Capital Structure and the Value of the Firm

by

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Abstract. The paper studies the relationship between the capital structure of two firms within a single risk class and their market values. Two primary results are derived. First, a change in one firm's capital structure generally does not alter the two firms' relative market values if and only if capital markets are perfect. Second, a change in one firm's capital structure generally alters the two firms' absolute market values if (a) capital markets are imperfect or (b) the firm's change in capital structure creates a new type of bond which had not previously been traded on the market.

Capital Structure and the Value of the Firm*

1. Introduction

In their 1958 article Modigliani and Miller [8] showed that if two firms are in the same risk class and in an economy with a perfect capital market having no transaction costs, taxes, and bankruptcy costs, then their relative market values are independent of their capital structures. This result, which conflicted with the conventional wisdom, has spawned a large theoretical literature that extends, praises, criticizes, and modifies their original results. Two groups within this literature are important with respect to this paper.

The first group includes papers of Stiglitz [12], Smith [11], Baron [1], Hagen [3], and Milne [7]. These papers, all of which have retained the perfect capital market assumption have defined with increasing precision the circumstances under which the Modigliani-Miller result is valid. In particular Hagen [3] and Milne [7] have shown that changes in capital structure may change the market values of all firms within a risk class. Thus, while a change in leverage does not alter the equality of the market values of firms within a risk class, it may alter their absolute values.

The second group includes the papers of Robichek and Myers [10] and Kraus and Litzenberger [5]. These papers relax the

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*The comments of David Baron are gratefully acknowledged. The errors that remain, however, are our own.

prefect market assumption and show that the tax deductability of interest and the costs that are associated with defaulting on contractual payments may cause firms in the same risk class to have unequal values. The important assumption of this second group of papers is that real costs are associated with defaulting on contractually agreed debt repayment schedules. They show that this assumption implies that a tradeoff exists between the expected tax savings and the expected bankruptcy costs which increased leverage brings. The existence of this tradeoff in turn implies that an optimal leverage exists at which the firm's market value is maximized.

Both of these groups of papers undermine the usual interpretation of Modigliani-Miller's result which states that a firm's market value is independent of its capital structure. In this paper we bring these two strands of the literature together into a unified treatment. Our purpose therefore is to analyze the effect of leverage under different market structures and specify the conditions under which a change in leverage affects (a) the absolute, but not the relative, values of firms within a risk class and (b) the absolute and relative values of firms within a risk class. Our results fall into two groups. The first group, which appears in section four and is an extension of Milne [7], and Hagen [3], is based on the assumption of perfect capital markets where transactions costs, taxes, and bankruptcy costs do not exist. These theorems show that perfect capital markets are a sufficient condition for the relative market values of firms within a risk class to be independent of leverage. They also show that perfect capital markets are not generally a sufficient condition for the absolute market value of firms within a risk class to be independent of leverage. In particular, if a change of leverage alters the variety of debt securities traded on the market, then generally the absolute market values, but not the relative market values, of all firms within the risk class change. The second group, which appears in section five, drops the perfect capital makret assumption and shows that changes in capital structure may change both the absolute and relative market values of firms within a risk class.

In order to emphasize the commonality that underlies these two groups of results, we have designed our proof to bring out those fundamental aspects of market equilibria on which all of the theorems depend. The proofs rely on showing that any securities market which contains levered securities is equivalent to a conceptually much simpler market that does not contain levered securities. Once this equivalence is formally established within the proof of Theorem 2, then proofs of the substantive results contained in Theorems 2 through 5 immediately follow essentially as corollaries of this equivalence. In addition, all of our results are framed within a simple multi-period model that shows more clearly than does the standard two-period model the effects which leverage, default on interest payments, and bankruptcy costs has on a securities payment stream and hence on its market value.

2. Formulation

Our model analyses the equilibrium prices within a securities market that contains five securities: a riskless asset, two distinct types of bonds, firm number one's unlevered stock, and firm number two's levered stock. Firms one and two are both assumed to belong to the same risk class. Our goal is to compare the equilibrium market values which firms one and two achieve in two contrasting situations. The securities are labeled as follows:

$\mathcal{A}$: the riskless security,

$\mathcal{B}$: bonds of the first type,

$\mathcal{B}^*$: bonds of the second type,

$\mathcal{S}$: firm one's unlevered stock,

$\mathcal{T}$: firm two's stock when it is levered by one unit of type $\mathcal{B}$ bonds,

$\mathcal{T}^*$: firm two's stock when it is levered by one unit of type $\mathcal{B}^*$ bonds.

In the initial situation firm two's stock is of type $\mathcal{T}$, i.e., it is levered by one unit of type $\mathcal{B}$ bonds. In the final situation firm two's stock is of type $\mathcal{T}^*$, i.e., it is levered by one unit of type $\mathcal{B}^*$ bonds.

Let the equilibrium values of the various securities in the initial situation be as follows:

1 = value of one unit of riskless security $\mathcal{A}$,

D = value of one unit of bond $\mathcal{B}$,

D^* = value of one unit of bond $\mathcal{B}^*$,

V_1 = value of firm one's stock $\mathcal{S}$,

V_2 = total value of firm two, which equals the value of its stocks $\mathcal{T}$ plus the value of one unit of bond $\mathcal{B}$.

Since V_2 represents the total value of firm two, V_2-D represents the value of its stock alone. In the final situation the value of each security is represented by the same symbol with the addition of a prime. Thus $V_2'-D^{*\prime}$ represents the value of firm two's stock $\mathcal{J}^*$ in the final situation. Similarly, V_1' represents the value of firm one's stock $\mathcal{J}$ in the final situation. Given this notation our two basic questions may be stated as follows. First, what conditions are necessary to guarantee that $V_1=V_2$ and $V_1'=V_2'$? Second, what conditions are necessary to guarantee that $V_1=V_1'$ and $V_2=V_2'$?

Further specification of our model requires introduction of a standard notation. Any variable with a bar above it, such as $\overline{Y}$ or $\overline{Q}$, is an infinite dimensional row vector. Functions with bars such as $\overline{B}(\cdot)$, $\overline{C}(\cdot)$, and $\overline{X}(\cdot)$ have as their images infinite dimensional row vectors. Let $\Omega = R \times R \times R \times \ldots$ and let $\Omega^+ = R^+ \times R^+ \times R^+ \times \ldots$ where R is the set of real numbers and R^+ is the set of non-negative real numbers. Let $\overline{C} = (C_1,C_2,C_3,\ldots,C_t,\ldots)$ and $\overline{X} = (X_1,X_2,\ldots,X_t,\ldots)$. The vector inequality $\overline{C} \leq \overline{X}$ means that, for all $t = 1,2,3,\ldots,C_t \leq X_t$. Inequalities may also be used with vector valued functions. For example, if $\overline{X}(\overline{Y}) = (X_1,X_2,X_3,\ldots,X_t,\ldots)$ and $\overline{C}(\overline{Q},\overline{Y}) = (C_1,C_2,C_3,\ldots,C_t,\ldots)$, then $\overline{X}(\overline{Y}) \geq \overline{C}(\overline{Q},\overline{Y})$ means that, for all $t = 1,2,3,\ldots,X_t \geq C_t$. Finally, $\overline{0}$ is the infinite dimensional vector of zeros.

Each firm has a stock of physical assets which produce income streams over all future periods. Since both firms are in the same risk class, both will have identical income streams over all future periods.[1] Let this stream of future net operating income be the random vector $\overline{Y} = (Y_1,Y_2,Y_3,\ldots,Y_t,\ldots) \in \Omega$. The vector is defined on Ω and not Ω^+ because the firm's net operating income may be negative during some future periods. Consequently negative components of $\overline{Y}$ are admissible. Stockholders and bondholders, however, do have limited liability. Therefore let the function $\overline{X}(\overline{Y}) = (X_1,X_2,X_3,\ldots,X_t,\ldots)$ be the actual amount of net operating income which firms one and two each has available to pay as dividends to its stockholders and as interest to its bondholders. This function has three properties: (a) it is non-negative in all of its components; (b) if a component Y_t of $\overline{Y} = (Y_1,Y_2,Y_3,\ldots)$ is greater than zero, then $Y_t \geq X_t$; and (c) it is non-decreasing in each component of its argument $\overline{Y}$. Property (a) follows directly from the limited liability of stockholders and bondholders. Property (b) is a consequence of the possibility that in some period τ preceding the period t in question a negative net operating income Y_τ may be realized. This means that the firm will have some obligations to suppliers and other short term creditors which must be retired using the income from periods subsequent to τ. Therefore, in period t, $X_t < Y_t$ is a possibility. Property (c) states that increased operating income can not lead to reduced payments to bondholders and stockholders in any period.

We assume a 100% payout ratio for both firms. Therefore for firm one $\overline{X}(\overline{Y})$ is the stream of dividends which it pays its stockholders. For firm two $\overline{X}(\overline{Y})$ is the amount which it has available to pay both its bondholders and its stockholders. Let $\overline{Q} = (Q_1,Q_2,Q_3,\ldots,Q_t,\ldots) \in \Omega^+$ be the stream of payments which firm two is obligated contractually to pay its bondholders in the initial situation when it is levered by one unit of bonds $\mathcal{B}$. Similarly, let $\overline{Q}^* = (Q_1^*,Q_2^*,Q_3^*,\ldots,Q_t^*,\ldots)$ be the stream of payments which it is obligated contractually to pay its bondholders in the final situation when it is levered by one unit of bonds $\mathcal{B}^*$. The streams $\overline{Q}$ and $\overline{Q}^*$ might represent bonds of different term structure and aggregate amount. For example, $\overline{Q}$ could be a ten year bond with a nominal interest rate of 8% and a face value of \$1,000,000. In that case $Q_\tau = \$80{,}000$ for $\tau = 1,2,\ldots,9$, $Q_{10} = \$1{,}080{,}000$, and $Q_\tau = 0$ for $\tau > 10$. $\overline{Q}^*$ might then be a twenty-five year bond with a nominal interest rate of 8.7% and a face value of \$2,000,000. Therefore $Q^*_\tau = \$174{,}000$ for $\tau = 1,2,\ldots,24$, $Q^*_{25} = \$2{,}174{,}000$, and $Q^*_\tau = 0$ for $\tau > 25$.

The vectors $\overline{Q}$ and $\overline{Q}^*$ are streams of planned payments, not actual payments. For example, if, in any period t, $Q_t > X_t$, then firm two defaults on its payments to its bondholders and incurs an obligation to pay the defaulted amount as soon as possible out of its income in period t + 1 and, if necessary, subsequent periods. Therefore we define $\overline{C}$ to be the rescheduling function which calculates the firm's actual payments to its bondholders: $\overline{C}(\overline{Q},\overline{Y}) = (C_1,C_2,C_3,\ldots,C_t,\ldots)$ where $\overline{Q}$ is the bond's scheduled stream of payments and C_t is the amount the owners of one unit of the bond actually receive in period t. This function has three properties:

a. for all $\overline{Q}\in\Omega^+$ and all $\overline{Y}\in\Omega$, $\overline{C}(\overline{Q},\overline{Y}) \geq \overline{0}$;

b. for all $\overline{Q}\in\Omega^+$ and all $\overline{Y}\in\Omega$ such that $\overline{X}(\overline{Y}) \geq \overline{Q}$, $\overline{C}(\overline{Q},\overline{Y}) = \overline{Q}$;

c. for all $\overline{Q}\in\Omega^+$ and all $\overline{Y}\in\Omega$, $\overline{X}(\overline{Y}) \geq \overline{C}(\overline{Q},\overline{Y})$.

Property (a) states that payments to bondholders cannot be negative, property (b) states that if income is sufficient, then bondholders are paid in full, and property (c) states that payments to bondholders cannot exceed income. These three properties are respectively implied by bondholders' lack of liability for the firm's debts, the contractual nature of the firm's payments to bondholders, and the assumed 100% payout ratio. They do not completely specify the function $\overline{C}(\overline{Q},\overline{Y})$, but any function satisfying all three is suitable for our purposes. For example, $\overline{C}(\overline{Q},\overline{Y})$ might or might not be specified to require that firm two pay interest on any bond payments it temporarily defaults.

The assumption of 100% payout and the definition of $\overline{C}(\overline{Q},\overline{Y})$ together imply that firm two's stream of dividends to its stockholders will be $\overline{X}(\overline{Y}) - \overline{C}(\overline{Q},\overline{Y})$. This formulation allows firm two to default on its scheduled bond payments $\overline{Q}$. Nevertheless this formulation does not allow firm two to be forced into bankruptcy and out of existence. This assures that both firms one and two earn and distribute to their stockholders and bondholders identical income streams, $\overline{X}(\overline{Y})$. In section five we introduce the concept of bankruptcy costs and thus allow for the case of firm two going bankrupt and ceasing to exist as a result of its leverage.[2]

The other income producing asset, in addition to $\mathscr{S}$, $\mathscr{T}$, $\mathscr{T}^*$, $\mathscr{B}$, and $\mathscr{B}^*$, which investors can own is $\mathscr{A}$, the riskless asset. It pays interest $r \geq 0$ for all future periods. Let $\overline{r}$ be the vector $(r,r,r,\ldots,r,\ldots) \in \Omega^+$. Thus if investor i invests a_i dollars in $\mathscr{A}$, he then receives the income stream $a_i\overline{r} = (a_ir, a_ir, a_ir,\ldots,a_ir,\ldots)$ in perpetuity.

From among those securities which are traded on the market each investor i chooses that portfolio $P_i = (a_i,b_i,c_i,d_i,e_i)$ which maximizes his expected utility subject to his budget constraint.[3] If in the initial situation the five securities $\mathscr{A}$, $\mathscr{B}$, $\mathscr{B}^*$, $\mathscr{S}$, and $\mathscr{T}$ are traded on the market, then the constraint is

$$a_i + b_i D + c_i D^* + d_i V_1 + e_i (V_2 - D) \leq B_i \tag{2.1}$$

where a_i is the amount of riskless asset $\mathcal{A}$ he purchases, b_i is the number of units of bonds $\mathcal{B}$ he purchases, c_i is the number of units of bonds $\mathcal{B}^*$ he purchases, d_i is the proportion of firm one's stock $\mathcal{S}$ he purchases, e_i is the proportion of firm two's stock $\mathcal{T}$ he purchases, and B_i is his endowment of wealth. If, as is the case in some of our theorems, security $\mathcal{B}^*$ is not traded in the initial situation, then c_i is set identically equal to zero for every investor. In the final situation, where the five securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, and $\mathcal{T}^*$ are traded on the market, the budget constraint becomes

$$a_i + b_i D' + c_i D^{*\prime} + d_i V_1' + e_i (V_2' - D^{*\prime}) \leq B_i \ . \tag{2.2}$$

Investors are not allowed to borrow at the riskless rate of interest or to sell stocks short. Nevertheless he can borrow on the same terms as firm two; in this case either $b_i < 0$ or $c_i < 0$. Therefore his portfolio choice must obey non-negativity constraints $a_i \geq 0$, $d_i \geq 0$, $e_i \geq 0$ while b_i and c_i are unrestricted as to sign.

If in the initial situation the five securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$ and $\mathcal{T}$ are traded, then the income stream which investor i receives from portfolio P_i is

$$\overline{I}(P_i,\overline{Y}) = a_i\overline{r} + b_i\overline{C}(\overline{Q},\overline{Y}) + c_i\overline{C}(\overline{Q}^*,\overline{Y}) + d_i\overline{X}(\overline{Y})$$
$$+ e_i[\overline{X}(\overline{Y}) - \overline{C}(\overline{Q},\overline{Y})]. \tag{2.3}$$

But if the security $\mathcal{B}^*$ is not traded, then c_i is set equal to zero. In the final situation when securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$ and $\mathcal{T}^*$ are traded his income stream is

$$\overline{I}(P_i,\overline{Y}) = a_i\overline{r} + b_i\overline{C}(\overline{Q},\overline{Y}) + c_i\overline{C}(\overline{Q}^*,\overline{Y}) + d_i\overline{X}(\overline{Y})$$

$$+ e_i[\overline{X}(\overline{Y}) - \overline{C}(\overline{Q}^*,\overline{Y})]. \tag{2.4}$$

Let $U_i(\overline{I})$ be his multiperiod von Neumann-Morgenstern utility function for the income stream $\overline{I}$ and let $f_i(\overline{Y})$ be the continuous subjective probability density function that describes his beliefs concerning the uncertain future values of $\overline{Y}$. We assume that U_i and f_i are both everywhere differentiable, that U_i is strictly increasing in all of its arguments $\overline{I}$, and that U_i is strictly concave and thus risk averse everywhere. Moreover we assume that the collection of investor utility functions and probability distributions $\{(U_1,f_1),(U_2,f_2),\ldots,(U_n,f_n)\}$ satisfy sufficient conditions for the existence of a competitive equilibrium.[4]

The expected utility of portfolio P_i for investor i is therefore

$$\mathcal{U}(P_i) = \int U[\overline{I}(P_i,\overline{Y})]\ f(\overline{Y})d\overline{Y}. \tag{2.5}$$

We assume that, for all finite portfolios P_i, $\mathcal{U}(P_i)$ is finite. Investor i prefers portfolio P_i to portfolio P_i' if $\mathcal{U}(P_i) > \mathcal{U}(P_i')$ and is indifferent if $\mathcal{U}(P_i) = \mathcal{U}(P_i')$. We represent preference and indifference between P_i and P_i' respectively by $P_i \succ_i P_i'$ and $P_i \sim_i P_i'$.

Each investor is assumed to be a price taker. The price vector $(1,D,D^*,V_1,V_2)$ is a competitive equilibrium for the initial situation if and only if a set of portfolios $\hat{P}_i = (\hat{a}_i,\hat{b}_i,\hat{c}_i,\hat{d}_i,\hat{e}_i)$ exist which, given these prices, are optimal for each investor i and which clear the market. The conditions for market clearance are: $\sum_i \hat{b}_i = 1$, $\sum_i \hat{c}_i = 0$, $\sum_i \hat{d}_i = 1$, and $\sum_i \hat{e}_i = 1$. If security $\mathcal{B}^*$ is not traded then recall that c_i is set equal to zero for every investor. Similarly, let the price vector $(1,D',D^{*\prime},V_1',V_2')$ be a competitive equilibrium for the final situation. The market clearance conditions for the final situation are: $\sum_i \hat{b}_i = 0$, $\sum_i \hat{c}_i = 1$, $\sum_i \hat{d}_i = 1$, and $\sum_i \hat{e}_i = 1$. The condition for $\sum_i \hat{b}_i$ is 1 in the initial situation and 0 in the final situation because firm two issues one unit of bonds $\mathcal{B}$ in the initial situation but not in the final situation. Nevertheless in the final situation investors are allowed to issue and trade bonds of type $\mathcal{B}$ themselves. Thus in the final situation the requirement is that net sales and net purchases cancel out. Identical reasoning applies to the conditions for $\sum_i \hat{c}_i$. No market clearing condition is appropriate for the riskless security $\mathcal{a}$ because investors can purchase it in any quantity at the exogenous price of one. We assume that an equilibrium set of prices does exist.

3. Basic Securities and Basic Economies

In this section we present a model of an idealized securities market which, as the succeeding sections show, mirrors the behavior of a more realistic securities market. In this idealized market only four securities exist. Three of these securities were defined in the previous section: $\mathcal{A}$, the riskless security; $\mathcal{B}$, the debt security with payment stream $\overline{C}(\overline{Q},\overline{Y})$ per unit; and $\mathcal{B}^*$, the debt security with payment stream $\overline{C}(\overline{Q}^*,\overline{Y})$ per unit. Let $\mathcal{S}$ be the pure equity security with value V and with payment stream $\overline{X}(\overline{Y})$ per unit. Note that one unit of $\mathcal{S}$ is equivalent to the stock $\mathcal{J}$ of firm one. In general, when all four secruties are traded on the market, each investor invests in a portfolio $\theta_i = (\alpha_i,\beta_i,\gamma_i,\delta_i)$ where α_i, β_i, γ_i, and δ_i respectively represent the number of units of securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, and $\mathcal{S}$. He chooses that portfolio θ_i which maximizes the expected utility of the income stream.

$$\overline{I}(\theta_i,\overline{Y}) = \alpha_i\overline{r} + \beta_i\,\overline{C}(\overline{Q},\overline{Y}) + \gamma_i\,\overline{C}(\overline{Q}^*,\overline{Y}) + \delta_i\,\overline{X}(\overline{Y}) \tag{3.1}$$

subject to the budget constraint

$$B_i \geq \alpha_i + \beta_i D + \gamma_i D^* + \delta_i V \tag{3.2}$$

and the non-negativity constraints $\alpha_i \geq 0$ and $\delta_i \geq 0$. Investors can hold either positive or negative quantities of $\mathcal{B}$ and $\mathcal{B}^*$. Let $\hat{\theta}_i = (\hat{\alpha}_i,\hat{\beta}_i,\hat{\gamma}_i,\hat{\delta}_i)$ be that portfolio which is optimal for investor i given prices of unity, D, D*, and V respectively for securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, and $\mathcal{S}$.

We assume that each of these four securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, and $\mathcal{S}$ generates an income stream which is not a linear combination of the income streams of the other three securities, i.e. no vector $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ exists such that, for all $\overline{Y} \in \Omega$, $\eta_1 \overline{r} + \eta_2 \overline{C}(\overline{Q}, \overline{Y}) + \eta_3 \overline{C}(\overline{Q}^*, \overline{Y}) + \eta_4 \overline{X}(\overline{Y}) = \overline{0}$ where $\overline{0}$ is the infinite dimensional vector of zeros. Such a set of securities with linearly independent income streams is called a set of basic securities. A composite security is a security which pays an income stream that is a linear combination of income streams from basic securities. For example, security $\mathcal{J}^*$, the stock of a firm that is levered by the debt security $\mathcal{B}^*$, is a composite security because its income stream of $\overline{X}(\overline{Y}) - \overline{C}(\overline{Q}^*, \overline{Y})$ is the difference of the income streams of the basic securities $\mathcal{S}$ and $\mathcal{B}^*$. These definitions of basic and composite securities are an adaptation of the "spanning" concept which Ekern and Wilson [2] introduced and which others including Radner [9] and Leland [6] have used profitably.

Within the framework of this model we can define two basic economies whose equilibrium properties play a critical role in the proofs of Theorems 2, 3, and 4.

Basic Economy Ψ. Investors trade securities $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{S}$. They do not trade $\mathcal{B}^*$, i.e. for all investors i, $\hat{\gamma}_i = 0$. The supplies of $\mathcal{B}$ and $\mathcal{S}$ are respectively zero units and two units. $\mathcal{A}$ is traded at the exogenous price of one. Therefore the market clearing conditions are: $\sum_i \hat{\beta}_i = 0$ and $\sum_i \hat{\delta}_i = 2$. Let D_Ψ and V_Ψ be the equilibrium prices of $\mathcal{B}$ and $\mathcal{S}$.

Basic Economy χ. Investors trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, and $\mathcal{S}$. The supplies of $\mathcal{B}$, $\mathcal{B}^*$ and $\mathcal{S}$ are respectively zero units, zero units, and two units. $\mathcal{A}$ is traded at the exogenous price of one. Therefore the market clearing conditions are: $\sum_i \hat{\beta}_i = 0$, $\sum_i \hat{\gamma}_i = 0$, and $\sum_i \hat{\delta}_i = 2$. Let D_χ, D^*_χ, and V_χ be the equilibrium prices of $\mathcal{B}$, $\mathcal{B}^*$, and $\mathcal{S}$.

The difference between basic economies Ψ and χ is that in Ψ security $\mathcal{B}^*$ has not been invented and is not yet traded but in χ security $\mathcal{B}^*$ is traded. Observation 1 points out that the invention of a new security may change the equilibrium prices of all securities.

Observation 1. In general, $V_\chi \neq V_\Psi$ and $D_\chi \neq D_\Psi$. Only in special circumstances will $V_\chi = V_\Psi$ and $D_\chi = D_\Psi$.

This observation follows from the elementary fact that a change in the variety of commodities available for trade within an economy normally alters the equilibrium prices of all remaining commodities.

4. Perfect Capital Markets

This section develops a theory appropriate for capital markets which operate costlessly and without distortion. We define a perfect capital market to mean that (a) individuals and firms can borrow funds on identical terms, (b) an alteration of either a firm's or an investor's financial structure causes no change in the total income stream available to investors in the stocks of firms, investors in the bonds of firms, and investors in the bonds of indicidual investors, and (c) no transaction costs exist. This definition of perfect capital markets needs elaboration. The condition that the investor can borrow on the same terms as the firm means that he and the firm pay the same interest rate and have identical privileges to reschedule payments depending on the realized earnings of the firm. For example, suppose firm two issues a bond which sells on the open market at price D and which contracts to pay the bond's purchasers an income stream $\overline{Q}$. Its actual payments are $\overline{C}(\overline{Q},\overline{Y})$, a function of the firm's realized income. Now suppose an investor wishes to borrow the amount $D' = \theta D$ where θ is a positive scalar. Our definition implies that if the individual can borrow on the same terms as firm two, then he can sell a bond at the price $D' = \theta D$. His contractual payments are then $\overline{Q} = \theta\overline{Q}$ and his actual payments are $\theta\overline{C}(\overline{Q},\overline{Y})$. Thus the investor's repayments on this debt depend not on his personal income, but on the income of the firms in which he may invest.[5]

The second condition that the total income stream available to equity and debt holders be invariant with changes in financial structure is most easily understood by specifying explicitly the type of income stream which is admissible under this condition. Let $\hat{P}_i = (\hat{a}_i, \hat{b}_i, \hat{c}_i, \hat{d}_i, \hat{e}_i)$ be the optimal portfolio held by investor i. If this second condition for perfect capital markets is met, then the income stream from this portfolio is

$$\overline{I}(\hat{P}_i, \overline{Y}) = \hat{a}_i \overline{r} + \hat{b}_i\ \overline{C}(\overline{Q}, \overline{Y}) + \hat{c}_i\ \overline{C}(\overline{Q}^*, \overline{Y})$$
$$+ \hat{d}_i\ \overline{X}(\overline{Y}) + \hat{e}_i [\overline{X}(\overline{Y}) - \overline{C}(\overline{Q}, \overline{Y})] \tag{4.1}$$

Specifically excluded from (4.1) are deductions or additions to the income stream that represent real payments which are made to or received from third parties and which vary as the financial structure of the firm varies, such as taxes and bankruptcy costs. This case is considered in section five.

We prove four theorems in this section. Theorem 1 is our analogue to Proposition 1 of Modigliani and Miller [8]: if capital markets are perfect, then firms one and two have equal market values regardless of the presence of default risk and regardless of firm two's leverage. This theorem is true because, as many authors have shown before, homemade leverage and corporate leverage are perfect substitutes within perfect capital markets. In Theorem 1, however, nothing is said about how the absolute market values of firms one and two vary when firm two changes its leverage. In Theorem 2 we show that if firm two's change in leverage from $\overline{Q}$ to $\overline{Q}^*$ does not alter the variety of bonds which investors are trading on the market, then the change in leverage does not affect the values of the two firms. In Theorem 3 we show that the assumption of an unchanged variety of bonds is critical. If the change in leverage of firm two does alter the variety of bonds which investors are trading on the market, then the change in leverage may affect the absolute values of the two firms. Similar results have been derived by Milne [7] and Hagen [3]. The section ends with Theorem 4. It demonstrates that in the presence of default risk even a simple change in the fact value of the bond issued by firm two, while leaving constant its term structure and nominal interest rate, may cause a change in the absolute values of firms one and two.

> Theorem 1 (Modigliani and Miller). If capital markets are perfect, then then market value of firm one equals the market value of firm two, i.e. $V_1 = V_2$ and $V_1' = V_2'$.

Proof. Consider, without any loss of generality, in economy where securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{J}$, and $\mathcal{T}$ are traded. Let the equilibrium prices of these securities be respectively 1, D, D*, V_1, and (V_2-D). Suppose that $V_1 > V_2$. Since V_1 is an equilibrium price, $\sum_i \hat{d}_i = 1$ where $\hat{d}_i$ is the optimal holdings of $\mathcal{J}$ by investor i. Pick an arbitrary investor j for whom $\hat{d}_j > 0$. Such an investor must exist because $\sum_i \hat{d}_i = 1$.

Let investor j sell his $\hat{d}_j$ units of $\mathcal{J}$ at market price V_1 and with the proceeds buy $\hat{d}_j(V_1/V_2)$ units of $\mathcal{T}$ and $\hat{d}_j(V_1/V_2)$ units of $\mathcal{B}$. This transaction does not violate his budget constraint because

$$\hat{d}_j \, V_1 = \hat{d}_j \, \frac{V_1}{V_2} \, (V_2-D) + \hat{d}_j \, \frac{V_1}{V_2} \, D. \tag{4.2}$$

Label his new portfolio $\tilde{P}_j$. We now compare his income stream from the two portfolios:

$$\bar{I}(\hat{P}_j,\bar{Y}) = \hat{a}_j\bar{r} + \hat{b}_j\ \bar{C}(\bar{Q},\bar{Y}) + \hat{c}_j\ \bar{C}(\bar{Q}^*,Y) + \hat{d}_j\ \bar{X}(\bar{Y}) + \hat{e}_j\ [\bar{X}(\bar{Y}) - \bar{C}(\bar{Q},\bar{Y})]$$

$$= \hat{a}_j\bar{r} + (\hat{b}_j-\hat{e}_j)\ \bar{C}(\bar{Q},\bar{Y}) + \hat{c}_j\bar{C}(\bar{Q}^*,\bar{Y}) + (\hat{d}_j+\hat{e}_j)\bar{X}(\bar{Y}); \quad (4.3)$$

$$\bar{I}(\tilde{P}_j,\bar{Y}) = \hat{a}_j\bar{r} + (\hat{b}_j+\hat{d}_j\ \frac{V_1}{V_2})\ \bar{C}(\bar{Q},\bar{Y}) + \hat{c}_j\bar{C}(\bar{Q}^*,\bar{Y}) + (\hat{e}_j+\hat{d}_j\ \frac{V_1}{V_2})\ [\bar{X}(\bar{Y}) - \bar{C}(\bar{Q},\bar{Y})]$$

$$= \hat{a}_j\bar{r} + (\hat{b}_j-\hat{e}_j)\ \bar{C}(\bar{Q},\bar{Y}) + \hat{c}_j\bar{C}(\bar{Q}^*,\bar{Y}) + (\hat{e}_j+\hat{d}_j\ \frac{V_1}{V_2})\ \bar{X}(\bar{Y}); \quad (4.4)$$

$$\bar{I}(\tilde{P}_j,\bar{Y}) - \bar{I}(\hat{P}_j,\bar{Y}) = \hat{d}_j(\frac{V_1}{V_2} - 1)\ \bar{X}(\bar{Y}) \quad (4.5)$$

Since $\hat{d}_j > 0$, $V_1/V_2 > 1$, $\bar{X}(\bar{Y}) \geq \bar{0}$ for all $\bar{Y}\in\Omega$ and $\bar{X}(\bar{Y}) > \bar{0}$ for some $\bar{Y}\in\Omega$, it follows that $\bar{I}(\tilde{P}_j,\bar{Y}) \geq \bar{I}(\hat{P}_j,\bar{Y})$ for all $\bar{Y}\in\Omega$ and that $\bar{I}(\tilde{P}_j,\bar{Y}) > \bar{I}(\hat{P}_j,\bar{Y})$ for some $\bar{Y}\in\Omega$. In other words, portfolio $\tilde{P}_j$ produces an income stream for investor j which dominates that income stream which portfolio P_j produces. Therefore, because his utility function $U_j(\bar{I})$ is strictly increasing in all components of $\bar{I}$, he strictly prefers $\tilde{P}_j$ over $\hat{P}_j$. This, however, contradicts our original assumption that $\hat{P}_j$ is his optimal portfolio.

Therefore, if $V_1 > V_2$, then no investor who holds a positive quantity of $\mathcal{S}$ is in equilibrium. Hence, V_1 can not be an equilibrium price since some investors must hold $\mathcal{S}$ in equilibrium. In an exactly analogous manner we can verify that $V_1 < V_2$ implies that no investor who holds positive quantities of $\mathcal{T}$ is in equilibrium. Therefore, a necessary condition for equilibrium is that $V_1 = V_2$. Similarly it can be shown that $V_1' = V_2'$. Q.E.D.

Theorem 1 implies that in perfect capital markets firms one and two have equal market values. In Theorem 2 we show that if investors trade both types of debt $\mathcal{B}$ and $\mathcal{B}^*$ in both the initial and final situations, then the change in the leverage of firm two from $\mathcal{B}$ to $\mathcal{B}^*$ affects neither the relative nor absolute values of firms one and two. To do this, we compare the market values attained by the two firms before and after the change in firm two's leverage. Let the initial situation where firm two is levered by one unit of $\mathcal{B}$ be called economy K. Let the final situation where firm two is levered by one unit of $\mathcal{B}^*$ be called economy K'. The market clearing conditions for economy K are: $\Sigma b_i = 1$, $\Sigma c_i = 0$, $\Sigma d_i = 1$, and $\Sigma e_i = 1$. The market clearing conditions for economy K' are: $\Sigma b_i = 0$, $\Sigma c_i = 1$, $\Sigma d_i = 1$, and $\Sigma e_i = 1$. Let the equilibrium values which firms one and two attain be $V_K = V_1 = V_2$ in economy K and $V_K' = V_1' = V_2'$ in economy K'. We show that economies K and K' are analogues of the basic economy χ which we defined in the preceding section. This means that an equilibrium price V_χ for $\mathcal{S}$ in economy χ is an equilibrium price for firms one and two in economies K and K'. Therefore, in equilibrium $V_K = V_\chi = V_K'$, the hypothesis which we seek toprove.

Theorem 2. Suppose capital markets are perfect. If in the initial situation investors trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{J}$, and $\mathcal{T}$ and in the final situation they trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{J}$, and $\mathcal{T}^*$, then the market values of firms one and two do not change between the initial and final situation, i.e. $V_1 = V_2 = V_1' = V_2'$.

Proof. Recall that V_χ, D_χ, and D^*_χ are equilibrium price for basic economy χ. Set $V_K = V_\chi$, $D_K = D_\chi$, and $D^*_K = D^*_\chi$. We now show that an equilibrium with these prices exists for economy K given that economy χ is in equilibrium. Within economy χ every investor i has an optimal portfolio $\hat{\theta}_i = (\hat{\alpha}_i, \hat{\beta}_i, \hat{\gamma}_i, \hat{\delta}_i)$ which, when aggregated with all other investors' optimal portfolios, results in market clearance, i.e. $\sum_i \hat{\beta}_i = 0$, $\sum_i \hat{\gamma}_i = 0$, and $\sum_i \hat{\delta}_i = 2$. The portfolio $\hat{\theta}_i$ generates the income stream.

$$\overline{I}^{\chi}(\hat{\theta}_i, \overline{Y}) = \hat{\alpha}_i \overline{r} + \hat{\beta}_i \overline{C}(\overline{Q}, \overline{Y}) + \hat{\gamma}_i \overline{C}(\overline{Q}^*, \overline{Y}) + \hat{\delta}_i \overline{X}(\overline{Y}). \qquad (4.6)$$

Investor i can generate an identical stream in economy K with portfolio $\tilde{P}_i = (\tilde{a}_i, \tilde{b}_i, \tilde{c}_i, \tilde{d}_i, \tilde{e}_i)$ where $\tilde{a}_i = \hat{\alpha}_i$, $\tilde{b}_i = \hat{\beta}_i + \frac{1}{2}\hat{\delta}_i$, $\tilde{c}_i = \hat{\gamma}_i$, $\tilde{d}_i = \frac{1}{2}\hat{\delta}_i$, and $\tilde{e}_i = \frac{1}{2}\hat{\delta}_i$. This stream is:

$$\overline{I}^{K}(\tilde{P}_i, \overline{Y}) = \tilde{a}_i r + \tilde{b}_i \overline{C}(\overline{Q}, \overline{Y}) + \tilde{c}_i \overline{C}(\overline{Q}^*, \overline{Y})$$

$$+ \tilde{d}_i \overline{X}(\overline{Y}) + \tilde{e}_i [\overline{X}(\overline{Y}) - \overline{C}(\overline{Q}, \overline{Y})]$$

$$= I^{\chi}(\hat{\theta}_i, \overline{Y}). \qquad (4.7)$$

Note that since we assumed $V_K = V_\chi$, $D_K = D_\chi$, and $D^*_K = D^*_\chi$ and since $\hat{\theta}_i$ satisfies investor i's budget constraint, then $\tilde{P}_i$ also satisfies investor i's budget constraint in economy K. Moreover, as substitution shows, the set of portfolios $\{\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_i, \ldots\}$ satisfy the market clearance conditions for economy K.

For economy K to be in equilibrium, given that the market clearing conditions are satisfied by the portfolios $\{\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_i, \ldots\}$, $\tilde{P}_i$ must be an optimal portfolio for each investor i. Suppose that

$\tilde{P}_i$ is not an optimal portfolio for an investor i in economy K when $V_K = V_\chi$, $D_K = D_\chi$, and $D_K^* = D_\chi^*$. Consequently another portfolio $\hat{P}_i$ exists which is optimal. Since $\hat{P}_i$ is optimal, it must satisfy investor i's budget constraint. Portfolio $\hat{P}_i$ generates income stream

$$\bar{I}^K(\hat{P}_i,\bar{Y}) = \hat{a}_i\bar{r} + \hat{b}_i\ \bar{C}(\bar{Q},\bar{Y}) + \hat{c}_i\ \bar{C}(\bar{Q}^*,\bar{Y})$$
$$+ \hat{d}_i\ \bar{X}(\bar{Y}) + \hat{e}_i\ [\bar{X}(\bar{Y}) - \bar{C}(\bar{Q},\bar{Y})]. \tag{4.8}$$

Investor i can generate an identical income stream in basic economy χ with proftolio $\tilde{\theta}_i = (\tilde{\alpha}_i,\tilde{\beta}_i,\tilde{\gamma}_i,\tilde{\delta}_i)$ where $\tilde{\alpha}_i = \hat{a}_i$, $\tilde{\beta}_i = \hat{b}_i - \hat{e}_i$, $\tilde{\gamma}_i = \hat{c}_i$, and $\tilde{\delta}_i = \hat{d}_i + \hat{e}_i$. It is easily verified that $\tilde{\theta}_i$ satisfies investor i's budget constraint and that, for all $\bar{Y}\in\Omega$, $\bar{I}^\chi\ (\tilde{\theta}_i,\bar{Y}) = \bar{I}^K\ (\hat{P}_i,\bar{Y})$.

Investors value portfolios only by their income streams. Consequently, because $\hat{\theta}_i$ and $\tilde{P}_i$ yield identical streams of income, $\hat{\theta}_i \sim_i \tilde{P}_i$. Similarly $\hat{P}_i \sim_i \tilde{\theta}_i$. By assumption, $\tilde{P}_i$ is not optimal and $\hat{P}_i$ is. Consequently $\hat{P}_i \succ_i \tilde{P}_i$. Transitivity of investor preferences therefore implies that $\tilde{\theta}_i \succ_i \hat{\theta}_i$, i.e. $\hat{\theta}_i$ is not an optimal portfolio. But this contradicts our original assumption that $\hat{\theta}_i$ is the optimal portfolio for i in economy χ. Therefore, $\tilde{P}_i$ is an optimal portfolio in economy K. Thus $V_K = V_\chi$, $D_K = D_\chi$, and $D_K^* = D_\chi^*$ are equilibrium prices for economy K.

A parallel argument can be constructed to show that if we assume $V_K' = V_\chi$, $D_K' = D_\chi$, $D_K^{*\prime} = D_\chi^*$, an equilibrium exists for economy K′ given economy χ is in equilibrium. Reversing the arguments, it is clear that any equilibrium prices V_K, D_K, and D^*_K for economy K or any equilibrium prices V'_K, D'_K, and D^*_K for economy K′ are equilibrium prices in economy χ. Therefore a set of equilibrium prices in any one economy is a set of equilibrium prices in the other two economies, which implies that $V_K = V'_K$. Q.E.D.

Above in Theorem 2 we showed that if the change in leverage of firm two does not increase the variety bonds investors trade, then the absolute and relative values of firms one and two remain constant. In Theorem 3 we show that if the change in leverage creates a new bond that was not previously traded, then this change may affect the absolute, but not relative, values of firms one and two.

Theorem 3. Suppose capital markets are perfect. If in the initial situation investors trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{T}$ and in the final situation they trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, and $\mathcal{T}^*$, then the market values of firms one and two do generally change between the initial and final situations, i.e. $V_1 = V_2$ and $V'_1 = V'_2$, but generally $V_1 \neq V'_1$ and $V_2 \neq V'_2$.

Proof. Let the initial situation where firm two's debt is one unit of $\mathcal{B}$ be labeled economy L. Let the final situation where firm two's debt is one unit of $\mathcal{B}^*$ be labeled economy L′. From Theorem 1 we know that $V_1 = V_2$ and $V'_1 = V'_2$. Let $V_L = V_1 = V_2$ and $V'_L = V'_1 = V'_2$. The argument which we used in Theorem 2 to show that economy K is an analogue of basic economy χ applies here: economies L and L′ are analogous to basic economies Ψ and χ respectively. That is, any set of equilibrium prices in economy Ψ is a set of equilibrium prices in economy L and conversely. The same is true for economies L′ and χ. Therefore $V_L = V_\Psi$ and $V_L' = V_\chi$. Observation one states that V_Ψ is not necessarily equal to V_χ; therefore, V_L is not necessarily equal to V_L' which means V_1 is not necessarily equal to V_1' and V_2 is not necessarily equal to V_2'. Q.E.D.

In Theorem 4 we examine the case where firm two increases its leverage by issuing more of its original bonds, that is the case where $\overline{Q}^* = (1+\varepsilon)\overline{Q}$. Thus the planned payment streams $\overline{Q}$ and $\overline{Q}^*$ have different absolute levels, but identical term structures. We show that in the presence of default risk such an increase causes a change in the absolute values of the firms. The reason is that increasing the absolute size of contractual payments from $\overline{Q}$ to $\overline{Q}^*$ increases the likelihood that a default will actually occur. If a default does occur, then it changes the term structure of actual payments which firm two makes to its bondholders and stockholders. This differentiates bond $\mathcal{B}^*$ from bond $\mathcal{B}$ in a substantive manner and, as Theorem 3 implies, may bring about a change in the absolute market values for the two firms. Nevertheless, if no default risk exists and if firm two increases its leverage by issuing more bonds of the same term structure as its original bonds, then this change can not cause any change in the values of firms one and two.

Theorem 4. Suppose capital markets are perfect. Let $\overline{Q}$ be the contractual stream of payments of bond $\mathcal{B}$ and let $\overline{Q}^* = (1+\varepsilon)\overline{Q}$, where ε is a positive scalar, be the contractual stream of payments of bond $\mathcal{B}^*$. If in the <u>initial</u> situation investors trade securities $\mathcal{a}$,

$\mathcal{B}$, $\mathcal{S}$, and $\mathcal{T}$ and in the final situation they trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, and $\mathcal{T}^*$, then the absolute market values of firms one and two may change between the initial and final situations if and only if default risk exists.

Proof. From Theorem 1 we know that firms one and two have identical values. Therefore $V_1 = V_2$ and $V_1' = V_2'$. If no default risk exists, then planned payments to bondholders equal actual payments to bondholders, i.e., for all $\overline{Y} \in \Omega$, $\overline{C}(\overline{Q}, \overline{Y}) = \overline{Q}$ and $\overline{C}(\overline{Q}^*, \overline{Y}) = \overline{Q}^* = (1+\varepsilon)\overline{Q}$. Consequently, if we choose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{S}$ to be basic securities, then $\mathcal{B}^*$ is not a basic security because its returns are a linear function of the returns of $\mathcal{B}$. Therefore both the initial and final situations are analogous to basic economy Ψ. The reasoning used in the proof of Theorem 2 therefore implies that $V_1 = V_2 = V_\Psi = V_1' = V_2'$.

Now assume default risk exists. By the continuity of $f(\overline{Y})$, there exists a $\overline{Y}' \in \Omega$ such that (a) $C_t(\overline{Q}^*, \overline{Y}') < Q_t^* = Q_t(1+\varepsilon)$ and $C_t(\overline{Q}, \overline{Y}') \geq Q_t$ for some time period $t > 0$, (b) $C_\tau(\overline{Q}^*, \overline{Y}') \geq Q_\tau^* = Q_\tau(1+\varepsilon)$ and $C_\tau(\overline{Q}, \overline{Y}') \geq Q_\tau$ for some time period τ, (c) and $f(\overline{Y}') > 0$. Such a $\overline{Y}'$ must exist because conditions (a), (b), and (c) require only that the income stream $\overline{Y}'$ be picked such that it is large enough in period t so that no default occurs when

The contractual payments are $\overline{Q}$ and that it is small enough that when the contractual payments are increased to $\overline{Q}^*$ default results in period t. The implication of this choice is that no scalar η exists such that $\overline{C}(\overline{Q},\overline{Y}') = \eta\,\overline{C}(\overline{Q}^*,\overline{Y}')$. Therefore, in addition to $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{S}$, security $\mathcal{B}^*$ is a basic security when default risk exists. As a consequence, the initial situation is analogous to the basic economy Ψ and the final situation is analogous to basic economy χ. Therefore $V_1 = V_2 = V_\Psi$ and $V_1' = V_2' = V_\chi$, thus generally $V_1 \neq V_1'$ and $V_2 \neq V_2'$. Q.E.D.

5. Imperfect Capital Markets

In defining perfect capital markets we required that an alteration of either a firm's or an investor's financial structure not change the total amount of income available to investors in the equity of firms, investors in the debt of firms, and investors in the debt of individual investors. Given this condition, the income stream from a representative portfolio $P_i = (a_i, b_i, c_i, d_i, e_i)$ made up of securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, $\mathcal{T}$ is

$$\overline{I}(P_i,\overline{Y}) = a_i\overline{r} + b_i\ \overline{C}(\overline{Q},\overline{Y}) + c_i\ \overline{C}(\overline{Q}^*,\overline{Y}) + d_i\ \overline{X}(\overline{Y}) + e_i[\overline{X}(\overline{Y}) - \overline{C}(\overline{Q},\overline{Y})]. \tag{5.1}$$

Generally, however, changes in financial structure do have consequences on the size of the real income stream available for distribution among the various investors. Thus a more realistic formulation of the income stream generated for investor i by portfolio P_i is

$$\overline{I}(P_i,\overline{Y}) = a_i\overline{r} + b_i\ \overline{C}(\overline{Q},\overline{Y}) + c_i\ \overline{C}(\overline{Q}^*,\overline{Y}) + d_i\ \overline{X}(\overline{Y}) + e_i[\overline{X}(\overline{Y}) - \overline{C}(\overline{Q},\overline{Y}) + \overline{Z}(\overline{Q},\overline{Y})]. \tag{5.2}$$

$\overline{Z}(\overline{Q},\overline{Y})$ is the function which describes the consequences of corporate leverage on firm two's net income.

The function $\overline{Z}$ represents the sum of all costs and revenues which leverage creates for firm two. For example, if firm two is levered by one unit of security $\mathcal{B}$, then its interest payments to the holders of $\mathcal{B}$ are a tax deductable cost. Here $\overline{Z}(\overline{Q},\overline{Y})$ represents an implicit payment from the government to firm two's stockholders which firm two earns simply by levering itself. The function $\overline{Z}(\overline{Q},\overline{Y})$ also represents effects other than tax saving. For example, if firm two does seriously default on its contractual debt payments, then, whether it formally goes bankrupt or not, it incurs a variety of extra costs. In the case of declared bankruptcy there are direct payments to third parties such as lawyers and trustees. In the case of a firm verging on bankruptcy, costs may be incurred in the form of opportunities foregone because creditors place limits on the firm's investment policy, operating policy, and borrowing capability.

An extreme example of bankruptcy costs is where, as a consequence of its leverage, firm two goes bankrupt and ceases to exist. Suppose the realized $\overline{Y}\in\Omega$ is such that firm two goes bankrupt and out of existence in period τ when it is levered by one unit of bonds $\mathcal{B}$. Further suppose that firm one continues in existence after period τ and earns, at least occasionally, profits for distribution to its stockholders. Let $\overline{X}(\overline{Y}) = (\ldots,X_\tau,\ldots,X_t,\ldots)$ and $\overline{Z}(\overline{Q},\overline{Y}) = (\ldots,Z_\tau,\ldots,Z_t,\ldots)$ where $t > \tau$. Pick t such that $X_t > 0$. This means that the stockholders of firm one collectively receive dividends of X_t in period t. The stockholders and bondholders (ex-holders) of firm two, however, receive nothing in period t because firm two no longer exists. Therefore $Z_t = -X_t$ for all $t > \tau$. In other words, bankruptcy costs in the case of liquidation are the negative of all positive earning which might have been realized in periods subsequent to τ.[6]

In general $\overline{\overline{Z}}(\overline{\overline{Q}},\overline{\overline{Y}})$ represents the sum of tax savings, bankruptcy charges, and other real costs or revenues attributable to firm two's leverage. Since $\overline{\overline{Z}}(\overline{Q},\overline{Y})$ represents revenue or cost attributable to leverage, if $\overline{Q} = \overline{0}$, then we set $\overline{Z}(\overline{Q},\overline{Y}) = \overline{0}$. This insures that if firm two eliminates its leverage, then its stockholder dividends become identical to firm one's stockholder dividends. In addition it means that the function $\overline{X}(\overline{Y})$ includes a provision for the bankruptcy costs which an unlevered firm can incur.

In this section we prove a single theorem. Its conclusion is that Theorems 1 and 2 are not valid when capital markets are imperfect. Thus Theorem 5 shows that in imperfect capital markets firms one and two may have different market values and that a change in firm two's leverage may cause changes in the market values of firms one and two even if securities $\mathcal{B}$ and $\mathcal{B}^*$ are traded in the initial and final situations.

> Theorem 5. Suppose capital markets are imperfect. If in the initial situation investors trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, and $\mathcal{T}$ and in the final situation they trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, and $\mathcal{T}^*$, then the market values of firms one and two (a) may be unequal in the initial situation, (b) may be unequal in the final situation, and (c) may change between the initial and final situation, i.e. generally $V_1 \neq V_2$, $V_1' \neq V_2'$, $V_1 \neq V_1'$, and $V_2 \neq V_2'$.

Proof. Our proof is to construct an example where an imperfection in the capital market allows the possibility that the equilibrium market values of firms one and two are neither identical nor invariant with respect to changes in the capital structure of firm two. Let the initial situation where debt of firm two is one unit of $\mathcal{B}$ be labeled economy M and let the final situation where its debt is one unit of $\mathcal{B}^*$ be labeled economy M'. Assume that the income streams $\overline{X}(\overline{Y}) + \overline{Z}(\overline{Q},\overline{Y})$ and $\overline{X}(\overline{Y}) + \overline{Z}(\overline{Q}^*,Y)$ are not linearly dependent on each other and the income streams of the basic securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, and $\mathcal{S}$. Therefore the streams $\overline{X}(\overline{Y}) + \overline{Z}(\overline{Q},\overline{Y})$ and $\overline{X}(\overline{Y}) + \overline{Z}(\overline{Q}^*,\overline{Y})$ are the income streams of two, new, basic securities. Label them $\mathcal{S}^o$ and $\mathcal{S}^{*o}$ respectively. Define additional basic economies, ν and ω, as follows.

Basic Economy ν. Investors trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, and $\mathcal{S}^o$. The supplies of $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, and $\mathcal{S}^o$ are respectively zero units, zero units, one unit, and one unit. $\mathcal{A}$ is traded at the exogenous price of one. Security $\mathcal{S}^{*o}$ is not traded. Let D_ν, D^*_ν, V_ν, and V^o_ν be the equilibrium prices of $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, and $\mathcal{S}^o$.

Basic Economy ω. Investors trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, and $\mathcal{S}^{*o}$. The supplies of $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, $\mathcal{S}^{*o}$ are respectively zero units, zero units, one unit, and one unit. $\mathcal{A}$ is traded at the exogenous price of one. Security $\mathcal{S}^o$ is not traded. Let D_ω, D^*_ω, V_ω, and V^{*o}_ω be the equilibrium prices of $\mathcal{B}$, $\mathcal{B}^*$, $\mathcal{S}$, and $\mathcal{S}^{*o}$.

The reasoning which we used to justify Observation 1 implies that generally $V_\nu \neq V^o_\nu$, $V_\omega \neq V^{*o}_\omega$, $V_\nu \neq V_\omega$, and $V^o_\nu \neq V^{*o}_\omega$.

The technique which we used in Theorem 2 to show that basic economy χ is analogous to economy K is applicable here. It shows that basic economy ν is analogous to economy M and basic economy ω is analogous to economy M'. Therefore it follows that $V_1 = V_\nu$, $V_2 = V^o_\nu$, $V'_1 = V^o_\omega$, and $V'_2 = V^{*o}_\omega$. Hence, generally $V_1 \neq V_2$, $V'_1 = V'_2$, $V_1 \neq V'_1$, and $V_2 \neq V'_2$. Q.E.D.

6. Conclusions

In this paper we have defined within the traditional Modigliani-Miller paradigm of two firms in a single risk class the circumstances under which changes in firm two's financial structure does affect the values of firms one and two. The inclusion, however, of firm one in the model is unncessary for the results which we have derived. Using exactly the same methods of proof that were used to prove Theorems 2, 3, and 5 the following three theorems which delete all reference to the unlevered firm can be proven. Let $\mathcal{J}$ represent the stock of the one firm in the initial situation when it is levered by one unit of type $\mathcal{B}$ bonds, let $\mathcal{J}^*$ be its stock in the final situation when it is levered by one unit of type $\mathcal{B}^*$ bonds, let V be its value in the initial situation, and let V′ be its value in the final situation.

> Theroem 2′. Suppose capital markets are perfect. If investors trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, and $\mathcal{J}$ in the initial situation, and if they trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, and $\mathcal{J}^*$ in the final situation, then the market value of the firm does not change between the initial and final situations, i.e. $V = V'$.

> Theorem 3′. Suppose capital markets are perfect. If investors trade securities $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{J}$ in the initial situation and if they trade $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, and $\mathcal{J}^*$ in the final situation, then the market value of the firm may change between the initial and final situation, i.e., generally, $V \neq V'$.

Theorem 5′. Suppose capital markets are imperfect. If investors trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, and $\mathcal{J}$ in the initial situation and if they trade securities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$, and $\mathcal{J}^*$ in the final situation, then the market value of the firm may change between the initial and final situations, i.e. generally $V \neq V'$.

These results give a feeling of the role which a successful financial manager must play. First he must take advantage of the market imperfections by balancing off the tax savings against the bankruptcy changes which increased leverage brings. Then, in addition, he must be alert for opportunities to create new securities which offer a pattern of returns that are both currently unavailable on the market and wanted by investors. This raises an interesting empirical question. How frequently do firms successfully identify and profitably introduce a new security into the market?

FOOTNOTES

1. We assume that all investors know with certainty that firms one and two are in the same risk class.

2. Firms one and two are allowed to go bankrupt simultaneously. If, for example, for some $\overline{Y}\in\Omega$, the firms both go bankrupt in period τ, then for all $t > \tau$, $X_t = 0$ where $\overline{X}(\overline{Y}) = (X_1, X_2, X_3, \ldots, X_t, \ldots)$. Our assumptions only prohibit firm two from going bankrupt, as a result of its leverage, while firm one survives and continues earning positive returns in at least some subsequent periods.

3. An assumption which is implicit within this formulation is that investors value a portfolio P_i only for the income stream which it generates, not for the possibilities of capital appreciation which some portfolios may have. Thus this model is a static multiperiod model, not a dynamic model.

4. Hart [4] has shown that if investors have heterogeneous expectations concerning the future net operating incomes of the two firms, then existence of an equilibrium requires that the maximal permissible degree of heterogeneity be directly proportional to the degree of the investors' risk aversion. In other words, if investors are more risk averse, then they can disagree more concerning the firms' future incomes without destroying the competitive equilibrium's existence.

5. This definition of identical borrowing terms for firms and investors is very similar to the type of loan which Baron [1] allows investors. Investors are permitted to pledge that stock of firm one which they purchase as full collateral for their outstanding loans. If at some time the value of an investor's stock holdings should fall below the value of his loan, then he can forfeit his stock to his creditor and be free of any additional liability.

6. Presumably in period τ, the period which firm two is liquidated, Z_{τ} would be positive due to the bond holders getting at least a partial return of their principle as a result of the sale of firm two's assets.

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