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Market Games with Production and Public Commodities

by

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1. Introduction

Shapley and Shubik [8] have shown that the class of side-payment games which arises from the class of exchange economies with transferable utility (under mild assumptions) is the class of totally balanced games. (A game is totally balanced if all of its subgames possess nonempty cores.) Our purpose is to show that their results can be extended to situations in which production of both public and private commodities is possible, under certain assumptions.

After details of the economies and some technical assumptions are specified in Section 2, it is shown in Section 3 that a totally balanced side-payment game is derived from any such economy in which the utilities are monotone nondecreasing in the public commodities. A counterexample is presented in which the monotonicity assumption is not satisfied.

In Section 4, it is established that every totally balanced game can arise from such an economy in which the utilities are linear, nondecreasing and do not depend on the private goods (which serve only as inputs to production). If the number of private goods in such an economy is limited to one, however, then not every totally balanced game can be obtained. A partial characterization is presented for this case, and an interesting new subclass of the totally balanced games emerges as a by-product.

2. Productive Economies with Transferable Utility

For our purposes, a productive economy is a sextuple $(N, \mathbb{R}^m, \mathbb{R}^p_+, \mathcal{U}, W, Z)$ where: $N = \{1,\ldots,n\}$ is the set of economic agents; $\mathbb{R}^m$ is the private commodity space; $\mathbb{R}^p_+$ is the public commodity space; $\mathcal{U} = \{u^i\colon i \in N\}$ is an indexed collection of continuous, concave utility functions from $\mathbb{R}^{m+p}_+$ into $\mathbb{R}^1$; $W = \{w^i\colon i \in N\}$ is an indexed collection of initial endowment vectors in $\mathbb{R}^m_+$

(there are no public commodities initially); and Z is a closed, convex production cone with vertex zero in $\mathbb{R}^m \times \mathbb{R}^p_+$ satisfying $Z \cap \mathbb{R}^{m+p}_+ = \{0\}$ (no free production). An allocation is a vector $(x,y) = (x^1, \ldots, x^n, y) \in \mathbb{R}^{nm+p}_+$ satisfying $(\sum_{i \in N} x^i - \sum_{i \in N} w^i, y) \in Z$. For each $S \subseteq N$, $S \neq \emptyset$ we denote by $A(S)$ the set of S-allocations: $\{(x,y) \in \mathbb{R}^{nm+p}_+ : x^i = 0 \text{ for } i \notin S \text{ and } (\sum_{i \in S} x^i - \sum_{i \in S} w^i, y) \in Z\}$. For each $S \subseteq N$, it follows that $A(S)$ is compact and convex. When there are no public goods and no production possibilities, each productive economy corresponds to a market in [8].

The productive economy $(N, \mathbb{R}^m, \mathbb{R}^p_+, \mathcal{U}, W, Z)$ generates the side-payment characteristic-function game (N,v) where $v: 2^N \to \mathbb{R}^1$ is defined by

$$v(\emptyset) = 0$$

$$v(S) = \max_{(\bar{x},\bar{y}) \in A(S)} \; \min_{(\tilde{x},\tilde{y}) \in A(S^c)} \; \sum_{i \in S} u^i(\bar{x}^i, \bar{y} + \tilde{y}), \qquad S \neq \emptyset.$$

(S^c denoting the complement of S in N). From the assumptions, the max and min are attained; and the resulting vector $(\bar{x} + \tilde{x}, \bar{y} + \tilde{y})$ is an allocation.

Several remarks are now in order. Firstly, the generation of a side-payment game presumes, as usual, the existence of an additional freely transferable medium of exchange; i.e., an additional private commodity which is useless in production, the quantity of which enters all utility functions as a separate term, and which is endowed to each agent in quantities sufficiently large to accomodate all desired exchanges. There is no need to introduce it explicitly in this model, however. Secondly, the particular characteristic function v above is derived according to the α-derivation, or equivalently, when there are side-payments, the β-derivation (see [1] and [9]); that is, v(S) represents the total utility of which S can assure itself, assuming that S^c might respond to S's choice of actions with that action which is most damaging to

S from among S^c's alternative responses. This is a conservative measure of S's power; unduly so, perhaps, if S^c's most damaging response to S is also damaging to itself, as is frequently the case with models involving economic externalities (see [2], [3], [4], for example). Thirdly, the relatively complicated form of v is partially due to the fact that public and private bads have not been ruled out. If, for example, all utility functions were nondecreasing in the public components, then the best response for S^c would always involve providing no public commodities for S, a considerable simplification. Finally, this formulation does not allow a coalition to dump private bads on individuals not in the coalition; nor does it allow for consumption and production externalities other than of the purely public sort. Such possibilities may be expressible within the current framework by creating additional public commodities.

3. Productive Economies Which are Totally Balanced

In this section we shall show that every productive economy in which each u^i is monotone nondecreasing in y gives rise to a totally balanced game. That some such additional assumption is necessary is indicated by the following example (which, incidentally, satisfies all assumptions used later in this paper with the exception of monotonicity).

Example 1: $m = 1.$ $n = p = 3.$

$$w^i = 1 \text{ for } i = 1,2,3.$$
$$u^1(x^1, y_1, y_2, y_3) = 2y_1 - y_2.$$
$$u^2(x^2, y_1, y_2, y_3) = 2y_2 - y_3.$$
$$u^3(x^3, y_1, y_2, y_3) = 2y_3 - y_1.$$
$$Z = \{(z,y) \in \mathbb{R} \times \mathbb{R}^3_+ : z + y_1 + y_2 + y_3 \leq 0\}.$$

It is a straightforward matter to verify that the characteristic function which is derived from this example is:

$$v(\{i\}) = 0 \qquad i = 1,2,3;$$

$$v(\{i,j\}) = 3 \qquad i \neq j,\ i,j \in \{1,2,3\};$$

$$v(\{1,2,3\}) = 3.$$

Clearly, the core of this game is empty.

We shall be using the following material on balancedness (see [5]) in the rest of the paper. A collection of subsets $\mathscr{S}$ of N is balanced if there exist nonnegative weights $\{\delta_S : S \in \mathscr{S}\}$ satisfying:

$$\sum_{\substack{S \ni i \\ S \in \mathscr{S}}} \delta_S = 1 \quad \text{for } i = 1,\ldots,n.$$

A side-payment game (N,v) is balanced if for every balanced collection $\mathscr{S}$ of N,

$$\sum_{S \in \mathscr{S}} \delta_S v(S) \leq v(N).$$

A side-payment game is balanced if and only if it has a nonempty core. A side-payment game is totally balanced if each of its subgames is balanced.

Theorem 1: In a productive economy, if each u^i is monotone nondecreasing in y then the associated side-payment game is totally balanced.

Proof: Let U be a nonempty subset of N. Let $\mathscr{T}$ be a balanced collection of subsets of U with balancing weights $\{\delta_T : T \in \mathscr{T}\}$.

Let $(x(T),y(T)) \in A(T)$ guarantee T at least v(T) against any response from $A(T^c)$; in particular then, the response in which T^c produces no public goods.

$$\sum_{T\in\mathcal{T}} \delta_T v(T) \le \sum_{T\in\mathcal{T}} \delta_T \sum_{i\in T} u^i(x^i(T),y(T)) = \sum_{i\in U} \sum_{\substack{T\ni i\\ T\in\mathcal{T}}} \delta_T u^i(x^i(T),y(T)) \le \sum_{i\in U} u^i\Big(\sum_{\substack{T\ni i\\ T\in\mathcal{T}}} \delta_T x^i(T), \sum_{\substack{T\ni i\\ T\in\mathcal{T}}} \delta_T y(T)\Big).$$

If each member $i\in T$ devotes δ_T of his resources to the productive activities which yield $(x(T),y(T))$, then the vector $\delta_T(x(T),y(T))$ results. If each coalition $T\in\mathcal{T}$ does this, the result is the U-allocation $\sum_{T\in\mathcal{T}} \delta_T(x(T),y(T))$. But $v(U) \ge \sum_{i\in U} u^i(\sum_{T\in\mathcal{T}} \delta_T x^i(T), \sum_{T\in\mathcal{T}} \delta_T y(T))$, and $\sum_{T\in\mathcal{T}} \delta_T x^i(T) = \sum_{\substack{T\ni i\\ T\in\mathcal{T}}} \delta_T x^i(T)$.

Hence, it follows from the monotonicity assumption that

$$\sum_{i\in U} u^i\Big(\sum_{T\in\mathcal{T}} \delta_T x^i(T), \sum_{T\in\mathcal{T}} \delta_T y(T)\Big) \ge \sum_{i\in U} u^i\Big(\sum_{\substack{T\ni i\\ T\in\mathcal{T}}} \delta_T x^i(T), \sum_{\substack{T\ni i\\ T\in\mathcal{T}}} \delta_T y(T)\Big).$$

Thus $v(U) \ge \sum_{T\in\mathcal{T}} \delta_T v(T)$.||

The proof is similar to others in the literature and is included only for the sake of completeness. An alternative proof through Lindahl equilibria is also possible. See [2] and [7], for this approach.

4. Special Productive Economies

That every totally balanced game can arise from some productive economy is established in [8], where there are no production possibilities. In this section we explore whether every totally balanced game can result from classes of productive economies in which the private goods are only useful as inputs to production. We hope thereby to focus our attention on the public-good aspects of the economies. We shall first show that every totally balanced game can arise from some productive economy in which all utility functions are monotone nondecreasing affine functions of the public-goods quantities alone.

Let (N,v) be any superadditive side-payment game. There are to be n private commodities with endowments $w^i = e_i$ (the i^{th} unit vector in $\mathbb{R}^n$) for $i = 1,\ldots,n$. There are 2^n-1 public commodities, one associated with each nonempty coalition S. The production set

$$Z = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}_+^{2^n-1} : \sum_{S \ni i} y_S \le -x_i,\ i = 1,\ldots,n\}.$$

The utility functions have the form:

$$u^i(x^i,y) = \sum_{S \ni i} \lambda_S^i y_S + v(\{i\}) \qquad i = 1,\ldots,n \quad .$$

$$\text{where} \quad \sum_{i \in S} \lambda_S^i = v(S) - \sum_{i \in S} v(\{i\}) \ \forall S \subseteq N,\ S \ne \emptyset;$$

$$\text{and} \qquad \lambda_S^i \ge 0 \qquad \text{for all } i \in S,\ S \ne \emptyset.$$

(For the special case in which for each $S \ne \emptyset$ at most one λ_S^i differs from zero, the public goods may also be thought of as private.)

No matter what action S takes, S^c's most harmful response is to produce nothing. If S produces one unit of good S, it receives total utility of v(S). S can produce none of good T for $T \supsetneq S$. The other alternative is to produce various amounts of goods T for $T \subseteq S$. But the form of the production set requires that if player $i \in T$ devotes δ_T of his input to good T, then $\sum_{T \ni i} \delta_T \le 1$. Thus S can achieve

$$\max \quad \sum_{T \subseteq S} \delta_T [v(T) - \sum_{i \in T} v(\{i\})] + \sum_{i \in S} v(\{i\})$$

$$\text{subject to} \quad \sum_{\substack{T \ni i \\ T \subseteq S}} \delta_T \le 1 \quad \text{all } i \in S.$$

$$\delta_T \ge 0 \quad \text{all } T \subseteq S.$$

This maximum is clearly $v(S)$ for all S if and only if (N,v) is totally balanced.

Although there is no need for exchange of private goods in the above productive economies, interactive effects between private commodities are still present through the production set. The importance of private commodities is least in those productive economies having only one private commodity which is not an argument in any utility function. Example 2 illustrates, however, that not every totally balanced game can arise from a productive economy in which the utility functions are monotone nondecreasing, affine functions of the public goods alone and in which there is only one private good.

<u>Example 2</u>: $N=\{1,2,3,4\}$;

$v(\{i\})=0 \quad i=1,\dots,4;$

$v(\{1,2\})=v(\{3,4\})=1,\ v(\{i,j\})=0$ otherwise;

$v(S)=1 \quad \text{if } |S|=3;$

$v(N)=2.$

Since the utility functions are affine, there must be a public good (say good 12) which $\{1,2\}$ can produce exclusively to attain $v(\{1,2\})$. Let a_1^{12} and a_2^{12} be the coefficients of y^{12} in the utility functions of players 1 and 2, respectively. Let k_1 and k_2 be the respective constant terms in their utility functions. Then

$$(a_1^{12}+a_2^{12})(w^1+w^2)+k_1+k_2=1$$

The values a_3^{34}, a_4^{34}, k_3, and k_4 are determined similarly for $\{3,4\}$, and

$$(a_3^{34}+a_4^{34})(w^3+w^4)+k_3+k_4=1$$

Since $v(\{1,2,3\}) = 1$ and the coefficients are all nonnegative, it must be that

$$(a_1^{12} + a_2^{12} + a_3^{12})(w^1 + w^2 + w^3) + k_1 + k_2 + k_3 \le 1,$$

or,

$$(a_1^{12} + a_2^{12})w^3 + k_3 \le 0.$$

Similarly,

$$(a_1^{12} + a_2^{12})w^4 + k_4 \le 0.$$

Hence,

(1) $$(a_1^{12} + a_2^{12})(w^3 + w^4) + k_3 + k_4 \le 0.$$

Similarly,

(2) $$(a_3^{34} + a_4^{34})(w^1 + w^2) + k_1 + k_2 \le 0.$$

But, if

$$(a_1^{12} + a_2^{12}) \ge (a_3^{34} + a_4^{34}),$$

then (1) is impossible; and if

$$(a_1^{12} + a_2^{12}) \le (a_3^{34} + a_4^{34})$$

then (2) is impossible.

It is perhaps interesting that Example 2 is a convex game. (The convex games are a well-studied subset of the totally balanced games. See, for example, [6]. A game (N,v) is convex if $v(S \cup T) + v(S \cap T) \ge v(S) + v(T)$ for all S, $T \subseteq N$.)

We have not been able to determine whether every totally balanced game can arise from a productive economy with one private good and with utility functions which are monotone nondecreasing (but not necessarily affine) in the public goods alone. Neither have we been able to characterize the class of games which can arise from the above economies with the additional affine restriction. Every game in the following interesting class can arise from such an economy, however. A superadditive game (N,v) is <u>strongly totally balanced</u> if there is an imputation d satisfying

$$(3) \qquad (\sum_S d_i - \sum_S v(\{i\}))(v(T) - \sum_T v(\{i\})) \le (\sum_T d_i - \sum_T v(\{i\}))(v(S) - \sum_S v(\{i\})$$

for all T, S such that $\phi \neq T \subseteq S \subseteq N$.

<u>Lemma 1</u>: Suppose (N,v) is strongly totally balanced. Let $\alpha > 0$, $\beta_1, \ldots, \beta_n$ be scalars, and let $v'(S) = \alpha v(S) + \sum_S \beta_i$ for all $S \subseteq N$, $S \neq \phi$. Then (N,v') is strongly totally balanced. (Strong total balancedness is invariant under strategic-equivalence).

<u>Proof</u>: Let $d_i' = \alpha d_i + \beta_i$ for $i = 1, \ldots, n$. It is a straightforward matter to check that (3) holds with d' and v' replacing d and v, respectively.||

Condition (3) is considerably simplified for 0-normalized games ($v(\{i\}) = 0$ for $i = 1, \ldots, n$). Namely,

$$(4) \qquad \sum_S d_i \, v(T) \le \sum_T d_i \, v(S) \qquad \text{for all T, S such that } \phi \neq T \subseteq S \subseteq N.$$

Note that any strongly totally balanced game is totally balanced and that the vector d, suitably restricted and normalized, provides a core point for each subgame. Note that Example 2, though convex, is not strongly totally balanced (although it can be shown that every symmetric convex game is strongly totally balanced). On the other hand, there are strongly totally balanced games which are not convex. (Every balanced 3-player game is strongly totally balanced.)

<u>Theorem 2</u>: Every strongly totally balanced game can be derived from a productive economy with one private good and utility functions which are monotone

nondecreasing affine functions of the public goods alone.

Proof: Let (N,v) be strongly totally balanced. Assume without loss of generality that (N,v) is 0-normalized. Let d satisfy (4). Consider the productive economy with 2^n-1 public goods (identified with nonempty coalitions S) in which d_i is i's endowment of the private good,

$$Z = \{(x, y_{\{1\}}, \ldots, y_N) \in \mathbb{R}^1 \times \mathbb{R}^{2^n-1} : \sum_{\substack{S \subseteq 2^N \\ S \neq \emptyset}} y_S \leq -x\}, \quad \text{and}$$

$$u^i(x_i, y) = \sum_{S \ni i} \lambda_S^i y_S - \hat{p} d_i,$$

$$\text{where } \lambda_S^i = \begin{cases} 0 & \text{if } \sum_{j \in S} d_j = 0 \\ \dfrac{d_i v(S)}{(\sum_{j \in S} d_j)^2} + \dfrac{\hat{p}}{|S|} & \text{otherwise} \end{cases} \quad \text{and } \hat{p} \text{ is a very large positive number.}$$

S can attain v(S) by producing $\sum_{j \in S} d_j$ units of good S. If $T \supsetneq S$, for each unit of endowment which S devotes to producing good T, it receives total utility of

$$\sum_{j \in S} \lambda_T^j \leq \sum_{j \in T} \lambda_T^j - \lambda_T^i \text{ where } i \in T/S$$

$$\leq \frac{v(T)}{\sum_{j \in T} d_j} + \hat{p} - \frac{\hat{p}}{|T|} < \frac{v(S)}{\sum_{j \in S} d_j} + \hat{p} \quad \text{for } \hat{p} \text{ sufficiently large}$$

$$= \sum_{j \in S} \lambda_S^j .$$

Hence, S would do better to devote the same resources to good S. On the other hand, if $T_1, T_2, \ldots, T_k \subseteq S$ and S devotes $\alpha_1, \ldots, \alpha_k$ units of private good to producing goods $T_1,\dots,T_k$, respectively (where $\sum_{\ell=1}^{k} \alpha_\ell \le \sum_{j\in S} d_j$), then S receives total utility of (assuming $\sum_{i\in T_\ell} \lambda^i_{T_\ell} > 0$, $\ell = 1,\dots,k$)

$$\sum_{\ell=1}^{k} \alpha_\ell \sum_{i\in T_\ell} \lambda^i_{T_\ell} - \hat{p} \sum_{j\in S} d_j = \sum_{\ell=1}^{k} \alpha_\ell \left(\frac{v(T_\ell)}{\sum_{i\in T_\ell} d_i} + \hat{p}\right) - \hat{p} \sum_{j\in S} d_j$$

$$\le \sum_{\ell=1}^{k} \alpha_\ell \left(\frac{v(S)}{\sum_{j\in S} d_j} + \hat{p}\right) - \hat{p} \sum_{j\in S} d_j \le v(S). \qquad \text{Q.E.D.}\|$$

There are such economies which do not give rise to strongly totally balanced games, however

Example 3: Two public goods,

$w^i = 1,\ i = 1,\dots,4;\ Z = \{(x,y_1,y_2) : y_1 + y_2 \le -x\}$

$u^1(x,y_1,y_2) = u^2(x,y_1,y_2) = \frac{1}{2}y_1 - \frac{1}{2}$

$u^3(x,y_1,y_2) = u^4(x,y_1,y_2) = y_3 - 1.$

The characteristic function which arises from this economy is:

$v(\{i\}) = 0 \quad i = 1,\dots,4;$

$v(\{1,2\}) = 1,\ v(\{1,3\}) = v(\{1,4\}) = v(\{2,3\}) = v(\{2,4\}) = \frac{1}{2},\ v(\{3,4\}) = 2;$

$v(\{1,2,3\}) = v(\{1,2,4\}) = 1,\ v(\{1,3,4\}) = v(\{2,3,4\} = 7/2.$

$v(\{1,2,3,4\}) = 5.$

If this game were strongly totally balanced, then consideration of the subgame played by $\{1,2,3\}$ would require $d_3 = 0$. Similarly, $d_4 = 0$ from the subgame $\{1,2,4\}$. But $v(\{2,4\}) > 0$, a contradiction.

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