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# PITFALLS IN THE THEORY OF FAIRNESS

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## I. INTRODUCTION

Egalitarian allocations are in general Pareto-inefficient; likewise, fair (envy free) allocations that are Pareto efficient need not exist under the standard assumptions on the economic environment. Thus, if one's values conform to either egalitarianism or to freedom of envy an inescapable first-best equity-efficiency tradeoff has to be confronted. The concept of egalitarian-equivalent allocations recently advanced in [ 7 ] provides a possible way out of this tradeoff for egalitarian oriented societies. This paper is concerned with mitigating the first-best equity-efficiency tradeoff for value systems based on the desirability of freedom of envy.

## II THE THEORY OF FAIRNESS

An allocation is said to be fair if no person in the economy prefers anyone else's consumption bundle over his own (see Foley [2]). In other words, a fair allocation is free of envy. The individualistic nature of the fairness idea is attractive and spiritwise consistent with Paretian welfare economics. The appealing features of the fairness criterion from a distributional equity viewpoint are that it treats economic agents symmetrically (in an obvious sense), is ordinal in nature and is devoid of interpersonal welfare comparisons (since only intrapersonal utility comparisons are involved). It should be mentioned though that from the viewpoint of

political philosophy it is not entirely clear whether a concept of equity based on envy relationships can be morally acceptable (see Rawls [8], pp. 530-514). Be that as it may, the concept of fairness is certainly interesting enough to warrant investigation of its analytical properties.

From the viewpoint of welfare economics, the major drawback of the fairness criterion lies in its being inconsistent with the Pareto-efficiency principle. Specifically, due to Pazner and Schmeidler [5], it is now known that even under the classical convexity and selfishness assumptions on the economic environment, allocations that are both fair and Pareto-efficient will not always exist in economies with production. In light of the general acceptance of the Pareto criterion this presents a fundamental difficulty with the concept of fairness.

This fact prompted several attempts at modifying the concept of fairness in the hope of arriving at a reasonable equity criterion which would be consistent with Pareto-efficiency. I turn now to a critical evaluation of the attempts made so far.

## III EXISTING CONCEPTS OF FAIRNESS

(1) The first concept I wish to discuss is Varian's [10] interesting notion of <u>wealth-fair</u> allocations (called fair* in [9]). An allocation is said to be <u>wealth-fair</u> if no individual prefers the consumption-output bundle of anyone else over his own. In other words, a wealth fair allocation is envy-free in the sense that no person envies the complete position (defined as including the goods and leisure consumed <u>and</u> output produced) of any other person. As shown in [ 9 ], wealth-fair allocations are consistent with Pareto-efficiency under the standard assumptions on the economic environment.

The problem of course is that unless the production technology is additively separable in each agent's labor time inputs, it is impossible to impute output according to individual productivity. In other words, production processes are more often than not of such a joint nature as to make it impossible to disentangle individual contributions to total (observed) output. But in linear production economies for instance, the concept is well defined and the efficiency result of interest.

Confining our attention to those very special cases in which the wealth-fairness concept is well defined, it becomes of interest to discuss its normative significance. As noted by Varian himself [10], if one agent cannot possibly produce what another agent produces then no envy complaint can ever be raised by the first agent against the second. This example illustrates nicely the implicit sanctification of productivity (a morally irrelevant characteristic in itself)[1/] underlying the wealth-fairness criterion. In other words, it is first and foremost a "to each according to productivity" kind of slogan. It somewhat mitigates this near libertarian precept by requiring that those consumption commodities that are not resultant upon production should (loosely speaking) be divided in a fair manner (according to the original definition of fairness). In any case, ceteris paribus, this criterion penalizes the unable (in the productive sense). Horizontal equity (Musgrave's [3] suggestive term for the equal treatment of equals) is taken to mean under it that only persons which are identical both preferencewise and productivitywise will be treated equally. Persons who are identical only preferencewise on the other hand will be treated differentially to an extent entirely dependent upon their differential productivities. If it is

agreed that the subject matter of economic equity relates to preferences only (i.e. if it is agreed that the proper axiom of horizontal equity states that people with identical preferences ought to enjoy the same welfare level in the sense of being assigned a bundle lying on the same indifference surface)[2] then the wealth fairness concept ought to be rejected on ethical grounds. Also, it would seem that wealth-fair allocations can under no circumstances be rationalized in terms of hypothetical contractual agreements in Rawls' [8] original position.

(2) The second concept of interest is that of income-fairness suggested by Pazner and Schmeidler [6] and further discussed by Varian [9, 10]. An allocation is said to be income-fair if at the efficiency prices supporting this allocation the value of each person's consumption-cum-leisure bundle is equal. In other words, an income-fair allocation calls for the perfect equalization of potential-income (sometimes called implicit-income).

Unlike the wealth-fairness concept under which each person has a full ownership right over its "natural" endowment of time, under the income-fairness criterion each person is thus effectively assigned an equal property right (or share) in everybody's endowment of time (including one's own). The distribution of skills is thus viewed as a common pool of productive resources to be shared, in a sense, equally among all members of the society. Unlike the pure private good nature of individual productive skills implied by the wealth-fairness criterion (under which redistribution of the fruits of labor is prohibited), the income-fairness criterion is thus consistent with the general Rawlsian viewpoint.

As shown in [6], under the standard assumptions on the economic environment the income-fairness criterion is always consistent with Pareto-efficiency, a desirable property. There is, however, one major drawback with the income-fairness concept. It does not satisfy horizontal equity as defined above, i.e. at an income-fair allocation two persons with identical preferences will not in general be assigned bundles that lie on the same indifference surface whenever their productivities differ (see [6]; for a clear illustration of this fact see Musgrave [4], fn.11, pp. 630-631). In general, the more able a person, ceteris paribus, the more penalized he is relative to an unable one. Income-fair allocations therefore discriminate among people in a manner that is diametrically opposed to the discrimination taking place at wealth-fair allocations. In any event, as I consider the property of horizontal-equity to be of importance, its violation under the income-fairness criterion is troublesome in my opinion.

(3) The last existing concept of fairness is that recently suggested by Daniel who offers the following definitions. If the number of people who envy a person is equal to the number of people that he envies, then he will be said to be balanced with respect to envy at that allocation. An allocation is said to be balanced if everyone is balanced at it. He then goes on to prove that under the standard assumptions on the economic environment (to which a so called nondegeneracy assumption is added) there always exist Pareto-efficient allocations that are balanced. (see Daniel [1] some misattributions in which are rectified in Varian [11]).

Thus the balancedness criterion of fairness is consistent with the Pareto-efficiency requirement, a result of some interest in itself. The question,

however, is how normatively appealing is the balancedness requirement in the first place. Consider the extreme case of a large society in which every individual envies every other at some (Pareto-efficient) allocation. This allocation clearly satisfies the balancedness requirement. Yet, in such a large society what every individual can at most know is that he envies every other. There is really no way (if we want to be reasonable about it) for him to know that every other individual also envies him unless he is told so (say by an "ethical observer"). Thus the very spirit of the fairness idea (according to which each person can evaluate his position relative to others without any outside help) is lost. Furthermore, it is dubious whether the individual would feel much better even if he were told about this mysterious property of universal mutual envy. True it is a phenomenon that treats all agents symmetrically in a sense; but, it would seem to me that it would very likely give rise to the no less symmetric situation in which everybody is at everybody else's throat. Hardly a stable social situation and certainly not one in agreement with our intuitive vision of the good society. In brief, it is hard for me to take seriously the balancedness requirement as a plausible distributional equity criterion.

## IV NEW CONCEPTS OF FAIRNESS

In light of the above discussion, the question arises whether one could advance new concepts of fairness which are free of the difficulties associated with earlier attempts. I shall now present two new concepts, the second of which I believe to deserve special attention.

(1) In the light of the tremendous informational requirements in large societies placed upon any individual who has to perform the intrapersonal envy comparisons called for by all the previous fairness criteria [3/] one

could think it operationally meaningful to advance the following simplified fairness test. Let each person at any allocation determine whether he is better off with his bundle as compared to the average bundle in the economy. The informational requirements here are minimal in the sense that all that each agent has to know is his own bundle, the aggregate bundle [4] and the number of agents in the economy. An allocation at which no individual prefers the average bundle over his own will be said to be per capita-fair. In terms of the information likely to be (costlessly) available to any individual, this fairness requirement is very sensible.

Do per-capita-fair allocations exist? Of course. Simply give to each person an identical (egalitarian) consumption-cum-leisure bundle.[5] Will this egalitarian bundle be Pareto-efficient? Generally not. In fact, the major difficulty with the concept of per-capita-fair allocations is that, like the original fairness criterion, it is inconsistent with the Pareto-efficiency requirement. To see that, observe simply that in any standard two-person production economy in which no fair and Pareto-efficient allocation exists, there can exist no per-capita-fair and Pareto-efficient allocation. For, under convex preferences, the fact that at every Pareto-efficient allocation in such an economy at least one of the individuals envies the other implies that the average bundle is also preferred to the bundle that he has. So no Pareto-efficient and per-capita-fair allocation can exist in such an economy and this simple counterexample is enough to rule out the interesting notion of per-capita-fairness on grounds of inconsistency with the Pareto principle.

(2) The second novel fairness concept which comes to mind is motivated by the concept of egalitarian-equivalence recently introduced by Pazner and

Schmeidler [7]. An allocation will be said to be fair-equivalent if there exists a fair allocation in some hypothetical economy in which each person enjoys the same welfare level as that enjoyed by him at the allocation under consideration. In other words, an allocation is fair-equivalent if and only if its underlying welfare distribution could have been generated by a fair allocation in some hypothetical economy.[6/] If such an allocation exists and if our conception of the "good society" is that of an envy-free society, the normative significance of any fair-equivalent allocation derives from the fact that each person is indifferent between living in the actual economy (at the said allocation) and living in the "good" (envy-free) reference economy.

Regarding the question of whether or not the fairness-equivalence criterion is consistent with the Pareto-efficiency principle recall first the following definition. An allocation is said to be egalitarian-equivalent if its underlying welfare distribution could have been generated by an egalitarian economy. Clearly, any egalitarian-equivalent allocation is also fair-equivalent. As shown in [7], Pareto-efficient and egalitarian-equivalent allocations always exist under (even weaker than) the standard assumptions on either exchange or production economies. This establishes the consistency of the fairness-equivalence requirement with the Pareto-efficiency criterion, an important property. No less important, horizontal equity is also satisfied.

It clearly is the case that the set of fair-equivalent allocations will always contain the set of egalitarian-equivalent allocations. Therefore, the restriction imposed on the Pareto-set by the egalitarian-equivalence criterion is more discriminating (i.e. rules out more Pareto-efficient allocations as being normatively admissible on equity grounds). This might be considered by some as a distinct advantage of this criterion. But if one's view of

the good society is that of freedom of envy, the fairness equivalence criterion will seem more basic. And since those who are egalitarian minded can always restrict their attention to the egalitarian-equivalent subset of the set of fair-equivalent allocations, I think it wisest to let the reader decide for himself which of these two concepts (if any) better conforms to his own values. The fact that they both satisfy the horizontal equity requirement is in my mind an important property when comparing either of them with the modified fairness criteria discussed in Section III.

Like the modified fairness criteria discussed in Section III the two equivalence criteria are however much more informationally demanding than the original fairness criterion. The appealing simplicity with which each individual could perform his envy comparisons under the original notion of fairness is thus lost. In this sense, it would seem that the original theory of fairness may have reached a dead end. From the viewpoint of the omniscient-planner approach to Paretian welfare economics though, the two equivalence criteria seem to offer a promising avenue out of what I like to think of as being the disturbing problem of the first-best equity-efficiency tradeoff.

## MATHEMATICAL APPENDIX TO SECTION IV

Since the concepts and propositions of this section are new, the following mathematical treatment intends to make precise the ideas and statements presented in the text. The notation is consistent with that in [7].

Let $T$ denote the finite set of economic agents and let $R^{\ell}_{+}$ denote the non-negative orthant of the Euclidean space of dimension $\ell$, the set of consumption commodity bundles. Each $t$ in $T$ has a preference relation $\succsim_t$ on $R^{\ell}_{+}$ which is assumed to be <u>strongly connected</u>, <u>transitive</u>, <u>continuous</u>, <u>monotonic</u> and <u>convex</u> (i.e., for all $x$, $y$, $z$ in $R^{\ell}_{+}$ the following hold:

$x \succsim_t y$ or $y \succsim_t x$; $x \succsim_t y$ and $y \succsim_t z$ imply $x \succsim_t z$; the sets $\{x' \in R^{\ell}_{+} \mid x' \succsim_t x\}$ and $\{x' \in R^{\ell}_{+} \mid x \succsim_t x'\}$ are closed in $R^{\ell}_{+}$; $x \succ_t y$ implies $\lambda x + (1-\lambda)y \succ_t y$ for $0 < \lambda < 1$; and $x > y$ implies $x \succ_t y$ where inequalities between vectors in $R^{\ell}_{+}$ hold coordinatewise by definition, and the relations $\succ_t$ and $\sim_t$ are induced by $\succsim_t$ in the usual way).

The set of technologically feasible production plans is denoted by $K$, $(K \subset R^{\ell})$. As usual, if $z \in K$ then the negative coordinates of $z$ denote inputs and the positive coordinates of $z$ denote outputs. Let $W = \{\omega + K\} \cap R^{\ell}_{+}$, where $\omega \in R^{\ell}_{+}$ is an aggregate initial commodity vector, denote the feasible aggregate final consumption vectors. It is assumed that $W$ is a compact and convex set with nonempty interior. Free disposal is also assumed, i.e. $x \leqq y \in W$ implies $x \in W$.

The <u>economy</u> is formally defined as the vector $(T, R^{\ell}_{+}, \{\succsim_t\}_{t \in T}, W)$. An <u>allocation</u> is a $T$-list of elements of $R^{\ell}_{+}$ whose sum belongs to $W$. An

assignment is a T-list of elements of $R_+^\ell$ (whose sum does not necessarily belong to W). In other words, while an allocation is a feasible T-list of consumption (cum-leisure) bundles, an assignment is a T-list of such bundles which is not necessarily feasible. An assignment is denoted by $\{z_t\}_{t\in T}$ or simply $\{z_t\}$. An allocation $\{x_t\}$ is Pareto-efficient if for any other allocation $\{y_t\}$ the implication: $(\forall\, t\in T, y_t \succsim_t x_t) \Rightarrow (\forall\, t\in T, y_t \sim_t x_t)$ holds. An allocation $\{x_t\}$ is fair if for all t and t' in T: $x_t \succsim_t x_{t'}$.

Definition 1: An allocation $\{x_t\}$ is said to be per-capita-fair if for all t in T: $x_t \succsim_t \bar{x}$, where $\bar{x} = \frac{\Sigma_{t\in T} x_t}{|T|}$.

Proposition 1: There exist economies as defined above in which no Pareto-efficient allocation is per-capita-fair.

Proof: By counterexample. Consider a two-person economy as in [5] where no Pareto-efficient allocation is fair. At any Pareto-efficient allocation $\{x_1, x_2\}$ in such an economy, either $x_2 \succ_1 x_1$ and/or $x_1 \succ_2 x_2$. Suppose without loss of generality that $x_2 \succ_1 x_1$. By the convexity of preferences

$$\bar{x} = \frac{x_1 + x_2}{2} \succ_1 x_1.$$

Q.E.D.

Definition 2: An allocation $\{x_t\}$ is said to be fair-equivalent if there exists an assignment $\{z_t\}$ such that for all t and t' in T: $x_t \sim_t z_t$ and $z_t \succsim_t z_{t'}$.

Thus a fair-equivalent allocation $\{x_t\}$ is agentwise indifferent to a fair allocation in an economy in which the assignment $\{z_t\}$ is feasible.

Proposition 2: In an economy as defined above there always exist Pareto-efficient allocations that are fair-equivalent.

For the proof of Proposition 2 we need a definition and a lemma.

Definition 3: An allocation $\{x_t\}$ is said to be egalitarian-equivalent if there is a bundle $z$ in $R^{\ell}_{+}$ such that for all $t$ in $T$, $x_t \sim_t z$.

Lemma: If an allocation $\{x_t\}$ is egalitarian-equivalent then it is fair-equivalent.

Proof: By the definition of an egalitarian-equivalent allocation, there exists a constant assignment $\{z\}_{t\in T}$ such that $x_t \sim z$, for all $t$ in $T$. Since $z \gtrsim_t z$ for all $t$ in $T$, $\{x_t\}$ is fair-equivalent.

Q.E.D.

Proof of Proposition 2: As shown in [7], in an economy as defined above there always exist Pareto-efficient allocations that are egalitarian-equivalent. By the Lemma, Proposition 2 then follows.

Q.E.D.

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