Vote Buying II: Legislatures and Lobbying

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Abstract

We examine the consequences of lobbying and vote buying, assuming this practice were allowed and free of stigma. Two "lobbyists" compete for the votes of legislators by offering up-front payments to the legislators in exchange for their votes. We analyze how the lobbyists' budget constraints and legislator preferences determine the winner and the payments.

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1 Introduction

Consider a legislature that will vote over two alternatives, where two opposing lobbyists compete by bidding for legislators' votes. We study how the legislative outcome depends on the lobbyists' budgets and preferences and the legislators' preferences. We show that the outcome generally fails to fully reflect legislators' preferences. Moreover, we find that lobbyists' budget constraints can play a critical role in determining the outcome and can change the outcome completely and in interesting ways from situations where lobbyists' budgets exceed their maximal willingness to pay.

We model the lobbying process via a complete-information game in which lobbyists alternate in increasing their offers to legislators. Legislators care about how they cast their vote, and any payments they receive from lobbyists, rather than about the eventual outcome. The idea is that legislators care about how their voting record is perceived by their constituency, regardless of the actual outcome. This assumption turns out to have profound consequences since lobbyists buy votes via up-front payments (that are not contingent upon the legislative outcome). If legislators care only about outcomes, their votes matter only when they are pivotal. However, the probability of being pivotal is often negligible, especially in the context of vote buying where the lobbyist can intentionally make the legislators non-pivotal. This renders the legislators' preferences over outcomes unimportant and they are thus willing to tender their vote to the highest bidder. In contrast, when the legislator cares about how the vote is cast, their preferences significantly affect their prices and in turn who ends up winning the vote-buying competition and what strategies are followed.

Naturally the difference in the budgets of the lobbyists plays a critical role in determining which lobbyist is successful when lobbyists are budget constrained, and the difference in their maximal willingness to pay plays an important role when they are not budget constrained. However, legislators' voting preferences enter into the determination of the winner in subtle ways, and are markedly different in how they matter depending on whether or not lobbyists are budget constrained.

The main analytical result (Proposition 2 in section 3.1) concerns the case where lobbyists are budget constrained. There we show that the winning lobbyist is the one whose budget plus *half* of the sum of the value that each legislator attaches to voting in favor of the winning lobbyist exceeds the corresponding magnitude calculated for the other lobbyist. The result that preferences are weighed half as much as budgets in determining the outcome stems from the strategic aspects of the vote-buying game. In making a bid for any given legislator's vote, the lobbyist cares not only about how much he or she must promise to pay, but also about how much this offer will free up for the other lobbyist to use in bidding for other votes.

In contrast, when budgets are unbounded, the role of legislator preferences is very different. What matter then are the lobbyists' valuations and the intensity of preferences of a particular "near-median" group of legislators. The lobbyist with a-priori minority support wins when its valuation exceeds the other lobbyist's valuation by more than a magnitude that depends on the preferences of that near-median group (Proposition 3).

Thus, the voting preferences of the legislators have quite different effect in the two scenarios. When budget constrants are important, the intensity of the preferences of all legislators matter; when budgets do not constitute the important constraints, only the intensity of preferences of a particular near-median group of legislators matter.

The discussion in section 4 collects a number of additional issues, among them the case of unknown legislators' preferences, welfare implications and related literature. It is noted there that, in general, the outcome of the vote buying game need not be efficient and might involve higher or lower total surplus than what will arise in its absence. It is also claimed that, when lobbyists' budgets are raised by a certain donation game in which all of the population participates, then the lobbyists' budgets reflect the population preferences and the overall outcome is efficient.

Much of the of the formal literature on lobbying is concerned with influencing a single decision maker (e.g., a regulator). Our works belongs to a somewhat different strand of the literature that examines the lobbying of a voting body like a legislature. In the fundamental contribution of Groseclose and Snyder's (1996) the lobbyists move sequentially and each makes only one final offer. Their analysis focuses on the advantage that this asymmetric procedure confers on the second mover—the first mover can win only by buying a sufficiently significant supermajority..Our model essentially removes this asymmetry by allowing the lobbyists to keep responding to each other with counter-offers. It is conceivable that in some scenarios, some formal procedure indeed creates asymmetry on which the work of Groseclose and Snyder focuses. However, in many other situations there is no such formal structure and the lobbying process resembles more a continuing bidding process like the one we model. Our analysis shows that this changes significantly the strategic interaction and the results. Our paper is also related to a companion paper Dekel, Jackson and Wolinsky (2006a), henceforth DJWa, that models a general-election scenario rather than a legislative setting. The main difference is that, in the lobbying setting that we examine here, legislators care about how they cast their vote, whereas in the companion paper voters care only about the outcome. This change is more natural for the scenario of lobbying in a legislative setting compared to more general elections.[1] A second difference is in the focus here on the effect of budget constraints, which does not appear in DJWa.[2] These differences in setting lead to very different conclusions regarding the structure of equilibria. Finally, the vote-buying model itself differs: in DJWa we consider a uniform-offer model where the vote buyers cannot make different offers to different voters.[3]

2 A Model of Vote Buying

Prior to an election two lobbyists, X and Y, try to influence the voting of a legislature with an odd number, N, of legislators by directly buying votes of legislators. To simplify matters, we assume that vote buying is an ordinary transaction: the lobbyist gets full control of the vote in exchange for an up-front payment to the legislator.[4]

2.1 The Lobbying Game

The lobbyists alternate in making offers. Lobbyist k in its turn announces an up-front offer $p_i^k \geq 0$ to each legislator i for her vote. A fresh offer (or promise) made to a legislator cannot be lower than those previously made by the same lobbyist to the same legislator. There is a small additional cost, $\gamma > 0$, incurred each round in which a lobbyist makes an offer.

As explained in Section 2.2, given the outstanding offers at any stage, for each legislator there is a unique lobbyist to which that legislator would tender her vote if the

[1] Voters in a general election might also care significantly about how they cast their votes, which is, of course, suggested by the fact that people vote despite it being costly and their pivot probability being negligible. To the extent that the voting preference are more important than preferences over outcomes, the present paper provides a more relevant model for general elections. The other paper pertains to cases where voters care predominantly about outcomes.

[2] Budget constraints do not have the same impact in settings where voters care only about outcomes, and so their role is only interesting in this paper.

[3] In situations where voting preferences do not matter, targeting specific voters is less consequential. In the legislative application, lobbyists have strong incentives to target certain legislators.

[4] In DJWa we also consider the possibility of offering indirect promises to voters that are only contingent on the outcome. These are only consequential if up-front payments are not possible. In that case, legislators' voting preferences would not matter and so the analysis would be as in DJWa. Thus, we do not consider those payments here.

process were to stop at that stage. Let I_t^k denote the set of legislators who would tender to lobbyist $k = X, Y$ if the process were to stop at the beginning of period t. The bidding ends at the beginning of period t with a win by k if both $I_t^k > I_t^j$ and $I_{t-1}^k > I_{t-1}^j$, i.e., if j passed up an opportunity to outbid k.

Once the bidding process ends, legislators simultaneously tender their votes to the lobbyists. The lobbyist who collects more than half the votes wins.

The lobbyists finance their payments out of budgets denoted B^X and B^Y. The total payments that lobbyist k would have to pay at any stage of the game, assuming that the game were to end at that stage, cannot exceed B^k. That is, at the beginning of every period t it has to be that $\sum_{i \in I_t^k} p_i^k + \gamma\tau^k(t) \leq B^k$, where $\tau^k(t)$ is the number of periods in which k has made an offer up to the beginning of t. It is important that at each stage the budget constraint has to hold only with respect to those obligations that are still relevant at that stage. If lobbyist k's up-front offer p_i^k has been outbid by the other lobbyist, so that at that point legislator i would sell her vote to the other lobbyist, then lobbyist k does not have to count this up-front offer against its budget.

Each lobbyist has a value W^k for winning. If the game ends in period $t < \infty$ then lobbyist k's payoff is $W^k - \sum_{i \in I_t^k} p_i^k - \gamma\tau^k(t)$ if k wins, and $-\sum_{i \in I_t^k} p_i^k - \gamma\tau^k(t)$ if k loses. The payoff is $-\infty$ if the game never ends.

The game between the lobbyists is one of perfect information. The lobbyists' budgets and valuations and the legislators' preferences are commonly known to the lobbyists. When a lobbyist makes offers, he or she observes the past offers and promises received by each legislator.

Strategies are defined in the obvious way, and the solution concept is subgame perfect equilibrium.

2.2 Legislator Behavior

There is an odd number N of legislators. Each legislator i is characterized by parameters V_i^X and V_i^Y. Legislators are not formally modeled as strategic players in this game. Instead, we assume that a legislator will sell her vote to Lobbyist X if and only if

$$p_i^X + V_i^X > p_i^Y + V_i^Y. \tag{1}$$

The parameters V_i^k are interpreted as the utility Legislator i gets from *voting* for the outcome supported by lobbyist $k = X, Y$. The focus is on the legislators' voting preferences rather than on their preferences over outcomes, since it is natural to assume that for reelection considerations legislators care a great deal about how they vote regardless of what the actual outcome is. Even if they have direct preferences over the outcomes, those would probably be of secondary importance as they would matter only when the legislator's vote is pivotal which might occur only with low probability. It is natural to think of the V_i^k's as being related to the preferences of i's constituency over the actual outcome. We will indeed make this connection later when discussing efficiency in Section *4.3*.

2.3 Further Assumptions and Notation

Let $V_i = V_i^X - V_i^Y$. The analysis that follows depends on the V_i^k's only through V_i and we will therefore represent the preferences in terms of V_i. We order the i's so that V_i is nonincreasing and let m be the median legislator ($m = (N+1)/2$). Without loss of generality we assume $V_m > 0$, so that the median prefers to vote for X. Therefore without any vote-buying X would prevail. Let $n = \arg\max\{i : V_i > 0\}$, i.e., n has the weakest preference for X over Y from among all those who prefer X over Y.

There is a smallest money unit $\varepsilon > 0$. Both the offers and the budgets are whole multiples of ε. To avoid dealing with ties, which add nothing of interest to the analysis, we assume that the V_i's and W^k's are not whole multiples of ε.

Given a number z, let $\lceil z \rceil^{\varepsilon}$ denote the minimal multiple of ε greater than z, and $\lfloor z \rfloor_{\varepsilon}$ the maximal multiple of ε smaller than z. Assuming as above that each legislator votes for X (respectively Y) if and only if V_i plus the amount of money that legislator receives for that vote is strictly positive (respectively negative), then Y must spend at least $\bar{V} = \sum_{i=m}^{n} \lceil V_i \rceil^{\varepsilon}$ to obtain a majority. We assume that both B^Y and W^Y are greater than $\bar{V}$ as otherwise the solution is trivial.

In Figure 1 the solid line is $\lceil V_i \rceil^{\varepsilon}$, the line crosses the axis at n, the long vertical segment is at m, and the marked (red) area is $\bar{V}$.

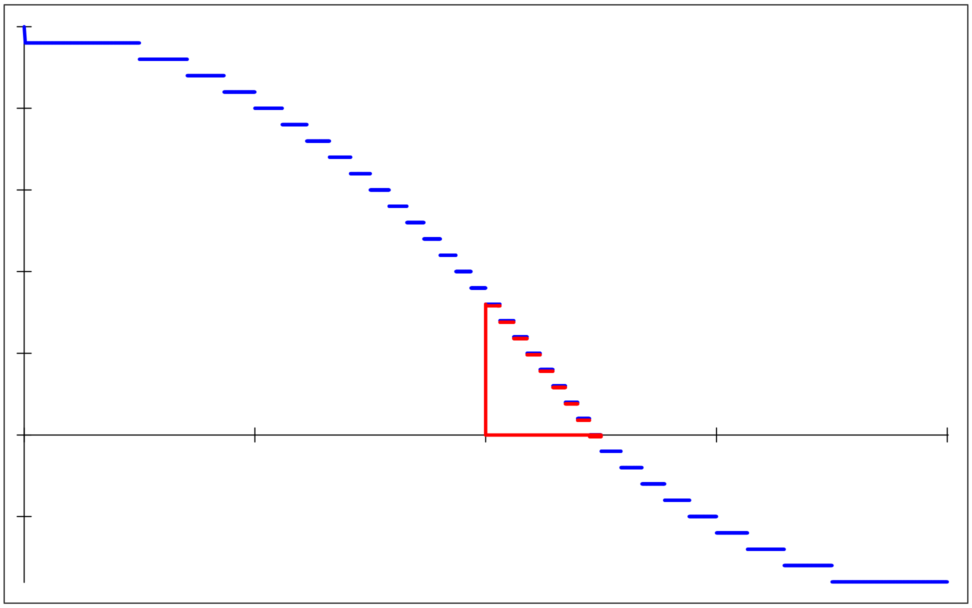

Figure 1: Voting preferences and related parameters

3 Vote-buying

The vote-buying game is a sort of a multi-unit auction with a special form of complementarity (only a bundle of more than half the units is valuable). It resembles an all-pay auction in that the loser may end up paying for some votes. But it is not a pure all-pay auction, since at most one lobbyist ends up paying for any given vote. If there were only one legislator, then this would be a complete information English auction (that allows jump-bidding).

We start with the following observation that applies to both constrained and unconstrained bidders.

PROPOSITION **1** *The vote-buying game has an equilibrium in pure strategies. In every equilibrium the same lobbyist wins, and the losing lobbyist never makes any offers.*

The existence of a unique winner when budgets are binding follows because this is a finite game of perfect information and ties are ruled out by assumption. In the unconstrained game, since offers are nondecreasing, they eventually reach a point where they must be greater than the value. While it is possible in principle that the bidder at that point expects to be outbid by the opponent and hence does not expect to pay that full amount, the fixed cost of making an offer ($\gamma > 0$) implies that such an offer will never be made. Thus the game is equivalent to a finite truncated version, and hence has a unique outcome. That the loser never makes offers also follows from positive the bidding cost γ.

3.1 Budget-constrained lobbyists

The winner is determined by a combination of the relative strengths in terms of the budgets and the intensity of the of the legislators' voting preferences. Roughly speaking, Y wins if its budget advantage, $(B^Y - B^X)$, exceeds a measure of the preference advantage of X measured by *one half* of the *total* utility advantage of X over Y, i.e., $\sum_i V_i/2$. To understand why the utilities of all legislators matter, but only count half as much as the size of the budgets, it is useful to understand the structure of the winning strategies. The following example helps developing the intuition for this problem by pointing out that the natural least expensive majority, LEM, strategy, which secures the least expensive minimal majority at each stage, may not be optimal.

EXAMPLE **1** *Optimal versus Naive Strategies - Why Utility has a Shadow Price of 1/2.*

There are three legislators with $V_1 = V_2 = 0.5$ and $V_3 = -30.5$. The grid size is $\varepsilon = 1$. Budgets are $B^X = 100$ and $B^Y = 80$.

Note that $B^X - B^Y = 20 < 29.5 = -\sum_i V_i$, so the total utility advantage for Y is greater than the absolute budget advantage of X. Nevertheless, as we show below in Proposition 2, X should win, because X's budget exceeds Y's budget plus half of the total utility difference. That is, basically what matters is the budget advantage relative to one half the total preference advantage (setting aside small corrections that are explained in the proof of the result). Let us see how X should play to win.

Suppose that X follows the naive LEM strategy of always spending the least amount necessary to guarantee a majority at any stage. Suppose (just for the purpose of illustration) that at the first stage Y makes offers of 55 to legislator 1 and 25 to legislator 3. The cheapest legislator for X to buy back is legislator 1 at a cost of 55. Assume Y now offers 55 to legislator 2. At this point X has 45 left in her budget, and cannot afford to buy back either legislator 2 or 3.

What was wrong with this strategy? The problem is that, while X bought the cheapest legislator in response to Y's offer, X also freed up a large amount of Y's budget for Y to spend elsewhere, while X's budget was committed. X needs to worry not only about what X is spending at any given stage, but also about how much of Y's budget is freed up. Effectively, freeing up a unit of Y's budget is "just" as bad for X as spending an extra unit of X's budget.

So, instead of following the naive LEM strategy of buying the cheapest legislators, let X always follow a strategy of measuring the "shadow price" of a legislator as the amount that X must spend *plus the amount of Y's budget that is freed up.* If X had followed that strategy, then in response to Y's first stage offer above, X would have purchased legislator 3 at a price of 56. Then Y would have 25 free, and could only spend it on legislators 1 and 2. Regardless of how Y spends this budget, X can always buy legislator 2 at the next stage at a price of at most 25, against which Y has no winning response. ■

The example shows that keeping track of the shadow price is a good strategy. In fact, for large budgets it guarantees a win for the winning candidate characterized in Proposition 2 below. Let us see how we get from this understanding of "shadow prices" to the expressions underlying Proposition 2.

Under the strategy suggested in the above example, X keeps track of the offer that X has to make to buy a legislator given the current offer of Y, plus the amount of Y's budget that is freed up. The amount that X has to offer to buy a given legislator i when Y has an offer of p_i^Y in place is $p_i^Y - V_i$. The amount of Y's budget that is freed up is p_i^Y. So the "shadow price" of buying legislator i is $2p_i^Y - V_i$. Dividing through by 2 gives us $p_i^Y - V_i/2$. In the proof this translates into the "strength" of Y being Y's budget less the sum of $V_i/2$ over legislators that prefer Y, X's "strength" being X's budget plus the sum of $V_i/2$ over those legislators that prefer X, and the winner being approximately the stronger lobbyist.

This is captured in Proposition 2 below, which includes some slight modifications to account for the grid size and some other details that are covered in the formal proof. The result requires that budgets be sufficiently large as specified next.

$$B^X > \left|\frac{mV_1}{2}\right| - \frac{\sum_{i=m+1}^{N} V_i}{2} - \frac{V_N}{2} + m\varepsilon \tag{2}$$

$$B^Y > \left|\frac{mV_N}{2}\right| + \frac{\sum_{i=1}^{m-1} V_i}{2} + \frac{V_1}{2} + m\varepsilon. \tag{3}$$

Proposition **2** *If the budgets are large enough so that (2) and (3) are satisfied, then, for sufficiently small γ, X wins at no cost if*

$$B^X > B^Y - \sum_i V_i/2 - V_N/2 + m\varepsilon \tag{4}$$

and Y wins at cost $\bar{V}$ paid to the legislators m (median) through n (almost-indifferent) if

$$B^Y > B^X + \sum_i V_i/2 + V_1/2 + m\varepsilon. \tag{5}$$

The interesting feature is that, very roughly, increasing a legislator's preference for a given lobbyist by \$1 is equivalent, in terms of who wins, to increasing the budget of that lobbyist by \$0.5. Thus money is worth much more to a lobbyist than having its bill being liked, as might be expected due to the use of funds being more flexible. Nevertheless, one of the implications of Proposition 2 is that a lobbyist with strong minority support can win despite having a lower budget than the opposition.

Note that if voting preferences are relatively unimportant, i.e., $\sum_i V_i$ is negligible relative to the budgets, then the comparison boils down to a comparison of the budgets. That is, the lobbyist with the highest budget wins. When this is the case, the optimal strategy simplifies to the strategy that seeks to obtain the least expensive majority at each point (LEM strategy), which is not optimal in general. A scenario with negligible voting preferences would arise, If legislators cared only about outcomes (and not how they vote) and the probability of being pivotal were negligible (as it would be in many plausible cases), since then the preferences over outcomes essentially do not affect the vote tendering considerations of the legislators.

As the proof makes clear, in fact only one large-budget condition is needed in each case. That is, X wins if equations (3) and (4) hold, and Y wins if (2) and (5) are satisfied. When budgets are not large enough (as given by (4) and (5)) the game becomes quite complex and the formula for determining the winner is involved. As we see little insight and great complication in such an analysis, we do not pursue it. The following example serves to show that an assumption of large enough budgets is necessary.

Example **2** *Large versus Small Budgets*

Let $B^Y = 0$, $B^X = 30.2$, $\varepsilon = 0.1$, $N = 3$, $V_1 = -10$, $V_2 = -20$, and $V_3 = -30$. Here X can win by buying legislators 1 and 2 at prices of 10.1 and 20.1.

In this example

$$B^X + \frac{\sum_i V_i}{2} + \frac{V_1}{2} = -5 < B^Y - m\varepsilon = -.2,$$

and so if we applied the expressions from Proposition 2, we would mistakenly conclude that Y should win. ■

If we did not assume small costs of making offers (i.e., if $\gamma = 0$), then the characterization of the winning lobbyist would be unchanged. There could potentially be multiple equilibria which differ from one another with respect to the total payments made by the winner and the identities of their recipients. The loser would still make no payments in equilibrium, but by making bids that will be outbid by the winner, the loser could force the winner to spend more than the minimum sum necessary to obtain a majority in the absence of active opposition.

3.2 Unconstrained lobbyists

We now analyze the case in which the budgets are not binding. The identity of the winner depends on the relative magnitudes of the lobbyists' valuations and the intensity of the voting preferences of the legislators whose index i falls between m (median) and n (weakest supporter of X). Recall that $\bar{V}$ is the sum that Y has to commit to the m through n legislators in order to outbid X in the first step in the least expensive way. Roughly speaking, Y wins at the cost $\bar{V}$ when Y's valuation, W^Y, exceeds W^X by a magnitude related to $\bar{V}$; since X enjoys a preference advantage, it wins at zero cost when $W^X > W^Y$; in the intermediate range in which W^Y exceeds W^X but is not sufficiently larger, the identity of the winner depends on who moves first.

Proposition **3** *There exists* $\lambda \in [\lfloor W^X \rfloor_\varepsilon, \lfloor W^X \rfloor_\varepsilon + \bar{V}]$ *such that, for sufficiently small* γ*, in any equilibrium:*

1. *If* $\lfloor W^Y \rfloor_\varepsilon > \lambda$*, then* Y *wins at cost* $\bar{V}$ *paid to the legislators* m *(median) through* n *(almost-indifferent).*
2. *If* $\lfloor W^Y \rfloor_\varepsilon < \lfloor W^X \rfloor_\varepsilon$ *then* X *wins at no cost.*
3. *If* $\lfloor W^Y \rfloor_\varepsilon \in \left(\lfloor W^X \rfloor_\varepsilon, \lambda\right)$ *then* Y *wins at cost* $\bar{V}$ *if it moves first , and* X *wins at positive cost if it moves first.*

The proof of this result (in the appendix) is somewhat related to the proof of Proposition 2 in Dekel, Jackson and Wolinsky (2006b), henceforth DJWb, which studies single-object all-pay-auctions, though the vote buying game of the present paper is not a pure all-pay auction.

The cutoff level λ has the following meaning. Suppose that X moves first and commits the maximal sum that does not exceed its value, $\lfloor W^X \rfloor_\varepsilon$, in a manner that makes it as costly as possible for Y to obtain the majority. Then λ is the minimal sum that Y would have to commit to voters in order to obtain a majority. The precise characterization of λ in terms of the parameters of the model is provided in the proof.

The following example clarifies the role of the bidding cost γ. The idea is that, with $\gamma = 0$, there are equilibria in which the higher value lobbyist, say Y, loses since, if Y tries to win, the other lobbyist, X, can make Y pay out substantial sums without X incurring any cost itself. This is accomplished by offers made by X that are later outbid by Y.

Example **3** *Bidding Costs*

$W^Y = 12.5$, $W^X = 9.5$, $\varepsilon = 1$, $N = 3$, $V_1 = V_2 = V_3 = 0.5$, $\gamma = 0$. Thus, $\bar{V} = 2$ and, if γ were positive, then by Proposition 3-(1) Y would win at the cost 2. To see that, with $\gamma = 0$, the situation might be different, suppose that X starts with $p_1^X = 9$ (the full offer is $p_1^X = 9$, $p_2^Y = p_3^Y = 0$ but for brevity here and hereafter we will often specify in each stage only the part of the outstanding offer that is being increased). We claim that there is an equilibrium in the ensuing subgame in which Y quits immediately, since it can win in the continuation only by paying more than 12.5. To construct such continuation, observe that in any equilibrium continuation Y would never commit more than W^Y in one step. This is because the expected incremental sum of payoffs of X and Y from that point on would be negative which is inconsistent with equilibrium continuation. Thus, Y responds to $p_1^X = 9$ with one of the following profiles of promises: (i) $p_1^Y = 10$, $p_2^Y = p_3^Y = 1$; (ii) $p_1^Y \geq 10$, $p_2^Y = 1$ or 2, $p_3^Y = 0$ (or the same with the roles of 2 and 3 interchanged). (iii) $p_1^Y = 0$, $p_2^Y \geq 1$, $p_3^Y \geq 1$ s.t. $p_2^Y + p_3^Y \leq 12$. The following is a SPE in the subgame following (i). X regains the majority with $p_2^X = 2$, $p_3^X = 2$, to which Y responds with $p_2^Y = p_3^Y = 3$ and X quits. If Y deviates to a cheaper offer like $p_2^Y = 3$, then on the path of the continuation X responds with $p_2^X = 9$ to which Y responds with $p_2^Y = 10$ and X quits. If, after $p_2^X = 9$, Y continued instead with $p_3^Y \in [2, 9]$, it would not save anything, since X would respond with $p_3^X = 9$ to which Y would respond with $p_2^Y = 10$ or $p_3^Y = 10$. Thus, if Y continues according to (i) and wins, it would end up spending more than W^Y. A SPE continuation after (ii) is essentially the same as in (i). That is, X responds with $p_2^X = 2$, $p_3^X = 2$ to which Y responds with $p_2^Y = p_3^Y = 3$ and X quits, etc. A SPE continuation following (iii) is as follows. Assuming that $p_2^Y \leq 9$, X responds with $p_2^X = 9$ (otherwise, $p_3^Y \leq 9$ and X would respond with $p_3^X = 9$) to which Y would respond with $p_2^Y = 10$. If at that point $p_2^Y + p_3^Y > 12.5$, then X would quit; If not, X would continue with $p_3^X = 9$, to which Y would respond with $p_3^Y = 10$ and X would quit. after $p_2^X = 9$, Y continued instead with $p_3^Y \in [2, 9]$, it would not save anything, since X would respond with $p_3^X = 9$ to which Y would respond with $p_2^Y = 10$ or $p_3^Y = 10$. Thus, if Y continues according to (iii) and wins, again it would end up spending more than W^Y.

Notice that, at any point along these continuations, X behaves optimally, since it expects to be relieved from any commitments that it makes by subsequent promises by

Y. This is why this construction requires $\gamma = 0$. With positive γ, X would not want to continue bidding when it is certain to lose, even if its commitments would be later annulled.

4 Discussion

4.1 Budget Constraints

At a first glance one might conjecture that the only difference between the scenarios with and without budget constraints is that in the constrained scenario the budgets play the same role that the valuations play in the unconstrained scenario. In some auction models this is indeed the case. However, it turns out that the outcomes of the two vote-buying scenarios with and without binding budget constraints are markedly different from one another. When the budget constraints are not binding only the preferences of the legislators whose index i falls between m (median) and n (weakest supporter of X) matter for the determination of the winner. These are the legislators whom Y must buy in order to outbid X in the least expensive way. In contrast, when budget constraints are the decisive element, the preferences of all the legislators affect the outcome. The weight given to the preferences that matter also differ across these two cases. In the case of budget constrained lobbyists, the preferences of the legislators enter with half the weight given to the budgets of the lobbyists. In the unconstrained case the preferences of the legislators indexed m to n enter with same weight as the lobbyists' valuations.

The important difference between budget constraints and valuations is that the former constitute hard constraints on the outstanding commitments while the latter can be exceeded despite it being unprofitable. In static scenarios, this distinction might not matter because bids in excess of the valuation are dominated. However, in a dynamic scenario in which past bids become sunk, the distinction between budgets and valuations might become very meaningful for behavior off the path of the equilibrium. When the budget constraints bind, a central strategic consideration concerns how much budget is being freed up for the opponent. Therefore, the most effective strategy does not necessarily minimize the payments promised to legislators at each stage and the preferences of those who are not the least expensive to acquire also enter the calculations. When the budget constrains do not bind, this consideration is irrelevant, as past offers are essentially sunk costs and the most effective strategy entails acquisition of the least expensive votes at each stage.

4.2 Negligible voting preferences

A special case of the results of propositions 2 and 3 is when the legislators' voting preferences are negligible. In such a case the lobbyist with the larger budget or larger value wins in the constrained and unconstrained scenarios respectively. This special case is interesting since in some scenarios the voters/legislators might not care about how they vote but still might care about the outcomes. However, if pivot considerations are negligible, the preferences over outcomes do not matter for the voting/tendering decision and the situation may be analyzed using the zero voting preferences case of the present model.

4.3 Efficiency

In the absence of any mechanism for buying and selling votes, the outcome of voting will in general be inefficient. There is simply nothing to make legislators take into account the effect of their vote on others. A natural hypothesis is that allowing the lobbyists to compete for the votes will help align the outcome with overall societal values for the alternatives, presuming that the lobbyists' budgets represent the utility of some (possibly unmodeled) agents. Our analysis shows that this is not always so.

Under what circumstances will vote buying result in efficiency? If budgets are binding, then equilibrium will be (approximately) efficient if for some reason the budgets are proportional to the true surpluses of the agents in the society, and the legislators' voting preferences are too. That is, let $V^X = \sum_i \lceil V_i \rceil^{\varepsilon}$, and $V^Y = \sum_i \lceil -V_i \rceil^{\varepsilon}$, then the equilibria will be efficient if $B^X/V^X = B^Y/V^Y$, and V^X and V^Y represent the preferences of the legislators' constituents. If budgets are raised by a donation game with forward-looking donors who can anticipate the willingness to pay in favor of each alternative, then the game essentially becomes an all-pay auction one of raising donations, where one side begins with an initial advantage. This is a variation on the games studied in DJWb. While certain such games could lead to an efficient outcome, it is clear that the set of circumstances in which the outcome would necessarily maximize total societal utility are quite stringent.

4.4 Unknown preferences

Our analysis so far has assumed that legislators' voting preferences are known. This seems reasonable in the lobbying scenario. Nevertheless, it is worthwhile exploring the effect of lobbyists' uncertainty over legislators' voting preferences.

Consider the up-front vote buying case. Suppose that, for all i, V_i is an independent draw from a continuous distribution F. We assume that F has a connected support and a continuous and positive density on its support, and is such that $z + F(z)/f(z)$ and $z + (F(z) - 1)/f(z)$ are both increasing on the support of F. There are many prominent distributions satisfying this, such as the uniform distribution. Let $\hat{V} = F^{-1}(0.5)$ be the median of the distribution F. In this environment we impose the constraint that parties' offers must in expectation be within their budgets at each point in the game, assuming it ends at that point.

PROPOSITION **4** *For any $\delta > 0$, there is $N(\delta)$ and $\bar{\varepsilon}$ such that for all $N > N(\delta)$ and all grids with $\varepsilon \in (0, \bar{\varepsilon})$ the following hold.*

- *If $B^Y > B^X + \hat{V}N/2 + \delta$, then Y wins with probability of at least $1 - \delta$.*
- *If $B^X > B^Y - \hat{V}N/2 + \delta$, then X wins with probability of at least $1 - \delta$.*

The result is almost a complete characterization of equilibria for large N, as the conditions cover budget differences except those that fall in an interval of size 2δ.

When δ is sufficiently small, the party who is likely to lose will not enter the bidding and the winning party will bid the minimum necessary to secure majority with sufficiently high probability. The reason for the minimum payment in equilibrium is clear. As in all other cases, the loser would like to avoid payment. Unlike the case of known voting preferences in which the loser can safely bid for some voters knowing that they will be bid away by the winner, here the uncertainty precludes such behavior as the loser does not know which voters will be bid away and hence might end up having to pay some of the bids it makes.

4.5 Indirect promises

Due to legal or ethical reasons or plainly because the voting is confidential, it might be the case that lobbyists cannot acquire legislators' votes directly or make payments contingent on how the legislator actually votes. Instead, a lobbyist can influence the voting only by making promises that will be fulfilled if and only if this lobbyist wins.

To model this, suppose that, in its turn to propose, Lobbyist k promises Legislator i a payment c_i^k (instead of the bribes p_i^k) that will be paid out if k wins, independently of how i voted. Again the process ends if two rounds go by without a change in who would be the winner.

Since the winner must pay *all* the promises it made, at any point along the process, it has to be that $\sum_{i=1}^{N} c_i^k \leq B^k$. This is in contrast with the up-front buying scenario where the payment offered to i counts against k's budget only if i prefers to tender to k. The payoff to Lobbyist k is $W^k - \sum_{i=1}^{N} c_i^k$ if k wins; 0 if k loses (and $-\infty$ if the game never ends). Thus, the winner honors its promises to all legislators regardless of how they cast their votes, while the loser is not making any payments.

Since they are not directly paid for their votes, they are assumed to vote according to their voting preferences V_i^k. Thus, Legislator i votes for Lobbyist X's proposal if and only if $V_i^X > V_i^Y$.

In the most compelling interpretation of this scenario, the lobbyist makes the promises to the constituency of legislator i. If, for example, the lobbyist can influence the structure of the bill being voted upon, the c_i^k's could represent "pork" to a given legislator's district. The V_i^k's are derived from the preferences of i's constituency over the actual outcomes including the promises (be it because the legislator cares about the constituency's benefit or because of reelection considerations). To formalize the connection between the promises and legislators' voting preferences, let U_i^k measure the benefit to i's constituency of Lobbyist k's win. The simplest way to think about it is that all the voters in i's district share the same preferences over the outcomes. We assume that V_i^k is an increasing function of $c_i^k + U_i^k$.Thus, legislator i will support Lobbyist X if and only if

$$c_i^X + U_i^X > c_i^Y + U_i^Y \tag{6}$$

The above is of course just an interpretation. Alternatively, one may simply think of U_i^k as Legislator i's personal utility of k's win the of the c_i^k's as promises that benefit i directly.

Other than the above, the game remains essentially as before. It is important to emphasize that the main difference is that here the legislator maintains control of the vote and payments are contingent only on the outcome, whereas in the up front buying scenario considered before payments were contingent on the individual's vote but not on the outcome of the vote.

Let $U_i = U_i^X - U_i^Y$ and relabel legislators so that U_i is non-increasing in i. Under this labeling, let $m = (N+1)/2$, suppose (w.l.o.g) that $U_m > 0$ and let $n = |\{i : U_i > 0\}|$. Also assume that for all i and k, the values U_i and W^k are not multiples of ε. Recall that, given a number z, $\lceil z \rceil^{\varepsilon}$ is the smallest multiple of ε greater than z, and let $\overline{U} =$

$\sum_{i=m}^{n} \lceil U_i \rceil^{\varepsilon} > 0$ be the minimal sum that Y has to promise to legislators in order to secure the support of a minimal majority, in case X does not promise anything..

The analysis is now the same as in the case where voters (legislators) care only about outcomes and not how they cast their vote. The results here are essentially the same as in DJWa, though the statement here also includes the possibility of binding budgets. We state here the results for completeness, but refer to DJWa for the proof.

PROPOSITION **5** *There exists an equilibrium in the indirect-promises game. In any equilibrium Y wins if and only if $min[B^Y, \lfloor W^Y \rfloor_{\varepsilon}] \geq min[B^X, \lfloor W^X \rfloor_{\varepsilon}] + \overline{U}$.*

The idea behind Proposition 5 is easily explained. Lobbyist Y must spend at least $\overline{U}$ in order to secure a majority. After that, X could try to obtain some of these votes back (or others, if Y has overspent on these marginal votes), with the competition back and forth leading to the winner being the lobbyist with the largest budget (or willingness to pay) once an expense of $\overline{U}$ has been incurred by Y.

This game has many equilibria because the loser's behavior is not pinned down, as it is certain to lose and will not have to honor the promises it makes. Introducing some uncertainty over the other lobbyist's budget/value singles out equilibria where lobbyists use Least Expensive Majority (LEM) strategies, in which each lobbyist purchases the least expensive majority in turn, provided that their total commitment does not exceed their budget or value. The identity of the winner would still be the same as above, but the total payment of the winner would be the loser's value adjusted by the magnitude $\overline{U}$, as spelled out in the following proposition.

The refinement we consider is "ex post perfect equilibrium:" a profile of strategies for each player (specifying a behavioral strategy for each realization of type) that form a subgame perfect equilibrium relative to any profile of realized types.[5]

The minimum of the budget and grid adjusted value ($\lfloor W \rfloor_{\varepsilon}$) of each lobbyist is distributed on a finite set $\mathcal{V} = \{0, \varepsilon, 2\varepsilon, \ldots, M\varepsilon\}$. The actual value of each lobbyist is not an integer multiple of ε.

PROPOSITION **6** *Consider the indirect-promises game with any full support distribution over $\mathcal{V}$.*

1. *LEM strategies constitute an ex post perfect equilibrium.*

[5] As discussed in Dekel, Jackson and Wolinsky (2006), the result also holds if we instead use ex post Nash equilibrium where players do not use weakly dominated strategies. That is neither a stronger nor weaker solution than ex post perfect equilibrium.

2. *In any ex post perfect equilibrium* Y *wins if* $min[B^Y, \lfloor W^Y \rfloor_\varepsilon] \geq min[B^X, \lfloor W^X \rfloor_\varepsilon] + \overline{U}$ *and* X *wins otherwise.*

3. *In any ex post perfect equilibrium if* Y *wins then* Y *promises exactly* $min[B^X, \lfloor W^X \rfloor_\varepsilon] + \overline{U}$ *and if* X *wins then* X *promises exactly* $\max\{min[B^Y, \lfloor W^Y \rfloor_\varepsilon] - \overline{U} + \varepsilon, 0\}$.

4. *In any ex post perfect equilibrium only voters between* $\hat{m} = \{\min i : \lceil U_i \rceil^\varepsilon = \lceil U_m \rceil^\varepsilon\}$ *and* $\hat{n} = \{\min i : U_i > -\varepsilon\}$ *can receive positive payments.*

To sum up, the lobbying competition with indirect promises has the flavor of an English Auction. Focusing on the refined equilibria of the perturbed game, the winner ends up paying the second highest budget or value (adjusted by a measure of the preference advantage that one has over the other among the legislators). Only the intensity of the preferences of a group of near median legislators affect the outcome and only members of this group get promises in equilibrium.

4.6 Related literature

The most closely related work is Groselose and Snyder (1996) that models the lobbying of a legislature by a targeted offers game where each party gets to move only once, and in sequence. The conclusions of Propositions 2 and 3 are quite different from theirs. Their model provides a significant second-mover advantage, which contrasts sharply with the open-ended sequential nature of our game. Specifically, in their game, in order to win, the first mover needs to be able to bid in such a way that it would be unprofitable for the second mover to buy any majority. In a game without an exogenously determined last mover, as the one we analyze, if one lobbyist is (temporarily) outbid for some legislator, it can remobilize those resources, which places lobbyists on a more equal footing. Also, owing to the single move that each lobbyist has in the Groseclose and Snyder model, the distinction between budgets is and values has no importance in their model, while in our model budgets and values have rather different effects on the outcome. It is conceivable that in some scenarios a formal procedure indeed creates asymmetry of the sort on which the work of Groseclose and Snyder focuses. However, in many other situations there is no such formal structure and the lobbying process resembles more a continuing bidding process like the one we model. Our analysis shows that this changes significantly the strategic interaction and the results.

Baron (2001) analyzes a game in which two competing lobbyists can make offers to legislators in repeated rounds. His game differs from ours in that he models agenda setting and the legislative game in much more detail (whereas we take two alternatives as fixed), and lobbyists pay to get their alternative proposed in addition to buying votes to get it passed. The agenda setting part of the game enriches the interaction substantially, but also makes it difficult to obtain characterizations of equilibrium. Nevertheless, Baron obtains some interesting results on the pattern of the resulting majority and how it relates to the proposal process. Given the difference in game structure and focus, his work and ours are complementary.

There are also related papers on lobbying that have roots in the common agency literature, such as Bernheim and Whinston (1986), Grossman and Helpman (1994), Dixit (1996), Le Breton and Salanie (2003), and Martimort and Semenov (2006), among others.[6] As such models generally look at a single voter (the politician or agent), the complete information solutions result in efficient outcomes (e.g., see Bernheim and Whinston (1986) and Le Breton and Salanie (2003)).[7] In particular the politician as well as each lobbyist ends up being pivotal; as if some lobbyist is making a payment that is not pivotal in swaying the politician, then they could lower their payment and not affect the outcome. This reinforces the idea that the inefficiencies that we uncovered are due to the fact that in many contexts at least some players end up not being pivotal in a vote buying game when the vote is not by unanimity.

Buchanan and Tullock (1962) discuss the rationale for the prohibition of vote buying. They observe that under unanimity voting rule, free trade in votes would lead to efficiency. They suggest however that this might not be the case when a simple majority rule is in force. They do not model the market for votes formally, but argue intuitively that a perfect market for votes would lead to efficiency, but that imperfections are likely to arise and might preclude efficiency. Our analysis provides in a sense a particular formal interpretation to these ideas. Neeman (1999) points out that, with some uncertainty over legislators' behavior, pivot considerations are of marginal importance and hence vote buying (by a single buyer) need not result in efficiency.[8] Our own analysis of efficiency focuses on the next step—it inquires about the efficiency consequences of competition

[6] There are also papers by Lizzeri (1999) who studies why lobbyists may create budget deficits, and Lizzeri and Persico (2001), study games where candidates can choose whether or not to offer a public good in addition to a redistribution. These are less related to the issues in our paper.

[7] As such, the focus of many of these models has been on various distributional issues such as taxation and redistribution, or the politics of protectionism and international trade.

[8] This and the point made by Buchanan and Tullock regarding efficiency of vote trading under unanimity are just alternative statements of the observation we made above that trading results in efficiency when every legislator is pivotal.

between vote buyers.

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6 Appendix

PROPOSITION **1:** *The vote-buying game with up-front payments has an equilibrium in pure strategies. In every equilibrium the same lobbyist wins, and the losing lobbyist never makes any offers.*

Proof of Proposition 1: The facts that the budget-constrained vote-buying game has an equilibrium in pure strategies follows from the fact that this is a finite game of perfect information, and hence we can find such an equilibrium via backwards induction.

The fact that in every equilibrium the same lobbyist wins, also follows from a backward induction argument. Each terminal node has a unique winner (as the V_i's are not a multiple of ε and so legislators are never indifferent), and lobbyists prefer to win regardless of the payments necessary. Thus, in any subgame, working by induction back from nodes whose successors are only terminal nodes, there is a unique winner. It then follows directly that the losing lobbyist never makes any offers, as they could otherwise deviate to offer nothing and guarantee no payment.

In the unconstrained game each period the offer to each legislator has to weakly increase, and it must strictly increase for at least one i. Therefore, after lN periods the minimal offer made to some legislator is $(l+1)\,\varepsilon$, and eventually is greater than $\max_k W^k$. An offer greater than W^k is made only if k is certain that $j \neq k$ will outbid k, but in equilibrium it cannot be that both X and Y are certain they will be outbid by the other. So in equilibrium both players quit in every period after some finite period, and hence the equilibrium is the same as if the game were truncated at any such period. Having reduced the game to a finite game we can complete argument as in the constrained case above. ▮

PROPOSITION **2:** *If the budgets are large enough so that (2) and (3) are satisfied, then, if γ is small enough, X wins if*

$$B^X > B^Y - \sum_i V_i/2 - V_N/2 + m\varepsilon \qquad (4)$$

and Y wins if

$$B^Y > B^X + \sum_i V_i/2 + V_1/2 + m\varepsilon. \tag{5}$$

Proof of Proposition 2

We prove the following result assuming $\gamma = 0$.

Lobbyist X has a strategy that guarantees winning at cost bounded by B^X if

$$\begin{aligned} B^X - B^Y &\geq -\sum_i V_i/2 - V_N/2 + m\varepsilon \text{ and} && (4)\\ B^X &\geq \left|\frac{mV_1}{2}\right| - \frac{\sum_{i=m+1}^N V_i}{2} - \frac{V_N}{2} + m\varepsilon && (2)\end{aligned}$$

and lobbyist Y has a strategy that guarantees winning at cost bounded by B^Y if

$$\begin{aligned} B^X - B^Y &\leq -\sum_i V_i/2 - V_1/2 - m\varepsilon \text{ and} && (5)\\ B^Y &\geq \left|\frac{mV_N}{2}\right| + \frac{\sum_{i=1}^{m-1} V_i}{2} + \frac{V_1}{2} + m\varepsilon. && (3)\end{aligned}$$

This immediately implies Proposition 2 because then for small enough γ when the inequalities are strictly satisfied the same strategies guarantee a win within the budget constraint.

Lobbyist X can guarantee a win using the strategy we describe next. Have X allocate offers as follows. Let t be the period. X will identify a set of legislators S_t to " buy" that has cardinality exactly m. X will make the minimal necessary offers to buy these votes.

To complete the proof we need only describe how X should select S_t, and then show that if X has followed this strategy in past periods, then X will have enough budget to cover the required payments regardless of the strategy of Y.

Let p_i^Y be the current offer that Y has to legislator i. Set this to 0 in the case where Y has never made a viable offer to the legislator, or in a case where X already has the best standing offer to the legislator. Similarly define p_i^X.

X selects to whom to make offers by looking for those with that minimize the sum of what X has to offer, plus what offers of Y's that X frees up. In particular, let S_t be the set of legislators than minimizes $\sum_{i \in S_t} 2p_i^Y - V_i$. This is equivalent to choosing the m legislators that have the smallest values of

$$p_i^Y - \frac{V_i}{2}.$$

In the case where there are some i's that are tied under the above criterion, let X lexicographically favor legislators with lower indices. To complete the proof, we simply need to show that this strategy is within X's budget in every possible situation, presuming that X has followed this strategy up to time t.[9]

Notice that the cost of a legislator $i \in S_t$ to X is at most

$$\lceil p_i^Y - V_i \rceil^{\varepsilon} + \varepsilon. \tag{7}$$

The expression $\lceil p_i^Y - V_i \rceil^{\varepsilon}$ captures the fact that it could be that $p_i^Y < V_i$ in which case no offer is necessary.

The amount that must be offered to a legislator can only rise or stay constant over time, and so if some legislators were "purchased" by X in the past and have not been subsequently purchased by Y, then these legislators are still among the cheapest m available in the current period time and would still be selected under X's strategy (including the lexicographic tie-breaking).

Let i^* denote the most "expensive" $i \in S_t$ in terms of the "adjusted price" $p_i^Y - \frac{V_i}{2}$. If there are several legislators tied for this distinction, pick the one with the lowest index. So, $i^* \in \arg\max_{i \in S_t} \left\{ p_i^Y - \frac{V_i}{2} \right\}$, and let $\overline{S}_t$ be the complement of S_t union $\{i^*\}$.

Given the algorithm followed by X, we know that

$$p_i^Y - \frac{V_i}{2} \leq p_{i^*}^Y - \frac{V_{i^*}}{2}$$

for every $i \in S_t$. This can be rewritten as

$$p_i^Y \leq p_{i^*}^Y - \frac{V_{i^*}}{2} + \frac{V_i}{2} \tag{8}$$

for each $i \in S_t$.

Equations (7) and (8) imply that the amount required by X to follow this strategy at this stage is at most

$$\sum_{i \in S_t} \left\lceil p_{i^*}^Y - \frac{V_{i^*}}{2} - \frac{V_i}{2} \right\rceil^{\varepsilon} + m\varepsilon \tag{9}$$

If we can get an upper bound on the expression $p_{i^*}^Y - \frac{V_{i^*}}{2}$, then we have an upper bound on how much X has to pay. So we want to maximize $p_{i^*}^Y - \frac{V_{i^*}}{2}$ subject to the following constraints:

(1) $p_i^Y - \frac{V_i}{2} \geq p_{i^*}^Y - \frac{V_{i^*}}{2}$ for every $i \notin S_t$,

[9] This implies the proposition, as it means that either Y will not respond and the game will end with X the winner, or else X will get to move again and can again follow the same strategy. As the game must end in a finite number of periods, this implies that X must win.

(2) $p_i^Y \geq \alpha V_i + p_i^X$, and

(3) $\sum_{i \in \overline{S}_t} p_i^Y \leq B^Y$.

To get an upper bound, we ignore (2), and relax (3) by replacing B^Y with $\bar{B}^Y = \max\left\{B^Y, \left|\frac{mV_1}{2}\right| + \frac{\sum_{i=1}^m V_i}{2}\right\}$. The solution then involves spending all of $\bar{B}^Y$ in a manner that equalizes $p_i^Y - \frac{V_i}{2}$ with $p_{i^*}^Y - \frac{V_{i^*}}{2}$ for each $i \notin S_t$. (This is feasible due to the lower bound imposed on B^Y; it is not necessarily feasible for B^Y, but still gives a bound.) Thus, we end up with

$$p_i^Y = x^Y\left(\overline{S}_t\right) + V_i/2,$$

for each $i \in \overline{S}_t$, where

$$x^Y(\overline{S}_t) = \frac{\bar{B}^Y - \sum_{i \in \overline{S}_t} \frac{V_i}{2}}{m} \tag{10}$$

From (9), for X's strategy to be feasible it is sufficient that

$$B^X \geq \sum_{i \in S_t} \left\lceil x^Y\left(\overline{S}_t\right) - V_i/2\right\rceil^{\varepsilon} + m\varepsilon.$$

Substituting for x^Y from (10), this becomes

$$B^X \geq \bar{B}^Y - \sum_{i \in S_t \cup \overline{S}_t} V_i/2 + m\varepsilon.$$

This simplifies to

$$B^X \geq \bar{B}^Y - \sum_i V_i/2 - V_{i^*}/2 + m\varepsilon,$$

which has an upper bound when $i^* = N$, and which then yields the claimed expressions by substituting the definition of B^Y. ∎

PROPOSITION **3:** *There exists value $\lambda \in [\lfloor W^X \rfloor_\varepsilon, \lfloor W^X \rfloor_\varepsilon + \bar{V}]$ such that, for sufficiently small γ, in any equilibrium*

1. *If $\lfloor W^Y \rfloor_\varepsilon > \lambda$, then Y wins at cost $\bar{V}$ paid to the legislators m (median) through n (almost-indifferent).*

2. *If $\lfloor W^Y \rfloor_\varepsilon < \lfloor W^X \rfloor_\varepsilon$ then X wins at no cost.*

3. *If $\lfloor W^Y \rfloor_\varepsilon \in \left(\lfloor W^X \rfloor_\varepsilon, \lambda\right)$ then Y wins at cost $\bar{V}$ if it moves first , and X wins at possibly non-zero cost if it moves first.*

Proof of Proposition 3: Define $\tilde{\imath}$ and $\tilde{z}$ as the solutions to $\sum_{i=\tilde{\imath}}^{n} (\tilde{z} - \lceil V_i \rceil^{\varepsilon}) = \lfloor W^X \rfloor_{\varepsilon}$ where $\tilde{z} \in [V_{\tilde{\imath}}, V_{\tilde{\imath}-1})$ and where $V_0 = \infty$. Now $d = \lfloor W^X \rfloor_{\varepsilon} - \sum_{i=\tilde{\imath}}^{n} (\lfloor \tilde{z} \rfloor_{\varepsilon} - \lceil V_i \rceil^{\varepsilon})$, and let $\kappa = d/\varepsilon$, where by construction $0 \leq \kappa \leq n - \tilde{\imath}$. Let $\lambda \equiv \min \{n - m, n - \tilde{\imath} - \kappa\} \times \lfloor z \rfloor_{\varepsilon} + \max \{0, \tilde{\imath} + \kappa - m\} \lceil z \rceil^{\varepsilon}$. To understand this notation observe that if X initially offers $\tilde{z} - \lceil V_i \rceil^{\varepsilon}$ to all legislators in $[\tilde{\imath}, n]$ then X would exhaust the value of winning. Moreover, subject to not offering more than the value, these offers maximize $\tilde{z} \times m$, the amount that Y would need to obtain a majority. However, $\tilde{z} - \lceil V_i \rceil^{\varepsilon}$ is not a feasible offer as it is not a multiple of ε. If X offers only $\lfloor \tilde{z} \rfloor_{\varepsilon} - \lceil V_i \rceil^{\varepsilon}$ to those legislators then X would have left over an amount d. Therefore to d/ε of these legislators X could offer ε more, i.e., $\lceil z \rceil^{\varepsilon} - \lceil V_i \rceil^{\varepsilon}$, without exceeding his value of winning. Then the minimal cost to Y to obtain a majority would be exactly λ.

Consider any node at which k must offer an additional amount that is more than W^k to obtain a majority. At such a node k will make such an offer only if both lobbyists are certain $j \neq k$ will overbid, which j will do only if both are certain j will win, in which case k looses $\gamma > 0$ by making the offer instead of quitting. So at any node where k must offer at least W^k to obtain a majority, k will quit.

Now assume w.l.o.g. that $\lfloor W^k \rfloor_{\varepsilon} < \lfloor W^j \rfloor_{\varepsilon}$. We argue by induction that, at any node where k must spend a strictly positive amount to obtain a majority, k will quit. Assume the inductive hypothesis that k will quit at any node where the minimal offer needed to obtain a majority is $W^k - l\varepsilon$. Consider a node α at which k must spend $W^k - (l+1)\,\varepsilon$. If k makes such an offer, leading to node β, consider a response of j of mirroring k's last bid and adding ε to m of the offers, leading to node α' at which the minimal required for k to obtain a majority becomes $W^k - l\varepsilon$ and hence k will quit at α'. Thus at β the continuation equilibrium must be one at which j wins, and hence k's offer at α leads to an additional loss to k of at least γ. Hence k would prefer to quit at α.

Thus we have the following.

1. If $\lfloor W^X \rfloor_{\varepsilon} > \lfloor W^Y \rfloor_{\varepsilon}$ then Y will not make an initial move and X wins without making any offer.

2. If $\lfloor W^Y \rfloor_{\varepsilon} > \lfloor W^X \rfloor_{\varepsilon}$ and Y is first to move and Y makes an offer of $\bar{V}$ to obtain a majority then X quits and Y wins.

3. If $\lfloor W^Y \rfloor_{\varepsilon} > \lfloor W^X \rfloor_{\varepsilon} + \bar{V}$ and X is the first to move, and X makes any offer less than W^X then Y can reply (at cost below W^Y) by mirroring X's offer and adding $\bar{V}$. At that point X will quit since a positive amount is required for a majority. Hence X's opening offer was not optimal, and the only outcome is for X not to

make an initial offer or to make an initial offer greater than W^X, which as already argued cannot be part of an equilibrium. Thus Y wins.

4. If $\lfloor W^X \rfloor_\varepsilon < \lfloor W^Y \rfloor_\varepsilon < \lfloor W^X \rfloor_\varepsilon + \bar{V}$ and X is the first to move, and can force Y to subsequently pay more than $\lfloor W^Y \rfloor_\varepsilon$ for a majority, and if X can do so at a cost less than W^X, then X will do so and win. When can this be done by X? Exactly when $\lfloor W^Y \rfloor_\varepsilon < \lambda$. Thus, if λ is greater than W^Y then X wins since, as argued above, Y must spend more than $\lfloor W^Y \rfloor_\varepsilon$ to obtain a majority after such an initial move by X and would prefer to quit. (The amount that X must spend to win will typically be less than W^X; we do not specify the exact amount as it is even more notationally cumbersome and not of great interest.) On the other hand if λ is less than W^Y then, whatever X does in the first move (so long as it is at a cost under W^X), Y can subsequently obtain a majority at a cost under W^Y whereupon X will need to spend a positive amount to obtain a majority while $W^Y > W^X$. Hence X will quit at this point, so that at any equilibrium X will not make any initial offer when $\lfloor W^Y \rfloor_\varepsilon < \lambda$.

This complete the proof of the proposition. ■

PROPOSITION **7** *Suppose that V^X satisfies (2) in the place of B^X, and V^Y satisfies (3) in the place of B^Y. In the large budget case, lobbyist X wins in the donations-based vote-buying game if*

$$V^X - V^Y \leq -V_N/3 + \frac{2}{3}m\varepsilon$$

and Y wins if

$$V^X - V^Y \leq -V_1/3 - \frac{2}{3}m\varepsilon.$$

The proof of Proposition 7 is also straightforward and is again omitted, noting simply that the above equations follow from (4) and (5) and a maximum willingness to donate of V_i, and that $\sum_i V_i = V^X - V^Y$).

PROPOSITION **4:** *For any $\delta > 0$, there is $N(\delta)$ and $\bar{\varepsilon}$ such that for all $N > N(\delta)$ and all grids with $\varepsilon \in (0, \bar{\varepsilon})$ the following hold.*

- *If $B^Y > B^X + \hat{V}N/2 + \delta$, then Y wins with probability of at least $1 - \delta$.*
- *If $B^X > B^Y - \hat{V}N/2 + \delta$, then X wins with probability of at least $1 - \delta$.*

Proof of Proposition 4:

Lemma 1 *Suppose that Party Y offers a constant price x to all voters, such that $1 > F(x) > 0$. The least expensive way for Party X to assure itself an expected share $\sigma \in [0,1]$ of the vote would be offering a constant price to all voters. The same is also true if the roles are reversed.*

Note that we do not assume here that the constant price offered by X is a multiple of ε. If that constraint were added, then the cost to X of obtaining a share σ would be at least as high (and might involve a different strategy).

Proof of Lemma 1: The problem of finding bids p_i^X that Party X can make to assure expected share σ at minimum cost is

$$\min_{\{p_i^X\}} \sum_i p_i^X[1 - F(x - p_i^X)] \text{ s.t. } \sum_i 1 - F(x - p_i^X) \geq N\sigma, p_i^X \geq 0. \tag{11}$$

The first order conditions to (11) can be written as

$$p_i^X f(x - p_i^X) + 1 - F(x - p_i^X) - \frac{\lambda}{N} f(x - p_i^X) - \mu_i = 0. \tag{12}$$

where λ and μ_i are nonnegative multipliers.

Given that the support of F is connected and f is positive on F's support, we have three possible ranges for solutions to (12): one where $f(x - p_i^X) = 0$ and $F(x - p_i^X) = 0$, one where $f(x - p_i^X) > 0$ and $0 < F(x - p_i^X) < 1$, and one where $f(x - p_i^X) = 0$ and $F(x - p_i^X) = 1$. The first order conditions cannot be satisfied in the first case, unless $\mu_i = 1$ in which case the non-negativity constraint is binding and $p_i^X = 0$. However, by hypothesis, $0 < F(x - 0)$, which is a contradiction of the presumption of the case that $F(x - p_i^X) = 0$. In the third case, for $f(x - p_i^X) = 0$ and $F(x - p_i^X) = 1$ to hold, since $1 > F(x)$ it must be that $p_i^X < 0$. However, this cannot be a solution given the non-negativity constraint. Thus all possible solutions must fall in the second case. In the second case, in order to satisfy the first order conditions, it must be that $p_i^X \leq \frac{\lambda}{N}$. [If $\mu_i = 0$ then this is clear since $(1 - F) > 0$. If $\mu_i > 0$, then the constraint that $p_i^X \geq 0$ must be binding, in which case $p_i^X = 0$ and again $p_i^X \leq \frac{\lambda}{N}$.] For this case, since $f(x - p_i^X) > 0$, we rewrite (12) as

$$x - p_i^X - \frac{1 - F(x - p_i^X)}{f(x - p_i^X)} - (x - \frac{\lambda}{N}) + \frac{\mu_i}{f(x - p_i^X)} = 0. \tag{13}$$

Suppose that there are two solutions, p_i^X and p_j^X to (13) in this range. Without loss of generality, letting $z^i = x - p_i^X > z^j = x - p_j^X$, we have

$$z^i - \frac{1 - F(z^i)}{f(z^i)} - (x - \frac{\lambda}{N}) + \frac{\mu_i}{f(z^i)} = 0 = z^j - \frac{1 - F(z^j)}{f(z^j)} - (x - \frac{\lambda}{N}) + \frac{\mu_j}{f(z^j)}.$$

Since $z - (1 - F(z))/f(z) = z + (F(z)) - 1)/f(z)$ is increasing (in this range where $f(z) > 0$), it follows that $0 = \mu_i < \mu_j$. (Note that μ_i takes on only two values.) But this implies $p_j^X = 0 < p_i^X$, which contradicts the fact that $z^i > z^j$.

Thus we have shown that any solution to (11) necessarily has identical prices offered to all agents.

The proof for Lemma 1 with the roles reversed for the parties has (11) replaced by

$$\min_{\{p_i^Y\}} \sum_i p_i^Y [F(p_i^Y - x)] \text{ s.t. } \sum_i F(p_i^Y - x) \geq N\sigma, p_i^X \geq 0,$$

with corresponding first order conditions

$$p_i^Y f(p_i^Y - x) + F(p_i^Y - x) - \frac{\lambda}{N} f(p_i^Y - x) - \mu_i = 0.$$

Working through similar cases as those above, and this time using the fact that $z + F(z)/f(z)$ is increasing on the support of F, yields the same conclusion. □

Lemma 2 *If* $(0.5+\eta)N[\frac{B^X}{(0.5-\eta)N} + F^{-1}(0.5-\eta)] < B^Y$*, then* Y *can obtain expected share* $(0.5+\eta)$ *of the vote at each stage. Similarly if,* $(0.5+\eta)N[\frac{B^Y}{(0.5-\eta)N} - F^{-1}(0.5+\eta)] < B^X$*, then* X *can obtain a share of* $(0.5+\eta)$ *at each stage.*

Proof of Lemma 2: We show the first claim, as the second is analogous. Suppose that it is Y's turn. If Y can offer all voters the same price $p = B^X/(0.5-\eta)N + F^{-1}(0.5-\eta)$, then Y can win in one step. This is so since, by the previous claim, X's least expensive way of getting at least $(0.5-\eta)N$ is by offering the same price to all voters. A constant price that suffices here is $B^X/(0.5-\eta)N$ which exactly exhausts X's budget (ignoring the constraint that X must make offers in multiples of ε, and more than exhausts it if the constraint is taken into account). Now, since $B^X \frac{0.5+\eta}{0.5-\eta} + (0.5+\eta)NF^{-1}(0.5-\eta) < B^Y$, the price p is feasible for Y when only $(0.5+\eta)N$ voters (or slightly more) accept it. Thus, if p is infeasible at that stage, then there are more than $(0.5+\eta)N$ voters who would prefer to sell to Y at that price. But this means that there is a lower price $p' < p$ that gives Y an expected majority of $(0.5+\eta)N$. Since $(0.5+\eta)Np' < (0.5+\eta)Np < B^Y$, the price p' is feasible. Clearly, if p' is not a multiple of ε then for any ε small enough there is a p'' that is slightly larger that also gives Y an expected majority of $(0.5+\eta)N$, and for a small enough grid size still more than exhausts X's budget. □

We now show (1) and (2) of the proposition. We concentrate on (1), as the other case is analogous, given the lemmas above. For $\delta > 0$, there exists sufficiently small $\eta > 0$ such that $(0.5+\eta)N[\frac{B^X}{(0.5-\eta)N} + F^{-1}(0.5-\eta)] < B^X + \alpha\bar{U}N/2 + \delta$. Therefore, if

η is sufficiently small, $B^Y > B^X + \alpha\bar{U}/2 + \delta$ together with Lemma 2 imply that Y can obtain an expected share of $(0.5 + \eta)$. When N is made sufficiently large (here we mean that B^X and B^Y increase proportionately with N), an expected share of $(0.5 + \eta)$ means an arbitrarily large probability of winning. Therefore, there exists $N(\delta)$ such that, for $N > N(\delta)$, Y's winning probability is above $1 - \delta$. This complete the proof of Proposition 4. ∎