_{\infty}$. Pick n^* such that $1/2^{n^*} < \lambda i$, and for each $n \geq n^*$ define

$$(4.9) \qquad T_n = \begin{cases} m/2^n & \text{on } \{(m-1)/2^n \leq T < m/2^n\}, \\ +\infty & \text{on } \{T = \infty\}. \end{cases}$$

Then, each $T_n \in st(\underline{\underline{G}}_t)$, the sequence (T_n) decreases to T, and therefore $\underline{\underline{G}}_{T+} = \bigcap_n \underline{\underline{G}}_{T_n}$. Hence, by the martingale convergence theorem, the first member of (4.7) is the limit of the same conditional expectation with $\underline{\underline{G}}_{T_n}$ replacing $\underline{\underline{G}}_T$. On the other hand, since every component interval of K_i has the form [ ) with length exceeding λi, the hypothesis $[T] \subset M \cap K_i$ implies that T is the left end point of such an interval and that the following hold on the set $\{T < \infty\} = \{T_n < \infty\}$; (recall that $T_n - T \leq 2^{-n} < \lambda i$);

$$(4.10) \qquad Y_{T_n} = Y_T = i, \quad S_{T_n} = T_n - T, \quad D_{T_n} = D_T;$$

(4.11) $$\theta_R \circ \theta_{T_n} = \theta_{D_{T_n}} = \theta_{D_T} = \theta_R \circ \theta_T.$$

Hence, a.s. on $\{T < \infty\}$,

(4.12) $$E[Z \circ \theta_R \circ \theta_T | \underline{\underline{G}}_{T+}] = \lim_n E[Z \circ \theta_R \circ \theta_{T_n} | \underline{\underline{G}}_{T_n}].$$

Let Λ be a set in $\underline{\underline{G}}_{T_n}$ and recall that $Y_{T_n} = i$ on $\{T_n < \infty\}$. Since T_n is countably valued, we have

(4.13) $$\begin{aligned} &E[Z \circ \theta_R \circ \theta_{T_n} ; \Lambda \cap \{T_n < \infty\}] \\ &= \sum_t E[Z \circ \theta_R \circ \theta_t ; \Lambda \cap \{T_n = t, Y_t = i\}] \\ &= \sum_t E_i[Z \circ \theta_R | R \leq \mu i]\, P[\Lambda \cap \{T_n = t, Y_t = i\}] \\ &= E_i[Z \circ \theta_R | R \leq \mu i]\, P[\Lambda \cap \{T_n < \infty\}], \end{aligned}$$

where the crucial step, the second equality, is justified below in Lemma (4.14). Proof follows from (4.12) and (4.13).

(4.14) LEMMA. Let $i \in \hat{D}$ be in the minimal state space of Y, let $t \in \mathbb{R}_0$, and take $Z \in b\underline{\underline{G}}_\infty$. Then, a.s. on $\{Y_t = i\}$,

$$E[Z \circ \theta_R \circ \theta_t | \underline{\underline{G}}_t] = E_i[Z \circ \theta_R | R \leq \mu i].$$

PROOF. Let $\underline{\underline{H}}$ be the collection of all $\Lambda \in \underline{\underline{G}}_t$ such that

(4.15) $$E[Z \circ \theta_R \circ \theta_t ; \Lambda \cap \{Y_t = i\}] = E_i[Z \circ \theta_R | R \leq \mu i] P[\Lambda \cap \{Y_t = i\}].$$

By the monotone class theorem, to show that $\underline{\underline{H}} = \underline{\underline{G}}_t$, it is sufficient to show that $\underline{\underline{H}}$ contains all sets of form $\Gamma \cap \{Y_t = j\}$ with $\Gamma \in \underline{\underline{F}}_t$ and $j \in \hat{D}$, because the latter generate $\underline{\underline{G}}_t$. Since $\{Y_t = j\} \cap \{Y_t = i\}$ is empty unless $i = j$, lemma will follow once we show that (4.15) holds for $\Lambda \in \underline{\underline{F}}_t$.

Let $\Lambda \in \underline{\underline{F}}_t$, note that $Z \in b\underline{\underline{G}}_\infty = b\underline{\underline{F}}_\infty$, and recall that $\{Y_t = i\} = \{X_t = \pi i,\ S_t + R \circ \theta_t \in (\lambda i, \lambda i + \mu i]\}$. Then, the ordinary Markov property yields

$$E[Z \circ \theta_R \circ \theta_t ; \Lambda \cap \{Y_t = i\}] = E[f(\pi i, S_t); \Lambda \cap \{X_t = \pi i\}] \tag{4.16}$$

where

$$f(j,a) = E_{ja}[Z \circ \theta_R;\ R \in (\lambda i,\ \lambda i + \mu i] - a]. \tag{4.17}$$

Next we evaluate $f(j,a)$ for $j = \pi i$ and $a \leq \lambda i$. Note that R is a perfect terminal time, and hence, $R = \lambda i - a + R \circ \theta_{\lambda i - a}$ on $\{R > \lambda i - a\}$; and further note that $(X_{\lambda i-a}, S_{\lambda i-a}) = (\pi i, \lambda i) = i$, $P_{\pi i,a}$ — almost surely on $\{R > \lambda i - a\}$. Hence, by the Markov property at $\lambda i - a$,

$$\begin{aligned} f(\pi i, a) &= E_{\pi i,a}[E_i[Z \circ \theta_R;\ R \leq \mu i];\ R > \lambda i - a] \\ &= E_i[Z \circ \theta_R \mid R \leq \mu i] P_i[R \leq \mu i] P_{\pi i,a}[R > \lambda i - a] \\ &= E_i[Z \circ \theta_R \mid R \leq \mu i] P_{\pi i,a}[R \in (\lambda i,\ \lambda i + \mu i] - a]. \end{aligned}$$

Putting the result obtained into (4.16) and using the Markov property for (X,S) once more, we see that the left side of (4.16) is equal to

$$E_i[Z \circ \theta_R \mid R \leq \mu i] P[\{R \circ \theta_t \in (\lambda i, \lambda i + \mu i] - S_t,\ X_t = \pi i\} \cap \Lambda],$$

which is equal to the right-hand side of (4.15). This completes the proof. $\square$

The following is the analog of Proposition (4.6) for stable and instantaneous states i of X. Since $X_t = Y_t$ on $\{Y_t \notin \hat{D}\}$, the result below is stronger than (4.6).

(4.18) PROPOSITION. Let $T \in st(\underline{\underline{G}}_{t+})$ be such that $[T] \subset M \cap K_i$ where $i \in E \setminus D = \hat{E} \setminus \hat{D}$ is fixed. Then, for any $Z \in b\underline{\underline{G}}_{\infty}$,

$$(4.19) \qquad E[Z \circ \theta_T | \underline{\underline{G}}_{T+}] = E_{i,0}[Z] \quad \text{a.s. on } \{T < \infty\}.$$

PROOF. Suppose i is instantaneous. Then, by Theorem (4.10) and (4.11) of [4], for a.e. ω, every $t \in K_i(\omega)$ is the limit of a decreasing sequence $(t_n) \subset M(\omega)$. Hence, $R_T = 0$ and $D_T = T$ almost surely. But for any $U \in st(\underline{\underline{G}}_{t+})$, $D_U \in st(\underline{\underline{F}}_t)$ and $\underline{\underline{G}}_{U+} \subset \underline{\underline{F}}_{D_U}$; (see MAISONNEUVE [9, p. 18] for a similar result). It follows that $T \in st\underline{\underline{F}}$ and $\underline{\underline{G}}_{T+} = \underline{\underline{F}}_T$, which imply (4.19) by the strong Markov property for (X,S).

Next suppose that i is stable, let $\Lambda \in \underline{\underline{G}}_{T+}$, and define U to be T on $\Lambda \cap \{T < \infty\}$ and $+\infty$ elsewhere. Then, $U \in st(\underline{\underline{G}}_{t+})$ also, and $[U] \subset M \cap K_i$ again; and to show that (4.19) holds, we need to show that

$$(4.20) \qquad E[Z \circ \theta_U; U < \infty] = E_{i,0}[Z]P[U < \infty].$$

Since i is stable, $M \cap K_i \subset \bigcup_n [V_n]$ where V_n is the time of n^{th} visit to i by X. Each V_n is a stopping time of $\underline{\underline{F}}$, and by Lemma (4.22) below

$$(4.21) \qquad \underline{\underline{G}}_{V_n+} = \underline{\underline{F}}_{V_n}.$$

Define

$$U_n = \begin{cases} U & \text{on } \{U = V_n\}, \\ +\infty & \text{elsewhere.} \end{cases}$$

Since U, $V_n \in st(\underline{\underline{G}}_{t+})$, the set $\{U = V_n\} \in \underline{\underline{G}}_{V_n+}$. Therefore, by (4.21), $\{U = V_n\} \in \underline{\underline{F}}_{V_n}$ and $U_n \in st\underline{\underline{F}}$. Since $U = \inf U_n$ and $\underline{\underline{F}}$ is right continuous, this implies that $U \in st\underline{\underline{F}}$. Now (4.20) follows from the strong Markov property for (X,S).

Finally, if i is absorbing, the result is obvious.

(4.22) LEMMA. Let T be the time of n^{th} visit to a fixed stable state i by X. Then, $\underline{\underline{G}}_{T+} = \underline{\underline{F}}_T$.

PROOF. Consider the σ-algebra $\underline{\underline{G}}_{(T+\varepsilon)-}$; see DELLACHERIE [5, p. 52] for the definition. Since $\underline{\underline{G}}_t = \underline{\underline{F}}_t \vee \sigma(Y_t)$, $\underline{\underline{G}}_{(T+\varepsilon)-}$ is generated by sets of form $\Lambda_t \cap \{T + \varepsilon > t, Y_t = j\}$ with $\Lambda_t \in \underline{\underline{F}}_t$ and $j \in \hat{D}$ as t runs through $\mathbb{R}_+$. But each such set is in $\underline{\underline{F}}_{T+\varepsilon} \vee \underline{\underline{H}}_\varepsilon$ where $\underline{\underline{H}}_\varepsilon$ is the σ-algebra generated by the sets $\{Y_{T+u} = j\}$ as j and u run through $\hat{D}$ and $[0,\varepsilon]$. Hence,

$$\underline{\underline{G}}_{T+} = \bigcap_{\varepsilon > 0} \underline{\underline{G}}_{(T+\varepsilon)-} \subset \bigcap_{\varepsilon > 0} (\underline{\underline{F}}_{T+\varepsilon} \vee \underline{\underline{H}}_\varepsilon). \tag{4.23}$$

Let $\Lambda \in \underline{\underline{G}}_{T+}$ be a subset of $\{T < \infty\}$. Since i is stable, $R_T > 0$ a.s. on $\{T < \infty\}$. So,

$$\Lambda = \lim_{\varepsilon} \Lambda \cap \{R_T > \varepsilon\}. \tag{4.24}$$

By (4.23), $\Lambda \in \underline{\underline{F}}_{T+\varepsilon} \vee \underline{\underline{H}}_\varepsilon$; and $\{R_T > \varepsilon\} \in \underline{\underline{F}}_{T+\varepsilon}$. So, $\Lambda \cap \{R_T > \varepsilon\}$ belongs to the trace of $\underline{\underline{F}}_{T+\varepsilon} \vee \underline{\underline{H}}_\varepsilon$ on the set $\{R_T > \varepsilon\}$; and the latter is simply the trace of $\underline{\underline{F}}_{T+\varepsilon}$ on $\{R_T > \varepsilon\}$, since $\{Y_{T+u} = j, R_T > \varepsilon\} = \emptyset$ for all $j \in \hat{D}$ and all $u \leq \varepsilon$. Hence, $\Lambda \cap \{R_T > \varepsilon\} \in \underline{\underline{F}}_{T+\varepsilon}$; and by (4.24) and the right continuity of $\underline{\underline{F}}$, we have that $\Lambda \in \underline{\underline{F}}_T$. We have shown that $\underline{\underline{G}}_{T+} \subset \underline{\underline{F}}_T$, and the reverse inclusion is trivial. □

We end this section by giving a slight modification of the sample paths of Y. Let i be an attractive instantaneous state, and let $L_i(\omega)$ be the set of all $t \in K_i(\omega)$ such that $(t,t+\varepsilon) \cap K_i(\omega) = \emptyset$ for some $\varepsilon > 0$. For a.e. ω, the set $L_i(\omega)$ is countable; if $t \in L_i(\omega)$, then there is $(t_n) \downarrow t$ such that $X_{t_n}(\omega) \to \phi$; and if $t \in L_i(\omega)$ and there is $(t_n) \downarrow t$ such that $X_{t_n}(\omega) \to j$ for some $j \in E$, then j is unstable; (see [4, Proposition (4.11)]).

We now give $\hat{E}$ the discrete topology and declare the point ϕ to be the point at infinity in the one point compactification of $\hat{E}$. Then, it follows that, for a.e. ω, we have $Y_s(\omega) \to \phi$ as $s \downarrow t$ for every $t \in L_i(\omega)$. We now define, for every $\omega \in \Omega$,

$$(4.25) \qquad \bar{Y}_t(\omega) = \begin{cases} \phi & \text{if } t \in \bigcup_{i \in A} L_i(\omega), \\ Y_t(\omega) & \text{otherwise,} \end{cases}$$

where A is the set of all attractive instantaneous states. The virtues of $\bar{Y}$ are summarized below.

(4.26) PROPOSITION. For a.e. ω, the path $\bar{Y}(\omega)$ is right lower semicontinuous everywhere on $\mathbb{R}_0 \setminus K_B(\omega)$ where B is the set of all repellent states (and $K_B(\omega) = \bigcup_{i \in B} K_i(\omega)$). All the results above remain true when Y is replaced by $\bar{Y}$ throughout.

PROOF. Pick ω such that the regularity condition (2.2a) on $Y(\omega)$ hold. If $t \in K_i(\omega)$ where i is stable or $i \in \hat{D}$, then $[t,t+\varepsilon) \subset K_i(\omega)$ for some $\varepsilon > 0$, and hence $Y(\omega)$ and $\bar{Y}(\omega)$ are right continuous at t. Next suppose that $t \in K_i(\omega)$ where i is attractive. If $\bar{Y}_t(\omega) = \phi$, we must have $t \in L_i(\omega)$ and then ϕ is the only limit from the right side by the discussion preceding (4.25). If $\bar{Y}_t(\omega) = i$ then t must not be in $L_i(\omega)$, which means that t is a right accumulation point of $K_i(\omega)$, and further, by the regularity (2.2a) and the discrete topology on $\hat{E}$, t cannot be a right accumulation point of any other $K_j(\omega)$ with $j \in \hat{E}$. Hence, if $t \notin K_B(\omega)$, then $\bar{Y}(\omega)$ is right lower semicontinuous at t.

To prove the second statement we only need to check the "strong Markov" property at stopping times T of $(\underline{\underline{G}}_{t+})$ such that $[T] \subset L_i$ for some attractive state i. Let T be such; then by the proof of Proposition (4.18), T is in fact a stopping time of $\underline{\underline{F}}$. But any $T \in st\underline{\underline{F}}$ such that $[T] \subset L_i$ is a.s. equal to $+\infty$ by Theorem (4.11) of [4]. So, the modification (4.25) does not affect the strong Markov property. □

5. CONVERSION TO A STRONG MARKOV PROCESS

In this section we will add some exponentially distributed random variables to the sample space Ω, replace the sojourns at holding states by these exponential variables, and show that the resulting process is strong Markov.

Let $(\Omega,\underline{\underline{M}},P)$, X, θ, $\bar{Y}$, M, etc. be as in Section 4. Throughout we will omit the bar on $\bar{Y}$ and simply write Y. Let C be the set of all stable and absorbing states of X; let $\hat{D}$ be as defined in (4.1); and set $\hat{C} = C \cup \hat{D}$; $\hat{C}$ will be the set of stable states for the new process.

For each $i \in \hat{C}$ and $n \in \mathbb{N} = \{1,2,\ldots\}$, let V_{in} be the time of n^{th} visit to i by Y; that is, V_{in} is the left end point of the n^{th} component interval of K_i. Then,

$$N_i(t) = \sum_{n\in\mathbb{N}} 1_{[0,t)} \circ V_{in} \tag{5.1}$$

is the number of visits to i by Y during $(0,t)$; note that $t \to N_i(t)$ is left continuous.

Let η be the exponential law on $(\mathbb{R}_+, \underline{\underline{R}}_+)$ with parameter 1, that is, $\eta(dt) = e^{-t}dt$, $t \in \mathbb{R}_+$. Define

$$(\Omega',\underline{\underline{M}}',P') = \prod_{i\in\hat{C}} (\mathbb{R}_+, \underline{\underline{R}}_+, \eta)^{\mathbb{N}}; \tag{5.2}$$

$$(\hat{\Omega},\hat{\underline{\underline{M}}},\hat{P}) = (\Omega,\underline{\underline{M}},P) \times (\Omega',\underline{\underline{M}}',P'); \tag{5.3}$$

and

$$\hat{P}_i = \begin{cases} P_i \times P' & \text{if } i \in \hat{D}, \\ P_{(i,0)} \times P' & \text{if } i \notin \hat{D}, \end{cases} \tag{5.4}$$

where the probabilities $P_{(j,a)}$ $(j \in E,\ a \in \mathbb{R}_+)$ are as described before, preceding Proposition (4.6).

For every sequence $\underset{\sim}{n} = (n_i)_{i \in \hat{C}}$ of non-negative integers n_i define the shift $\theta'_{\underset{\sim}{n}}$ on Ω' by

$$(5.5) \qquad \theta'_{\underset{\sim}{n}}((\omega_{im})_{m \in \mathbb{N}})_{i \in \hat{C}} = ((\omega_{i,n_i+m})_{m \in \mathbb{N}})_{i \in \hat{C}},$$

and for every $\hat{\omega} = (\omega, \omega') \in \hat{\Omega}$ and $t \in \mathbb{R}_+$ put

$$(5.6) \qquad \hat{\theta}_t \hat{\omega} = (\theta_t \omega,\ \theta'_{N(t,\omega)} \omega')$$

where $N(t,\omega)$ is the sequence $(N_i(t,\omega))_{i \in \hat{C}}$.

Finally, let W_{in} be the (i,n)-coordinate variable on Ω', and for $t > 0$ and $\hat{\omega} = (\omega, \omega') \in \hat{\Omega}$ put

$$(5.7) \qquad B^{\circ}_t(\hat{\omega}) = \sum_{i \in \hat{C}} \sum_{n \in \mathbb{N}} m_i W_{in}(\omega') 1_{[0,t)} \circ V_{in}(\omega)$$

where $m_i = \lambda i$ if $i \in \hat{D}$ (see (4.2) for the notation) and m_i is a median of the sojourn distribution at i for X if $i \in C$; (that sojourn distribution is $L(i, \bar{E}, \cdot)$ in the notation of (3.4) of [4]; also note that $m_i = \infty$ if i is absorbing).

(5.8) LEMMA. The process (B°_t) is increasing and left continuous. For each t, B°_t is finite valued almost surely $(\hat{P})$ on $\{\zeta > t\}$ where ζ is the time of entrance to absorbing states. For almost every $\hat{\omega} = (\omega, \omega')$

$$(5.9) \qquad B^{\circ}_{t+u}(\hat{\omega}) = B^{\circ}_t(\hat{\omega}) + B^{\circ}_u(\hat{\theta}_t \hat{\omega})$$

for all $t \in M(\omega)$ and all $u > 0$.

PROOF. That B° is increasing and left continuous is obvious. The finiteness of B_t° on $\{\zeta > t\}$ follows by comparing

$$\hat{E}[B_t^\circ | \underline{\underline{G}}_\infty \times \{\emptyset, \Omega'\}] = \sum_i m_i \sum_n 1_{[0,t)} \circ V_{in}$$

with the total length of time Y spends in $\hat{C}$: for $i \in \hat{D}$, m_i is less than any sojourn of Y at i; and for stable $i \in C$, m_i is a median of the distribution of the sojourn of Y at i, and said sojourn lengths are independent and identically distributed.

To see the additivity (5.9), let $\hat{\omega} = (\omega, \omega')$ be such that $W_{in}(\omega') < \infty$ for all i and n and $N_i(s,\omega) < \infty$ for all $i \in \hat{C}$ and all rational finite s; (the set of all such $\hat{\omega}$ is of full measure). Let $t \in M(\omega)$ and $u > 0$. Note that

$$(5.10) \qquad V_{in}(\theta_t \omega) = V_{i,N_i(t,\omega)+n}(\omega) - t, \qquad n \in \mathbb{N};$$

(this is not true for $t \notin M(\omega)$ if $X_t(\omega) \in C$ for instance). Hence,

$$\begin{aligned} B_u^\circ(\hat{\theta}_t \hat{\omega}) &= \sum_i \sum_{n \in \mathbb{N}} m_i W_{in}(\theta'_{N(t,\omega)} \omega') 1_{[0,u)} \circ V_{in}(\theta_t \omega) \\ &= \sum_i \sum_n m_i W_{i,N_i(t,\omega)+n}(\omega') 1_{[t,t+u)}(V_{i,N_i(t,\omega)+n}(\omega)) \\ &= \sum_i \sum_{n > N_i(t,\omega)} m_i W_{in}(\omega') 1_{[t,t+u)} \circ V_{in}(\omega). \end{aligned}$$

Moreover, by the way N_i is defined, $V_{in}(\omega) < t$ for all $n \leq N_i(t,\omega)$, and hence the summation with respect to n can be taken over all $\mathbb{N}$. Then, (5.9) is immediate. □

We now extend the definitions of X, S, W, M, K, etc. onto $\hat{\Omega}$ in the obvious fashion, and denote the extended variables by the same letters as before; for example, $X_t(\hat{\omega})$ is the old $X_t(\omega)$ if $\hat{\omega} = (\omega, \omega')$. Similarly, instead of $\underline{\underline{G}}_t \times \{\emptyset, \Omega'\}$ we simply write $\underline{\underline{G}}_t$, and let $\underline{\underline{H}}$ be the right continuous complete (relative to the P_i) history generated on $(\hat{\Omega}, \hat{\underline{\underline{M}}})$ by

(5.11) $$\underline{\underline{H}}^\circ_t = \underline{\underline{G}}_t \vee \sigma(B^\circ_u;\ u \leq t), \qquad t \in \mathbb{R}_0.$$

The following merely expresses the independence of the W_{in} from (X,S).

(5.12) PROPOSITION. Proposition (4.6) remains true with the new definitions of M, K_i, R, and with $\underline{\underline{H}}$, $\hat{\theta}$, $\hat{E}$, and $\hat{E}_i$ replacing $\underline{\underline{G}}$, θ, E, and E_i respectively. Similarly for Proposition (4.18).

We are now ready for the transformation to a Markov process $\hat{X}$: Define

(5.13) $$B_t = B^\circ_t + t - \int_0^t 1_{\hat{C}}(Y_s)\,ds, \qquad t \in \mathbb{R}_0,$$

(5.14) $$\tau_t = \inf\{u \geq 0:\ B_u > t\}, \qquad t \in \mathbb{R}_0.$$

It is clear that (B_t) is an increasing process adapted to $\underline{\underline{H}}^\circ$; therefore each τ_t is a stopping time of $(\underline{\underline{H}}^\circ_{t+}) \subset \underline{\underline{H}}$. Moreover, the process (τ_t) is increasing and right continuous. Define

(5.15) $$\hat{X}_t = Y_{\tau_t}, \qquad t \in \mathbb{R}_0;$$

let $\hat{\underline{\underline{F}}}$ be the history generated by $\hat{X}$; and introduce

$$(5.16) \qquad K(t,i,j) = \begin{cases} \hat{P}_i[\hat{X}_t \circ \hat{\theta}_R = j] & \text{if } i \notin \hat{D}, \\ \hat{P}_i[\hat{X}_t \circ \hat{\theta}_R = j \mid R \le \mu i] & \text{if } i \in \hat{D}, \end{cases}$$

for $i,j \in \hat{E}$ and $t \in \mathbb{R}_0$. The following is the main result.

(5.17) THEOREM. The process $\hat{X}$ is progressively measurable with respect to $\hat{\underline{F}}$. For any stopping time T of $(\hat{\underline{F}}_{t+})$, any $t \in \mathbb{R}_0$, and any $j \in \hat{E} \cup \{\phi\}$

$$(5.18) \qquad \hat{P}[\hat{X}_{T+t} = j \mid \hat{\underline{F}}_{T+}] = P_t(\hat{X}_T, j) \qquad \hat{P} \text{ a.s. on } \{\hat{X}_T \in \hat{E},\ T < \infty\},$$

where

$$(5.19) \qquad P_t(i,j) = K(t,i,j) \qquad \text{if } i \in \hat{E} \setminus \hat{C},$$

and if $i \in \hat{C}$,

$$(5.20) \qquad P_t(i,j) = I(i,j)\exp(-t/m_i) + \int_0^t m_i^{-1}\exp(-u/m_i)K(t-u,i,j)du.$$

Proof will be through several lemmas. Throughout, $\hat{M}$, $\hat{K}_i$, etc. have the same meaning relative to $\hat{X}$ as did M, K_i, etc. relative to X.

(5.21) LEMMA. The process $\hat{X}$ satisfies the regularity condition (2.2a). In fact, for a.e. $\hat{\omega}$, the path $\hat{X}(\hat{\omega})$ is right lower semicontinuous everywhere except on $\hat{K}_B(\hat{\omega}) = \{t\colon \hat{X}_t(\hat{\omega}) \in B\}$ where B is the set of all repellent states of X.

PROOF. For a.e. $\hat{\omega} = (\omega,\omega')$, the path $B(\hat{\omega})$ is strictly increasing on $\mathbb{R}_+ \setminus K_{\hat{C}}(\omega)$; $K_{\hat{C}}(\omega)$ is the union of countably many intervals of form [ ); $B(\hat{\omega})$ has a jump at the left end point of each such component interval [a,b), and remains constant on (a,b). It follows that the qualitative properties of the path $\hat{X}(\hat{\omega})$ are the same as those of $Y(\hat{\omega})$, and the lemma now follows from Proposition (4.26); (recall that we are omitting the bars).

(5.22) LEMMA. The process $\hat{X}$ is progressive with respect to $\underline{\underline{F}}$.

PROOF. Let B be the set of repellent states again. For $i \in \hat{E} \setminus B$, the constancy set $\hat{K}_i$ is progressive with respect to $\underline{\underline{F}}$ by the right lower semicontinuity proved in the preceding lemma; cf. CHUNG [1, p. 162]. For $i \in B$, $\hat{K}_i$ is the union of the graphs of countably many (only finitely many in a finite interval) stopping times of $\underline{\underline{F}}$. Hence $\hat{K}_i$ is well-measurable (and in particular progressive) relative to $\underline{\underline{F}}^\circ$ for each $i \in B$. □

(5.23) REMARK. Progressiveness of $\hat{X}$ with respect to $\underline{\underline{F}}$ does not follow from Theorem 57 in MEYER [10, p. 73] as before in (3.10). That theorem merely implies that $\hat{X}$ is progressive relative to $(\underline{\underline{H}}_{\tau_t})$; whereas we have $\underline{\underline{F}}_t \subset \underline{\underline{H}}_{\tau_t}$, and the inclusion is strict.

(5.24) PROOF of Theorem (5.17). The first statement is shown as Lemma (5.22). Let $T \in st(\underline{\underline{F}}_{t+})$. By the progressiveness of $\hat{X}$, $\hat{X}_T$ is $\underline{\underline{F}}_{T+}$ measurable; and hence, for any $i \in \hat{E}$, putting T_i to be T on $\{\hat{X}_T = i, T < \infty\}$ and $+\infty$ elsewhere, we obtain another stopping time T_i of $\underline{\underline{F}}$. Clearly, it is sufficient to prove (5.18) for T_i.

Accordingly, let $T \in st\underline{\underline{F}}$, $t \in \mathbb{R}_0$, and $j \in \hat{E} \cup \{\phi\}$ be fixed and suppose that $[T] \subset \hat{K}_i$ for some fixed $i \in \hat{E}$.

Suppose i is instantaneous. Then, $[T] \subset \hat{K}_i$ implies that $[\tau_T] \subset K_i$ and thus

$$(5.25) \qquad R \circ \hat{\theta}_{\tau_T} = 0, \quad \hat{X}_{T+t} = \hat{X}_t \circ \hat{\theta}_{\tau_T} = \hat{X}_t \circ \hat{\theta}_R \circ \hat{\theta}_{\tau_T}$$

a.s. on $\{\tau_T < \infty\} = \{T < \infty\}$. On the other hand, by Propositions (4.18) and (5.12) and the definition (5.16),

$$(5.26) \qquad \hat{P}[\hat{X}_t \circ \hat{\theta}_R \circ \hat{\theta}_{\tau_T} | \underline{\underline{H}}_{\tau_T}] = K(t,i,j) \quad \text{a.s. on } \{\tau_T < \infty\}.$$

In view of (5.25), (5.19), and the fact that $\underline{\underline{F}}_{T+} \subset \underline{\underline{H}}_{\tau_T}$, (5.26) implies that

$$(5.27) \qquad \hat{P}\{\hat{X}_{T+t} = j | \hat{\underline{\underline{F}}}_{T+}] = P_t(i,j) \qquad \text{a.s. on } \{T < \infty\}.$$

Next suppose that i is not instantaneous, that is, $i \in \hat{C}$. Now $[T] \subset \hat{K}_i$ implies that $[\tau_T] \subset M \cap K_i$ and that τ_T is a time of jump for B, and thus

$$(5.28) \qquad \hat{R}_T = B_{\tau_T +} - T > 0 \qquad \text{on } \{T < \infty\}.$$

By the definition of $\hat{X}$,

$$(5.29) \qquad \hat{X}_{T+t} = \begin{cases} i & \text{on } \{\hat{R}_T > t\} \\ \hat{X}_{t-\hat{R}_T}(\hat{\theta}_R \circ \hat{\theta}_{\tau_T}) & \text{on } \{\hat{R}_T < t\}, \end{cases}$$

(note that $(W_Z(\theta_V))(\omega) = W_{Z(\omega)}(\theta_V\omega) \neq W_{Z(\theta_V\omega)}(\theta_V\omega) = W_Z \circ \theta_V(\omega))$. Since $\underline{\underline{H}}$ is right continuous B_{τ_T+} is in $\underline{\underline{H}}_{\tau_T}$, and obviously so is T. So, by (5.28), $\hat{R}_T$ is in $\underline{\underline{H}}_{\tau_T}$. Now (5.29), Propositions (4.6) and (5.12), and the definition (5.16) imply that

$$(5.30) \qquad \hat{P}[\hat{X}_{T+t} = j | \underline{\underline{H}}_{\tau_T}] = I(i,j)I_{\{\hat{R}_T > t\}} + K(t - \hat{R}_T, i,j)I_{\{\hat{R}_T < t\}} + UI_{\{\hat{R}_T = t\}},$$

where the random variable U will prove immaterial very shortly.

On $\{\tau_T < \infty\} \supset \{T < \infty\}$, τ_T is a jump time for B, and the jump is of size $m_i W_{i,N_i(\tau_T)+1} = m_i W_{i,1} \circ \hat{\theta}_{\tau_T}$. Using the "memorylessness" of the exponential distribution (which W_{i1} has) along with the independence of W_{i1} from Y, we obtain that

(5.31) $$\hat{P}[\hat{R}_T > u | \hat{F}_{T+}] = \exp(-u/m_i) \quad \text{a.s. on } \{T < \infty\}.$$

Taking conditional expectations of both sides of (5.30) given $\hat{F}_{T+}$, using (5.31) on the right-hand side, and noting the definition (5.20), we obtain

(5.32) $$\hat{P}[\hat{X}_{T+t} = j | \hat{\underline{F}}_{T+}] = P_t(i,j) \quad \text{a.s. on } \{T < \infty\}.$$

The proof of (5.17) follows from (5.27) and (5.32).

The strong Markov property we have just shown does not yet imply that $\hat{X}$ is a Markov process, because we do not yet know if $\hat{P}\{\hat{X}_t = \phi\}$ is zero for all t. We will settle this matter, and bring the tale back to the introduction in the next section. Before leaving this section we merely point out the following.

(5.33) REMARK. $\hat{X}$ can be obtained from Y by a strictly increasing continuous time change as well: Consider the path $B(\hat{\omega})$ and let [a,b) be a component interval of some $K_i(\hat{\omega})$ for some $i \in \hat{C}$. At a, the path $B(\hat{\omega})$ jumps from its value $B_a(\hat{\omega})$ to the right hand limit $B_{a+}(\hat{\omega})$, and then remains constant over (a,b). Modify the path $B(\hat{\omega})$ on [a,b) by replacing it by a straight line from the point $(a, B_a(\hat{\omega}))$ to the point $(b, B_b(\hat{\omega})) = (b, B_{a+}(\hat{\omega}))$. Let this modification be done over each component interval of $K_i(\hat{\omega})$ and for all $i \in \hat{C}$, and let $\hat{B}(\hat{\omega})$ be the modified path. It is clear that $\hat{B}(\hat{\omega})$ is strictly increasing and continuous, and therefore, its inverse $\hat{\tau}(\hat{\omega})$ is continuous and strictly increasing. Moreover, the alteration described does not alter the image of Y, and so, we also have

(5.34) $$\hat{X}_t = Y_{\hat{\tau}_t}, \quad Y_t = \hat{X}_{\hat{B}_t}.$$

6. FROM STRONG MARKOV TO MARKOV AND CHUNG

Throughout this section we are working on $(\hat{\Omega},\hat{\underline{\underline{M}}},\hat{P})$ by having all the processes, etc. of Sections 2, 3, 4 extended onto $\hat{\Omega}$ in the natural manner, and then drop the "^" over $(\hat{\Omega},\hat{M},\hat{P})$. Recall the succession of processes X, $\bar{X}$, Y, $\bar{Y}$, $\hat{X}$: By (3.5), (3.9), (4.4), (4.1), (4.25) and (5.34) we have

$$(6.1) \qquad X_t = \bar{X}_{C_t}, \quad \bar{X}_u = \pi(Y_u), \quad Y_u = \bar{Y}_u \text{ a.s.}, \quad \bar{Y}_u = \hat{X}_{\hat{B}_u}.$$

So, if we put

$$(6.2) \qquad A_t = \hat{B}_{C_t}, \quad t \geq 0,$$

we obtain a strictly increasing and continuous process A, and

$$(6.3) \qquad X_t = \pi(\hat{X}_{A_t}) \text{ a.s.}$$

We next examine the essential range and the minimal state space of the process $\hat{X}$; see CHUNG [1] for the terms. We may, and do, assume that each $i \in E$ is in the essential range of X. Let A, B, C, D be the respective sets of all attractive, repellent, stable, unstable states of X; then $E = A \cup B \cup C \cup D$. Recall the definition (4.1) of $\hat{D}$ and $\hat{E}$, and recall again that $\hat{C} = C \cup \hat{D}$. Let

$$(6.4) \qquad D^* = \{i \in \hat{D}: P_i[R \leq \mu i] > 0\},$$

$$(6.5) \qquad E^* = (E \setminus D) \cup D^*, \quad E^0 = E^* \setminus B.$$

(6.6) PROPOSITION. The essential range of $\hat{X}$ is $E^* \cup \{\phi\}$. For each $j \in D$, there are infinitely many $i \in D^*$ with $\pi i = j$.

PROOF. It is clear that the essential range of $\hat{X}$ is contained in $\hat{E} \cup \{\phi\}$; hence, to show the first statement, we need to show that

$$(6.7) \qquad P[K_i = \emptyset] = 1 \qquad \text{if } i \in \hat{D} \setminus D^*.$$

Fix $i \in \hat{D} \setminus D^*$, and let $Q_1, Q_2, \ldots$ be the times of successive visits to the state $i = (\pi i, \lambda i)$ by the process $(\bar{X}, \bar{S})$, and note that

$$(6.8) \qquad \{K_i = \emptyset\} = \bigcap_n \{\bar{R} \circ \bar{\theta}_{Q_n} > \mu i\};$$

(note that $\bar{R}$ and $\bar{\theta}$ remain the same for $\bar{X}$ and $\bar{Y}$; so, there can be no confusion). Hence, the strong Markov property for $(\bar{X}, \bar{S})$ applied at its stopping times Q_n yields

$$1 - P[K_i = \emptyset] \leq \sum_n P[\bar{R} \circ \bar{\theta}_{Q_n} \leq \mu i] \leq \sum_n P_i[\bar{R} \leq \mu i] P[Q_n < \infty] = 0,$$

which is the desired result (6.7).

On the other hand, for any $j \in D$ and $a \in \mathbb{R}_0$, $P_{(j,a)}\{\bar{R} > b\} = n(j, a+b)/n(j,a)$ for some right continuous decreasing function $n(j, \cdot)$ whose limit $n(j, 0+)$ is infinite; see [4, Definition (3.16)]. It follows that $n(j, 1/k) - n(j, 1/(k-1)) > 0$ for infinitely many k in $\{1, 2, \ldots\}$, which means that there are infinitely many $(j, 1/k) \in D^*$ for each fixed $j \in D$.

For $i \in E^*$ consider the constancy set $\hat{K}_i$; it is right closed (Lemma (5.21)), is progressive relative to $\hat{\underline{\underline{F}}}$, and for any stopping time T of $\hat{\underline{\underline{F}}}$ such that $[T] \subset \hat{K}_i$ the strong Markov property applies; hence, $\hat{K}_i$ is a regeneration set. By the characterization theorem of MAISONNEUVE [8], then, $\hat{K}_i$ is the image of an increasing Lévy process. Note that if $i \notin B$, the Lévy process in question has a positive drift rate (this is because $\ell eb\ \hat{K}_i > 0$);

whereas for $i \in B$, $\hat{K}_i$ is almost surely discrete. The following proposition is immediate from these through a well known theorem in NEVEU [11, p. 41] which states that $A \to E[\text{Leb}\, K_i \cap A] = R(A)$, $A \in \underline{\underline{\mathbb{R}}}_+$, has a continuous derivative r and that $r(t) \to 1$ as $t \downarrow 0$.

(6.9) PROPOSITION. Let P_t be as defined in Theorem (5.17). For each $i \in E^* \setminus B$, $t \to P_t(i,i)$ is continuous strictly positive, and $\lim P_t(i,i) = 1$ as $t \to 0$.

Incidentally, so far we do not know that $P_t(i,j)$ are transition functions since $\hat{X}$ is not yet known to be a Markov process. Otherwise, the above result would have been standard. We finally settle this matter of Markovness.

(6.10) PROPOSITION. The minimal state space of $\hat{X}$ is $E^0 = E^* \setminus B$. Therefore, in particular, $\hat{X}$ is a Markov process.

PROOF. We have already shown for the process $\bar{X}$ that $P\{\bar{X}_t = \phi\} = 0$ for (Lebesgue) almost every t; note that $\{\bar{X}_t = \phi\} = \{Y_t = \phi\}$; and $\bar{Y}_t(\omega)$ differs from $Y(\omega)$, for a.e. ω, only at countably many points. Hence, the Lebesgue measure of $\{t: \bar{Y}_t(\omega) = \phi\}$ is zero for a.e. ω, and since $\hat{X}$ can be obtained from $\bar{\bar{Y}}$ by a strictly increasing absolutely continuous time change (see (5.34)), we have that the same is true for $\{t: \hat{X}_t(\omega) = \phi\}$. Therefore, by FUBINI,

$$P\{\hat{X}_t = \phi\} = 0 \quad \text{a.e. } t. \tag{6.11}$$

We next show that (6.11) is in fact true for all $t > 0$. Let P_t be as in Theorem (5.17). Pick $i \in E^0$; (6.11) implies that $P_t(i,E^*) = P_i\{\hat{X}_t \neq \phi\} = 1$ for a.e. t. Now pick $t_n \uparrow t$ such that $P_{t_n}(i,E^*) = 1$ for each n. Then,

$$P_t(i,E^*) \geq P_i\{\hat{X}_{t-t_n} = i,\ \hat{X}_t \in E^*\}$$
$$\geq P_{t-t_n}(i,i)P_{t_n}(i,E^*) = P_{t-t_n}(i,i);$$

and hence, by Proposition (6.9),

$$(6.12) \qquad P_t(i,E^*) = 1, \quad i \in E^0, \quad t \in \mathbb{R}_0.$$

Next let $i \in B$; since $\hat{K}_B(\omega)$ is countable for a.e. ω, by FUBINI again, $P_i[\hat{X}_s \in B] = 0$ for a.e. s. Hence, for a "good" $s < t$,

$$(6.13) \qquad P_t(i,E^*) = \sum_{j \in E^0} P_s(i,j)P_{t-s}(j,E^*) = P_s(i,E^0) = 1$$

by (6.12). We have thus shown that $P_t(i,E^*) = 1$ for all $t > 0$ for all $i \in E^*$; and now by the strong Markov property at t we get

$$(6.14) \qquad P_{t+s}(i,k) = \sum_{j \in E^*} P_t(i,j)P_s(j,k), \quad i,k \in E^*.$$

In other words, $\hat{X}$ is a Markov process, and (P_t) is a transition function on E^*. Since $\hat{X}$ is progressive, each function $t \to P_t(i,j)$ is measurable, and therefore (see CHUNG [1, p. 120]) is continuous. Since $P_t(i,j) = 0$ for a.e. t whenever $j \in B$, this implies that

$$(6.15) \qquad P_t(i,j) = 0 \qquad i \in E^*,\ j \in B,\ t \in \mathbb{R}_0.$$

It follows that $\{P_t(i,j);\ i,j \in E^0\}$ is a transition matrix, and that the minimal state space is E^0. □

The corollary below is immediate from the preceding proposition together with Proposition (6.9).

(6.16) COROLLARY. The matrix valued function $t \to [P_t(i,j);\ i,j \in E^0]$ is a standard transition function.

We summarize the facts concerning $\hat{X}$ below, and include the statement of main result (1.2) as well. We have already proved this.

(6.17) THEOREM. The process $\hat{X}$ (with its natural history $\hat{\underline{\underline{F}}}$) satisfies Conditions (2.2a,c,d,e); and hence is a semimarkov process in the strict sense. In fact, $\hat{X}$ is a Markov process, and enjoys the strong Markov property at every stopping time of $\hat{\underline{\underline{F}}}$ such that $\hat{X}_T \in \hat{E}$ a.s. on $\{T < \infty\}$. The essential range of process $\hat{X}$ is $E^* \cup \{\phi\} = A \cup B \cup C \cup D^* \cup \{\phi\}$; every i in A is heavy attractive, every i in B is repellent; every $i \in C \cup D^*$ is stable. The minimal state space of $\hat{X}$ is $E^0 = E^* \setminus B$. Finally, for every $t \in \mathbb{R}_0$, $X_t = \pi(\hat{X}_{A_t})$ a.s., where A is defined by (6.2) and is strictly increasing and continuous. □

The process $\hat{X}$ is almost a Chung process; all that we need do is to modify the paths so that the essential range does not contain any repellent states. So, we define

$$(6.18) \qquad X'_t(\omega) = \begin{cases} \hat{X}_t(\omega) & \text{if } \hat{X}_t(\omega) \notin B, \\ \phi & \text{if } \hat{X}_t(\omega) \in B. \end{cases}$$

(6.19) PROPOSITION. X' is a Chung process.

PROOF. The process X' has E^0 as its minimal state space, and $E^0 \cup \{\phi\}$

as its essential range. The transition function $[P_t(i,j)]$ on E^0 is standard; see Corollary (6.16).

For a.e. ω, the path $\hat{X}(\omega)$ is right lower semicontinuous at every $t \notin \hat{K}_B(\omega)$; (see Lemma (5.21) for this); and $X'(\omega) = \hat{X}(\omega)$ outside $\hat{K}_B(\omega)$. And for $t \in \hat{K}_B(\omega)$, by Theorem (4.10) of [4], the only possible limit $\hat{X}(\omega)$ has from the right side is ϕ (since $\hat{E}$ is given the discrete topology), and we have $\phi = X'_t(\omega)$ for such t. Hence, for a.e. ω, the path $X'(\omega)$ is right lower semicontinuous. □

The preceding implies Result (1.4) of the Introduction. If one's point of view is that probabilities are all that matter, then the preceding Proposition is the final result. However, from the point of view of sample paths and the strong Markov property, the preceding result is a step in the wrong direction. For, we have removed some states, and thereby lost the strong Markov property at stopping times T of $\underline{\underline{\hat{F}}}$ with $[T] \subset \hat{K}_B$. Indeed, when studying the boundary behavior of X', each point b in B would be re-introduced as a repellent boundary atom.

Therefore, it would be desirable to obtain a process $\tilde{X}$ whose minimal state space contains B and which is still a Chung process. This can be achieved at the cost of altering the looks of the sample paths radically: for, each $i \in B$ will have to become a stable state for $\tilde{X}$. This is because each i is entered by X only finitely often during a finite interval.

To obtain $\tilde{X}$ from $\hat{X}$ all that needs to be done is to replace each point $t \in \hat{K}_i$ by an interval whose length has an exponential distribution with mean m_i, where these m_i are so selected that

$$(6.20) \qquad \sum_{i \in B} m_i N_i(t) < \infty \quad \text{a.s.}$$

for all $t < \infty$; ($N_i(t)$ is the cardinality of $K_i \cap [0,t]$).

Formally, this is achieved via a further random change similar to that of Section 5. In fact, $\tilde{X}$ can be obtained directly from $\bar{Y}$ if we replace the set $\hat{C}$ by $\hat{B} = B \cup \hat{C}$ throughout Section 5, the only additional proof we require being the finiteness of B^0 defined by (5.7) (with $\hat{B}$ replacing $\hat{C}$ there). In other words, we need to show how the means m_i are to be selected in order to satisfy (6.20).

We end this note by describing this selection. For each $i \in B$, the set K_i (for Y or for X or $\bar{X}$; but we choose to work relative to $\bar{X}$) is discrete; its points (T_n^i) form a renewal process; and $N_i(t)$ appearing in (6.20) is the number of renewals occurring during $(0,t)$.

Let B_0 be the set of all transient repellent states; then $B \setminus B_0$ is the set of all recurrent ones. We say that j can be reached from i if $P_i\{K_j = \emptyset\} < 1$. If i is recurrent and j can be reached from i, then j must be recurrent and i can be reached from j; (these facts are the familiar arguments of Markov chains). It follows that $B \setminus B_0$ can be decomposed into classes $B_1, B_2, \ldots$ each of which is composed of (recurrent repellent) states which can be reached from each other almost surely.

Consider one such class B_n, $n \geq 1$, and let i and j be in B_n. Let $T_0, T_1, T_2, \ldots$ be the times of successive visits to the set $\{i,j\}$ by $\bar{X}$, and let $Z_0 = \bar{X}_{T_0}$, $Z_1 = \bar{X}_{T_1}$, $Z_2 = \bar{X}_{T_2}$, and so on. By the strong Markov property for $(\bar{X},\bar{S})$, recalling that $\bar{S}_{T_n} = 0$ for every n, we see that (Z_n) is a Markov chain with state space $\{i,j\}$. Since both i and j are recurrent and communicate with each other, the chain (Z_n) is ergodic. Therefore, as is well known, it has a limiting distribution (ν_i, ν_j), $(\nu_i > 0,\ \nu_j > 0,\ \nu_i + \nu_j = 1)$, and ν_j/ν_i is the expected number of visits to j in between two visits to i.

Fix a state $b \in B_n$ and define $\mu_j = \nu_j/\nu_b$ for every $j \in B_n$. Then,

$$(6.21)\qquad E_i[N_j(t)] \le \frac{\nu_j}{\nu_i} E_i[N_i(t)+1] = \frac{\mu_j}{\mu_i} E_i[N_i(t) + 1]$$

for all $i,j \in B_n$. Now pick $m_j \in (0,\infty)$ for each $j \in B_n$ such that

$$(6.22)\qquad \sum_{j \in B_n} m_j \mu_j = 1.$$

Finally, let all these be done for each B_n, $n = 1,2,\ldots$; and in addition, let m_j be selected for $j \in B_0$ such that

$$(6.23)\qquad \sum_{j \in B_0} m_j E[N_j(\infty)] = 1.$$

We show next that (6.20) holds with this selection of the m_j. Let T be the time of first entrance to $B \setminus B_0$. Note that

$$(6.24)\qquad \sum_{j \in B} m_j N_j(t) \le \sum_{j \in B_0} m_j N_j(\infty) + \sum_{j \in B\setminus B_0} m_j N_j(T + t).$$

Now,

$$(6.25)\qquad \sum_{j \in B_0} m_j N_j(\infty) < \infty \quad \text{a.s.}$$

since its expectation is finite by (6.23). Next, the second term on the right side of (6.24) is in fact a sum over the class B_n to which $\bar{X}_T$ belongs. Hence, on the set $\{\bar{X}_T = i\}$, letting B' denote the class to which i belongs,

$$E\Big[\sum_{j \in B\setminus B_0} m_j N_j(T+t) \,\Big|\, \underline{\underline{\bar{F}}}_T\Big] = \sum_{B'} m_j E_i[N_j(t)] \le \frac{1}{\mu_i} E_i[N_i(t) + 1]$$

by (6.21) and (6.22). The finiteness of the last term implies that

$$(6.26) \qquad \sum_{j \in B \setminus B_0} m_j N_j(T + t) < \infty$$

almost surely on $\{\bar{X}_T = i\}$. Since this holds for every $i \in B \setminus B_0$, (6.26) is true almost surely. Now (6.24),(6.25) and (6.26) imply (6.20).

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