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PROPORTIONAL SOLUTIONS TO BARGAINING SITUATIONS: INTERPERSONAL UTILITY COMPARISONS

by

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Abstract

A bargaining situation is described by a set of alternatives which are feasible to n-individuals when they do cooperate, and an alternative which comes about when they do not cooperate. The paper addresses the question of which cooperative outcome will be chosen. A Nash-type approach is used to prove that, under plausible axioms describing the underlying bargaining process, the individuals must be doing interpersonal comparison of utility. The model and the solution overcome some difficulties recently described by Nydegger and Owen.

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1. Introduction

Given n fixed individuals we consider the bargaining situations that they may encounter. We assume that such a situation is described by two components: the set of outcomes acheivable for them when they cooperate, and an outcome which occurs when they do not cooperate. In social choice problems the second component may also be thought of as the status quo situation. We do not deal with situations in which cooperation among subcoalitions may be feasible (n-person games instead of bargaining situations), however for the case n=2 the two concepts coincide.

The approach that we take is the same as in [Nash,1950] (see also [Luce & Raiffa,1957] and [Owen,1968] for a general discussion). We consider two principles (axioms) about the outcome of the bargaining process and we show that each of these principles is sufficient to imply that the players must be doing interpersonal comparison of utility among themselves.

The solution suggested here is different from the interpersonal utility comparison in Harsanyi's solution (see [Harsanyi, 1955]) as follows. Harsanyi shows that after suitable normalization the players will choose the outcome which maximizes the sum of the Von-Neumann Morgenstern utilities of the individuals. Our solution suggests that after the suitable normalization of the utilities, the players will maximize their utilities subject to the restriction that they all gain "equally" in the given situation. This is related to Rawls' approach in A Theory of Justice as we discuss in the last section of the paper.

The first principle that we consider is one of monotonicity (see [Owen,1968]). It says that if additional options were made available to the individuals in a given situation then no one of them should lose utility because of the availability of these new options. Thus this is a principle of justice. However it may also be viewed as a bargaining principle since a player who is asked to lose utility because of the new options may have a very convincing case in threatening to break cooperation. Theorem 1 states that under the principle of monotonicity, and other standard axioms, the only possible solutions are the ones which employ interpersonal utility comparisons.

A second principle that we investigate is one of negotiation by stages. It restricts the set of solutions to ones which are invariant under decomposition of the bargaining process into stages. Thus if the individuals consider first a subset of the set of feasible alternatives, reach an agreement on the subset, use the agreed point as a threat point (the noncooperative outcome) for a second stage where they would consider the remaining alternatives, and then reach a final agreement, then this final outcome should be the same as the outcome reached in one step (for a more detailed example from Economics see section 3).

This principle is observed in actual negotiations (e.g. Kissinger's step-by-step) and it is attractive since it makes the implementation of a solution easier. It is also attractive because we can view every bargaining situation that we encounter in life as a first step in a sequence of predictable or unpredictable bargaining situation that may still arise. Thus the outcome of the current bargaining situation will be the threat point for the future ones. Theorem 2 states that solutions which satisfy the condition of bargaining by stages must be of the interpersonal comparison of utility type.

Finally Theorem 3 shows that while the Nash condition of independence of irrelevant alternatives (see [Nash 1950]) and a condition of individual monotonicity [see [Kalai-Smorodinsky,1975] and [Rosenthal,1976]) are incompatible in the usual Nash model they are compatible in our model and together again lead to interpersonal comparison of utility.

We defer further discussion, motivation and criticism to the last section of the paper.

2. The Model and Definitions

We let 1,2,...,n represent n fixed individuals. We let R^n denote the n-dimensional Euclidean space, and R^n_+ is the non-negative orthant of R^n. A 0-normalized bargaining game (a game for short) of these n individuals is a set S, subset of R^n_+, which satisfies the following three conditions.

1. S is convex and compact.
2. S is comprehensive, i.e. if $x \in S$, $y \in R^n_+$ and $y \leq x$ then $y \in S$.
3. There is an $x \in S$ such that $x^i > 0$ for $i = 1,2,\ldots,n$.

The points in S represent the feasible Von-Neumann Morgenstern utility levels that the n individuals can reach simultaneously in a particular bargaining situation, i.e. $x = (x^1, X^2, \ldots, x^n) \in S$ if and only if there is a possible outcome in this bargaining situation which would assign individual 1 a utility level x^1, would assign 2 a utility level x^2, etc. If the individuals fail to agree on an outcome in S then they each receive a zero level of utility, and in this sense the game is 0-normalized.

The convexity assumption is made because we assume that randomization on different outcomes is possible (most of the results can be proved without this assumption). The comprehensiveness assumption is made because we assume that individuals can freely dispose of utility, and assumption 3 is simply that bargaining can prove worthwhile to all the players. It is also assumed implicitly in the model ($S \subset R^n_+$) that outcomes which are less favorable to one of the individuals than the noncooperative outcome can be disregarded.

We let β denote the set of all games of the n individuals. We assume that in all the games in β each individual is using the same multiplicity scale for his utility function. Thus a gain of one unit of utility for individual 1 in one game is as significant for him as a gain of one unit in a different game.

For a game S and a point $x \in S$ we say that x is (weakly) Pareto optimal if there is no $y \in S$ such that $y^i > x^i$ for $i = 1,2,\ldots,n$. x is strongly Pareto optimal if there is no $y \in S$ such that $y^i \geq x^i$ for $i = 1,2,\ldots,n$, and $y \neq x$.

A solution is a function $\mu : \beta \to R^n_+$ which satisfies the following conditions for every $S \in \beta$.

1. $\mu(S) \in S$ and is Pareto optimal.
2. Homogeneity, for every $c > 0$ $\quad \mu(cS) = c\mu(S)$.
3. Strong individual rationality, $\quad (\mu(S))^i > 0$ for $i = 1,2,\ldots,n$

Conditions 1 and 3 are self-explanatory. Condition 2 can be justified in different ways. First it is a very weak version of Nash's axiom in "The Bargaining Problem." Nash required that even when one applies a different linear transformation to every utility function of the different players the solution should be invarient. Here we make the same requirement but only for the cases when all the different utility functions are changed linearly by the same factor.

A second way to justify this condition is through the expected utility interpretation (recall that these are Von-Neumann Morgenstern utility functions). Imagine a bargaining situation given by a set S. A different bargaining situation T is defined as follows. With probability c ($0 \leq c \leq 1$) all the cooperative feasible outcomes in S are available and with probability 1-c no cooperative outcome is possible, thus the outcome is 0. If the players agree that $\mu(S)$ is the outcome whenever S is possible it follows that the expected utility level for the randomized game is $c\mu(S) + (1-c)0 = c\mu(S)$. The expected feasible utility combinations in the randomized game T is given by $T = cS$. Thus it is reasonable to require that $\mu(cS) = c\mu(S)$ for $0 \leq c \leq 1$. If $T = cS$ and $c > 1$ the same argument can be made by viewing S as a randomized game resulting from T (S is the lottery between the cooperative outcomes of T with probability $\frac{1}{c}$ and the non-cooperative outcome 0 with probability $\frac{c-1}{c}$). Thus $\mu(S) = \frac{1}{c}\,\mu(T) = \frac{1}{c}\,\mu(cS)$ or $c\mu(S) = \mu(cS)$.

A final argument for the condition of homogeneity is that the outcome of the bargaining should depend on the shape of the feasible set S. Since cS is just a c-factor blow-up of the shape of S it is reasonable to require that the solution of S, $\mu(S)$, should carry over to the corresponding point in the blown up picture, $c\mu(S)$.

It may be interesting to note that the condition of homogeneity is not necessary for theorem 2 (because step-by-step negotiation implies homogeneity). Also weaker versions of it would have sufficed for theorems 1 and 3. But since we feel that it is a natural condition and in order to keep the exposition simple we leave it in the definition of a solution.

3. Main Results

We say that a solution μ is proportional if there are strictly positive constants $p^1, p^2, \ldots, p^n$ such that for every $S \in \beta$

$\mu(S) = \lambda(S)p$ where $p = (p^1, p^2, \ldots, p^n)$

and $\lambda(S) = \max\{t : tp \in S\}$.

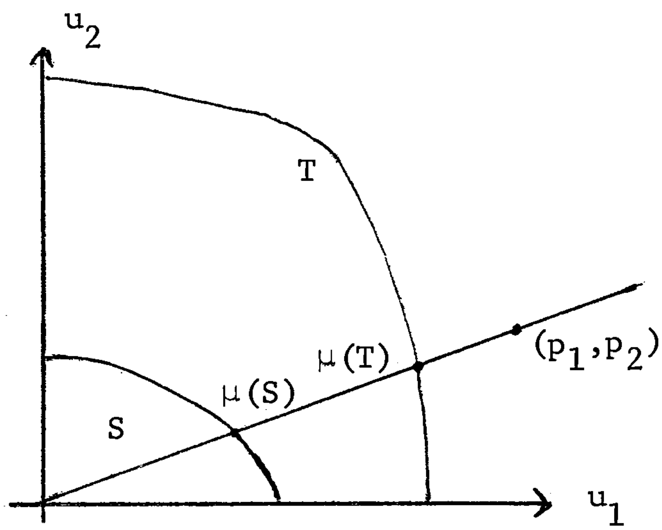

Fig. 1. a proportional solution for two individuals

We say that a solution μ satisfies the axiom of monotonicity (or μ is monotonic) if it satisfies the following condition. Let T and S be bargaining games, if $T \supset S$ then for $i = 1,2,\ldots,n$ $(\mu(S))^i \leq (\mu(T))^i$.

Theorem 1. A solution is monotonic if and only if it is proportional.

Proof. Clearly every proportional solution is monotonic. Assume that μ is any monotonic solution. Let $\Delta = \{x \in R^n_+ : \sum_{i=1}^{n} x^i \leq 1\}$, let $p = \mu(\Delta)$ and for every $S \in \beta$ let $\lambda(S) = \max\{t : tp \in S\}$. We will show that for every $S \in \beta$ the following three conditions are true.

1. $\mu(S) \geq \lambda(S)p$.
2. If $\lambda(S)p$ is strongly Pareto optimal then $\mu(S) = \lambda(S)p$.
3. $\mu(S) \leq \lambda(S)p$.

Since (1) and (3) imply that μ is a proportional solution with the weights in p the proof will be completed.

For every $\varepsilon > 0$ such that $\varepsilon < \min\{1-p^i : 1 \leqq i \leqq n\}$ let q_i be the point defined by $q_i^j = 0$ if $j \neq i$ and $q_i^i = p^i + \varepsilon$. Let V_ε be the game defined by $V_\varepsilon = \text{conv. hull } (\{0,p,q_1,q_2,\ldots,q_n\})$. Notice that $V_\varepsilon \subset \Delta$ therefore $\mu(V_\varepsilon) \leq \mu(\Delta) = p$. Since p is strongly Pareto optimal in V_ε this implies that $\mu(V_\varepsilon) = p$ (if not then $\mu(V_\varepsilon)$ must have for one coordinate $\mu(V_\varepsilon)^i > p^i = (\mu(\Delta))^i$ contradicting monotonicity). To see that condition (1) is true observe that by the comprehensiveness of S, for every $\delta < 1$ there exists an ε, $0 < \varepsilon < \min\{1-p^i : 1 \leqq i \leqq n\}$, for which $\delta\lambda(S)V_\varepsilon \subset S$. Therefore, by the monotonicity and homogeneity of μ, $\mu(S) \geqq \mu(\delta\lambda(S)V_\varepsilon) \geqq \delta\lambda(S)p$ and (1) follows. Condition (2) follows from (1) by the definition of strong Pareto optimality. To show that condition (3) is true let δ be any positive number which is greater than 1. Let S_δ be the game generated by $S \cup \{\delta\lambda(S)P\}$, i.e.

$$S_\delta = \{x \in R_+^n : \text{ for some } y \in \text{ conv. hull } (S \cup \{\delta\lambda(S)p\}), x \leqq y\}.$$

$S_\delta \supset S$, $\lambda(S_\delta) = \delta\lambda(S)$ and $\delta\lambda(S)$ is strongly Pareto optimal in S_δ. Hence by condition (2) $\mu(S_\delta) = \delta\lambda(S)p$, by monotonicity $\mu(S) \leq \delta\lambda(S)p$ and since δ was arbitrary $\mu(S) \leqq \lambda(S)p$. Q.E.D.

A solution satisfies the axiom of step-by-step negotiations if whenever $U, S \in \beta$, $U \subset S$, and $(S-\mu(U)) \cap R_+^n \in \beta$ then $\mu(S) = \mu(U) + \mu((S-\mu(U)) \cap R_+^n)$. Notice that $(S-\mu(U)) \cap R_+^n$ are the individually rational options that are left after $\mu(U)$ was agreed upon as a first step. One has to be careful in the interpretation of this condition. This is a condition on the feasible utility levels and not directly on the underlying game. For example, suppose the game under consideration involves the division of certain quantities of two commodities A and B among two individuals (no agreement between the individuals results in both individuals receiving zero quantities of both commodities). This game may be decomposed into two stages in a very natural way. In step I the individuals agree on the division of commodity A. In step II the division of commodity B is determined. However,the division of B is allowed to involve reallocations of commodity A. Our intuitive notion of a process which involves bargaining by stages is in the following two requirements. (i) if the individuals fail to come to an agreement in the second stage then the final outcome of step I will be the final outcome of the whole game. In this sense the outcome of the first step is the threat point for the second step. (ii) if an outcome for step II is reached then this outcome should increase the utility levels which the individuals obtained at the end of step I. This is because step II is a new bargaining game and the cooperation of both individuals is required for a successful outcome. The axiom of step-by-step negotiation is that our scheme should be such that it can be implemented by such two stages i.e. playing the two games described above should lead to the same outcome as playing the game in one shot or in any other decomposition to stages (e.g. first divide B and then A).

Theorem 2. A solution satisfies the step-by-step negotiation condition if and only if it is proportional.

Proof. It is easy to see that every proportional solution satisfies the step-by-step negotiation condition. So it suffices to show, by theorem 1, that every solution which satisfies the step-by-step condition is monotonic. Suppose that $U \subset S$. For every $0 < \delta < 1$ the game $(S - \mu(\delta U)) \cap R^n_+ \in \beta$ and hence, by strong individual rationality, step-by-step negotiation implies that $\mu(S) \geq \mu(\delta U) = \delta\mu(U)$. Since δ can be chosen arbitrarily close to 1 it follows that $\mu(S) \geq \mu(U)$. Q.E.D.

Notice that the assumption of homogeneity in the definition of a solution is not necessary for theorem 2 since either one of the conditions, step-by-step negotiations, or proportionality imply homogeneity.

A solution satisfies the condition of independence of irrelevant alternatives (IIA) if whenever U, $S \in \beta$, $U \subset S$ and $\mu(S) \in U$ then $\mu(U) = \mu(S)$.

For $i = 1,2,\ldots,n$ we let $R^{n|i} = \{x \in R^n : x^i = 0\}$. A solution satisfies the individual monotonicity (IM) condition if whenever U, $S \in \beta$ and for some i $U \cap R^{n|i} = S \cap R^{n|i}$, then $S \supset U$ implies that $(\mu(U))^i \leqq (\mu(S))^i$.

Continuity means that if $\{S_j\}_{j=1}^{\infty}$ is a sequence of games and S is a game such that $S_j \to S$ (in the Hausdorff topology) then $\mu(S_j) \to \mu(S)$.

Theorem 3. A solution satisfies the conditions of IIA, IM and continuity if and only if it is proportional.

Sketch of Proof. It is easy to see that any proportional solution satisfies the three conditions of the theorem. Now let μ be any solution satisfying IIA, IM and continuity. Again let $\Delta = \{x \in R^n_+ : \sum_{i=1}^n x^i \leqq 1\}$, let $p = \mu(\Delta)$ and for every $S \in \beta$ let $\lambda(S) = \sup\{t : tp \in S\}$. Let S be any game, we will show that $\mu(S) = \lambda(S)p$. Assume first that S has the property that $\lambda(S)p$ is strongly Pareto optimal in S. Let $C = \{x \in R^n_+ :$ for $i = 1,2,\ldots,n,\ \sum_{j=1}^n x^j - x^i \leqq 1\}$. By homogeneity and IIA $\mu(S \cap \lambda(S)\Delta) = \lambda(S)p$. By IM $\mu(S \cap \lambda(S)C))^i \geqq (\mu(S \cap \lambda(S)\Delta))^i$ for $i = 1,2,\ldots,n$ and since $\lambda(S)p$ is strongly Pareto optimal in $S \cap \lambda(S)\Delta$ it follows that

$$\mu(S \cap \lambda(S)C) = \lambda(S)p.$$

Now by IIA, by the fact that $\lambda(S)p$ is interior to $\lambda(S)C$, and by continuity it follows that $\mu(S) = \lambda(S)p$. If $\lambda(S)p$ is not strongly Pareto optimal in S then we can estimate S by a sequence of games S_j for which $\lambda(S_j)p$ is strongly Pareto optimal in S_j and the conclusion, that $\lambda(S)p = \mu(S)$, follows by continuity. Q.E.D.

4. Discussion

The results presented in section 3 are descriptive and not applicative in their nature. They show that given fixed n individuals with fixed scales utility functions they will compromise in different bargaining situations so as to keep their proportions of utility gains fixed. A more difficult problem is to find what these proportions should be. Either they can be found from games which have been played, or we can use the results of section 3 to make this problem easier. These results imply that it suffices to find the proportions for a "simple" game and these same proportions will be kept (as long as the players are using the same multiplicative scale for their utility functions) for more complicated games. For example if we can find out how these n players would divide a dollar among themselves then we would know how they should compromise in all other situations. (Obviously in our approach we are ignoring the very difficult questions of misrepresentation of utility functions by individuals.)

Not using Nash's axiom of independence of scale of utility, is the main difference between Nash's approach and ours. We feel that Nash's axiom is too strong and that although it overcomes the difficulty of the indeterminancy of the scale of the utility functions it implies other undesirable conclusions as we describe by the following example. (See [Nydegger-Owen, 1975] for further discussion and experimental results.) We consider the following two 2-person games. Individuals 1 and 2 have linear utility for money and they are given one hundred chips to divide among themselves. In the first game each player can cash in each chip for one dollar. In the second game player 1 can cash in each chip for 3 dollars while player 2 can still cash in each chip for 1 dollar. Nash's independence of scale of utility axiom implies that the two players should divide the chips in the same way in these two different situations. Thus if their division in the first game was 50 chips to player 1 and 50 chips to 2, then the same division should hold in the second game. The Nydegger-Owen experiment implies that while the players would divide the chips 50-50 in the first situation, they would divide them 25-75 to players 1 and 2 respectively in the second situation giving each player 75 dollars. This is precisely in agreement with the model described here. The experimental result can be strengthened by the following argument. If a division of 50-50 is proposed, player 2 would threaten not to cooperate pointing out that he is bound to lose 50 dollars while player 1 will lose 150 dollars.

Our solution also overcomes the difficulty of the indeterminacy of the scale of the utility functions because it leaves the agreement proportions of the n individuals as open parameters. Thus if the individuals were to agree on proportions $(p^1,p^2,\ldots,p^n)$ and in a different representation player 1 were to use a different scale, then the proportions should be adjusted accordingly. For example if player 1 uses a utility function which equals twice his original utility function then the agreement proportions should be $(2p^1,p^2,\ldots,p^n)$ and the same solution will be obtained under the two different representations.

It would be very natural to normalize the utility functions of the individuals so that <u>in a symmetric bargaining situation they would compromise on an outcome that would assign them all an equal amount of utility gains</u>. If such a normalization were made then the proportional solution suggested in this paper is to give all individuals an equal amount of utility gains in every bargaining situation.

A natural question that arises as a result of this study is whether one can use the approach presented here in order to find solutions to n-person cooperative games without sidepayments (see [Aumann 1967]) . In other words is any one of the conditions described in the theorems of section 3 sufficient to define uniquely a reasonable family of solutions to cooperative games when coalitions other than the grand coalition can also gain by being formed.

A final remark which may be of interest is on the connection between the solution presented here for bargaining situation and John Rawls (1971) max-min approach to the theory of justice. We just point out that if one was to define max-min social welfare function on R^n_+ by $x > y$ if and only if $\min_{1 \leq i \leq n} x^i > \min_{1 \leq i \leq n} y^i$ then the most prefered outcome from any set of alternatives S which are feasible to a society (relative to the starting position 0) is the proportional solution suggested in this paper. We believe that this intuitive observation can be made into a more formal one to show that an implication in the other direction is also true, i.e. proportional solutions to bargaining situations imply that social welfare functions on R^n_+ should be of the max-min type.

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