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AN ADMISSIBLE SET OCCURRING IN VARIOUS BARGAINING SITUATIONS

by

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1. Introduction and a Definition.

Given a set of alternatives, S, we are interested in finding the possible choices that a society, or a group of individuals, would make from this set. We assume that there is a binary relation, M on S, that describes the valid arguments in the social bargaining process that leads to the final choice. For example, in Social Choice Theory xMy may stand for a majority of the individuals prefers alternative x to alternative y. Let $\hat{M}$ denote the transitive closure of M(i.e. $x\hat{M}y$ if there is a finite sequence of alternatives $x=x_0, x_1, x_2, \ldots, x_n=y$ for which $x_{i-1}Mx_i$ for $i=1,2,\ldots,n$) then $x\hat{M}y$ means that whenever the social bargaining process is at y it may shift to x in a finite number of steps. This approach leads naturally to the following definition.

The <u>admissible set of the pair</u> (S,M) is the set

$A(S,M) = \{x \in S \mid y \in S \text{ and } y\hat{M}x \text{ imply } x\hat{M}y\}$.

This concept was introduced in a paper by Kalài-Pazner-Schmeidler and they proved the straightforward result:

<u>Theorem 1</u>: <u>If</u> S <u>is finite and nonempty then</u> A(S,M) <u>is nonempty.</u>

One may think of the pair (S,M) as a (bargaining) Markov process where xMy if and only if there is a positive probability of transition from y to x when $x \neq y$. It was shown in [K-P-S] that if S is finite then with probability one the bargaining process will enter to and stay in A(S,M) after a finite number of steps. Furthermore A(S,M) is the smallest subset of S with this property.

The purpose of this work is to study further the properties of the admissible set, and to include the cases where S is infinite. We show that the admissible set is characterized as the set of final outcomes in a qualitative (bargaining) Markov process (Theorem 5). Two general existence theorems are presented; these theorems are general enough to guarantee existence in all the examples that we have considered. We discuss examples where the admissible set is a new solution concept and we show that the admissible set approach leads to a unified treatment of cooperative and non-cooperative game theory. More specifically, with the right interpretations, the admissible set coincides with the core of an n-person cooperative game without sidepayments, with the Nash equilibria of a game in the normal form, and it contains the competitive equilibria prices in the case of an exchange economy with finitely many commodities and traders.

2. Existence and Characterization.

A straightforward extention of theorem 1 to the infinite case is

Theorem 2: *If* S *is a nonempty compact topological space and if for every* $x \in S$ *the set* $\{y \in S \mid xMy\}$ *is open, then* A(S,M) *is nonempty.*

Proof: If there exists a $y \in S$ such that for no $x \in S$ xMy then $y \in A(S,M)$ and the proof is completed. So assume that this is not the case then the collection $\{0_x\}_{x \in S}$ where $0_x = \{y \in S | xMy\}$ is an open cover of S. Therefore there exists a finite subset of S, $X = \{x_1, x_2, \ldots, x_n\}$, such that $\{0_{x_i}\}_{i=1}^n$ is a finite subcover. $A(X,\hat{M})$ is not empty and $A(X,\hat{M}) \subset A(X,M)$. Q.E.D.

In case that the binary relation is transitive, the corresponding admissible set consists of the maximal elements (i.e. those that are not strictly dominated). The special continuity condition of the next theorem is useful for applications.

Theorem 3: Let S be a nonempty compact topological space and let R be a transitive, reflexive, binary relation such that for every $x \in S$ the set $\{y \in S | yRx\}$ is closed, then $A(S,R)$ is nonempty.

Proof: We have to show that S contains an R-maximal element (i.e. an element $y \in S$ such that for every $x \in S$, xRy implies yRx). By Zorn's lemma it suffices to show that every R-chain, T, is bounded. For every $x \in T$ let $Rx = \{z \in S : zRx\}$. Clearly $Rx \neq \emptyset$ and if zRx then $Rz \subset Rx$. So by the finite intersection property $\bigcap_{x \in T} Rx \neq \emptyset$ and any element in the intersection is a bound for T. Q.E.D.

An interesting mathematical question is whether the use of Zorn's lemma can be omitted from this proof.

In the case of a finite S (as discussed in [K-P-S]) A(S,M) has the property of "outer stability" in the following sense. If $y \notin A(S,M)$ then there is an $x \in A(S,M)$ such that $x\hat{M}y$ This property is lost in the infinite case even when the conditions of Theorem 2 are met. (As an example consider the set of imputations of a symmetric, non convex, 0-1 normalized, 3-person game with sidepayments and the domination relation.) However if we deal with an admissible set A(S,R) where R satisfies the conditions of Theorem 3 then it is true that:

Theorem 4: *If* $y \notin A(S,R)$ *then there is an* $x \in A(S,R)$ *such that* xRy (*provided that* R *and* S *satisfy the conditions of Theorem 3*).

Proof: For every $y \in S$, $Ry = \{x \in S | xRy\}$ is compact. So $A(Ry,R) \neq \emptyset$ and, by the transitivity of R, $A(Ry,R) \subset A(S,R)$. Q.E.D.

Another property of A(S,M) which is carried over from the finite case to the arbitrary S is that of A(S,M) being the union of the minimally M-closed subsets of S. For an arbitrary (S,M) a subset T of S is said to be *M-closed* if it is nonempty and for every $y \notin T$ and every $x \in T$ not yMx. T is *minimally M-closed* if it is M-closed and no proper subset of T is M-closed.

Theorem 5: A(S,M) *is the union of the minimally* M-*closed subsets of* S.

Proof: Let T be a minimally M-closed subset of S, $y \in T$ and suppose, per absurdum, that $y \notin A(S,M)$. Then there is an $x \in T$ such that $x\hat{M}y$ and not $y\hat{M}x$. Hence it follows that the M-closed set $\{x\} \cup \{z \in S | z\hat{M}x\}$ is a proper subset of T, which is the desired contradiction.

Conversely, it is shown that if $y \in A(S,M)$ then y belongs to some minimally M-closed subset. Specifically the set $T \equiv \{y\} \cup \{x \in S | x\hat{M}y\}$. The set T is clearly M-closed. If for some $x \neq y$, $x\hat{M}y$ then $y\hat{M}x$ because $y \in A(S,M)$. Hence, any M-closed subset of T contains y and therefore T is minimally M-closed. Q.E.D.

The characterization of the admissible set given in Theorem 5 shows that the admissible set is a union of pairwise disjoint sets. The elements of each such set are symmetrically connected by the relation $\hat{M}$ but two elements belonging to two distinct sets are $\hat{M}$-incomparable.

3. Applications to Games in Characteristic Function Form

Example 1: Let V be an n-person cooperative game (without sidepayments). That is $N = \{1,2,\ldots,n\}$ is the set of players and for each $E \subset N$, $E \neq \emptyset$, V_E is a closed nonempty subset of R_+^N, the nonnegative orthant of the n-dimensional Euclidian space. It is further assumed that if $0 \leqq x \leqq_E y \in V_E$ then $x \in V_E$. (Inequalities in R^N are assumed to hold coordinatewise unless specified otherwise like in the self explanatory notation $\leqq_E$.) Finally, V_N is assumed to be compact.

In order to apply Theorem 2, let $S = V_N$ and M be the dominance relation, i.e. xMy if $x \in V_E$ for some $E \subset N$ and

$y <_E x$.

<u>Corollary 1</u>: $A(S,M)$ <u>is nonempty</u>.

<u>Proof</u>: It is easy to check that the domination relation is open (in the sense of Theorem 2) in the relative topology of V_N, thus by Theorem 2 $A(S,M) \neq \emptyset$. Q.E.D.

The next proposition shows that under some weak assumptions on the game the admissible set coincides with the core of the game whenever the latter is not empty.

<u>PROPOSITION 1</u>: <u>Let</u> V <u>be any game with a non-empty core and in which for every</u> $E \subset N$ V_E <u>contains a point with strictly positive coordinates</u>. <u>Then</u> $A(S,M)$ <u>coincides with the core of</u> V.

<u>Proof</u>: Recall that the core of V $C(V) = \{x \in V_N |$ for every $y \in V_N$ not $yMx\}$. Let $y \in V_N$ such that $y \notin C(V)$. If there exists a $z \in V_N$ such that zMy through a subset E of N for which $E \neq N$ then we proceed in the following way: There exists a positive number ϵ such that $(\epsilon,\epsilon,\ldots,\epsilon) \in (\bigcap_{i=i}^{n} V_{\{i\}}) \cap V_N$.

There exists a $u \in V_N$ such that uMy through E and $u^i < \epsilon$ for every $i \notin E$. There exists a $w \in V_N$ such that wMu through some $\{j\}$ where $j \in \overline{E} = N - E$ and for which $w^i < \epsilon$ for $i = 1,2,\ldots,n$. For every point $x \in C(V)$ we have xMw and thus $x\hat{M}y$. If the M-domination on y can be done only through N and wMy then we let $x = w + t(w-y)$ where $t = \max\{r: w+r(w-y) \in V_N\}$ thus x is on the boundary of V_N and all of its coordinates are greater than these of y. It follows that $x \in C(V)$ and that xMy. In either case we can find a point $x \in C(V)$ such that $x\hat{M}y$ thus $A(S,M) = C(V)$. Q.E.D.

A probabilistic analogy of this result was presented in [G] and later in [N]. It was shown there that if the transitions between payoffs in V_N are governed by a Markov process compatible with the relation M, then each trajectory converges to the core with probability one.

An interpretation of the relation M, relevant in this context, is that of qualitative probability. More specifically we distinguish only between probable and improbable events. xMy means that the transition from y to x is probable (has a positive quantitative probability) and not xMy means that the probability of transition from y to x is null. Proposition 1 shows that the qualitative approach suffices to obtain the convergence to the core via the dominance relation.

4. Applications to Games in Normal Form.

Let S be the set of n-lists of strategies in an n-person game in the normal form. $S = \overset{n}{\underset{i=1}{X}} S^i$ where S^i is the simplex of mixed strategies of player i. We can define various relations M on S, for the cooperative and non-cooperative cases, which yield new and old solution concepts.

Example 2: Define xMy if every player i for whom $x^i \neq y^i$ gets a higher payoff at x than at y. The admissible set A(S,M) is a new cooperative solution concept for n-person games in the normal form. (It contains, of course, the strong Nash equilibria.)

However, existence of A(S,M,) in this case is not implied by either Theorem 2 or 3 and it is still an open question. On the other hand A(S,M) may be very large for some games and may even coincide with S(for example, in the prisoner's dilemma game).

A natural way to extend the relation M is by considering its closure. More precisely, define the relation R on S by xRy if and only if $x \in$ closure $(\{z \in S \mid z\hat{M}x\} \cup \{y\})$. In this case R satisfies the conditions of Theorem 3 and $A(S,R) \neq \emptyset$. The proof of this fact is similar to the proof of corollary 2 which follows.

<u>Example 3</u>: xMy if for some player j $x^j \neq y^j$, $x^i = y^i$ for every $i \neq j$, and j^{th} payoff at x is higher than his payoff at y. In other words, we restrict the bargaining to be done non-cooperatively.

With this definition A(S,M) may still be too large. For example in the 2-person zero sum matching pennies game the admissible set is identical to all of S. This example, provided to us by R.J. Aumann, motivated the following definition of the relation R. xRy if and only if $x \in$ closure $(\{z \in S \mid z\hat{M}y\} \cup \{y\})$. With this new definition, the existence of the admissible set is assured and the admissible set exhibits interesting properties as demonstrated by the following corollary and proposition.

<u>Corollary 2</u>: <u>The relation</u> R <u>fulfills the conditions of Theorem</u> 3 (<u>Hence</u> $A(S,R) \neq \emptyset$).

Proof: The only nontrivial step of the proof is to show that R is transitive. Suppose that xRy and yRz. We want to show that xRz. Let 0_x be any neighborhood of x, we will show that the set $\hat{M}^{-1}(0_x) = \{w \in S | x'\hat{M}w$ for some $x' \in 0_x\}$ is a neighborhood of y. This would mean that every neighborhood of x contains a point which $\hat{M}$ dominates z, thus xRz and the proof would be completed.

It suffices to show that if $u\hat{M}y$ and 0_u is any neighborhood of u then $\hat{M}^{-1}(0_u)$ is a neighborhood of y. Since the M domination is done by a finite number of M dominations it suffices to show that if uMv and 0_u is any neighborhood of u then $M^{-1}(0_u) = \{w \in S | u'Mw$ for some $u' \in 0_u\}$ is a neighborhood of v.

So assume that uMv and 0_u is a neighborhood of u and assume further without loss of generality that the domination is done through player 1 (i.e. $u^i = v^i$ for $i \neq 1$ and 1^{th} payoff at u is greater than his payoff at v). It can be easily shown by the continuity of 1^{th} payoff that $M^{-1}(0_u \cap \{t \in S | t^1 = u^1\})$ contains a neighborhood of v. Q.E.D.

Proposition 2: *In example 3, A(S,R) contains the Nash equilibria and in the case of a 2-person zero sum game A(S,R) coincides with the Nash equilibria (minimax strategies).*

Proof: It is obvious that A(S,R) contains the Nash equilibria since for every equilibrium point x there is no $y \neq x$ such that yRx. For the second part of the propositon we assume that the game is a 2-person zero sum game and that (x_o, y_o) is any pair of mixed strategies. We will show that there is a pair (x_1,y_1) of minimax strategies such that $(x_1,y_1)R(x_o,y_o)$. Let (u,v) be a fixed pair of minimax strategies. If the payoff to player I at (x_o,y_o), $V(x_o,y_o)$, equals $V(u,v)$ then either (x_o,y_o) is a minimax pair or one of the players can improve by changing his strategy. This will result in a new point which R-dominates (x_o,y_o). So we assume without loss of generality that $V(x_o,y_o) > V(u,v)$. $V(x_o,v) \leq V(u,v)$ and if strict inequaltiy holds then the proof is completed because then $(u,v)M(x_o,v)M(x_o,y_o)$. So we assume that $V(x_o,v) = V(u,v)$. If (x_o,v) is an equilibrium point then we can stop and if not, then we can find a strategy y, close as we wish to v in which $V(x_o,y) < V(u,v)$. Now exchanging the roles of the two players we can either find an equilibrium point (x,y) such that $(x,y)R(x_o,y)$, or we find a point (x,y) for which x is as close to u as we wish and $V(x,y) > V(u,v)$. Continuing to exchange the roles of the two players we are either stopped by reaching an equilibrium point (x_1,y_1) which R-dominates (x_o,y_o), or we produce a sequence of points which converges to (u,v), and then $(u,v)R(x_o,y_o)$. In either case we find a minimax point which R-dominates (x_o,y_o). Q.E.D.

A somewhat different relation, L, on the space of mixed strategies, S, yields an admissible set which always coincides with the Nash equilibria. We define L in the following Example.

Example 4: For a given $x \in S$ we define the set of possible replies to x ($PR(x)$) as follows. Consider subsets of S, B, which satisfy the following three conditions.

(1) $B = \overset{n}{\underset{i=1}{X}} B^i$ where $B^i \subset S^i$ (i^{th} mixed strategies).

(2) $x \in B$

(3) If $y \in B$, $z^i \in S^i$ and player i^{th} payoff at $(y^1, \ldots, y^{i-1}, z^i, y^{i+1}, \ldots, y^n)$ is higher than his payoff at y then $z^i \in B^i$.

It is easy to check that intersections of sets satisfying (1), (2), and (3) also satisfy these conditions. Also S itself satisfies these three conditions thus we can and we do define the $PR(x)$ to be the minimal set satisfying (1), (2), and (3).

Intuitively one may think of a possible reply as a list of strategies of the n-player's were each player can rationalize his strategy by considering possible rationalizations of the other players. (A constructive way of defining $PR(x)$ which may be more intuitive, is given in the proof of the next lemma).

We define the relation L on S by $y \, L \, x$ if $y \in$ closure $(PR(X))$.

Proposition 3: For every game, $A(S,L)$ consists of precisely the Nash equilibrium strategies. Moreover for every $x \in S$ there is a Nash equilibrium point y such that $y \in PR(x)$.

Proof: We first observe that a point is an equilibrium point if and only if it is the only possible reply to itself or equivalently the only point that L dominates it is itself. Thus it suffices to prove the second part of the proposition.

Next we show that if $y \in S$ is an equilibrium point "relative to" $Lx = \{z \in S | z \; Lx\}$ (no player can improve his payoff by changing the point y to a point z in Lx) then y is an equilibrium point relative to all of S (or just an equilibrium point). If y is an equilibrium point relative to Lx and it is not an equilibrium point then there is a $z \in (S-Lx)$ and a player j such that $z^i = y^i$ for every $i \neq j$ and j^{th} payoff at z is higher than his payoff at y. But then for every neighborhood of z, O_z, the set $\{w \in S|$ for some $u \in O_z \; u^i = w^i$ for $i \neq j$ and j^{th} payoff at u is higher than at $w\}$ is a neighborhood of y. This implies that $z \in Lx$ which is a contradiction.

Now it suffices to show that Lx contains an equilibrium point relative to Lx. We proceed by the method outlined by Nash (1953). By the lemma that follows Lx is convex and compact. For every $y \in Lx$, let $\Phi(y) = \{z \in Lx | z$ is a best reply to y in $Lx\}$ where z is a best reply to y in Lx if for every player i i^{th} payoff at $(y^1,\ldots,y^{i-1}, \hat{z}^i, y^{i+1},\ldots y^n)$ is maximized, over Lx, at $\hat{z}^i = z^i$. It follows that for every $y \in Lx$, $\Phi(y)$ is convex, compact, and nonempty. Also $\Phi(y)$ is an upper semi-continuous correspondence so by Kakutani's fixed point theorem there is a $y \in Lx$ such that $y \in \Phi(y)$. But any point which is a best reply to itself must be an equilibrium point, Q.E.D.

Lemma 1: For every $x \in S$ $PR(x)$ is convex.

Proof: We use the following alternative way of defining $PR(x)$.

$$PR_1(x) = \mathop{X}_{i=1}^{n} PR_1^i\ (x) = \mathop{X}_{i=1}^{n}\{x^i\} = \{x\}$$

For j-1, 2, 3,...; for i=1,2,...,n

$$PR_{j+1}^i\ (x) = \{z^i \in S^i \mid \text{for some } y \in PR_j(x)\ i^{th} \text{ payoff at } (y^1,\dots,y^{i-1}, z^i, y^{i+1},\dots,y^n) \text{ is higher than his payoff at } y\}$$

$$PR_{j+1}(x) = PR_j(x) \cup \mathop{X}_{i=1}^{n} PR_{j+1}^i(x).$$

It is clear that $\bigcup_{j=1}^{\infty} PR_j(x)$ is a subset of $PR(x)$ and that it satisfies conditions (1),(2) and (3) in the definition of $PR(x)$, hence $PR(x) = \bigcup_{j=1}^{\infty} PR_j(x)$.

To show the convexity of $PR(x)$, it suffices to show the convexity of $PR^i(x)$ for i=1, 2,...,n (recall that $PR(x) = \mathop{X}_{i=1}^{n} PR^i(x)$) and therefore it suffices to show that:

For j=1, 2, 3,...

for i=1, 2,...n Convex Hull $(PR_j^i(x)) \subset PR^i(x)$.

We use induction on j. For j=1, the statement is trivial since $PR_1^i(x) = \{x^i\}$. In order to prove the statement for an integer $j > 1$ we make the following claim.

> If $y \in PR(x)$, $(y^i+\delta^i) \in S^i$ and i^{th} payoff at $(y^1,\ldots,y^{i-1}, y^i+\delta^i, y^{i+1},\ldots,y^n)$ is higher than his payoff at y then for every $z^i \in PR^i(x)$ and every $\lambda > 0$ whenever $z^i+\lambda\delta^i \in S^i$ $z^i+\lambda\delta^i \in PR^i(x)$.

We leave the proof of this claim to the end and first complete the inductive proof. Assuming that $j > 1$, i is any player, $y^i, z^i \in PR_j^i(x)$, α, $\beta > 0$ and $\alpha + \beta = 1$ we want to show that $\alpha y^i+\beta z^i \in PR^i(x)$. There are u and w in $PR_{j-1}(x)$ such that i^{th} payoff at $(u^1,\ldots,u^{i-1},y^i,u^{i+1},\ldots,u^n)$ is strictly better than his payoff at u and similarly for z^i and w.

$$\alpha y^i+\beta z^i = \alpha u^i+\beta w^i+ \alpha(y^i-u^i) + \beta(z^i-w^i).$$

By induction hypothesis $\alpha u^i+\beta w^i \in PR^i(x)$. Also $\alpha u^i+\beta w^i+\alpha(y^i-u^i) \in S^i$ because it equals $\alpha y^i+\beta w^i$.
So by our claim, with δ^i being y^i-u^i, λ being α, y being u, and z being $\alpha u^i+\beta w^i$, it follows that $\alpha u^i+\beta w^i+\alpha(y^i-u^i) \in PR^i(x)$.
Now applying the claim again, with δ^i being z^i-w^i, y^i being w^i, λ being β and z^i being $\alpha u^i+\beta w^i+\alpha(y^i-u^i)$ we conclude that $\alpha y^i+\beta z^i \in PR^i(x)$ which completes the inductive proof.

To prove our claim, we observe that i^{th} payoff at a vector of the type $(y^1,\ldots,y^{i-1},w^i,y^{i+1},\ldots,y^n)$ is $w^i \cdot V(y^1,\ldots,y^{i-1},y^{i+1},\ldots,y^n)$ where V is a vector which does not depend on y^i. Thus the hypothesis in the claim shows that $\delta^i \cdot V(y) > 0$ which implies that $\lambda\delta^i \cdot V(y) > 0$ for every $\lambda > 0$.

It follows that i^{th} payoff at $(y^1,\ldots,y^{i-1},z^i+\lambda\delta^i,y^{i+1},\ldots,y^n)$ is higher than his payoff at $(y^1,\ldots,y^{i-1},z^i,y^{i+1},\ldots,y^n)$ which implies that $z^i+\lambda\delta^i \varepsilon PR^i(x)$ and completes the proof of the claim, Q.E.D.

The complexity of the relation L points out that even under assumptions of costless communication among players, the solution concept of Nash equilibrium requires considerable computational ability from the players.

5. Concluding Remarks

Other solution concepts can be shown to coincide with the admissible set that arises from natural definitions of S and M. These include the α-core and the β-core for games in the normal form. For games with a continuum of players (with or without sidepayments) there are several ways of defining S and M so that we are assured of the existence of A(S,M) and so that A(S,M) contains the cores of such games.

Another example which differs from those mentioned or discussed previously is the Walrasian economy.

Example 5: Let S be the simplex of normalized price vectors in an exchange economy with ℓ commodities and let $f: S \rightarrow R^{\ell}$ be the excess demand function in this economy. Define pMq if: $p^i > q^i$ if and only if $f^i(q) > 0$ and $p^i < q^i$ if and only if $f^i(q) < 0$. Under standard assumptions on excess demand functions $q^i = 0$ implies that $f^i(q) > 0$ hence M is well defined. The admissible set A(S,M) contains all the Walras equilibria.

Defining the relation R by: xRy iff $x \in$ closure $(\{z \in S | z\hat{M}y\})$ leads to smaller admissible set, A(S,R), which may still contain cycles in addition to Walras equilibria. It seems an interesting problem to find a binary relation on S, derived from M, so that the corresponding admissible set will coincide with the Walras equilibria. (An analogue to Example 4 and Proposition 3 of Section 5.) This approach to stability may prove to be less restrictive than the classical stability analysis.