Optimal Dynamic Auctions*

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Abstract

We consider a dynamic auction problem motivated by the traditional single-leg, multi-period revenue management problem. A seller with C units to sell faces potential buyers with unit demand who arrive and depart over the course of T time periods. The time at which a buyer arrives, her value for a unit as well as the time by which she must make the purchase are private information. In this environment, we derive the revenue maximizing Bayesian incentive compatible selling mechanism.

KEYWORDS: dynamic mechanism design, optimal auctions, virtual valuation, revelation principle

JEL CLASSIFICATION NUMBERS: D44, C72, C73

1 Introduction

The problem of optimally selling a finite number of indivisible goods to buyers arriving over time has a long pedigree (see for example Stokey (14) and Bulow (4)). One setting restricts the seller to using posted prices and capacity controls. Such mechanisms are widely used in the airline, hotel, and car-rental industries. With the rise in Internet commerce, many sellers have begun experimenting with alternative pricing mechanisms such as auctions. This has inspired a number of theoretical investigations into such dynamic auctions, for example Vulcano et al (15) and Lavi and Nisan (7).

This paper involves a risk neutral seller seeking to sell C identical and indivisible units over T discrete time periods, indexed by t. Buyers with unit demand (also called bidders hereafter) arrive over time, and have private valuations for a unit of

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the good. In particular, buyer i has a valuation v_i for one unit of the good, excess units are worthless. Furthermore, each bidder has an arrival time t_i and a deadline $\bar{t}_i$. This means bidder i cannot participate in the auction before time t_i or after time $\bar{t}_i$. Goods assigned to agent i outside the interval $[t_i, \bar{t}_i]$ have no value to him. The 3-tuple $(v_i, t_i, \bar{t}_i)$, called bidders i's type, are assumed to be the private information of bidder i. Thus bidder i is in a position to claim a valuation smaller or larger than v_i, an arrival later than t_i and a deadline earlier than $\bar{t}_i$.

The airline industry is an example which corresponds closely to the model we analyze. The separation of buyers over time is also typical to this industry. The two major customer segments are leisure travelers and business travelers, the former generally making travel plans well in advance of actual travel (due to the need to coordinate this with other arrangements), while the latter may make travel plans just a few days before departure. The no-early-entry assumption can be justified on the grounds that travelers will not seek tickets until they know that they actually need to travel. The no-late-exit assumption can be justified on the grounds that the bidder may need time to make other arrangements (hotel accommodations, take time off work etc.), and getting the ticket later may not leave the bidder with sufficient time to make these arrangements.

1.1 Overview of the Main Results

Under the assumption that agents types are independent draws from a common distribution, satisfying a suitable hazard rate condition, we derive the revenue maximizing Bayesian Incentive compatible mechanism. The intuition is the same as in Myerson's (10) classic optimal auction paper, but our result is not a corollary of it.

Myerson's paper considers a static problem where the type of a bidder is one dimensional, i.e. a bidder's only private information is his valuation. The seller solicits bidders types, and then uses this information to compute each bidder's 'virtual value'. It is then shown that under a suitable hazard rate assumption, allocating greedily according to this 'virtual value' is optimal in terms of expected revenue amongst all mechanisms that are incentive compatible.

Myerson's results should properly be viewed as a statement about the equivalence of two classes of optimization problems. The first is the problem of optimally allocating the good when the seller has full information. The second is when the buyers' valuations are private information (and therefore incentive compatibility constraints must also be met). What Myerson shows is that under a suitable hazard rate assumption, the second problem is computationally equivalent to the first. In other words, the additional incentive constraints to do not make the problem 'harder'.

Our setting departs from Myerson (10) in that types are multi-dimensional, but we obtain a similar equivalence result. The multi-dimensional setting introduces incentive compatibility constraints not present in the one dimensional type case. So our main result does not follow from Myerson (10). Nevertheless, we show how to compute a modified 'virtual value' for each type. Then one can use use a dynamic program based on the virtual values to determine the optimal allocation and payments. The resulting allocation and payments are incentive compatible.

The general problem of finding a revenue maximizing Bayesian incentive compatible problem when types are multi-dimensional is well known to be difficult (see Rochet and Choné (13)) and solved cases are rare (for example, Armstrong (1)) Typically, this literature assumes the types come from a continuum, and employs the tools of vector calculus to derive the results. Here we assume that types are discrete. This allows us to use a network flow interpretation of the problem developed in Malakhov and Vohra (8). In our view this makes the analysis both more transparent and comprehensible. We know of no good argument to prefer a continuum type space to a discrete one. The choice should be based on whichever is the most mathematically convenient.

1.2 Related Literature

The idea to use auctions for revenue management is not new. Vulcano *et al* (15) consider a special case of our model where each buyer is present for at most one period (in terms of our model $\bar{t}_i = t_i$ for all i). The only private information is the buyer's valuation for a single unit.

Gallien (5) studies a monopolist selling a finite number of identical items over an infinite horizon to time sensitive buyers with unit demands. Buyers arrive over time via a renewal process. He shows that a direct revelation mechanism where an arriving buyer is offered a unit at a price depending on his valuation, the time of arrival and past history, is the expected revenue optimal incentive compatible mechanism. Time sensitivity of buyers is modeled using a discount factor common to all buyers that is known to the seller. The valuation of the bidder and his time of entry are his private information.

Board (2) studies a durable goods monopolist with infinite supply. In each period $T = 1, 2, \ldots, \infty$, a measure of consumers with unit demand enter the market. The measure of consumers entering and the distribution of their valuations for the good vary stochastically over time. Consumers can delay their purchases, but are time sensitive via a discount rate that is common knowledge. Board calculates a price path that maximizes expected revenue. The bidder's valuation and his time of entry are his private information.

Lavi and Nisan (7) consider the same model as Vulcano *et al* but in a setting where there is no prior distribution over types. In this setting they propose a mechanism and perform a worst case analysis of the revenue achieved.

Ng, Parkes and Seltzer (11) consider a closely related model where the seller has C identical indivisible units of a good. These units are re-usable, i.e. a sale in this case is one unit being allotted for one period. They demonstrate a dominant strategy mechanism, and perform a worst case analysis of the revenue achieved. Hajiaghayi, Kleinberg, Mahdian and Parkes (6) consider the same model but achieve a better competitive ratio. They also look to characterize the class of *deterministic* allocation rules that can be part of a dominant strategy incentive compatible mechanism.

1.3 Organization of this Paper

In Section 2 we introduce notation and describe our model. In Section 3, we describe the incentive compatibility constraints that a mechanism must meet, and then simplify them. In Section 4 we formulate and analyze the seller's decision problem, and derive the (expected) revenue maximizing auction. In this section, we also discuss the difficulties associated with exit time. Section 5 discusses some possible extensions to this model and concludes.

2 Model and Notation

A seller wishes to sell C units of a good (identical and indivisible) over T time periods, numbered $1, 2, \ldots, T$.

Buyers arrive over time. In particular, buyers learn their types at some point $t \leq T$. There is a finite set of agents' types $\mathbb{T} = I \times T^2$. A 3-tuple $(i, t, \bar{t}) \in \mathbb{T}$ is interpreted as follows:

- $i \in I$ is the reservation value of the agent for his desired quantity, where $I = \{1, 2, \ldots, I\}$.
- $t \leq T$ is the time of entry of the agent into the system.
- $\bar{t} \leq T$ is his time of exit from the system, $\bar{t} \geq t$.

In each period t, N_t risk neutral potential buyers arrive. N_t is a non-negative, discrete valued random variable, distributed according to a known probability mass function $g(\cdot)$, with support $\{0, 1, \ldots, \infty\}$ (our reasons for requiring full support are discussed in footnote 8).

There is a common prior (a probability distribution with p.d.f. f) on the space of types.[1] Agents arriving at time t receive i.i.d. draws from the posterior of this distribution (i.e. given time of arrival is type t). Formally, the probability that an agent arriving at time t has type $(i, t, \bar{t})$ is

$$\mathbb{P}\{\text{type} = (i, t, \bar{t}) | \text{arrivaltime} = t\} = \frac{f(i, t, \bar{t})}{\sum_{i' \in I, \bar{t}' \geq t} f(i', t, \bar{t}')}. \tag{1}$$

Thus we assume that the valuation of an agent arriving at some time t is independent of both the number of agents arriving at that time and the valuation of other agents arriving at that time.

We impose a partial order $\succeq$ on the space $\mathbb{T}$, defined as

$$(i', t', \bar{t}') \succeq (i, t, \bar{t}) \equiv (i' \geq i) \wedge (t' \geq t) \wedge (\bar{t}' \geq \bar{t}). \tag{2}$$

[1] We restrict our attention to pdf's that have a 'monotone hazard' rate, appropriately redefining this property for our current environment. A formal definition is in (3).

Intuitively, a type will be partially ordered above another if it has a (weakly) higher valuation, arrives earlier and has a later deadline. The monotone hazard rate condition here is that:

$$(i', t', \bar{t}') \succeq (i, t, \bar{t}) \Rightarrow \frac{f_{(t',\bar{t}')}(i')}{1 - F_{(t',\bar{t}')}(i')} \geq \frac{f_{(t,\bar{t})}(i)}{1 - F_{(t,\bar{t})}(i)}. \quad (3)$$

Here $f_{(t,\bar{t})}(\cdot)$ and $F_{(t,\bar{t})}(\cdot)$ are the pdf and cdf respectively conditional on entering at time t and exiting at $\bar{t}$. An interpretation of the condition is given in Section 3.2.

An inventory of bids I is a list of bids that arrived over the the T time periods, with their entry and exit times. The set of all possible inventories is denoted $\mathbb{I}$. At any time t, the system only knows I^t, the bids that arrived at time t or earlier. Let $I^t_{t'}$ be the multi-set of bids of agents who arrived at time t or earlier whose bids expire at or after t'. Let A^t be the multi-set of bids of agents who arrive at time t (unknown, random at time $t-1$). Given an inventory of bids I, let $L_t(I)$ be the set of bids in I that expire at t. Therefore, we have that

$$I^{t+1}_{t+1} = \left(I^t_t \backslash L_t(I^t_t)\right) \bigcup A^{t+1}.$$

Let I_t be the bids in I that expire at or after t. Finally, given inventories I and I', we say that $I^t_t \succeq I'^t_t$ if the vector of bids in I^t_t dominates the vector of bids in I'^t_t. In other words, for every bid in I'^t_t, there exists a distinct bid in I^t_t which has a higher (virtual) valuation and a later exit time, intuitively making I^t_t a 'better' inventory of bids to have (at time t).

An agent of type $(i, t, \bar{t})$, and given a probability of allotment a assigns a monetary value to this allocation, given as: [2]

$$v(a|(i, t, \bar{t})) = i.a \quad (4)$$

By the revelation principle, we can restrict attention to direct revelation mechanisms.[3] Each agent is asked to reveal her type. The auctioneer, as a function of the announcements, decides on the allocation, and the payment each agent is to make.

An allocation rule is a sequence of functions, one for each time period $\tau \in T$, $a^\tau : \mathbb{T} \times \mathbb{I}^\tau \rightarrow [0, 1]$.[4] In other words, $a^\tau_{(i,t,\bar{t})}(I^\tau)$ is the allocation that an agent who announces type $(i, t, \bar{t})$ gets in time period τ, given that the inventory of bids in the system up to that point in time are I^τ.

If a type $(i, t, \bar{t})$ is not present in the profile I^τ, we take $a^\tau_{(i,t,\bar{t})}(I^\tau) = 0 \, \forall \tau \in T$. Further we let $P : \mathbb{T} \times \mathbb{I} \rightarrow \mathbb{R}_+$ be the payment function, i.e. an agent who announces type $(i, t, \bar{t})$ when the inventory turns out to be I makes a payment of $P_{(i,t,\bar{t})}(I)$.

We further posit that an agent derives utility if he is allocated a unit somewhere

[2] Since the agent here is sensitive to when he gets allotted the object, strictly speaking we should point out that this is if he gets the allotment in any period t', $t \leq t' \leq \bar{t}$.

[3] A proof that the Revelation Principle applies to this setting is in Appendix B.

[4] There are various feasibility constraints that we impose on allocation rules which we introduce later.

over the time periods he is present in the system. The idea is that an agent of type $(i,t,\bar{t})$ cares for exactly 1 unit, allotted to him sometime between t and $\bar{t}$. Let $\mathcal{A}_{(i,t,\bar{t})}(I) = \max\{a^{\tau}_{(i,t,\bar{t})}(I^{\tau}) : t \leq \tau \leq \bar{t}\}$; we refer to this as his allotment. Define $P_{(i,t,\bar{t})}$ as the interim expected payment that an agent who announces type $(i,t,\bar{t})$ must make.

We can now define (by yet another abuse of notation)

$$a_{(i,t,\bar{t})} = \mathbb{E}_I\left[\mathcal{A}_{(i,t,\bar{t})}(I)|[(i,t,\bar{t}) \in I]\right]$$

as the interim allocation probability (for type$(i,t,\bar{t})$). The expected monetary gain for an agent whose type is $(i,t,\bar{t})$ is $i.a_{(i,t,\bar{t})} - P_{(i,t,\bar{t})}$.

3 Incentive Compatibility

Incentive Compatibility requires that no agent should have an incentive to misreport his type. In other words, for every type $(i,t,\bar{t})$, and any misreport $(i',t',\bar{t}')$:

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq v(a_{(i',t',\bar{t}')}|(i,t,\bar{t})) - P_{(i',t',\bar{t}')}.$$

However, we can relax the constraints corresponding to some of these misreports based on the context. An agent of type $(i,t,\bar{t})$ can only misreport his type as $(i',t',\bar{t}')$, where

- $t' \geq t$: This captures the fact that the agent can announce his type only after he is in the system.[5]
- $\bar{t}' \leq \bar{t}$: This says that an agent cannot claim to be in the system any longer than he actually is, alternately, receiving the object after the agent's true deadline is worthless to the agent.

The relevant (Bayesisan) incentive compatibility (IC) constraints are listed below.

1. Misreport Value: An agent can misreport his valuation.

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq v(a_{(i',t,\bar{t})}|(i,t,\bar{t})) - P_{(i',t,\bar{t})}. \tag{5}$$

2. Under-report Presence: An agent present in the system from t through $\bar{t}$ periods may choose to report he is in the system for some strict contiguous subset of that, i.e. $t' \geq t$, $\bar{t}' \leq \bar{t}$.

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq v(a_{(i,t',\bar{t})}|(i,t,\bar{t})) - P_{(i,t',\bar{t})}, \tag{6}$$

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq v(a_{(i,t,\bar{t}')}|(i,t,\bar{t})) - P_{(i,t,\bar{t}')}. \tag{7}$$

3. Misreport Value and Presence:

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq v(a_{(i',t',\bar{t}')}|(i,t,\bar{t})) - P_{(i',t',\bar{t}')}. \tag{8}$$

[5] Alternately, an agent can announce his type only after he knows it himself.

Further, we have the Individual Rationality constraint, that no agent can have a strictly negative expected surplus:

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq 0. \tag{9}$$

3.1 Implications of Incentive Compatibility

In this section we show that the only IC constraints that matter are the **adjacent** misreports of value, over-reports of requirement and misreports of presence.

Recall that given an allocation of a, an agent of type $(i,t,\bar{t})$ derives utility from it only if this allotment is made in a period $t' \in [t,\bar{t}]$.

Observation 1 Any incentive compatible mechanism in this model, represented by (a,P) (i.e. an allotment rule and a payment rule), can be represented by an allocation rule a' and the same payment rule P, where the allocation rule a' allots type $(i,t,\bar{t})$ at time $\bar{t}$ if at all.

Proof: Consider a modified allocation rule a' constructed as follows: For each type $(i,t,\bar{t})$, reallocate the maximum probability of getting a unit during any period t through $\bar{t}$ to getting exactly at time $\bar{t}$. As long as a is feasible, a' will also be, since each agent gets weakly fewer units. Also a' will be incentive compatible with the same pricing rule since the agent, by our assumptions, will be indifferent between a and a' regardless of his type. □

Note that this observation does not require that certain misreports are forbidden. In particular, it applies even when reporting a later exit time is allowed. To be precise, we have the following observation.

Observation 2 Suppose the pricing rule is a non-negative function of types, and that the allotment rule is of the type identified in Observation 1. Then the IC constraint corresponding to an agent reporting a late-departure is redundant.

Proof: Consider an agent of type $(i,t,\bar{t})$ misreporting type as $(i,t,\bar{t}')$, where $\bar{t} < \bar{t}'$. The relevant IC constraint is:

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq v(a_{(i,t,\bar{t}')}|(i,t,\bar{t})) - P_{(i,t,\bar{t}')}.$$

By Observation 1, an agent reporting an exit time of $\bar{t}'$ gets allotted at $\bar{t}'$. By our assumptions on the functional form of $v(.)$ $v(a_{(i,t,\bar{t}')}|(i,t,\bar{t})) = 0$. Therefore,

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq -P_{(i,t,\bar{t})}.$$

This is redundant given Individual Rationality (9) and the fact that $P \geq 0$. □

The following proposition is based on the case of one dimensional types.

Observation 3 If a mechanism (a, P) is incentive compatible, then:

$$\forall t, \bar{t} : i' > i \Rightarrow a_{(i',t,\bar{t})} \geq a_{(i,t,\bar{t})}.$$

Proof: Suppose not, i.e. suppose for some $t, \bar{t}$ and $i' > i$, we have that $a_{(i',t,\bar{t})} < a_{(i',t,\bar{t})}$. From the fact that i has no incentive to say that he is i', we have that:

$$i \cdot (a_{(i,t,\bar{t})} - a_{(i',t,\bar{t})}) \geq P_{(i,t,\bar{t})} - P_{(i',t,\bar{t})}.$$

Since $i' > i$, we have that:

$$i' \cdot (a_{(i,t,\bar{t})} - a_{(i',t,\bar{t})}) > P_{(i,t,\bar{t})} - P_{(i',t,\bar{t})}.$$

Rewriting, we see:

$$v(a_{(i',t,\bar{t})}|(i', t, \bar{t})) - P_{(i',t,\bar{t})} < v(a_{(i,t,\bar{t})}|(i', t, \bar{t})) - P_{(i,t,\bar{t})}.$$

This contradicts incentive compatibility for type $(i', t, \bar{t})$. □

Next we identify a subset of these IC constraints that are redundant. As a notational shorthand we refer to the IC constraint corresponding to where a type $(i, t, \bar{t})$ misreports his type as $(i', t', \bar{t}')$ by the notation $(i, t, \bar{t}) \rightarrow (i', t', \bar{t}')$.

Lemma 1 *The IC constraint (8) is implied by (5), (6) and (7).*

Proof: Add the ICs $(i, t, \bar{t}) \rightarrow (i, t', \bar{t}')$ and $(i, t', \bar{t}') \rightarrow (i', t', \bar{t}')$. This yields:

$$\begin{aligned} P_{(i,t,\bar{t})} - P_{(i',t',\bar{t}')} \leq \; & v(a_{(i,t,\bar{t})}|(i, t, \bar{t})) + v(a_{(i,t',\bar{t}')}|(i, t', \bar{t}')) - \\ & v(a_{(i,t',\bar{t}')}|(i, t, \bar{t})) - v(a_{(i',t',\bar{t}')}|(i, t', \bar{t}')). \end{aligned} \tag{10}$$

Recall that by our assumptions on the functional form of v;

$$v(a_{(i,t',\bar{t}')}|(i, t, \bar{t})) = v(a_{(i,t',\bar{t}')}|(i, t', \bar{t}')), \tag{11}$$

$$v(a_{(i',t',\bar{t}')}|(i, t', \bar{t}')) = v(a_{(i',t',\bar{t}')}|(i, t, \bar{t})). \tag{12}$$

Substituting from (11) and (12) into (10), we see that (10) implies the IC (8). □

We now show that of the remaining IC constraints only the 'adjacent' ones matter.

Theorem 1 *Suppose that v is of the form described in section 2. Further suppose that $i > i' \Rightarrow a_{(i,t,\bar{t})} \geq a_{(i',t,\bar{t})}$ (follows from Observation 3). Then all IC constraints are implied by the following adjacent IC constraints:*

1. $(i, t, \bar{t}) \rightarrow (i + 1, t, \bar{t})$
2. $(i, t, \bar{t}) \rightarrow (i - 1, t, \bar{t})$
3. $(i, t, \bar{t}) \rightarrow (i, t + 1, \bar{t})$
4. $(i, t, \bar{t}) \rightarrow (i, t, \bar{t} - 1)$

Proof: To prove item 1, we show that the following pair of inequalities:

$$\begin{aligned} v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} &\geq v(a_{(i-1,t,\bar{t})}|(i,t,\bar{t})) - P_{(i-1,t,\bar{t})}, \\ v(a_{(i-1,t,\bar{t})}|(i-1,t,\bar{t})) - P_{(i-1,t,\bar{t})} &\geq v(a_{(i-2,t,\bar{t})}|(i-1,t,\bar{t})) - P_{(i-2,t,\bar{t})} \end{aligned}$$

imply

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq v(a_{(i-2,t,\bar{t})}|(i,t,\bar{t})) - P_{(i-2,t,\bar{t})}.$$

The rest follows by induction.Adding the two inequalities above, and adding and subtracting $v(a_{(i-2,t,\bar{t})}|(i,t,\bar{t}))$; we get

$$\begin{aligned} v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq\ & [v(a_{(i-1,t,\bar{t})}|(i,t,\bar{t})) - v(a_{(i-2,t,\bar{t})}|(i,t,\bar{t}))] - \\ & [v(a_{(i-1,t,\bar{t})}|(i-1,t,\bar{t})) - v(a_{(i-2,t,\bar{t})}|(i-1,t,\bar{t}))] + \\ & v(a_{(i-2,t,\bar{t})}|(i,t,\bar{t})) - P_{(i-2,t,\bar{t})}. \end{aligned}$$

Recall that by increasing differences, and monotonicity of the allocation rule:

$$\begin{aligned} 0 \leq\ & [v(a_{(i-1,t,\bar{t})}|(i,t,\bar{t})) - v(a_{(i-2,t,\bar{t})}|(i,t,\bar{t}))] - \\ & [v(a_{(i-1,t,\bar{t})}|(i-1,t,\bar{t})) - v(a_{(i-2,t,\bar{t})}|(i-1,t,\bar{t}))]. \end{aligned}$$

Therefore, we have that

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq v(a_{(i-2,t,\bar{t})}|(i,t,\bar{t})) - P_{(i-2,t,\bar{t})}.$$

Item 2 can also be proven in a similar manner, and the proof is omitted. We now proceed to show item 3. To show this we prove that the following pair of inequalities:

$$\begin{aligned} v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq v(a_{(i,t+1,\bar{t})}|(i,t,\bar{t})) - P_{(i,t+1,\bar{t})}, \\ v(a_{(i,t+1,\bar{t})}|(i,t+1,\bar{t})) - P_{(i,t+1,\bar{t})} \geq v(a_{(i,t+2,\bar{t})}|(i,t+1,\bar{t})) - P_{(i-2,t+2,\bar{t})} \end{aligned}$$

imply that

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq v(a_{(i,t+2,\bar{t})}|(i,t,\bar{t})) - P_{(i,t+2,\bar{t})}.$$

To see this add the two ICs and recall that based on our assumptions about functional form of v.

$$v(a_{(i,t+1,\bar{t})}|(i,t,\bar{t})) = v(a_{(i,t+1,\bar{t})}|(i,t+1,\bar{t})) \tag{13}$$

$$v(a_{(i,t+2,\bar{t})}|(i,t+1,\bar{t})) = v(a_{(i,t+2,\bar{t})}|(i,t,\bar{t})) \tag{14}$$

Making the appropriate substitution via (13)&(14), we get the desired inequality. The proof of item 4 is similar and omitted. □

Now, we show that under certain conditions, one need not even consider one of the under-and over-report of value ICs (5).

Lemma 2 *If either the under-report of value $((i,t,\bar{t}) \to (i-1,t,\bar{t}))$ or over-report of value $((i-1,t,\bar{t}) \to (i,t,\bar{t}))$ IC binds, the other is satisfied.*

Proof: If the downward adjacent IC binds, then we have that

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t}) - v(a_{(i-1,t,\bar{t})}|(i,t,\bar{t}) = P_{(i,t,\bar{t})} - P_{(i-1,t,\bar{t})}.$$

Invoking monotonicity of the allocation rule (which follows from Observation 3), followed by increasing differences, we have that

$$v(a_{(i,t,\bar{t})}|(i-1,t,\bar{t}) - v(a_{(i-1,t,\bar{t})}|(i-1,t,\bar{t}) \leq P_{(i,t,\bar{t})} - P_{(i-1,t,\bar{t})}.$$

This implies that the corresponding over-report IC is satisfied. The other direction can be shown similarly. □

3.2 Monotonicity

When types are one dimensional, an allocation rule is said to be monotonic if higher types have a higher (interim) probability of getting allotted than lower types. We modify this definition to account for the fact that in this setting, types are multi-dimensional.

Definition 1 *An allocation rule a is said to be monotonic if:*

1. $\forall i, i', t, \bar{t} : (i \geq i') \Rightarrow (a_{(i,t,\bar{t})} \geq a_{(i',t,\bar{t})})$, *i.e. a higher valuation increases probability of allotment (all other things being equal).*
2. $\forall i, t, \bar{t}, t' : (t \geq t') \Rightarrow (a_{(i,t,\bar{t})} \leq a_{(i,t',\bar{t})})$, *i.e. an earlier entry into the system increases probability of allotment (all other things being equal).*
3. $\forall i, t, \bar{t}, \bar{t}' : (\bar{t} \leq \bar{t}') \Rightarrow (a_{(i,t,\bar{t})} \leq a_{(i,t,\bar{t}')})$, *i.e. a later exit from the system increases probability of allotment (all other things being equal).*

Note that this is equivalent to the allocation rule being monotonic with respect to the partial order we imposed on the space of types, i.e.

$$(i,t,\bar{t}) \succeq (i',t',\bar{t}') \Rightarrow a_{(i,t,\bar{t})} \geq a_{(i',t',\bar{t}')}. \tag{15}$$

Also, note that in our setting, part 1 of the definition follows from Observation 3. However parts 2 & 3 are not implications of incentive compatibility (as opposed to the classical 1-D types case where monotonicity is an implication of IC).

We now state a more general theorem:

Proposition 1 *Suppose the distribution of types meets the monotone hazard rate described earlier (3). Consider a class of mechanisms that satisfy (5) & (6), i.e. the agent can misreport his valuation and entry time but where the agent must report his* *exit time truthfully. There exists a mechanism with an allocation scheme that meets parts* 1 *&* 2 *of Definition 1 which maximizes expected revenue within the class of all feasible, incentive compatible and individually rational mechanisms. Furthermore if this allocation rule satisfies part* 3 *of Definition 1, then the mechanism will be Incentive Compatible, i.e. it will satisfy the IC constraints (5-7).*

The allocation scheme proceeds as follows: the valuation of each type is replaced with a 'virtual valuation', defined below in (20). Any type is considered for allocation only at its stated exit time. It is allocated only if its 'virtual valuation' exceeds the expected marginal value of that unit given that the optimal policy is followed on all other currently received bids and future arrivals. The allocation rule is formally described and discussed in Appendix A. The payment rule to accompany this allocation rule is described in Theorem 2.

If this allocation rule is not monotonic with respect to exit time, the (expected) revenue maximizing mechanism is hard to specify. We provide sufficient conditions for the allocation rule to be monotonic below, along with a discussion of possible interpretations. Recall that we required the inverse hazard rate to be ordered as the types, i.e.

$$(i', t', \bar{t}') \succeq (i, t, \bar{t}) \Rightarrow \frac{f_{(t',\bar{t}')}(i')}{1 - F_{(t',\bar{t}')}(i')} \geq \frac{f_{(t,\bar{t})}(i)}{1 - F_{(t,\bar{t})}(i)}.$$

This condition can be broken into two parts: firstly, fixing entry and exit times, the distribution of valuations with that entry and exit time has a increasing hazard rate- this is a standard assumption in auction design since Myerson (10). The other part of this condition has a natural interpretation: fix $(t, \bar{t})$ and $(t', \bar{t}')$ such that $t' \geq t$ and $\bar{t}' \leq \bar{t}$. The condition implies that

$$\mathbb{E}[i|(t, \bar{t}), i \geq \nu] \geq \mathbb{E}[i|(t', \bar{t}), i \geq \nu]$$

Thinking of users who stay in the system shorter as 'business' travelers, and users who stay in the system longer as 'leisure' travelers, we are effectively saying that the latter is expected to have higher valuations than the former.

A sufficient condition for allocations to be monotonic in exit time is that types with lower exit times ceteris paribus have a substantially lower hazard rate, ie

Definition 2 *A distribution is said to have sufficiently increasing hazard rate if, for any two types* $(i, t, \bar{t})$ *and* $(i, t, \bar{t} + 1)$*, we have that:*

$$\frac{f_{(t,\bar{t})}(i)}{1 - F_{(t,\bar{t})}(i)} < \frac{f_{(t,\bar{t}+1)}(i)}{1 - F_{(t,\bar{t}+1)}(i)} + c(i, T, n, t, \bar{t}, F) \tag{16}$$

where $c(.)$ *is a non-negative function which depends on the problem parameters.*

It appears impossible to get a closed form expression for c. However, in the appendix we give an example to show how the existence of such a c can be verified in a particular instance.

4 A Formulation

We now look to formulate the problem of finding the revenue maximizing mechanism as a linear program. Before we do this, we identify the constraints that any such program must satisfy.

Let $n_{(i,t,\bar{t})}(I)$ be the number of agents of type $(i, t, \bar{t})$. Now, let us enumerate the constraints our mechanisms must satisfy:

1. Incentive Compatibility: We have already shown that the only relevant ICs are adjacent versions of equations (5), (6) and (7).
2. Individual Rationality: All types must have non-negative expected surplus from participating in the auction.

$$v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} \geq 0$$

3. Monotonicity: Recall that monotonicity was defined as:

$$(i,t,\bar{t}) \succeq (i',t',\bar{t}') \Rightarrow a_{(i,t,\bar{t})} \geq a_{(i',t',\bar{t}')}.$$

4. Feasibility of allocation:

$$\forall I : \sum_{\tau \in T} \sum_{(i,t,\bar{t}) \in \mathbb{T}} n_{(i,t,\bar{t})}(\pi) a^{\tau}_{(i,t,\bar{t})}(I^{\tau}) \leq C. \tag{17}$$

5. No Clairvoyance: This says that an online allocation rule cannot base the allotment at time τ on the types of agents who enter the system after time τ. We have implicity assumed this by specifying the allocation at time τ as a function of the inventory up to time τ, I^{τ}.
6. Allot Agents at exit: This says that only agents exiting the system at the time can be allocated units, i.e.

$$\forall I; \forall (i,t,\bar{t}) \in \mathbb{T}, \forall \tau \neq \bar{t} : \mathcal{A}^{\tau}_{(i,t,\bar{t})}(I^{\tau}) = 0 \tag{18}$$

This is without loss of generality by Observation 1.

Now suppose an allocation rule is given. We characterize a revenue maximizing IC payment rule that supports it. Our methodology is based on Malakhov and Vohra (8). They show in their model that the problem of finding the optimal payment rule given an allocation rule is the dual of finding the shortest path in a network. They also show that if the allocation rule is monotonic, then one can characterize these shortest paths.

So suppose that the allocation rule given to us meets the constraints listed above. Recall that by Lemma 2, if the under-report of value IC is met as an equality, then the corresponding over-report constraint is satisfied. We therefore proceed by relaxing the over-report ICs. We show that the solution to our relaxed optimization problem does in fact satisfy under-report ICs at equality, thus justifying relaxing over-report

ICs.

Therefore, fix a (feasible) allotment rule $\mathcal{A}$. The relaxed linear program to maximize expected revenue from a single agent is [OPTPRICE]:

$$\max_{P_{(i,t,\bar{t})}} \sum_{(i,t,\bar{t})\in\mathbb{T}} f_{(i,t,\bar{t})} P_{(i,t,\bar{t})}$$

$$\text{s.t.}$$

$$\begin{aligned}
v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} &\geq v(a_{(i-1,t,\bar{t})}|(i,t,\bar{t})) - P_{(i-1,t,\bar{t})} \\
v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} &\geq v(a_{(i,t+1,\bar{t})}|(i,t,\bar{t})) - P_{(i,t+1,\bar{t})} \\
v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} &\geq v(a_{(i,t,\bar{t}-1)}|(i,t,\bar{t})) - P_{(i,t,\bar{t}-1)} \\
v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - P_{(i,t,\bar{t})} &\geq 0
\end{aligned}$$

We now describe a network representation of this linear program. Fix the allocation rule as suggested above. Introduce a vertex for each type in $\mathbb{T}$. Introduce one more vertex as a dummy vertex. The dummy vertex corresponds to a dummy type that always gets allotted nothing, and pays 0, which allows us to represent the individual rationality constraint as an extra IC constraint. We need to introduce 4 types of edges corresponding to four classes of adjacent ICs in this model (over-report of valuation is relaxed based on Lemma 2).

1. IC $(i,t,\bar{t}) \to (i-1,t,\bar{t})$: Introduce an edge from $(i-1,t,\bar{t})$ to $(i,t,\bar{t})$, of length $v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - v(a_{(i-1,t,\bar{t})}|(i,t,\bar{t})) = i.(a_{(i,t,\bar{t})} - a_{(i-1,t,\bar{t})})$.
2. IC $(i,t,\bar{t}) \to (i,t+1,\bar{t})$: Introduce an edge from $(i,t+1,\bar{t})$ to $(i,t,\bar{t})$, of length $v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - v(a_{(i,t+1,\bar{t})}|(i,t,\bar{t})) = i.(a_{(i,t,\bar{t})} - a_{(i,t+1,\bar{t})})$.
3. IC $(i,t,\bar{t}) \to (i,t,\bar{t}-1)$: Introduce an edge from $(i,t,\bar{t}-1)$ to $(i,t,\bar{t})$, of length $v(a_{(i,t,\bar{t})}|(i,t,\bar{t})) - v(a_{(i,t,\bar{t}-1)}|(i,t,\bar{t})) = i.(a_{(i,t,\bar{t})} - a_{(i,t,\bar{t}-1)})$.

Further, introduce an edge from the dummy node to each vertex $(i,t,\bar{t})$ of length $i.a_{(i,t,\bar{t})}$.

[OPTPRICE] is the dual of a min-cost flow problem on the graph specified above, where a flow of $f_{(i,t,\bar{t})}$ needs to be sent to vertex $(i,t,\bar{t})$ from the dummy node. (see for instance Papadimitriou and Steiglitz (12)). Let $P_{(i,t,\bar{t})}$ be the cost of the relevant flow. We show that the shortest path in such a network is well defined. Monotonicity of the allocation rule implies all edges in this network are positive in length, ruling out negative cycles.We show that the shortest path from type $(1,\tau,\tau)$ to a generic type $(i,t,\bar{t})$ (where $t \leq \tau \leq \bar{t}$) is of the form $(1,\tau,\tau) \to (1,t,\bar{t}) \to (2,t,\bar{t}) \ldots \to (i,t,\bar{t})$.

The following theorem shows that the payments of any type $(i,t,\bar{t})$ is a linear function of its own allocation, and the allocations of types $(i',t,\bar{t})$, $i' < i$.

Theorem 2 *The optimal payment rule given a monotonic allocation rule is:*

$$P_{(i,t,\bar{t})} = i.a_{(i,t,\bar{t})} - \sum_{k=1}^{i-1} a_{(k,t,\bar{t})}. \tag{19}$$

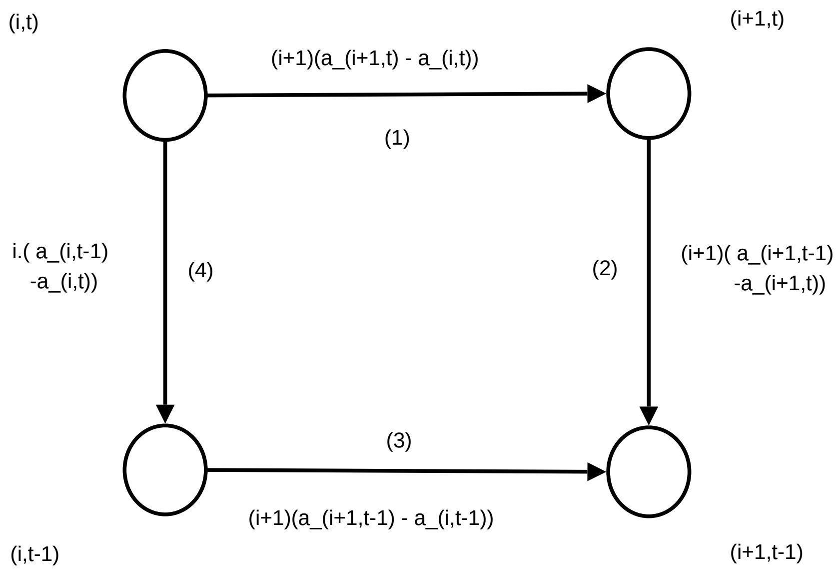

FIGURE 1: The Shortest Path Graph

PROOF: To begin, suppose all agents have exit time T. We therefore drop exit times from the agents' type wlog. Consider the 4 adjacent types $(i,t),(i+1,t),(i,t-1),(i+1,t-1)$, as shown in Figure 1. There are 2 possible paths from (i,t) to $(i+1,t-1)$- edge 1 followed by 2 and edge 4 followed by 3. The length of the former path is $(i+1)(a_{(i+1,t-1)} - a_{(i,t)})$. The length of the latter path is $(i+1).a_{(i+1,t-1)} - i.a_{(i,t)} - a_{(i,t-1)}$. By monotonicity $a_{(i,t-1)} \geq a_{(i,t)}$ and therefore the latter path is shorter than the former. By the Principle of Optimality, in a setting where all agents had exit time T, the shortest path from $(1,T)$ to a generic type (i,t) would be $(1,T) \to (1,T-1) \ldots \to (1,t) \to (2,t) \ldots \to (i,t)$.

Now, we drop the assumption that all agents have the same exit time T. Suppose we wish to find the shortest path from $(i,t,\bar{t})$ to $(i,t-1,\bar{t}+1)$. There are 4 possible paths: [6]

1. $(i,t,\bar{t}) \to (i,t,\bar{t}+1) \to (i,t-1,\bar{t}+1) \to (i+1,t-1,\bar{t}+1)$.
2. $(i,t,\bar{t}) \to (i,t-1,\bar{t}) \to (i+1,t-1,\bar{t}) \to (i+1,t-1,\bar{t}+1)$.
3. $(i,t,\bar{t}) \to (i,t-1,\bar{t}) \to (i,t-1,\bar{t}+1) \to (i+1,t-1,\bar{t}+1)$.
4. $(i,t,\bar{t}) \to (i+1,t,\bar{t}) \to (i+1,t,\bar{t}+1) \to (i+1,t-1,\bar{t}+1)$.

The lengths of these paths are:

1. Paths 1 and 3: $(i+1)a_{(i+1,t-1,\bar{t}+1)} - i.a_{(i,t,\bar{t})} - a_{(i,t-1,\bar{t}+1)}$.
2. Path 2: $(i+1)a_{(i+1,t-1,\bar{t}+1)} - i.a_{(i,t,\bar{t})} - a_{(i,t-1,\bar{t})}$.
3. Path 4: $(i+1)(a_{(i+1,t-1,\bar{t}+1)} - a_{(i,t,\bar{t})})$.

By monotonicity $a_{(i+1,t-1,\bar{t}+1)} \geq a_{(i,t-1,\bar{t})} \geq a_{(i,t,\bar{t})}$, and therefore path 1 is the shortest path. Once again, applying the principle of optimality, the shortest path is of they

[6]This list is not exhaustive, but other paths can be ruled out by the previous argument.

type described, and Equation (19) follows. $\square$

By Theorem 2 we can now write the problem of finding the revenue maximizing auction [OPTAUC] as:

$$
\begin{aligned}
\max_{\{a\}} \sum_{(i,t,\bar{t})\in\mathbb{T}} f_{(i,t,\bar{t})}(i.a_{(i,t,\bar{t})} - \sum_{k=1}^{i} a_{(k,t,\bar{t})}) & & \\
\text{s.t.} \quad \forall (i,t,\bar{t}) \succeq (i',t',\bar{t}') : \quad a_{(i,t,\bar{t})} & \geq & a_{(i',t',\bar{t}')} \\
\sum_{\pi^{n-1}} \mathbb{P}[\pi = ((i,t,\bar{t}),\pi^{n-1})|(i,t,\bar{t}) \in \pi] \mathcal{A}^{\bar{t}}_{(i,t,\bar{t})}(((i,t,\bar{t}),\pi^{n-1})) & = & a_{(i,t,\bar{t})} \\
\forall \pi \in \Pi : \sum_{\tau\in T} \sum_{(i,t,\bar{t})\in\mathbb{T}} n_{(i,t,\bar{t})}(\pi) a^{\tau}_{(i,t,\bar{t})}(\pi^{\tau}) & \leq & Q \\
\forall \pi; \forall \tau \in T; \forall (i,t,\bar{t}) \in \mathbb{T} : \mathcal{A}^{\bar{t}}_{(i,t,\bar{t})}(\pi^{\tau}) & \in & \{0, q\} \\
\forall \pi; \forall (i,t,\bar{t}) \in \mathbb{T}, \forall \tau \neq \bar{t} : \mathcal{A}^{\tau}_{(i,t,\bar{t})}(\pi^{\tau}) & = & 0
\end{aligned}
$$

Let $F_{t\bar{t}}(r) = \sum_{k=1}^{r-1} f_{(t,\bar{t})}(k)$. Then we can re-write the objective function of [OPTAUC] as

$$\sum_{(i,t,\bar{t})\in\mathbb{T}} f_{(i,t,\bar{t})} a_{(i,t,\bar{t})} \nu_{i,t,\bar{t}}$$

where

$$\nu_{i,t,\bar{t}} = \Big(i - \frac{1 - F_{t\bar{t}}(i)}{f_{(t,\bar{t})(i)}} \Big) \tag{20}$$

is the type $(i,t,\bar{t})$'s virtual valuation (in the sense of Myerson). Actually solving this program requires a dynamic programming approach that is outlined in Appendix A.

We conclude this section by giving an example where the optimal allocation rule is non-monotonic, but the payment rules turn out to be the same as those computed by Theorem 2. This shows that monotonicity of the allocation rule is sufficient but not necessary for our pricing rule to be incentive compatible. Also, the allocation rule fails part 3 of Definition 1; and this can be traced back to the fact that the setting below does not meet Definition 2.

An Example Let $T = 2$, $C = 1$. There are 2 possible valuations, i.e. $I = \{1, 2\}$. Therefore there will be 6 possible types, corresponding to the 3 possible entry-exit times $(1, 1)$, $(1, 2)$ and $(2, 2)$ and 2 possible valuations. Let the arrival rate be such that in each period 1 agent arrives with probability 0.49 (and no-agent arrives otherwise). Suppose that conditional on arrival at time 2, type $(2, 2, 2)$ has probability 1. Further suppose that conditional on arrival at time 1, types $(2, 1, 2)$ and $(2, 1, 1)$ have probability 0. It is easy to show that this distribution meets the monotone hazard rate condition. However it does not meet Definition 2, since the types $(i, 1, 2)$ and $(i, 1, 1)$ have the same virtual valuations. Finally consider the following allocation rule: $a_{(1,1,1)} = 1$, $a_{(2,2,2)} = 1$, $a_{(1,1,2)} = .51$. Note that this rule is clearly not monotonic in the sense of Definition 1. However, coupled with the payment rule $P_{(1,1,1)} = 1$, $P_{(1,1,2)} = .51$ and $P_{(2,2,2)} = 2$, this is incentive compatible.

To see that this is the optimal allocation rule: if type $(1, 1, 1)$ arrives at time 1, allot him for a payment of 1 (in expectation this is better than not allotting him). On the other hand if $(1, 1, 2)$ arrives, then it is optimal to wait, since, potentially in period 2 an agent of higher valuation arrives. Finally note that the payments are the same as would have been calculated from equation (19).

5 Conclusion

In this paper, we formulate and solve a multi-period dynamic mechanism problem for the sale of multiple identical items, with the novel feature that we allow agents to be strategic with respect to the revelation of their arrival times. Remarkably, despite the generality of the problem, the solution turns out to be intuitive in that it is the natural generalization of the optimal auction for the static case to this dynamic framework. Our use of a discrete type space has the advantage of transparency of analysis, and allows us to approach this problem from an intuitive graph theoretic perspective.

As for additional work, we can see several important extensions to this model that are potentially interesting. Intuitively, the assumption driving our results is the preferences of the bidder. We assume throughout that the bidder desires exactly 1 unit of a homogenous good, and would get no utility from being allotted outside his time in the system. This cuts down severely on the number of ways a bidder may misreport his private information. Recall that we motivated this problem (rather, this entire model has been motivated) as particularly relevant to the airline and hotel industries. Given that several companies and online retailers in this sector are trying to sell 'package deals', where they sell, for example, airline tickets and hotel reservations together, we believe it may be of interest to study how this can be done optimally, while relaxing our rather strong assumption on preferences. Models of selling heterogenous goods dynamically have been studied, but there has been no work we are aware of where buyers are allowed to be strategic with respect to their time preferences.

References

M. Armstrong. Multiproduct nonlinear pricing. *Econometrica*, 64(1):51–75, January 1996.

S. Board. Durable-goods monopoly with varying demand. University of Toronto, 2005.

K. Border. Implementation of reduced form auctions: A geometric approach. *Econometrica*, 59(4):1175–1187, 1991.

J. Bulow. Durable-goods monopolists. *Journal of Policial Economy*, 90(2):314–332, 1982.

J. Gallien. Dynamic mechanism design for online commerce. *Operations Research*, 54(2):291–310, 2006.

M. Hajiaghayi, R. Kleinberg, M. Mahdian, and D. Parkes. Online auctions with re-usable goods. In *Proc. 6th ACM Conf. on Electronic Commerce (EC'05)*, pages 165–174, 2005.

R. Lavi and N. Nisan. Competitive analysis of online auctions. In *Proceedings of the 2nd ACM Conference on Electronic Commerce*, 2000.

A. Malakhov and R. Vohra. Single and multi-dimensional optimal auctions- a network approach. CMS-EMS DP No.1397, Northwestern University, December 2004.

H. Moulin. On scheduling fees to prevent merging, splitting and tranferring of jobs. Rice University, December 2005.

R. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6:58–73, 1981.

C. Ng, D. Parkes, and M. Seltzer. Virtual Worlds: Fast and Strategyproof Auctions for Dynamic Resource Allocation. Technical report, Harvard University, 2002.

C. Papadimitriou and K. Steiglitz. *Combinatorial Optimization: Algorithms and Complexity.* Prentice-Hall, Edgewood Cliffs, N.J., 1982.

J.-C. Rochet and P. Chone. Ironing, sweeping, and multidimensional screening. *Econometrica*, 66(4):783–826, July 1998.

N. Stokey. Intertemporal price discrimination. *Quarterly Journal of Economics*, 93(3):355–371, 1979.

G. Vulcano, G. van Ryzin, and C. Maglaras. Optimal dynamic auctions for revenue management. *Management Science*, 48(11):1388–1407, 2002.

A Monotonicity

In this appendix we give a proof for Proposition 1. In our opinion this proof is of independent interest. Since the proof is reasonably elaborate, we build intuition by examining the 1-D case ($C = 1$, $T = 1$) where agents cannot over-report their valuation. We believe this case captures the crux of the argument. In the one-dimensional types case, a variant of the classical Spence-Mirrlees condition tells us that if an agent can misreport his type both up and down, only monotonic allocation rules can be Incentive Compatible. When misreports can only take place in one direction, monotonicity is not implied by Incentive Compatibility.

In the first subsection, we consider the 1-D case as described above (i.e. agents cannot over-report their valuation). Then we prove prove Proposition 1 for our model.

A.1 Single Dimensional Types

There are n risk-neutral bidders bidding for a single object to be sold by a risk-neutral seller. Bidders have private valuations $i \in I = \{1, 2, \ldots, I\}$. In terms of our model, $C = 1$, $T = 1$, $Q = \{1\}$. Types are i.i.d. by some distribution f, which satisfies the monotone hazard rate condition, i.e. $\frac{f(i)}{1-F(i)}$ is increasing in i. There are n agents/bidders. Possible realizations of types are $\pi \in \Pi = I^n$. Suppose, no agent can lie 'upward', i.e. an agent of type i cannot report that he is of type $i' > i$. Hence only the downward IC constraints need be imposed. Border (3) shows how to write the space of feasible (interim) allocation rules as a set of linear constraints.

The revenue maximizing program can be rewritten as $[P]$

$$
\begin{aligned}
& \max_{a,p} \sum_{i=1}^{T} f(i) p_i \\
& \text{s.t. } i.a_i - p_i \geq i.a_{i'} - p_{i'} \quad \forall i, i' < i \\
& \quad i.a_i - p_i \geq 0 \forall i \\
& \quad a \text{ feasible} \\
& \quad a_i \in [0,1] \qquad \forall i \in T
\end{aligned}
$$

To solve $[P]$ consider a relaxed program where agent of type i can only report his type as i or $i-1$: $[P']$

$$
\begin{aligned}
& \max_{a,p} \sum_{i=1}^{T} f(i) p_i \\
& \text{s.t. } i.a_i - p_i \geq i.a_{i'} - p_{i'} \quad \forall i, i' = i-1 \\
& \quad i.a_i - p_i \geq 0 \forall i \\
& \quad a \text{ feasible} \\
& \quad a_i \in [0,1] \qquad \forall i \in T
\end{aligned}
$$

Claim 1 Given any (feasible) allocation rule a, the prices that maximize revenue for $[P']$ will be $p_i = i a_i - \sum_{j=1}^{i-1} a_j$.

Proof: We prove this by induction. Consider the lowest type i that gets allotted. Clearly in any revenue maximizing auction it will pay i. Induction Hypothesis: $(i+k)$ pays : $(i+k)a_{i+k} - \sum_{j=1}^{i+k-1} a_j$.

Induction Step: In the optimal solution, $i+k+1$ is indifferent to being $i+k$ (else increase his price till he is: recall that in this relaxed program, an agent with valuation i can only report his type as i or $i-1$).

$$
(i+k+1)a_{i+k+1} - p_{i+k+1} = (i+k+1)a_{i+k} - p_{i+k} = \sum_{j=1}^{i+k} a_j .
$$

□

Claim 2 Therefore, given a, the net revenue is $\sum_{j=1}^{T} f(i)a_i(i - \frac{1-F(i)}{f(i)})$.

Pick the following solution to program $[P']$. In each profile allot the highest type as long as it has a positive virtual value. Call this allocation rule a^*. Note that a^* is clearly monotonic. Given the monotone hazard rate assumption, a^* is optimal for $[P']$. Monotonicity of a^* implies that it is feasible in $[P]$. Therefore it must also be optimal $[P]$.

Claim 3 The allocation rule a^* and the prices $p_i = ia_i^* - \sum_{j=1}^{i-1} a_j^*$ are optimal for $[P]$ given our monotone hazard rate assumption.

A.2 A Dynamic Allocation Rule

The idea of the proof is the same as above. First, we relax the IC constraints corresponding to misreports of time i.e. (6),(7). By the machinery in Section 4; solving the problem is equivalent to solving a maximization problem where valuations in the objective function are replaced by 'virtual valuations' as defined in (20), as long as the resulting allocation is monotonic with respect to Part 1 of Definition 1. We relax this monotonicity condition as well; and show that if virtual valuations are monotonic, the solution of the program meets parts 1 and 2 of Definition 1. This proves the first part of Proposition 1. Finally, we show that if the virtual valuations also meet the extra condition in Definition 2; the allocation rule will meet part 3 of Definition 1 as well, therefore proving the second part of Proposition 1. The proof is convoluted, and to aid understanding we begin by considering the case $C = 1$, i.e. there is only 1 unit for sale.

A.2.1 The 1 Unit Case

With monotonicity, the incentive constraints and the individual rationality condition relaxed, the problem is a standard dynamic allocation problem: At any time t, n_t users enter, each having valuation and exit time $(\nu_{i,t,\bar{t}}, \bar{t})$ with probability $\frac{f_{t,\bar{t}}(i)}{\sum_{i',\bar{t}'} f_{t,\bar{t}'}(i')}$. Further any unallocated users with exit time t exit. If a unit has not already been allotted, the program can choose to allot it to any user currently in the system (including the ones exiting the system in that period). Standard arguments show that in any period, the expected value maximizing program will allot if at all to agents exiting the system in that period. Further, at each time t, there exists a function $R_{t+1}(\cdot)$, whose argument is the types of users in the system arriving at or before period t, and departing in period $t+1$ or later (denoted by I_{t+1}^t). [7] The optimal policy at time t allots to the highest virtual valuation exiting at time t conditional on it being higher than the cutoff $R_{t+1}(\cdot)$. This cutoff, R_{t+1} represents the expected value of following the optimal policy from $t + 1$ onward, given the users currently

[7] We do not formally specify the domain of the function R_t for ease of notation.

in the system. One can easily specify the family of functions $\{R_t\}_{t=1}^T$ by backward induction. The notation $()^+$ in the sequel refers to the maximum element in a set.

$$\begin{aligned} R_{T+1} &= 0; & (21)\\ R_T(I) &= \mathbb{E}_{A^T}[(I \bigcup A^T)^+]; & (22)\\ R_t(I) &= \mathbb{E}_{A^t}\Big[\mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I))^+ > R_{t+1}(A^t_{t+1}\bigcup I_{t+1})\}}(L_t(A^t)\bigcup L_t(I))^+ \\ &\quad + \mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I))^+ \le R_{t+1}(A^t_{t+1}\bigcup I_{t+1})\}} R_{t+1}(A^t_{t+1}\bigcup I_{t+1})\Big]. & (23)\end{aligned}$$

Equation (21) says that the value of the unit after period T is 0. Equation (22) says that the value of the unit in the last period is the expected maximum virtual valuation among the remaining inventory and the users that arrive in that period. Equation (23) inductively defines the value of the unit in period t as a function of the inventory. The allocation rule can allot either to the agent leaving in period t with the highest virtual valuation, which it will do if this is larger than $R_{t+1}(A^t_{t+1}\bigcup I_{t+1})$. Alternately, it can let those bids expire and pick up $R_{t+1}(A^t_{t+1}\bigcup I_{t+1})$ (in expectation).

Before we can study properties of this allocation rule, we prove some (intuitive) properties of the cutoff rule $R_t(\cdot)$. Our first observation confirms our intuition that a better inventory leads to a higher future expected payoff $R_t(\cdot)$.

Observation 4 If $I_t \succeq I'_t$, $R_t(I_t) \ge R_t(I'_t)$.

Proof: The proof is by induction on t. Clearly if $I_T \succeq I'_T$, we have that $R_T(I_T) \ge R_T(I'_T)$. Assume that for all $t' \ge t+1$,

$$I_{t'} \succeq I'_{t'} \ \Rightarrow\ R_{t'}(I_{t'}) \ge R_{t'}(I'_{t'}).$$

We show this is true for t.

We know that $I_t \succeq I'_t$, let $S_t \subseteq I_t$ be the set of bids that dominate $L_t(I'_t)$. Then for any set of people arriving in period t, A^t; $I_{t+1}\bigcup A^t_{t+1}\backslash S_t \succeq I'_{t+1}\bigcup A^t_{t+1}$. By our inductive hypothesis,

$$R_{t+1}(I_{t+1}\bigcup A^t_{t+1}\backslash S_t) \ge R_{t+1}(I'_{t+1}\bigcup A^t_{t+1}). \quad (24)$$

It remains to show that $R_t(I_t) \ge R_t(I'_t)$. However,

$$\begin{aligned} R_t(I_t) \ \ge\ & \mathbb{E}_{A^t}\Big[\mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I_t)\bigcup S_t)^+ > R_{t+1}(A^t_{t+1}\bigcup I_{t+1}\backslash S_t)\}}(L_t(A^t)\bigcup L_t(I_t)\bigcup S_t)^+ \\ & + \mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I_t)\bigcup S_t)^+ > R_{t+1}(A^t_{t+1}\bigcup I_{t+1}\backslash S_t)\}} R_{t+1}(A^t_{t+1}\bigcup I_{t+1}\backslash S_t)\Big],\end{aligned}$$

where the right hand side of the inequality above corresponds to the suboptimal policy of allotting either the leaving agents at time t or agents in S_t. By definition,

$$R_t(I'_t) = \mathbb{E}_{A^t}\Big[\mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I'_t))^+ > R_{t+1}(A^t_{t+1}\bigcup I'_{t+1})\}}\big(L_t(A^t)\bigcup L_t(I'_t)\big)^+ + \mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I'_t))^+ \leq R_{t+1}(A^t_{t+1}\bigcup I'_{t+1})\}} R_{t+1}(A^t_{t+1}\bigcup I'_{t+1})\Big]$$

Consider the terms inside the expectation operator in the two inequalities above. It is easy to see that

$$\big(L_t(A^t)\bigcup L_t(I_t)\bigcup S_t\big)^+ \geq \big(L_t(I^t_t)\bigcup L_t(I'_t)\big)^+,$$

combining this with equation (24), we get our desired result. □

Armed with this result, we can pursue our original goal of showing that the interim allocation probabilities are monotonic as per Definition 1. We will prove each of the 3 parts of Definition 1 separately, hence proving Proposition 1

Part 1 Since the allocation rule is a cutoff rule, any sequence of bids along which a type $(i, t, \bar{t})$ gets allotted will also result in the allocation of $(i', t, \bar{t})$ for $i' > i$. ■

Part 2 This requires that the interim probability of getting allotted is decreasing in entry time, holding exit time and valuation constant. To see this, fix two types, $(i, t, \bar{t})$ and $(i, t+1, \bar{t})$ (by Theorem 1, monotonicity in adjacent entry times is necessary and sufficient). Consider an I such that $(i, t, \bar{t})$ is in I^t_t. If $(i, t+1, \bar{t})$ is in I^{t+1}_{t+1}, it will never be allotted, since at $\bar{t}$, $(i, t, \bar{t})$ will also be present, but it has a higher virtual valuation. So suppose $(i, t+1, \bar{t})$ is not in I^{t+1}_{t+1}.

Consider the inventory I' constructed as:

$$\begin{aligned} I'^t_t &= I^t_t \backslash \{(i, t, \bar{t})\}, \\ I'^{t+1}_{t+1} &= I^{t+1}_{t+1} \bigcup \{(i, t+1, \bar{t})\}, \end{aligned}$$

with I'^{t+2}_{t+2} through $I'^{\bar{t}}_{\bar{t}}$ defined appropriately. The set of all I''s as constructed above is the set of all inventories that contain the bid $(i, t+1, \bar{t})$, but not $(i, t, \bar{t})$. [8] We show that whenever $(i, t+1, \bar{t})$ is allotted in I',$(i, t, \bar{t})$ is allotted in I. To see that, $(i, t+1, \bar{t})$ will be allotted along this sequence if:

1. The good was not allotted before $t + 1$.
2. The good was not allotted from time $t + 1$ through $\bar{t}$.

[8]It is at this precise step that we need the property that n_t has full support on $\mathbb{Z}_+$. Without it, on some inventories, an agent misreporting his entry time could produce an inventory that was impossible under the problem parameters. For example if in a given period, n_t is such that at most n people can arrive at time t, then a misreport of entry time by an agent who arrived earlier could lead to $n + 1$ agents claiming a t arrival; which the auctioneer knows is impossible. This would make misreports easier to detect and punish- something a full support assumption would rule out.

3. $(i, t+1, \bar{t})$ has the highest virtual valuation among $I_t'^{\bar{t}}$; and this is greater than $R_{\bar{t}+1}(I_{\bar{t}+1}'^{\bar{t}})$.

Further note that since $I_{t'}^{t'} \succeq I_{t'}'^{t'}$ for any t', it must be that $R_{t'}(I_{t'}^{t'}) \geq R_{t'}(I_{t'}^{t'})$. Therefore:

1. If the good was not allotted before $t+1$ along I', it would not have been allotted up to $t+1$ along I (all bids stay the same except that I has the bid $(i, t, \bar{t})$ which I' does not, which weakly increases $R_{t+1}(.)$).
2. If the good was not allotted from time $t+1$ through $\bar{t}$ along I', it would not have been allotted along I either, by the same token.
3. Finally, by assumption the good was allotted to $(i, t+1, \bar{t})$ at $\bar{t}$. Therefore this was the highest virtual valuation among $I_{\bar{t}}'^{\bar{t}}$; and this is greater than $R_{\bar{t}+1}(I_{\bar{t}+1}'^{\bar{t}})$. But then $(i, t, \bar{t})$ has the highest virtual valuation among $I_{\bar{t}}^{\bar{t}}$ (by construction); and this is greater than $R_{\bar{t}+1}(I_{\bar{t}}^{\bar{t}})$ since $I_{\bar{t}}'^{\bar{t}} = I_{\bar{t}}^{\bar{t}}$.

Therefore $a_{i,t,\bar{t}} \geq a_{i,t+1,\bar{t}}$. ■

Part 3 Fix two types $(i, t, \bar{t})$ and $(i, t, \bar{t}+1)$, we refer to them in the rest of this proof as τ and τ' respectively. Denote their virtual valuations by ν_τ and $\nu_{\tau'}$. We need to show that $a_{(i,t,\bar{t})} \leq a_{(i,t,\bar{t}+1)}$. Let $I = (I_1^1, \ldots, I_T^T)$ be a sequence of inventories as before. The probability that the good is allotted to type $(i, t, \bar{t})$, $a_{(i,t,\bar{t})} =$

$$\begin{aligned} & \mathbb{P}[\text{Good not allotted before } t] && (25) \\ \times\ & \mathbb{P}[\text{Good not allotted up to } \bar{t}-1 | \text{ Good not allotted before } t, \tau \in A^t] && (26) \\ \times\ & \mathbb{P}[\nu_\tau \text{ is largest virtual valuation in } L_{\bar{t}}(I_{\bar{t}}^{\bar{t}}), \nu_\tau \geq R_{\bar{t}+1}(I_{\bar{t}+1}^{\bar{t}}) \\ & \quad | \text{ Good not allotted up to } \bar{t}-1, \tau \in A^t]. && (27) \end{aligned}$$

Define $\nu'_{\tau'} = \nu_\tau + c$ for $c > 0$. Compute R' as in Equations (21-23) with the virtual valuation of τ' taken as $\nu'_{\tau'}$. Recompute the allocation rule with this R'. Note that for any time $t' \geq \bar{t}+2$, $R'_{t'} = R_{t'}$. The probability that this good is allotted to type $(i, t, \bar{t}+1)$, $a'_{(i,t,\bar{t}+1)}$, with its virtual valuation defined as $\nu'_{\tau'}$ is:

$$\begin{aligned} & \mathbb{P}[\text{Good not allotted before } t] && (28) \\ \times\ & \mathbb{P}[\text{Good not allotted up to } \bar{t}-1 | \text{ Good not allotted before } t, \tau' \in A^t] && (29) \\ \times\ & \mathbb{P}[\text{Good not allotted at } \bar{t} | \text{ Good not allotted before } \bar{t}, \tau' \in A^t] && (30) \\ \times\ & \mathbb{P}[\nu'_{\tau'} \text{ is largest virtual valuation in } L_{\bar{t}+1}(I_{\bar{t}+1}^{\bar{t}+1}), \nu_{\tau'} \geq R_{\bar{t}+2}(I_{\bar{t}+2}^{\bar{t}+1}) \\ & \quad | \text{ Good not allotted up to } \bar{t}+1, \tau' \in A^t.] && (31) \end{aligned}$$

We note that $a'_{(i,t,t\bar{+}1)}$ is increasing in c:

1. (28) is weakly increasing in c- as the value of a possible future arrival increases; the allocation rule is more conservative about allotting at earlier times.
2. (29) and (30) are weakly increasing in c- as the value of a current type already present in the system increases, the allocation rule is more conservative about

allotting to another type.

3. (31) is weakly increasing in c- as the value of one of the exiting types increases, it is more likely to be the highest valued exiting type. Further it is more likely that it is larger than the expected future payoff from not allotting in that period, $R_{\bar{t}+2}(I_{\bar{t}+2}^{\bar{t}+1})$.

Further for some c large enough; clearly $a'_{(i,t,\bar{t}+1)} \geq a_{(i,t,\bar{t})}$ (for very large values of c, $a'_{(i,t,\bar{t}+1)} \to 1$). Let c^* be the lowest such c. If

$$\nu_{\tau'} \geq \nu_{\tau} + c^*, \tag{32}$$

it follows that $a_{(i,t,\bar{t}+1)} \geq a_{(i,t,\bar{t})}$. By Definition 2, the distribution is such that (32) is satisfied. As we pointed out earlier, we are unable to analytically characterize c^*. Example 1 shows that there are meaningful distributions that meet this condition. ■

Example 1 There is 1 unit for sale, over 2 time periods, 1 and 2. In period 1, 1 or 2 agents arrive with equal probability. In period 2, only 1 agent arrives. Conditional on arriving in period 1, agents have an exit time of 1 or 2 with equal probability.

Agents with entry-exit time combinations $(1, 2)$ and $(2, 2)$ have valuations drawn from a uniform distribution on $[1, 2]$. This implies that the virtual valuation of an agent of type $(v, ., 2) = 2v - 2$; and virtual valuations are uniform on $[0, 2]$

Agents with entry and exit time 1 have valuations drawn from a uniform distribution $[1, 2.5]$. Therefore, the virtual valuation of an agent of type $(v, ., 2) = 2v - 2$; and virtual valuations are uniform on $[-0.5, 2.5]$.

It is easy to see that

$$\begin{aligned} R_2(\phi) &= 1, \\ R_2((v, 1, 2)) &= v^2 - 2v + 2. \end{aligned}$$

If a type $(v, 1, 1)$ arrives at time 1-

- (With probability $\frac{1}{2}$) No other agent shows up in that period, and it is allotted as long as $2v - 2.5 > 1 \Rightarrow v > 1.75$.
- (With probability $\frac{1}{4}$) Another agent of type $(v', 1, 1)$ shows up, it is allotted as long as $v > v'$ and $v > 1.75$; i.e. wp $\frac{v-1}{2.5-1} = \frac{2(v-1)}{3}$.
- (With probability $\frac{1}{4}$) Another agent of type $(v', 1, 2)$ shows up, and our agent is allotted as long as $2v - 2.5 > v'^2 - 2v' + 2$, i.e. $v' < 1 + \sqrt{2v - 3.5}$; therefore w.p. $\sqrt{2v - 3.5}$.

Therefore

$$a_{(v,1,1)} = \begin{cases} \frac{1}{2} + \frac{(v-1)}{6} + \frac{\sqrt{2v-3.5}}{4} & \text{if } v > 1.75 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly if a type $(v, 1, 2)$ arrives at time 1-

- (With probability $\frac{1}{2}$) No other agent shows up in that period, and it is allotted as long as the agent who arrives in period 2 has a lower valuation, i.e. $v - 1$.

- (With probability $\frac{1}{4}$) Another agent of type $(v', 1, 2)$ shows up, and our agent is allotted as long as $v > v'$, and the valuation of the agent who arrives in period 2 is lower i.e. w.p. $(v-1)^2$.
- (With probability $\frac{1}{4}$) Another agent of type $(v'1, 1)$ shows up; and our agent is allotted as long as $2v' - 2.5 < v^2 - 2v + 2 \Rightarrow v' < \frac{v^2-2v+4.5}{2}$ and the agent who arrives in period 2 has a lower valuation; i.e. wp $(v-1)\frac{(v^2-2v+2.5)}{3}$.

Therefore

$$a_{v,1,2} = \frac{v-1}{2} + \frac{(v-1)^2}{4} + \frac{(v-1)(v^2-2v+2.5)}{3}.$$

We only need ensure that $a_{(v,1,2)} \geq a_{(v,1,1)}$ in the range $[1.75, 2]$. To see this note that $a_{(2,1,1)} \approx 0.843$; while $a_{(1.75,1,2)} \approx 0.906$.

A.3 The C-unit case

In this case, the optimal allocation policy will depend also on the number of units k left in the system. Therefore the cutoffs will be of the form $\{R_t^k(\cdot)\}_{k=1}^C$, which are defined inductively as:

$$\forall t \qquad R_t^0(I) = 0; \tag{33}$$

$$\forall k \qquad R_{T+1}^k = 0; \tag{34}$$

$$R_T^1(I) = \mathbb{E}_{A^T}[(I \bigcup A^T)^+]; \tag{35}$$

$$\forall k > 1 \qquad R_T^k(I) = \mathbb{E}_{A^T}[(I \bigcup A^T)^+ + R_T^{k-1}(I \backslash (I \bigcup A^T)^+)]; \tag{36}$$

$$\begin{aligned} \forall k \qquad R_t^k(I) = \mathbb{E}_{A^t}\Big[& \mathbb{I}_{\{(L_t(A^t) \bigcup L_t(I))^+ > R_{t+1}^k(A_{t+1}^t \bigcup I_{t+1})\}} \\ & \times \big(\big(L_t(A^t) \bigcup L_t(I)\big)^+ + R_t^{k-1}(I \backslash \big(L_t(A^t) \bigcup L_t(I)\big)^+ \big) \big) \\ & + \mathbb{I}_{\{(L_t(A^t) \bigcup L_t(I))^+ \leq R_{t+1}(A_{t+1}^t \bigcup I_{t+1})\}} \\ & \times R_{t+1}^k(A_{t+1}^t \bigcup I_{t+1}) \Big] \end{aligned} \tag{37}$$

It is easily noted the appropriate version of Observation 4 remains true in this case- if $I_t \succeq I_t'$ then for any k, t, we have that $R_t^k(I_t) \geq R_t^k(I_t')$. Further, we have that $R_t^k(I_t) \geq R_t^{k-1}(I_t)$. Therefore the proofs of Parts 1 and Part 2 carry over as before.

To obtain the correct version of our proof of part 3 requires one to notice that the number of units allotted up to time $\bar{t}$ when the inventory contains $(i, t, \bar{t})$ is weakly more than the number of units allotted when the inventory contains $(i, t, \bar{t}+1)$. Conditional on any number of units allotted up to $\bar{t}$, we once again need the virtual value to drop sufficiently to ensure that the probabilities of being allotted for for $(i, t, \bar{t})$ are less than $(i, t, \bar{t}+1)$.

B The Revelation Principle

This section outlines a revelation principle for this environment. We first formally state and prove the revelation principle as it applies here. We then discuss why this is, in our opinion, not restrictive in the sense that any Bayesian-Nash Equilibrium where our revelation principle does not apply can be 'transformed' into one where it does (i.e. all the bidders are indifferent between the two equilibria, as is the seller). We conclude with an example which illustrates why we cannot prove a revelation principle in full generality.

Fix the environment as described in Section 2. Recall that the game takes place over periods 1 through T. The game form we propose is as follows: in each period t, there is a message space M_t. An agent with entry time t and exit time $\bar{t}$ can send messages over any contiguous subset of periods t through $\bar{t}$. We denote the history of all messages received by the seller from each player up to (and including) time t by h_t, and the set of all possible histories by H_t.

The allocation rule is a sequence of functions $\{f_t\}_{t=1}^{T}$, where f_t is a function from an element $h_t \in H_t$ to allocations up to time t (the formal definitions can be seen in Section 2, we do no repeat notation here). Further suppose that this allocation rule is feasible, in that over any possible history, the allocation rule allots no more than C units, and does not allot players in a period in which they do not send a message. A payment rule P associates with the string of messages sent by a player, a non-negative payment to be made by him.

A Bayesian-Nash Equilibrium has its usual meaning: suppose that in equilibrium each type $(i, t, \bar{t})$ sends messages $m = (m_1, m_2, \ldots, m_k)$, starting some time $t' \geq t$ and finishing some time $t' + k \leq \bar{t}$. Then it must be the case that

$$\mathbb{E}[v(f_{t'+k}(h_{t'+k})|(i, t, \bar{t}))] - P(m) \geq \mathbb{E}[v(f_{\bar{t}}(h'_{\bar{t}})|(i, t, \bar{t}))] - P(m').$$

for any other sequence of messages that a player can send, m'. (The expectation here is over messages sent by other agents in equilibrium.)

We show the revelation principle for equilibria in which each type $(i, t, \bar{t})$ is allotted exactly q units at time $\bar{t}$ if at all.

Lemma 3 *Suppose we have an equilibrium such that each type $(i, t, \bar{t})$ is allotted q units at time $\bar{t}$ if at all. Then there exists a direct revelation mechanism (DRM) which gives the same expected utility to all buyers, and the same expected revenue to the seller.*

Proof: Let the allocation and payment rules for the direct revelation mechanism be the same as for the original game. We are left to show that this mechanism is incentive compatible. To this end suppose that the DRM is not incentive compatible, and suppose instead type $(i, t, \bar{t})$ can profitably misreport his type as $(i', t', \bar{t}')$. Then:

- $\bar{t}' \leq \bar{t}$, since by assumption, a buyer reporting exit time t' gets allotted at time t', and a buyer of true type $(i, t, \bar{t})$ gets no utility from objects allotted after $\bar{t}$.

- $t' \geq t$, since an agent arriving at time t cannot report his type as one with entry time strictly less than t.

But this implies that type $(i, t, \bar{t})$ could have sent the messages corresponding to $(i', t', \bar{t}')$, and deviated profitably in the original game, contradicting our hypothesis that the messages in the original game constituted a Bayesian Nash equilibrium. □

Next we show that this result is in some sense generic. We show that any other equilibrium (i.e. where some types get allotted in quantities other than their requirement and/or before their exit time) can be converted into one where all types get allotted with the same (interim) probability. However in this new equilibrium, each type gets allotted exactly at their exit time. Further in this new equilibrium, each type has the same expected utility and the seller has the same expected revenue. However we may need to expand the message space in order for our construction to work.

PROPOSITION 2 *Suppose there is an equilibrium in the original game where an agent of type $(i, t, \bar{t})$ sends messages $m_{(i,t,\bar{t})}$ in equilibrium. Then there is an equilibrium (in a potentially modified game with an expanded message space) such that in this equilibrium:*

1. *Each type $(i, t, \bar{t})$ sends messages $m'_{(i,t,\bar{t})}$ starting period t through $\bar{t}$.*
2. *Messages are prefix-free: for any two types $(i, t, \bar{t})$, $(i', t, \bar{t}')$ s.t. $\bar{t} \geq \bar{t}'$, the messages sent by the two types from period t through $\bar{t}'$ are not the same.*
3. *Each type $(i, t, \bar{t})$ is allotted at $\bar{t}$. Further all types have the same expected utility and the seller has the same expected revenue as in the original equilibrium.*

PROOF: We shall construct a potential equilibrium with the properties described above, and proceed to show that it is in fact an equilibrium. So (expanding the message space if necessary) construct a sequence of prefix-free messages, one for each possible type. Note that this set of equilibrium messages is 'fully separating', in the sense that at no stage can a type be confused for another.

For each type $(i, t, \bar{t})$, and any possible profile of types of other agents, let the allocation rule in our new equilibrium reallocate all probabilities of getting a unit during any period t through $\bar{t}$ to getting it at exactly $\bar{t}$, and discard the rest of the allocations. Doing this for each type gives us an allocation rule, which is feasible since it allots weakly fewer units than the old rule (which was feasible by assumption). Further let the payment of each type be the same as in the original mechanism. It is clear that each type will be indifferent between the old allotment and the new one.

Therefore it is left to show that this new 'equilibrium' is in fact one. Once again suppose not, i.e. suppose that some type $(i, t, \bar{t})$ can profitably deviate by sending messages corresponding to $(i', t', \bar{t}')$. By the construction of the allocation rule, it must be the case that $t \leq t'$ and $\bar{t} \geq \bar{t}'$. But then clearly agent $(i, t, \bar{t})$ could also have profitably deviated by sending messages corresponding to $(i', t', \bar{t}')$ in the original game as well. Contradiction. □

Finally we show why there cannot be a revelation principle for equilibria where types get allotted at times other than their exit time.

Suppose $T = 2$, $C = 1$. Suppose there is exactly 1 potential buyer, who has two possible types: $(1, 1, 1)$ and $(1, 1, 2)$, i.e. the first type has valuation 1 and enters and exits at time 1; the second type also has valuation 1 and enters at time 1 but exits at time 2. Suppose $M_1 = \{1, 2\}$, and $M_2 = \{1\}$. Suppose further that

$$f_1(1) = 0;\ f_1(2) = 1;\ f_2(1, 1) = f_2(2, 1) = 0;$$

i.e. the allocation rule allots the unit to an agent announcing 2 in the first period, and nothing otherwise. Further suppose that

$$P(1) = P(1, 1) = 0;\ P(2) = 3;\ P(2, 1) = 0.99.$$

Consider the following equilibrium: type $(1, 1, 1)$ announces 1 in the first period, type $(1, 1, 2)$ announces 2 in the first period and 1 in the second. The only deviation $(1, 1, 1)$ has is to announce 2 in the first period. But this gets him the unit at the price of 3, which is not profitable. Similarly one can check that $(1, 1, 2)$ cannot profitably deviate.

However, the direct revelation mechanism that implements this allocation rule will not be incentive compatible- type $(1, 1, 1)$ can report $(1, 1, 2)$, get the unit in period 1 and pay 0.99, which gets him a surplus of 0.01, whereas announcing $(1, 1, 1)$ gets him a surplus of 0.

Note however that is easy to construct an ‘equivalent’ equilibrium (all types and the seller are indifferent)- simply allot type $(1, 1, 2)$ in period 2 rather than period 1.