Monopoly with Resale*

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Abstract

This paper illustrates the intricacies associated with the design of revenue-maximizing mechanisms for a monopolist who expects her buyers to resell in a secondary market. We consider two modes of resale: the first is to a third party who does not participate in the primary market; the second is inter-bidders resale, where the winner in the primary market resells to the losers.

The main contribution is in showing how revenue-maximizing mechanisms can be designed investigating the optimal informational linkage with the secondary market. To control the price in the resale game, the monopolist must design an allocation rule and a disclosure policy that optimally fashion the beliefs of the participants in the resale market. We show that it is generically impossible to maximize revenue through deterministic selling procedures and disclosing only the decision to trade with a particular buyer. To create the optimal informational linkage, the monopolist may need to induce stochastic allocations and disclose also the price paid in the primary market. The optimal allocation rule and disclosure policy maximize the expected sum of the bidders' *resale-augmented virtual valuations* under the constraints imposed by the sequential rationality of the bidders' offers in the resale game.

Keywords: optimal information linkage between primary and secondary markets, disclosure policy, stochastic allocations, resale-augmented virtual valuations.

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1 Introduction

Durable goods are typically traded both in primary and secondary markets. Indeed, auctions for real estates, artwork and antiques are often followed by resale. The same is true for licenses, patents, Treasury bills, emission and spectrum rights. Similarly, IPOs and privatizations generate ownership structures which change over time as a consequence of active trading in secondary markets.

Resale may have different explanations. First, it may be a consequence of the fact that not all potential buyers participate in the primary market.[1] For example, a buyer may value a good only if it goes first into the hands of another agent. Intermediation is typical for instance in IPOs and Treasury bills markets and is often believed to be value-enhancing. Alternatively, participation only in secondary markets may be due to a change in the environment: At the time the government decides to sell spectrum rights, a company may not bid in the auction because at that point it does not formally exist yet or it attaches a low value to the rights, possibly because of its current position in the market, or because of the business strategy of its management. After a merger, a privatization, or a successful takeover, the same company may develop interest in possessing the rights and decide to buy them from the winner in the primary market.[2] Furthermore, participation only in secondary markets may also be strategic as indicated in McMillan (1994) and Jehiel and Moldovanu (1996).

Second, resale may be a direct consequence of misallocations in the primary market. As shown first in Myerson (1981), optimal auctions are typically inefficient when the distributions of the bidders' valuations are asymmetric. By committing to a policy that misplaces the good into the hands of a buyer who does not value it the most, a seller can induce more aggressive bidding and raise higher expected revenues. When resale is possible, bidders may thus attempt to correct misallocations in the auction by further trading in a secondary market.[3]

As indicated in Haile (1999, 2003), with resale, buyers' willingness to pay in the primary market is endogenous for it incorporates the surplus expected in the secondary market which in turn depends on the information filtered by the monopolist through her selling procedure.

In this paper, we examine the intricacies associated with the design of revenue-maximizing mechanisms for a monopolist who expects her buyers to resell.[4] The revenue-maximizing mechanisms are obtained by investigating the optimal informational linkage with the secondary market and showing how this linkage can be implemented through the design of a proper allocation rule and the adoption of an optimal disclosure policy. We analyze a simple two-stage game of incom-

[1] Bikhchandani and Huang (1989), Haile (1999), and Milgrom (1987) consider auctions followed by resale where the set of bidders in the primary market does not include all potential buyers.

[2] Haile (2003) and Schwarz and Sonin (2001) consider models where bidders' valuations change over time.

[3] See also Gupta and Lebrun (1999) for an analysis of first-price asymmetric sealed bid auctions followed by resale where trade in the secondary market is motivated by the inefficiency of the allocation in the primary market.

[4] Revenue-maximizing mechanisms without resale have been examined, among others, by Bulow and Roberts (1989), Harris and Raviv (1981), Maskin and Riley (1984), Myerson (1981), Riley and Samuelson (1981).

plete information where in the first stage, a monopolist sells a durable and indivisible good in a primary market, whereas in the second stage, the primary buyer resells the good in a secondary market. Trade in the resale game is the result of a simple ultimatum bargaining procedure in which players make take-it-or-leave-it offers with a probability distribution that reflects their relative bargaining abilities. Although stylized, the model illustrates the dependence of the resale surplus on the information disclosed in the primary market and is sufficiently tractable to allow for a complete characterization of the optimal allocation rule and disclosure policy from the monopolist's viewpoint.

The first part of the paper considers the case where resale is to a third party who participates only in the secondary market. In this case, the monopolist uses the primary buyer as an intermediary to extract surplus also from those buyers who are not willing, or able, to contract directly with her in the primary market. The study of this simple environment illustrates the main intricacies associated with the design of revenue-maximizing mechanisms. Two results are highlighted in the analysis.

First, the monopolist may find it optimal to sustain *stochastic allocations* in the primary market, for example, using lotteries to allocate the good and/or offering menus that induce the buyer to follow mixed strategies. By selling with different probability to different types, the monopolist uses the decision to trade as a signal of the buyer's valuation to induce the third party to offer a higher price in the resale game. Contrary to deterministic allocations, stochastic selling procedures give the monopolist a better control over the beliefs of the participants in the secondary market and do not require to exclude completely those buyers with a lower willingness to pay. To illustrate, suppose the buyer in the primary market has either a high or low valuation and assume the third party's prior beliefs are unfavorable to the buyer in the sense she is expected to offer a low price in the event she learns nothing about the value the latter attaches to the good. Then, if the monopolist uses a deterministic mechanism that sells to either type with certainty, the third party does not learn anything from the decision to trade and thus offers a low price. If, on the other hand, the monopolist sells only to the high type, the third party perfectly learns the value the buyer attaches to the good and offers a price equal to his high valuation, which again leaves no surplus to the buyer. In contrast, with a stochastic mechanism, the monopolist can sell to the high type with certainty and to the low type with probability positive but low enough to induce the third party to offer a high resale price, increasing the surplus the low type expects from resale and thus his willingness to pay in the primary market.

Second, the optimal mechanism may require the adoption of a *disclosure policy* richer than the simple announcement of the decision to trade. In the example above, the monopolist could disclose two signals, the first with a higher probability when the buyer reports a high valuation, the second with a higher probability when he reports a low valuation. By making the two signals sufficiently informative (but not perfect), the monopolist may thus induce the third party to offer a high resale price also to the low type. The advantage of disclosing information in addition to the decision to trade comes from the possibility to increase the level of trade without necessarily reduce the price in the secondary market. In the limit, if the monopolist knew the buyer's valuation, she could sell with certainty to either type and use only the stochastic disclosure policy to sustain a high resale price.

Things are however more complicated when the buyer's valuation is not known to the initial seller. In this case, disclosure enhances the level of trade but does not allow the monopolist to sell with certainty, for the combination of a certain allocation rule and a stochastic disclosure policy is not incentive compatible. Indeed, if trade were certain, the high type would always select the contract with the lowest price, irrespective of the associated disclosure policy. But then the low type would have an incentive to mimic the high type, paying the same price and inducing the monopolist to disclose the most favorable signal with a higher probability. The only way the monopolist can sort the two types and at the same time disclose informative signals to the third party is by making the high type pay a higher price than the low type, which is possible only if a higher price is associated with a lottery that delivers the good with a higher probability.

We show how the optimal disclosure policy can be obtained as part of a direct revelation mechanism in which the seller commits to disclose abstract signals with a distribution that depends on the information revealed in the primary market. After characterizing the optimal mechanism, we propose a natural implementation in which these additional signals are simply the prices the buyer pays for lotteries that deliver the good with different probabilities. The optimal menu of price-lottery pairs may either perfectly separate the buyer's types or induce the latter to play a mixed strategy with type-dependent randomizations. In the first case, the monopolist discloses only the decision to trade, whereas in the second also the price paid.

The analysis of the single-buyer case is extended in two directions. First, we allow for multiple bidders in the primary market, but maintain the assumption resale is to a third party who participates only in the secondary market. In this case, resale increases the willingness to pay and reduces the differences across types and hence the informational rents the monopolist must leave to the buyers to induce truthful information revelation. On the other hand, resale also creates new incentives for low types to misrepresent their valuations for the purpose of obtaining a higher price in the secondary market, which in turn may interfere with the monopolist's ability to sustain the same allocations as in the absence of resale. However, we show that by committing not to disclose any information in addition to the identity of the winner, it is always possible to implement exactly the same allocation rule as in a Myerson optimal auction in which bidders do not have the option to resell. It follows that a monopolist always benefits from resale when she can not contract with all potential buyers and is able to prevent the winner in the primary market from reselling to the losers.

Finally, in the last part of the paper we examine revenue-maximizing mechanisms for a monopolist who expects inter-bidders resale. To the best of our knowledge, this problem has been examined only by Ausubel and Cramton (1999) and Zheng (2002). Ausubel and Cramton assume perfect resale markets and show that if all gains from trade are exhausted through resale, then it is strictly optimal for the monopolist to implement an efficient allocation directly in the primary market. The case of perfect resale markets, although a benchmark, abstracts from important elements of resale. First, when bidders trade under asymmetric information, misallocations are not necessarily corrected in secondary markets (Myerson and Satterthwaite (1983)). Second, and more important, efficiency in the secondary market is endogenous as it depends on the information revealed in the primary market which is optimally fashioned by the monopolist through her choice of an allocation rule and a disclosure policy.

Zheng assumes it is always the winner in the auction who offers the price in the secondary market and suggests a mechanism that, under a few assumptions on the distributions of the bidders' valuations, gives the monopolist the same expected revenue as in a standard optimal auction where resale is prohibited. Instead of selling to the bidder with the highest virtual valuation, the monopolist sells to the bidder who is most likely to implement in the secondary market the same final allocation as in the optimal auction of Myerson (1981).

Although an important contribution, Zheng's result crucially relies on the assumption it is always the seller who offers the resale price. This assumption, justified on the grounds of symmetry between primary and secondary markets, has a lot of bite since it implies the monopolist can perfectly control the distribution of bargaining power in the secondary market through the allocation of the good in the primary market. However, that the original seller has full bargaining power rarely implies that any seller of the same good will necessarily have full bargaining power when contracting with prospective buyers. In general, the distribution of bargaining power is likely to be a function of the allocation of the good, but also of the individual characteristics of the players, such as their personal bargaining abilities. What is more, assuming it is always the seller who offers the resale price eliminates strategic effects which are important for the analysis of the design of revenue-maximizing mechanisms. First, when the monopolist can not perfectly control the distribution of bargaining power in the secondary market, it is generically impossible to achieve Myerson's expected revenue without prohibiting resale. Second, to create the optimal informational linkage the monopolist can not limit attention to deterministic selling procedures: the revenue-maximizing mechanism in the primary market may require the use of stochastic allocations and possibly the adoption of a disclosure policy richer than the announcement of the identity of the winner. We show how the optimal allocation rule and disclosure policy can be designed maximizing the expected sum of the bidders' *resale-augmented virtual valuations*, taking into account the constraints on the sequential rationality of the bidders' offers in the secondary market.

As anticipated, auctions followed by an opportunity for resale have been analyzed also by Haile (1999, 2003) who studies equilibria in standard auction formats such as English, first-price, and second-price sealed bid. His results well illustrate how the option to resell creates endogenous valuations and induces signaling incentives that may revert the revenue ranking obtained in the literature that assumes no resale. Our analysis builds on some of his insights, but differs from his in that we assume the monopolist is not constrained to use any specific auction format. Furthermore, the focus is on the design of the *optimal informational linkage* between primary and secondary markets from the monopolist's viewpoint and on the possibility to implement it through the adoption of an appropriate disclosure policy.

Finally, another strand of literature related to our work considers bidding behavior in auctions followed by product market competition. Jehiel and Moldovanu (2000) focus on the effect of positive and negative externalities stemming from subsequent market interaction on equilibrium bidding strategies in second-price sealed bid auctions. They also study the effects of reserve prices and entry fees on the seller's revenue and on welfare. More recently, Katzman and Rhodes-Kropf (2002) and Zhong (2002) study the effect of different bid announcement policies on the seller's expected revenue in standard auctions followed by Bertrand and Cournot competition. Our paper differentiates from this literature for the analysis of resale instead of imperfect product market competition and for the use of mechanism design to derive the monopolist's *optimal selling procedure* and *disclosure policy*. To the best of our knowledge, this is new to the literature.

The rest of the paper is organized as follows. Section 2 studies the case where the monopolist sells to a buyer in the primary market who then resells to a third party in a secondary market; this section illustrates the main intricacies of optimal mechanism design for a monopolist who expects resale. Section 3 extends the analysis to the case where the monopolist can contract with all potential buyers but can not prohibit the winner in the primary market from reselling to the losers. Finally, Section 4 concludes. All proofs are relegated to the Appendix.

2 Resale to Third Parties

2.1 The environment

Consider an environment where in the primary market a monopolistic *seller* (S hereafter) trades a durable and indivisible good with a (representative) *buyer,* B. If B receives the good from S, he can either keep it for himself, or resell it to a (representative) *third party*, T, in a secondary market.[5]

S can not sell directly to T, nor can S or B communicate with T before they trade. For example, T may represent a prospective future buyer that does not exist at the time S needs to sell (such as in the case of a new firm that is expected to be created as a result of a merger or a privatization). Alternatively, T may value the good only if it goes first into the hands of B, as with

[5] We adopt the convention of using masculine pronouns for B and feminine pronouns for S and T.

an intermediate product that needs to be processed before it can be used also by T. In this case, S is likely to lack the necessary bargaining power to extract money from T without selling anything to her, as indicated in Milgrom (1987). Finally, there can be legal or political impediments that prevent S from contracting with T, such as in the case of a privatization in which the government is constrained to sell only to domestic firms.

Let $x_B^i \in \{0,1\}$ represent the decision to trade between B and player i, with $i = S, T$. When $x_B^i = 1$, the good "changes hands". For example, for $i = S$, $x_B^S = 1$ means that B obtains the good from S. Similarly, for $i = T$, $x_B^T = 1$ means that T obtains the good from B. On the contrary, if $x_B^i = 0$, there is no trade between B and player i. A trade outcome $\left\{x_B^i, t_B^i\right\}$, consists of the allocation of the good x_B^i and a monetary transfer $t_B^i \in \mathbb{R}$ between B and player i.

We denote with θ_i the value $i = B, T$ attaches to the good and with $\boldsymbol{\theta} := (\theta_B, \theta_T) \in \boldsymbol{\Theta} := \Theta_B \times \Theta_T$ a profile of valuations for B and T. We also assume the value to S is common knowledge and equal to zero. All players have quasi-linear preferences, respectively equal to

$$\begin{aligned}
u_S &= t_B^S, \\
u_B &= \theta_B x_B^S (1 - x_B^T) - t_B^S + t_B^T, \\
u_T &= \theta_T x_B^S x_B^T - t_B^T.
\end{aligned}$$

We make the following assumptions on valuations.

A1: For $i \in \{B, T\}$, $\Theta_i = \left\{\overline{\theta}_i, \underline{\theta}_i\right\}$ with $\Delta\theta_i := \overline{\theta}_i - \underline{\theta}_i \geq 0$, $\underline{\theta}_i > 0$, and $\Pr(\overline{\theta}_i) = p_i$.

A2: For any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, $\Pr(\boldsymbol{\theta}) = \Pr(\theta_B) \cdot \Pr(\theta_T)$.

A3: B is the only player who knows θ_B and T is the only player who knows θ_T.

A4: $\underline{\theta}_B \leq \underline{\theta}_T$ and $\overline{\theta}_B \leq \overline{\theta}_T$.

Assumptions A1-A4 identify two markets in which (i) agents have *discrete independent private values*, (ii) trade occurs under *asymmetric information,* and (iii) there are *gains from trade in either market*. Assumption A4 leads to two possible cases:

A4.1: $\underline{\theta}_B \leq \underline{\theta}_T \leq \bar{\theta}_B \leq \overline{\theta}_T$,

A4.2: $\underline{\theta}_B \leq \bar{\theta}_B \leq \underline{\theta}_T \leq \overline{\theta}_T$.

In all other cases, the outcome in the resale bargaining game does not depend on the beliefs either B or T have about the rival's valuation.

Assuming two types is a limitation, but is needed to keep the analysis tractable. The problem with a continuum of types stems from the difficulty to optimize over the space of all possible posterior beliefs for T without imposing ad hoc restrictions. A similar difficulty arises in the literature of dynamic contracting where a principal needs to control for the belies of his future selves in the downstream interaction with the agent; although a complete characterization is available in the two-type case (Laffont and Tirole 1988, 1990), it is well known that the generalization to the continuum is hard to settle. The same problem also emerges in Zheng (2002) if one relaxes the assumptions on the distributions and on the specific bargaining game which are necessary for the optimality of his winner-selection rule.

Primary Market

In the primary market, S designs a mechanism which consists in a trading procedure along with a disclosure policy. As proved in Pavan and Calzolari (2002), there is no loss of generality in restricting attention to direct revelation mechanisms

$$\phi_S : \Theta_B \to \mathbb{R}\times\triangle(\{0,1\} \times Z)$$

in which B reports a message $\theta_B \in \Theta_B$, pays a transfer $t^S_B(\theta_B) \in \mathbb{R}$ and with probability $\phi_S(1, z|\theta_B)$ receives the good and information $z \in Z$ is disclosed to T in the secondary market. Since players have quasi-linear payoffs, it is without loss of generality to restrict attention to mechanisms ϕ_S : $\Theta_B \to \mathbb{R} \times \triangle(\{0,1\} \times Z)$ instead of $\phi_S : \Theta_B \to \triangle(\mathbb{R} \times \{0,1\} \times Z)$, for $t^S_B(\theta_B)$ can always be read as the *expected* transfer between B and S. Note that we allow S to choose any menu (indexed by Θ_B) of joint distributions over $\{0,1\} \times Z$. Imposing restrictions on the correlation between the (marginal) distribution over x^S_B and that over z (for example, assuming the lottery over the decision to trade is independent from that used to disclose information to T) may preclude the possibility to characterize the optimal mechanism.

We do not assign any precise meaning to the set Z at this point, but we assume it is sufficiently rich to generate any desired posterior beliefs in the secondary market. As we prove in Lemma 1, with a finite number of players and types, Z can be restricted to be a finite set. This abstract representation of information transmission between the two markets allows to replicate with a direct revelation mechanism fairly general disclosure policies. The implementation of the optimal mechanism at the end of the section will suggest a natural interpretation of Z as the set of possible prices paid by B in the primary market.

Note that the disclosure policy is stochastic for two reasons. First, trade between B and S may be subject to uncertainty which may be reflected into the signal z when this is correlated with the decision to trade. Second, it may well be in the interest of S to commit not to fully disclose to T all the information that has been revealed in the primary market. Indeed, we assume S is not exogenously compelled to release any particular information and can commit to any disclosure policy of her choosing (the case where S can not commit to her disclosure policy is discussed at the end of Section 2.4). Furthermore, S can not charge T for the information disclosed, which is consistent with the assumption that S can not contract with T. We relax this assumption in Section 3, where we allow S to extract money also from secondary buyers that do not obtain the good in the primary market but receive valuable information about rivals' valuations. Finally, S can not make the price contingent on the outcome in the secondary market, nor can she design the resale game, for example by assigning bargaining power to one of the two bidders. If this were the case, S could also indirectly control the final allocation and the analysis of the constraints imposed by resale would be uninteresting.

Secondary Market

The resale price is assumed to be the result of a random ultimatum bargaining procedure: With probability λ_B, B makes a take-it-or-leave-it offer to T, whereas with the complementary probability $\lambda_T = 1 - \lambda_B$, T makes a take-it-or-leave-it offer to B. Restricting the two players to make take-it-or-leave-it offers that consist of a single price instead of more general mechanisms is without loss of generality in this environment with quasi-linear preferences, private values and finite types (Maskin and Tirole, 1990, Prop. 11). Hence, one can interpret λ_B also as the probability a player designs the resale mechanism in the secondary market. This bargaining procedure is stylized, yet common in the contract theory literature. It has the advantage of being particularly tractable and at the same time it allows for an explicit analysis of the dependence of the resale outcome on the information disclosed by the monopolist in the primary market, which is the interest of our analysis. Note that $\lambda_B = 1$ corresponds to the bargaining game examined in Zheng (2002), where it is always the seller who makes the offer in the secondary market. We believe this assumption may not be appropriate for two reasons. First, it implies the initial monopolist can perfectly control the distribution of bargaining power in the secondary market which in turn eliminates important strategic effects associated with resale (For example, disclosure becomes irrelevant when it is always B who makes the price in the resale game). Second, we do not find the justification of symmetry with the primary market fully convincing: That the initial seller has full bargaining power rarely implies that any seller of the same good will necessarily have full bargaining power when contracting with prospective buyers. In general, the distribution of bargaining power is likely to be a function of the allocation of the good, but also of the individual characteristics of the players. By assuming the distribution of bargaining power is entirely identity-dependent (and parametrized by the exogenous probability each bidder offers the resale price), we isolate in this paper strategic effects that are new to the literature and show how they affect the design of optimal selling procedures.[6]

Timing

- At $t = 1$, S publicly announces a selling mechanism $\phi_S \in \Phi_S$, where Φ_S is the set of all possible feasible (direct revelation) mechanisms.[7] If B refuses to participate in ϕ_S, the game ends and all players enjoy their reservation payoffs which are equal to zero. If B accepts, he reports θ_B, pays an expected transfer $t_B^S(\theta_B)$ and with probability $\phi_S(1, z|\theta_B)$ receives the good and information $z \in Z$ is disclosed to T in the secondary market.

[6] In addition to Zheng's representation of the resale game, the other modelizations considered in the literature are the following: (a) resale is the result of any possible bargaining procedure which leads to valuations in the primary market satisfying a set of reasonable conditions (cfr Haile, 1999); (b) the resale game is exogenous, but assumed to lead to efficient allocations with certainty (Ausubel and Cramton, 1999).

[7] Pavan and Calzolari (2002) formally prove that assuming S publicly announces her mechanism corresponds to the most favorable case for the initial seller. It is important to note that although T can observe the direct mechanism ϕ_S, she does not necessarily observe its realization, i.e. the announcement θ_B, the decision to trade, x_B^S, or the transfer t_B^S.

- At $t = 2$, if $x_B^S = 1$, bargaining between B and T takes place according to the procedure described above. Otherwise, the game is over.

Figure 1 summarizes the trading environment.

[Insert Figure 1 here]

The game described above can be solved by backward induction examining first how the price in the secondary market is influenced by the outcome in the primary market and then choosing the selling procedure that maximizes revenue taking into account the effect of disclosure on the resale surplus.

2.2 The outcome in the secondary market

Consider first the case where T **offers the price** in the resale game. Given information $z \in Z$, T updates her beliefs about θ_B using Bayes rule and the fact that B received the good in the primary market so that

$$\Pr(\bar{\theta}_B | x_B^S = 1, z) = \frac{\phi_S(1, z|\bar{\theta}_B) p_B}{\phi_S(1, z|\bar{\theta}_B) p_B + \phi_S(1, z|\underline{\theta}_B)(1 - p_B)}.$$

Note that, even if T does not directly observe x_B^S, she always makes her offer contingent on the event $x_B^S = 1$: indeed, trade between B and T in the secondary market is possible only if B received the good from S. For any $z \in Z$ and $\theta_T \in \Theta_T$, T then offers a price[8]

$$t_B^T(\theta_T, z) = \begin{cases} \bar{\theta}_B & \text{if } \Pr(\bar{\theta}_B | x_B^S = 1, z) \geq \frac{\Delta\theta_B}{\theta_T - \underline{\theta}_B}, \\ \underline{\theta}_B & \text{if } \Pr(\bar{\theta}_B | x_B^S = 1, z) < \frac{\Delta\theta_B}{\theta_T - \underline{\theta}_B}. \end{cases}$$

which is obtained comparing the surplus $\theta_T - \bar{\theta}_B$ with the surplus $(\theta_T - \underline{\theta}_B)\left[1 - \Pr(\bar{\theta}_B | x_B^S = 1, z)\right]$ that T can achieve by offering a lower price at the risk of ending up without buying. It follows that the surplus B obtains from reselling to T when it is the latter to offer the price is

$$r_B(\bar{\theta}_B, \theta_T | z) = 0 \text{ for any } \theta_T \in \Theta_T \text{ and } z \in Z.$$

$$r_B(\underline{\theta}_B, \theta_T | z) = \begin{cases} \Delta\theta_B \text{ if } t_B^T(\theta_T, z) = \bar{\theta}_B, \\ 0 \text{ otherwise.} \end{cases}$$

[8] In the case T is just indifferent between offering a high and a low price, we assume she offers a high price. In addition, we assume B sells to T when he is indifferent between accepting T's offer and retaining the good. These assumptions are not needed and are introduced just to simplify the exposition.

Letting $r_B(\theta_B|z) := \mathbb{E}_{\theta_T}\left[r_B(\theta_B, \theta_T|z)\right]$, we then have that the difference in the surplus expected by the two types is $\Delta r_B(z) := [r_B(\overline{\theta}_B|z) - r_B(\underline{\theta}_B|z)] \in [-\Delta\theta_B, 0]$, with

$$\Delta r_B(z) = \begin{cases} 0 & \text{if } \Pr(\bar{\theta}_B|x_B^S = 1, z) < \frac{\Delta\theta_B}{\overline{\theta}_T - \underline{\theta}_B}, \\ -p_T\Delta\theta_B & \text{if } \Pr(\bar{\theta}_B|x_B^S = 1, z) \in \left[\frac{\Delta\theta_B}{\overline{\theta}_T - \underline{\theta}_B}, \frac{\Delta\theta_B}{\underline{\theta}_T - \underline{\theta}_B}\right), \\ -\Delta\theta_B & \text{if } \Pr(\bar{\theta}_B|x_B^S = 1, z) \geq \frac{\Delta\theta_B}{\underline{\theta}_T - \underline{\theta}_B}. \end{cases}$$

Next, consider the case where B **asks the price.** When his personal value for the good is θ_B, B asks a resale price

$$t_B^T(\theta_B) = \begin{cases} \overline{\theta}_T & \text{if } p_T > \frac{\underline{\theta}_T - \theta_B}{\overline{\theta}_T - \theta_B}, \\ \underline{\theta}_T & \text{if } p_T \leq \frac{\underline{\theta}_T - \theta_B}{\overline{\theta}_T - \theta_B}. \end{cases}$$

which is obtained comparing the surplus $\underline{\theta}_T - \theta_B$ that B can guarantee himself asking a low price with the expected surplus $p_T(\bar{\theta}_T - \theta_B)$, that can be obtained by asking a higher price $\overline{\theta}_T$. It follows that the surplus B expects from resale when he makes the offer is

$$s_B(\theta_B, \theta_T) = \begin{cases} \bar{\theta}_T - \theta_B & \text{if } t_B^T(\theta_B) = \bar{\theta}_T, \text{ and } \theta_T = \bar{\theta}_T, \\ \underline{\theta}_T - \theta_B & \text{if } t_B^T(\theta_B) = \underline{\theta}_T, \\ 0 & \text{otherwise.} \end{cases}$$

Letting $s_B(\theta_B) := \mathbb{E}_{\theta_T}\left[s_B(\theta_B, \theta_T)\right]$, we then have that $\Delta s_B := [s_B(\overline{\theta}_B) - s_B(\underline{\theta}_B)] \in [-\Delta\theta_B, 0]$, with

$$\Delta s_B = \begin{cases} -\Delta\theta_B & \text{if } p_T \leq \frac{\underline{\theta}_T - \overline{\theta}_B}{\overline{\theta}_T - \overline{\theta}_B}, \\ p_T(\overline{\theta}_T - \overline{\theta}_B) - (\underline{\theta}_T - \underline{\theta}_B) & \text{if } p_T \in \left(\frac{\underline{\theta}_T - \overline{\theta}_B}{\overline{\theta}_T - \overline{\theta}_B}, \frac{\underline{\theta}_T - \underline{\theta}_B}{\overline{\theta}_T - \underline{\theta}_B}\right], \\ -p_T\Delta\theta_B & \text{if } p_T > \frac{\underline{\theta}_T - \underline{\theta}_B}{\overline{\theta}_T - \underline{\theta}_B}. \end{cases}$$

Note that both Δs_B and Δr_B are negative: resale (to third parties) not only increases the value the primary buyer attaches to the good, but also reduces the differences between types. As illustrated below, this affects the monopolist's ability to extract surplus in the primary market as well as the structure of the optimal mechanism.

2.3 The optimal mechanism in the primary market

At $t = 1$, both S and B have correct expectations on how the resale outcome depends on the mechanism adopted in the primary market. Taking these expectations into account, the monopolist then chooses a mechanism $\phi_S \in \Phi_S$ which maximizes her expected revenue subject to the buyer's individual rationality and incentive compatibility constraints

$$U_B(\theta_B) := \sum_{z \in Z} \phi_S(1, z|\theta_B) \left\{\theta_B + \lambda_B s_B(\theta_B) + \lambda_T r_B(\theta_B|z)\right\} - t_B^S(\theta_B) \geq 0, \qquad (IR(\theta_B))$$

$$U_B(\theta_B) \geq \sum_{z \in Z} \phi_S(1, z|\hat{\theta}_B) \{\theta_B + \lambda_B s_B(\theta_B) + \lambda_T r_B(\theta_B|z)\} - t_B^S(\hat{\theta}_B), \text{for any } \hat{\theta}_B \in \Theta_B. \quad (IC(\theta_B))$$

Formally, the program for the optimal mechanism can be written as

$$\mathcal{P}_S : \begin{cases} \max\limits_{\phi_S \in \Phi_S} \mathbb{E}_{\theta_B}\left[t_B^S(\theta_B)\right] \\ \text{subject to } IR(\theta_B) \text{ and } IC(\theta_B) \text{ for any } \theta_B \in \Theta_B. \end{cases}$$

Given the price formation process in the resale (sub)game, there is no loss of generality in assuming S chooses a mechanism that discloses only three signal, say z_l with $l = 1, 2, 3$, such that

$$\begin{array}{l} t_B^T(\theta_T, z_1) = \bar{\theta}_B \text{ for any } \theta_T, \\ t_B^T(\theta_T, z_2) = \underline{\theta}_B \text{ for any } \theta_T, \\ t_B^T(\theta_T, z_3) = \bar{\theta}_B \text{ if } \theta_T = \bar{\theta}_T \text{ and } t_B^T(\theta_T, z_3) = \underline{\theta}_B \text{ otherwise.} \end{array}$$

In words, signals z_1 and z_2 correspond to information that induces T to offer respectively a high and a low price independently of her valuation, whereas signal z_3 induces $\bar{\theta}_T$ to offer a high price and $\underline{\theta}_T$ a low price. Indeed, if ϕ_S were to disclose more than three signals, then at least two signals would induce the same offer from T in the secondary market. But then S could simply replace ϕ_S with another mechanism ϕ_S' in which these two signals are replaced with a single one, preserving the outcome in the resale game and hence generating the same revenue in the primary market. The formal argument is in Lemma 1.

Lemma 1 (Cardinality of the signal space) *Let*

$$\#_{\phi_S} Z := \#\left\{z \in Z : \phi_S(1, z|\theta_B) > 0 \textit{ for some } \theta_B \in \Theta_B\right\}.$$

For any mechanism ϕ_S such that $\#_{\phi_S} Z > 3$, there exists another mechanism ϕ_S' such that $\#_{\phi_S'} Z \leq 3$ which is payoff-equivalent for all players.

Also note that since the offer $t_B^T(\theta_T, z_1)$ is always contingent on the event trade occurred in the primary market, the information S discloses to T when $x_B^S = 0$ has no value. Hence, with a slight abuse of notation, in what follows we will assume that S does not disclose any signal when $x_B^S = 0$. This also implies that for each message $\theta_B \in \Theta_B$, $\sum_{l=1}^3 \phi_S(1, z_l|\theta_B) = 1 - \phi_S(0|\theta_B)$, where $\phi_S(0|\theta_B)$ is the probability of no trade.

As in standard screening models, since the high type can always guarantee himself at least the same payoff as the low type by announcing $\widehat{\theta}_B = \underline{\theta}_B$, at the optimum the two constraints $IC(\overline{\theta}_B)$ and $IR(\underline{\theta}_B)$ necessarily bind, which implies that[9]

$$t_B^S(\underline{\theta}_B) = \sum\nolimits_{l=1}^3 \phi_S(1, z_l|\underline{\theta}_B) \{\underline{\theta} + \lambda_B s_B(\underline{\theta}_B) + \lambda_T r_B(\underline{\theta}_B|z)\},$$

[9] The proof is in the Appendix – Lemma 2.

$$\begin{array}{rcl} t_B^S(\overline{\theta}_B) & = & \sum_{l=1}^{3}\phi_S(1,z_l|\overline{\theta}_B)\left\{\overline{\theta}+\lambda_B s_B\left(\overline{\theta}_B\right)+\lambda_T r_B(\overline{\theta}_B|z)\right\}+ \\ & & -\sum_{l=1}^{3}\phi_S\left(1,z_l|\underline{\theta}_B\right)\left[\Delta\theta_B+\lambda_B\Delta s_B+\lambda_T\Delta r_B(z_l)\right]. \end{array}$$

In other words, $\overline{\theta}_B$ must be given a price discount (informational rent)

$$U_B(\overline{\theta}_B)=\sum_{l=1}^{3}\phi_S\left(1,z_l|\underline{\theta}_B\right)\left[\Delta\theta_B+\lambda_B\Delta s_B+\lambda_T\Delta r_B(z_l)\right]$$

to truthfully reveal his type. Note that $\Delta\theta_B+\lambda_B\Delta s_B+\lambda_T\Delta r_B(z_l)$ is positive since both Δs_B and $\Delta r_B(z_l)$ are clearly smaller than $\Delta\theta_B$. Other way stated, resale reduces the differences in valuations, but does not invert the ranking across types. Also note that contrary to standard screening mechanisms without resale, the informational rent $U_B(\overline{\theta}_B)$ depends not only on the probability of receiving the good when announcing $\widehat{\theta}_B=\underline{\theta}_B$, but also on the information disclosed by the monopolist to T when B reports a low type. Substituting $t_B^S(\underline{\theta}_B)$ and $t_B^S(\overline{\theta}_B)$ into $U_S := \mathbb{E}_{\theta_B}\left[t_B^S(\theta_B)\right]$, the revenue can be written in terms of *resale-augmented virtual valuations*, which are defined as the sum of the standard virtual valuations as in Myerson (1981), that is $M(\overline{\theta}_B):=\bar{\theta}_B$ and $M(\underline{\theta}_B):=\underline{\theta}_B-\frac{p_B}{1-p_B}\Delta\theta_B$, augmented by the virtual surplus each type expects from resale. The latter consists of the real surplus $\lambda_B s_B(\theta_B)+\lambda_T r_B(\theta_B|z)$, adjusted by the effect of resale on the informational rent for the high type.

Definition 1 (Resale-augmented virtual valuations) *Let*

$$V(\overline{\theta}_B|z_l):=M(\overline{\theta}_B)+\lambda_B s_B(\overline{\theta}_B)+\lambda_T r_B(\overline{\theta}_B|z_l),$$

$$V(\underline{\theta}_B|z_l):=M(\underline{\theta}_B)+\lambda_B\left[s_B(\underline{\theta}_B)-\frac{p_B}{1-p_B}\Delta s_B\right]+\lambda_T\left[r_B(\underline{\theta}_B|z_l)-\frac{p_B}{1-p_B}\Delta r_B(z_l)\right],$$

be the resale-augmented virtual valuation *for a buyer with private value θ_B, conditional on S disclosing information z_l to T in the secondary market, with $l=1,...,3$.*

Note that since $\Delta r_B(z_1)=-\Delta\theta_B$, $\Delta r_B(z_2)=0$, and $\Delta r_B(z_3)=-p_T\Delta\theta_B$, we have that

$$r_B(\underline{\theta}_B|z_l)-\frac{p_B}{1-p_B}\Delta r_B(z_l)=\begin{cases}\Delta\theta_B+\frac{p_B}{1-p_B}\Delta\theta_B & \text{if } z_l=z_1,\\ 0 & \text{if } z_l=z_2,\\ p_T\left[\Delta\theta_B+\frac{p_B}{1-p_B}\Delta\theta_B\right] & \text{if } z_l=z_3,\end{cases}$$

which implies that the virtual valuations of the low type can be ranked as follows

$$V(\underline{\theta}_B|z_1)\geq V(\underline{\theta}_B|z_3)\geq V(\underline{\theta}_B|z_2).$$

On the other hand, the high type does not expect any surplus from resale when it is T who offers the price in secondary market, and hence $V(\overline{\theta}_B|z_l)=\overline{\theta}_B+\lambda_B s_B(\overline{\theta}_B)$ for any z_l.[10]

[10]Note that by substituting the expressions for $M(\theta_B)$, $s_B(\theta_B)$ and $r_B(\theta_B|z_l)$ into $V(\theta_B|z_l)$, the virtual valuations can be read as functions of the exogenous variables only.

Using Definition 1, the optimal selling mechanism for S can thus be obtained by choosing an allocation rule and a disclosure policy that solve the following (reduced) program.

Lemma 2 (Beliefs-constrained reduced program) *The optimal mechanism in the primary market* ϕ_S^* *maximizes*

$$U_S = \mathbb{E}_{\theta_B}\left[\sum_{l=1}^{3} V(\theta_B|z_l)\phi_S\left(1, z_l|\theta_B\right)\right]$$

subject to

$$\begin{array}{l}\sum_{l=1}^{3}\phi_S\left(1, z_l|\bar{\theta}_B\right)\left[\Delta\theta_B+\lambda_B\Delta s_B+\lambda_T\Delta r_B(z_l)\right]\geq \\ \sum_{l=1}^{3}\phi_S\left(1, z_l|\underline{\theta}_B\right)\left[\Delta\theta_B+\lambda_B\Delta s_B+\lambda_T\Delta r_B(z_l)\right]\end{array} \qquad (\mathcal{M})$$

and

$$\begin{array}{ll}\Pr(\bar{\theta}_B|x_B^S=1, z_1)\geq\frac{\Delta\theta_B}{\underline{\theta}_T-\underline{\theta}_B}, & (1)\\ \Pr(\bar{\theta}_B|x_B^S=1, z_2)\leq\frac{\Delta\theta_B}{\bar{\theta}_T-\underline{\theta}_B}, & (2)\\ \Pr(\bar{\theta}_B|x_B^S=1, z_3)\in\left[\frac{\Delta\theta_B}{\bar{\theta}_T-\underline{\theta}_B},\frac{\Delta\theta_B}{\underline{\theta}_T-\underline{\theta}_B}\right]. & (3)\end{array}$$

Constraints $(1)-(3)$ guarantee that, given the mechanism ϕ_S^* and the information z_l, it is indeed sequentially optimal for T to offer the equilibrium price in the secondary market. Constraint $(\mathcal{M})$ is a disclosure-augmented *monotonicity condition* which guarantees that the low type does not gain from mimicking the high type. Note that $(\mathcal{M})$ reduces to the standard monotonicity condition according to which trade must occur with a higher probability when the buyer reports a higher type only if $\lambda_B = 1$, in which case the information disclosed in the primary market has no effect on the resale price. The remaining constraints, $(\overline{IC})$, $(\underline{IR})$ and $(\overline{IR})$, have already been embedded into the reduced program via the resale-augmented virtual valuations.

In what follows, we first construct a solution to the program of Lemma 2. Next, we discuss the implications for the optimality of stochastic allocations and richer disclosure policies (Proposition 1). Finally, we propose a simple implementation where the signals are the prices B pays in the primary market (Proposition 2).

The solution to the reduced program of Lemma 2 depends on the sign of the virtual valuations $V(\theta_B|z_l)$, which in turn depends on the severity of the adverse selection problem as well as on T's prior beliefs about the value B attaches to the good. For simplicity, consider case A4.1 (overlapping supports). The results for A4.2 are similar and are omitted for brevity. Note that under A4.1, there are no signals z_1 that can induce T to offer a high price when she has a low valuation and hence necessarily $\phi_S^*(1, z_1|\theta_B) = 0$ for any θ_B. On the other hand, $\bar{\theta}_T$ may be willing to offer a high price, but only if she believes B assigns a high value to the good in which case an offer at a low price is likely to be rejected. In what follows, we will say that T's prior beliefs are *favorable* to the buyer when $\bar{\theta}_T$ is expected to offer a high price even in the event she learns nothing from the outcome in the primary market. This is clearly the most favorable case also for the monopolist who does not have any incentives in changing T's behavior in the resale game. At the optimum, S sells to either type if $V(\underline{\theta}_B|z_3) \geq 0$ and only to $\bar{\theta}_B$ otherwise. Indeed, when $V(\underline{\theta}_B|z_3) < 0$, the rent S must leave to $\bar{\theta}_B$ in case she sells also to $\underline{\theta}_B$ is so high that she is better off excluding the low type from trade.

Things are more difficult for S when T's prior beliefs are *unfavorable*, in which case $\overline{\theta}_T$ is expected to offer a low price when her posterior beliefs are close to her prior. The monopolist may then attempt to change T's behavior in the resale game by disclosing a signal z_3 with sufficiently higher probability when B announces $\widehat{\theta}_B = \bar{\theta}_B$ than when he reports $\widehat{\theta}_B = \underline{\theta}_B$ so as to induce $\overline{\theta}_T$ to offer a high price conditional on receiving information z_3. Disclosure has however a cost, for it precludes the possibility to sell with certainty to either type. To see this, suppose trade occurs with probability one both when B reports $\widehat{\theta}_B = \underline{\theta}_B$ and $\widehat{\theta}_B = \overline{\theta}_B$ (i.e. $\phi_S(1, z_2|\theta_B) + \phi_S(1, z_3|\theta_B) = 1$ for any θ_B). From $\overline{IC}$, S must give $\overline{\theta}_B$ a price discount (informational rent)[11]

$$U_B(\overline{\theta}_B) \geq U_B(\underline{\theta}_B) + \Delta\theta_B + \lambda_B \Delta s_B + \lambda_T(-p_T\Delta\theta_B)\,\phi_S^*(1, z_3|\underline{\theta}_B).$$

But then if the low type pretends he has a high valuation and reports $\widehat{\theta}_B = \overline{\theta}_B$, he gets

$$\begin{aligned} &U_B(\overline{\theta}_B) - \left[\Delta\theta_B + \lambda_B\Delta s_B + \lambda_T(-p_T\Delta\theta_B)\,\phi_S^*\left(1, z_3|\overline{\theta}_B\right)\right] = \\ &= U_B(\underline{\theta}_B) + \lambda_T p_T \Delta\theta_B\left[\phi_S^*\left(1, z_3|\overline{\theta}_B\right) - \phi_S^*(1, z_3|\underline{\theta}_B)\right] > U_B(\underline{\theta}_B) \end{aligned}$$

since necessarily $\phi_S^*\left(1, z_3|\overline{\theta}_B\right) > \phi_S^*(1, z_3|\underline{\theta}_B)$. In other words, if the menu offered by the monopolist is such that trade is certain whatever the buyer's choice, the high type, who is not interested in the information disclosed to T, would always select the contract with the lowest price. But then the low type would have an incentive to mimic the high type, paying the same price and inducing the monopolist to disclose the most favorable signal, z_3, with a higher probability. The only way the monopolist can sort the two types when she discloses information is by making the high type pay a higher price than the low type, which is possible only if a higher price is associated with a lottery that delivers the good with a higher probability. It follows that to be incentive compatible, any mechanism which discloses both signals z_2 and z_3, must necessarily be associated with a stochastic allocation rule.

With unfavorable beliefs, S thus faces a trade-off between selling with certainty but inducing a low resale price, or sustaining a higher resale price but at the cost of not being able to trade with certainty. To solve this trade–off, first note that $\overline{\theta}_T$ is willing to offer a high price conditional on z_3 if and only if $\Pr(\bar{\theta}_B|x_B^S = 1, z_3) \geq \frac{\Delta\theta_B}{\overline{\theta}_T - \underline{\theta}_B}$, or equivalently $\phi_S(1, z_3|\underline{\theta}_B) \leq J(\overline{\theta}_T)\phi_S\left(1, z_3|\overline{\theta}_B\right)$, where $J(\overline{\theta}_T) := \frac{p_B(\overline{\theta}_T - \overline{\theta}_B)}{(1-p_B)\Delta\theta_B} < 1$ in case of unfavorable beliefs. Since U_S is increasing in $\phi_S\left(1, z_3|\overline{\theta}_B\right)$ and since a higher $\phi_S\left(1, z_3|\overline{\theta}_B\right)$ allows S to disclose signal z_3 with a higher probability also when B reports $\underline{\theta}_B$, whenever at the optimum S decides to disclose signal z_3, then necessarily $\phi_S^*\left(1, z_3|\overline{\theta}_B\right) = 1$ and $\phi_S^*(1, z_3|\underline{\theta}_B) = J(\overline{\theta}_T)$. Substituting $\phi_S^*\left(1, z_3|\overline{\theta}_B\right)$ and $\phi_S^*(1, z_3|\underline{\theta}_B)$ into the monotonicity

[11] Recall that $\Delta r_B(z_2) = 0$ and $\Delta r_B(z_3) = -p_T\Delta\theta_B$.

condition ($\mathcal{M}$), we then have that the upper bound on $\phi_S(1, z_2|\underline{\theta}_B)$ is given by $\phi_S(1, z_2|\underline{\theta}_B) = 1 - J(\overline{\theta}_T)/K < 1 - \phi_S(1, z_3|\underline{\theta}_B)$, where $K \in [0, 1)$ is a "discount" factor which captures the cost of not being able to sell with certainty to $\underline{\theta}_B$ when disclosing both signals z_2 and z_3.[12] It follows that the (virtual) value of trading with $\underline{\theta}_B$ and disclosing both signals is

$$\phi_S(1, z_3|\underline{\theta}_B)V(\underline{\theta}_B|z_3) + \phi_S(1, z_2|\underline{\theta}_B)V(\underline{\theta}_B|z_2) = J(\overline{\theta}_T)V(\underline{\theta}_B|z_3) + \left[1 - J(\overline{\theta}_T)/K\right] V(\underline{\theta}_B|z_2).$$

To determine the optimal mechanism, S thus compares the above with the value $V(\underline{\theta}_B|z_2)$ of selling to $\underline{\theta}_B$ with certainty and disclosing only the less favorable signal z_2. It follows that when $V(\underline{\theta}_B|z_2) > K\ V(\underline{\theta}_B|z_3) > 0$, S finds it optimal to favor trade over a higher resale price and at the optimum $\phi_S^*\left(1, z_2|\overline{\theta}_B\right) = \phi_S^*\left(1, z_2|\underline{\theta}_B\right) = 1$. When instead $V(\underline{\theta}_B|z_2)$ is positive but less than $K\ V(\underline{\theta}_B|z_3)$, the optimal mechanism consists in selling with certainty to the high type and with probability $\phi_S^*\left(1, z_2|\underline{\theta}_B\right) + \phi_S^*\left(1, z_3|\underline{\theta}_B\right) < 1$ to the low type. Finally, when $V(\underline{\theta}_B|z_2) < 0$, but $V(\underline{\theta}_B|z_3) \geq 0$, the rent S must leave to the high type in case she sells to $\underline{\theta}_B$ and $\overline{\theta}_T$ offers a low price is so high that S does not find it profitable to disclose signal z_2, so that at the optimum $\phi_S^*\left(1, z_2|\underline{\theta}_B\right) = 0$ and $\phi_S^*(1, z_3|\underline{\theta}_B) = J(\overline{\theta}_T)$. In this case, the allocation rule is stochastic, but the optimal disclosure policy is deterministic and consists in disclosing only signal z_3 whatever the buyer's type.

We summarize the above results in the following Lemma and in Figure 2.

Lemma 3 (Optimal mechanism) *Assume A1-A4.1.*

- *When beliefs are* ***favorable****, the optimal allocation rule and disclosure policy are deterministic: S discloses only signal z_3 and sells to the low type if and only if $V(\underline{\theta}_B|z_3) > 0$.*
- *When instead beliefs are* ***unfavorable****, S discloses only signal z_3 when $V(\underline{\theta}_B|z_2) < 0$, both z_2 and z_3 when $V(\underline{\theta}_B|z_2) \in [0, K\ V(\underline{\theta}_B|z_3)]$, and only z_2 when $V(\underline{\theta}_B|z_2) > K\ V(\underline{\theta}_B|z_3) > 0$. Furthermore, S excludes the low type when $V(\underline{\theta}_B|z_3) < 0$, and sells to $\underline{\theta}_B$*
    1. *with probability $J(\overline{\theta}_T)$ when $V(\underline{\theta}_B|z_2) < 0 < V(\underline{\theta}_B|z_3)$,*
    2. *with probability $J(\overline{\theta}_T) + [1 - J(\overline{\theta}_T)/K]$ when $V(\underline{\theta}_B|z_2) \in [0, K\ V(\underline{\theta}_B|z_3)]$,*
    3. *with probability one when $V(\underline{\theta}_B|z_2) > K\ V(\underline{\theta}_B|z_3) > 0$.*

In all cases, the high type always receives the good with certainty.

[Insert Figure 2 here]

[12] Note that $K = \frac{[\Delta\theta_B + \lambda_B \Delta s_B] J(\overline{\theta}_T)}{[\Delta\theta_B + \lambda_B \Delta s_B] J(\overline{\theta}_T) + [1 - J(\overline{\theta}_T)] \lambda_T p_T \Delta\theta_B}$ is increasing in $J(\overline{\theta}_T)$, that is, it is higher the more favorable the initial prior of T.

Two important insights can be derived from Lemma 3. First, the presence of a secondary market significantly changes the properties of revenue-maximizing allocations in the primary market. In particular, contrary to standard mechanisms that do not explicitly account for resale and for which trade is deterministic, a monopolist who expects her buyers to resell may need to induce *stochastic allocations* in the primary market (either through lotteries, or mixed strategies). This follows from the fact that the decision to trade is an informative signal of the buyer's valuation; as a consequence, deterministic mechanisms in which the monopolist sells with certainty to a subset of types and excludes the others, typically fail to induce the desired beliefs in the resale game. To see how stochastic allocations may lead to higher revenues than deterministic ones, consider area BEFD in Figure 2, where for simplicity we restricted attention to $\lambda_T = 1$, which is the most interesting case for the analysis of the manipulation of beliefs in the secondary market. Note that in this case, T's prior beliefs are unfavorable – $J(\overline{\theta}_T) < 1$ – and $V(\underline{\theta}_B|z_2) < 0 < V(\underline{\theta}_B|z_3)$, implying that S finds it optimal to trade with $\underline{\theta}_B$ if and only if $\overline{\theta}_T$ offers a high resale price. With deterministic mechanisms, then neither B, nor S, would benefit from resale. Indeed, if S were to sell to both types, then T would learn nothing about B's valuation and since her prior beliefs are unfavorable, she would offer a low price in the resale game. If, on the other hand, S were to sell only to the high type, then the third party would offer a price equal to the buyer's high valuation, once again leaving no surplus to the buyer. In contrast, with a stochastic allocation, S can trade with positive probability also with the low type and at the same time induce T to offer a high resale price in the secondary market.

Second, to increase the level of trade in the primary market without reducing the probability of a high resale price, the monopolist may need to disclose information in addition to the decision to trade. To see this, consider area HBDI in Figure 2 in which beliefs are unfavorable and $V(\underline{\theta}_B|z_2) > 0$. If S were to sustain a high resale price disclosing only the decision to trade, then the best she could do is selling to the high type with certainty and to the low type with probability $J(\overline{\theta}_T)$. In contrast, by adopting a richer disclosure policy, S can increase the level of trade with $\underline{\theta}_B$ above $J(\overline{\theta}_T)$ and at the same time guarantee that $\overline{\theta}_T$ continues to offer a high resale price with probability $J(\overline{\theta}_T)$. It follows that when trade with the low type is profitable even in the event T offers a low price, that is when $V(\underline{\theta}_B|z_2) > 0$, combining lotteries with a stochastic disclosure policy leads to a higher revenue.[13]

We conclude that

Proposition 1 (Stochastic allocations and disclosure policy) *In the presence of resale, it is generically impossible to maximize revenue with deterministic allocations. Furthermore, to optimally fashion the beliefs of the participants in the secondary market, a monopolist may need to adopt*

[13] Recall that a stochastic disclosure policy is incentive compatible only if it is associated with a stochastic allocation rule.

a disclosure policy richer than the simple announcement of the decision to trade with a particular buyer.

We now turn attention to the implementation of the optimal mechanism. To create the desired informational linkage, the monopolist has two natural instruments: (1) she can sell to different types with different probabilities so that the decision to trade is itself informative of the buyer's valuation; (2) she can disclose the price B pays in the primary market as an additional signal. The next proposition indicates a simple implementation based on these two instruments. We limit attention to the interesting case in which in the direct mechanism S discloses both signals z_2 and z_3. In all other cases, z does not convey any information and the optimal disclosure policy is simply the announcement of the decision to trade.

Proposition 2 (Prices as signals) *When the optimal informational linkage can not be implemented disclosing only the decision to trade, the monopolist uses the price as an additional signal and offers a menu of lotteries which induce the buyer to follow a mixed strategy.*

Let the menu consists of two price-lottery pairs: if B pays τ_H he receives the good with certainty, whereas if he pays $\tau_L < \tau_H$, he receives the good with probability positive, but less than one. The menu is designed so as to induce the high type to pay τ_H and the low type to randomize choosing τ_H with probability $J(\overline{\theta}_T)$ and τ_L with probability $1 - J(\overline{\theta}_T)$. Given B's mixed strategy, it is optimal for $\overline{\theta}_T$ to offer a high price when she observes τ_H and a low price when she observes τ_L, so that the signals z_2 and z_3 in the direct mechanism are simply the prices B pays in the primary market.

Note that S needs to combine *lotteries* with *mixed strategies*. To see this, suppose S tries to induce B to randomize over τ_L and τ_H without using lotteries, i.e. by selling with certainty whatever the price. Then, the high type – who does not care about the information disclosed to T – would always pay the lowest price τ_L. Anticipating that T will never offer a high price if she observes τ_H, the low type would also pay τ_L. To sort the two types and create the desired informational linkage, S must thus associate τ_L with a lottery that delivers the good with probability strictly less than one.

Next, suppose S tries to implement the optimal mechanism with lotteries, but without making B play a mixed strategy. If S perfectly separates the two types and discloses the price, she fully informs T about B's valuation, which is clearly never optimal. If, on the other hand, she separates the two types and keeps the price secret, then the only thing S can do to sustain a high resale price is assign τ_L to a lottery that gives the good with probability at most equal to $J(\overline{\theta}_T)$. On the contrary, by inducing the buyer to play a mixed strategy and using the price as a signal, the monopolist can increase the level of trade with the low type and induce T to offer a high resale price with the same probability.

2.4 Discussion

In the rest of this section, we discuss (1) the possibility of collusion between B and S and its influence on the optimal informational linkage; (2) the extension to multiple bidders in the primary market and the positive effect of resale on revenue when the monopolist can prohibit the winner from selling to the losers.

Collusion. The optimal mechanism of Lemma 3 has been derived assuming in the primary market S and B do not collude at the expenses of T. Collusion possibilities arise from the fact that S could publicly announce a mechanism ϕ_S and then offer B a secret side contract so that she discloses only the most favorable signals with probability one.[14] When S lacks of the commitment not to collude with B, the only credible information that can be disclosed to the secondary market is the decision to trade. Furthermore, without commitment, the possibility for S to make ϕ_S public has no strategic effect so that ϕ_S must be a best response to the strategy T is expected to follow in the secondary market. As in the case with full commitment, the optimal mechanism can be designed by looking at the value of the (collusion proof) resale-augmented virtual valuations. Let $\xi = \mathbb{E}_{\theta_T}\left[\Pr(t_B^T(\theta_T) = \bar{\theta}_B)\right]$ be the probability T is expected to offer a high price in the resale game. Without commitment, no signals are disclosed to T and the (collusion proof) *resale-augmented virtual valuations* reduce to

$$\begin{aligned} V(\overline{\theta}_B|\xi) &:= \overline{\theta}_B + \lambda_B \bar{s}_B, \\ V(\underline{\theta}_B|\xi) &:= \underline{\theta}_B - \frac{p_B}{1-p_B}\Delta\theta_B + \lambda_B\left[\underline{s}_B - \frac{p_B}{1-p_B}\Delta s_B\right] + \lambda_T\xi\left[\Delta\theta_B + \frac{p_B}{1-p_B}\Delta\theta_B\right]. \end{aligned}$$

The seller's optimal (collusion-proof) mechanism then maximizes $U_S := \mathbb{E}_{\theta_B}\left[V(\theta_B|\xi)\phi_S(1|\theta_B)\right]$ under the monotonicity condition $\phi_S(1|\overline{\theta}_B) \geq \phi_S(1|\underline{\theta}_B)$, where $\phi_S(1|\theta_B)$ is the probability of trade when B reports θ_B.

It is important to note that even when she can not commit to a credible disclosure policy, S can still fashion the *informational linkage with the secondary market*, but this has to be done entirely through a stochastic allocation rule. To see this, assume T's prior beliefs are unfavorable and A4.1 holds, so that $\xi \in [0, p_T]$. Also, suppose $V(\underline{\theta}_B|\xi = 0) < 0$, but $V(\underline{\theta}_B|\xi = p_T) > 0$. Then, S sells to $\underline{\theta}_B$ with probability $J(\overline{\theta}_T)$ and to $\overline{\theta}_B$ with certainty. $\overline{\theta}_T$ is then indifferent between offering a high and a low price and in equilibrium randomizes offering $\overline{\theta}_B$ with probability $\xi^* \in (0,1)$ and $\underline{\theta}_B$ with probability $1-\xi^*$, where ξ^* solves $V(\underline{\theta}_B|p_T\bar{\xi}^*) = 0$ and thus makes S indifferent between selling to the low type and retaining the good.

Multiple Bidders and expected revenue. The effect of resale on the monopolist's revenue depends on whether the secondary buyer is a third party or a bidder who did not win in the

[14] An alternative form of collusion is between S and T. This would lead to standard ratchet-effect results as in the literature of dynamic contracting.

primary market. This second possibility is examined in the next section. Here, we maintain the assumption that resale is to a third party who participates only in the secondary market, but we allow for multiple bidders in the primary market. This corresponds to an environment where S has the possibility to prohibit the winner from reselling to the losers, but is not able to contract with all potential buyers at the time she needs to sell.

With multiple bidders and independent types, an optimal auction maximizes the expected sum of the bidders' resale-augmented virtual valuations, subject to the sequential rationality constraints for the offers T makes in the secondary market (the reduced program is in the Appendix – proof of Remark 1). The only difference with respect to the single bidder case is that now S compares the bidders' resale-augmented virtual valuations $V(\theta_i|z_l)$ (as in Definition 1) for each state $\boldsymbol{\theta}_B := (\theta_1, ..., \theta_N) \in \Theta_B := \prod_i^N \Theta_i$ where N is the number of bidders and Θ_i the set of private values for bidder i.[15] However, contrary to standard auctions without resale, S *does not necessarily assign the good to the bidder with the highest virtual valuation*. Indeed, this would be the case if the resale price were exogenous. When, instead, the price in the secondary market depends on the beliefs about the valuation of the primary buyer, S may find it optimal to assign the good to a bidder with a lower virtual valuation in state $\boldsymbol{\theta}_B$ if this allows S to sell to the same bidder also in another state $\boldsymbol{\theta}'_B$ without reducing the price offered by T in the resale game. Assume, for example, that when two bidders B_i and B_j both report a high type, B_i has a larger resale-augmented virtual valuation than B_j. If the constraint for T offering a high resale price to B_j binds, then assigning the good to B_j when both bidders report a high type is more effective in relaxing this constraint than assigning the good to B_i. By giving the good to B_j in state $\boldsymbol{\theta}_B$, S can then increase the probability of selling to the same bidder also when he reports a lower valuation, say in state $\boldsymbol{\theta}'_B$, which in turn boosts revenue if the probability of state $\boldsymbol{\theta}'_B$ is significantly high compared to that of $\boldsymbol{\theta}_B$. In the next section , we will show that similar incentives for (virtual) misallocations arise in auctions followed by inter-bidders resale.

Consider now the effect of resale on revenue. The option to resell increases the value each bidder assigns to the good by the surplus expected in the secondary market. Furthermore, resale reduces the difference between high and low valuation buyers and hence the rents S must leave to the high types to induce truthful information revelation in the primary market. It follows that the *resale-augmented virtual valuations* are higher than the corresponding *Myerson virtual valuations* for auctions without resale. Nevertheless, this alone does not imply that resale is revenue-enhancing, for the monopolist may not be able to implement the same allocations as in the absence of resale without violating the incentive compatibility constraints of the low types (note that the simple monotonicity condition for standard mechanisms does not guarantee that $\underline{IC}$ are satisfied when bidders can resell). However, one can show that through a policy that discloses only the identity of

[15] Clearly, Θ_i need not be the same across bidders.

the winner, the monopolist can always implement exactly the same allocation rule as in a Myerson optimal auction without resale. Along with the effect on the virtual valuations discussed above, it implies that indeed[16]

Remark 1 *A monopolist benefits from resale when she is not able to contract with all potential buyers and can prohibit the winner in the primary market from reselling to the losers.*

Assuming the initial seller is able to prevent the losers from bidding in the secondary market is often unrealistic. When this is the case, the effect of resale must be reconsidered, as we show next.

3 Inter-Bidders Resale

Suppose now the monopolist can contract with both B and T, but is not able to prohibit the winner from reselling to the loser. Let the allocation of the good in the primary market (equivalently, the identity of the winner) be denoted by

$$\mathbf{x}_S := (x_T^S, x_B^S) \in \mathbf{X}_S := \left\{(x_T^S, x_B^S) \in \{0,1\}^2 \text{ such that } x_T^S + x_B^S \leq 1\right\}.$$

A direct mechanism (with an embedded disclosure policy) is now a mapping

$$\phi_S : \mathbf{\Theta} \to \mathbb{R}^2 \times \Delta(\mathbf{X}_S \times Z)$$

such that when T and B report $\boldsymbol{\theta} := (\theta_T, \theta_B) \in \mathbf{\Theta} := \Theta_T \times \Theta_B$, they pay $\mathbf{t}_S = (t_T^S, t_B^S) \in \mathbb{R}^2$ to S and with probability $\phi_S(\mathbf{x}_S, \mathbf{z}|\boldsymbol{\theta})$ the allocation of the good is $\mathbf{x}_S \in \mathbf{X}_S$ and information $\mathbf{z} := (z^T, z^B) \in \mathbf{Z} := Z^T \times Z^B$ is disclosed to the two bidders at the end of the auction.[17] The payoff of bidder $i \in \{B, T\}$ is now $u_i = \theta_i x_i^S(1 - x^r) + \theta_i x_j^S x^r - t_i^S + t^r$, where $j \neq i \in \{B, T\}$, and $x^r = 1$ if the good changes hands in the secondary market and $x^r = 0$ otherwise. t^r denotes the resale price and is positive when i wins the auction and negative when he, or she, loses.

Bargaining in the resale game takes place according to the stochastic ultimatum bargaining procedure described in the previous section. Now that also T participates in the primary market, S learns information from both bidders and hence can now affect not only the price offered by a buyer, but also the price asked by a resale seller in the secondary market. In what follows, we refer to $s_i(\boldsymbol{\theta}|x_h^S = 1, z^i)$ and $r_j(\boldsymbol{\theta}|x_h^S = 1, z^i)$ as the *equilibrium* resale surplus of bidder i (respectively j) in state $\boldsymbol{\theta}$, conditional on i setting the price in the resale game and bidder h winning the good in the primary market, with $i, j, h \in \{T, B\}$, and $j \neq i$. Formally, $s_i(\boldsymbol{\theta}|x_h^S = 1, z^i) = s_i(\boldsymbol{\theta}|\widehat{\theta}_i, x_h^S = 1, z^i)$

[16]The following result clearly does not depend on the discreteness of the type space. The proof is available upon request.

[17]Note that B observes only z^B and T observes only z^T. Furthermore, there are no exogenous constraints that oblige S to disclose any particular information apart from $\mathbf{x}_S$. Hence, by examining the case where S sends private (possibly correlated) signals to T and B, we are *de facto* considering the most favorable case for the monopolist.

and $r_j(\boldsymbol{\theta}|x_h^S = 1, z^i) = r_j(\boldsymbol{\theta}|\widehat{\theta}_i, x_h^S = 1, z^i)$ for $\widehat{\theta}_i = \theta_i$, where $s_i(\boldsymbol{\theta}|\widehat{\theta}_i, \cdot)$ and $r_j(\boldsymbol{\theta}|\widehat{\theta}_i, \cdot)$ denote the resale surplus when the true state is $\boldsymbol{\theta}$ and bidder i announces $\widehat{\theta}_i$ to S in the primary market (these functions are similar, yet not identical, to those in the previous section; their precise characterization is in the Appendix – proof of Proposition 3). Note that conditional on i setting the price, the resale surplus $s_i(\boldsymbol{\theta}|\cdot)$ and $r_j(\boldsymbol{\theta}|\cdot)$ does not depend on the behavior of bidder j, that is on $\widehat{\theta}_j$.

To be individually rational and incentive compatible, an auction followed by inter-bidders resale must satisfy the following constraints:

$$\begin{aligned} U_i(\theta_i) := \mathbb{E}_{\theta_j} \Big\{ \sum_{h=i,j} \sum_{\mathbf{z}\in\mathbf{Z}} \big[\theta_i \mathbb{I}(h=i) + \lambda_i s_i(\theta_i, \theta_j | x_h^S = 1, z^i) + \\ + \lambda_j r_i(\theta_i, \theta_j | x_h^S = 1, z^j)\big] \phi_S(x_h^S = 1, \mathbf{z}|\theta_i, \theta_j)\Big\} - t_i^S(\theta_i) \geq 0, \end{aligned} \qquad (IR(\theta_i))$$

and

$$\begin{aligned} U_i(\theta_i) \geq \mathbb{E}_{\theta_j} \Big\{ \sum_{h=i,j} \sum_{\mathbf{z}\in\mathbf{Z}} \Big[\theta_i \mathbb{I}(h=i) + \lambda_i s_i(\theta_i, \theta_j | \widehat{\theta}_i, x_h^S = 1, z^i) + \\ + \lambda_j r_i(\theta_i, \theta_j | x_h^S = 1, z^j)\big] \phi_S(x_h^S = 1, \mathbf{z}|\widehat{\theta}_i, \theta_j)\Big\} - t_i^S(\widehat{\theta}_i) \end{aligned} \qquad (IC(\theta_i))$$

for any $(\theta_i, \widehat{\theta}_i) \in \Theta_i^2$, where $\mathbb{I}(h = i)$ is an indicator function, assuming value one if $h = i$ and zero otherwise.[18] As in Section 2, constraints $(\underline{IR})_i$ and $(\overline{IC})_i$ necessarily bind at the optimum, which in turn leads to the following revenue equivalence result.

Remark 2 (Revenue equivalence) *Let ϕ_S and ϕ_S' be any two individually rational and incentive compatible mechanisms such that $(\underline{IR})_i$ and $(\overline{IC})_i$ bind for every bidder. Then, ϕ_S and ϕ_S' generate the same revenue if they are characterized by the same allocation rule and the same disclosure policy, that is if $\phi_S(\mathbf{x}_S, \mathbf{z}|\boldsymbol{\theta}) = \phi_S'(\mathbf{x}_S, \mathbf{z}|\boldsymbol{\theta})$ for any $\mathbf{x}_S$, $\mathbf{z}$ and $\boldsymbol{\theta}$.*

Contrary to standard revenue equivalence theorems, when bidders have the option to resell, two mechanisms that generate the same allocations in the primary market need not be revenue-equivalent, unless they also disclose the same information. Indeed, what matters for the bidders' willingness to pay is the expectation of the surplus in the two markets. Furthermore, since the surplus in the resale game depends on the bidders' posterior beliefs about rivals' valuations, their willingness to pay in the primary market is a function of the information filtered by the monopolist through the choice of her disclosure policy.

Using the result that $(\underline{IR})_i$ and $(\overline{IC})_i$ necessarily bind, the program for the revenue-maximizing mechanism can be rewritten in terms of *resale-augmented virtual valuations*. Let $V_i(\boldsymbol{\theta}|x_h^S = 1, \mathbf{z})$

[18] That $t_i^S(\theta_i)$ does not depend on θ_j gurantees that in the direct mechanism bidder i does not learn any information about θ_j from the price he pays to S, in which case his posterior beliefs are uniquely determined by the allocation of the good $\mathbf{x}^S$and the signal z^i. However note that although in the direct mechanism the transfers are uninformative, in indirect mechanisms, one can possibly use them as signals to implement the desired informational linkage with the secondary market (as indicated in Proposition 2).

denote bidder i's virtual valuation in state $\boldsymbol{\theta}$, conditional on bidder h winning the auction and information $\mathbf{z} = (z^T, z^B)$ being disclosed by the monopolist to the two bidders; as in the previous section, $V_i(\boldsymbol{\theta}|x_h^S = 1, \mathbf{z})$ is the sum of Myerson virtual valuation $M(\theta_i)$ with the virtual surplus that bidder i obtains from resale. We then have that[19]

Proposition 3 (Optimal auctions with inter-bidders resale) *An optimal auction followed by inter-bidders resale maximizes the* ***expected sum*** *of the bidders' resale-augmented virtual valuations,*

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{h=T,B}\sum_{\mathbf{z}\in\mathbf{Z}}\left(\sum_i V_i(\boldsymbol{\theta}|x_h^S = 1, \mathbf{z})\right)\phi_S(x_h^S = 1, \mathbf{z}|\boldsymbol{\theta})\right]$$

subject to monotonicity conditions and the constraints on the sequential rationality of the bidders' offers in the resale game.

Note that contrary to the case examined in the previous section, S now maximizes the *sum* of the virtual valuations; indeed, now that the monopolist can contract with all potential buyers, she can make not only the winner, but also the loser, pay for the surplus he, or she, expects from resale in the secondary market.

In what follows, instead of describing the details of the solution to the program in Proposition 3, we discuss directly the effect of inter-bidders resale on the monopolist's expected revenue and on the structure of the optimal allocation rule.

3.1 Expected revenue

Proposition 4 (Impossibility to replicate Myerson) *Suppose the monopolist can not perfectly control the distribution of bargaining power in the resale game through the allocation of the good in the primary market. Then, it is generically impossible to obtain Myerson's expected revenue without prohibiting inter-bidders resale.*

That the expected revenue of any auction followed by resale is never higher than in a Myerson optimal auction where resale is prohibited is immediate when the monopolist can contract with all potential buyers.[20] On the contrary, that S *strictly* suffers from the impossibility to prohibit resale is not obvious in the light of the results in the literature that assumes it is always the winner in the primary market who has full bargaining power in the resale game (cfr Zheng, 2002).

To illustrate, assume the supports of the two bidders' valuations overlap, that is $\underline{\theta}_B \leq \underline{\theta}_T \leq \bar{\theta}_B \leq \overline{\theta}_T$, and consider the Myerson optimal auction. Recall that the latter is a mechanism that

[19] The constraints are similar to those in Section 2 and are relegated to the Appendix.

[20] Indeed, resale imposes additional constraints and thus limits the seller's ability to extract surplus from the bidders.

in each state $\boldsymbol{\theta}$ assigns good to the bidder with the highest *virtual valuation* $M(\theta_i)$, provided that $\max_i \{M(\theta_i)\} \geq 0$, where $M(\overline{\theta}_i) := \bar{\theta}_i$ and $M(\underline{\theta}_i) := \underline{\theta}_i - \frac{p_i}{1-p_i}\Delta\theta_i$.

When $M(\underline{\theta}_B) \geq \max\{M(\underline{\theta}_T), 0\}$, Myerson auction prescribes that S should give the good to T if the latter has a high valuation and to B otherwise with an expected revenue of $p_T\overline{\theta}_T + (1-p_T)\underline{\theta}_B$.

Suppose now the monopolist can not prohibit resale, but assume it is always the winner in the primary market who sets the price in the secondary market (more generally who designs the resale mechanism). As shown in Zheng (2002), the impossibility to prohibit resale then does not hurt the monopolist. Indeed, S can simply sell to B at a price $p_T\overline{\theta}_T + (1-p_T)\underline{\theta}_B$ and use the latter as a *middleman* to extract surplus from T in the secondary market. Since in this case B learns nothing about T's valuation, he asks a price $\overline{\theta}_T$ independently of his type (indeed, $M(\underline{\theta}_T) \leq M(\underline{\theta}_B)$ implies $p_T > \frac{\underline{\theta}_T - \theta_B}{\overline{\theta}_T - \theta_B}$ for any $\theta_B \in \Theta_B$). Through resale, S thus implements the same final allocation and obtains exactly the same expected revenue as in a Myerson optimal auction where resale is prohibited.

The assumption it is always the winner who makes a take-it-or-leave-it offer is however restrictive, for it implies the monopolist can perfectly control the distribution of bargaining power in the resale game through the allocation of the good in the primary market. When instead, the distribution of bargaining power is a function of the players' individual characteristics, then it is generically impossible to replicate Myerson revenue with resale. Indeed, when $M(\underline{\theta}_T) \leq M(\underline{\theta}_B)$ and $\lambda_T > 0$, any mechanism in which B obtains the good with positive probability must necessarily leave some rent to $\bar{\theta}_T$, contrary to Myerson or Zheng. In this case, the impossibility to prohibit resale leads to a loss of revenue for the monopolist, as we formally prove in the Appendix.

A similar result holds for $M(\underline{\theta}_T) > M(\underline{\theta}_B)$ and $\lambda_B > 0$. To see this, assume $\lambda_B = 1$ and suppose $M(\underline{\theta}_T) > 0 > M(\underline{\theta}_B)$, but $M(\underline{\theta}_T) - \frac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T) < 0$. In this case, the monopolist finds it optimal to withhold the good when the two bidders report a low valuation, whereas she would have sold to T if the latter did not have the option to resell. Indeed, without resale, selling to T in state $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$ requires to increase the rent for $\overline{\theta}_T$ by $(1-p_B)\Delta\theta_T$, but has no effect on the rent for $\overline{\theta}_B$. It follows that if $p_T[(1-p_B)\Delta\theta_T] < (1-p_T)(1-p_B)\underline{\theta}_T$, or equivalently $M(\underline{\theta}_T) > 0$, the monopolist prefers to sell to T than withholding the good. On the contrary, with resale, if S sells to T when $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$, she also has to increase the rent for $\overline{\theta}_B$ by $(1-p_T)(\overline{\theta}_B - \underline{\theta}_T)$, for the latter can always pretend to have a low valuation, lose the auction, and then purchase the good from T in the secondary market by offering $\underline{\theta}_T$. It follows that when $p_B[(1-p_T)(\overline{\theta}_B - \underline{\theta}_T)] > (1-p_T)(1-p_B)M(\underline{\theta}_T)$, i.e. when $M(\underline{\theta}_T) - \frac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T) < 0$, S is better off withholding the good in state $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$, in which case the impossibility to prohibit resale is again revenue decreasing.

3.2 Manipulation of bidders' beliefs

The inability to transfer bargaining power to a bidder along with the good, not only results in a loss of revenue, but it also affects significantly the structure of the optimal allocation rule, as we show next.

Proposition 5 (Stochastic allocations) *Suppose a monopolist can not perfectly control the distribution of bargaining power in the resale game through the allocation of the good in the primary market. Then, it is generically impossible to maximize revenue with a deterministic selling procedure.*

As in the case where resale is to third parties, the monopolist uses the decision to trade with a buyer as a signal of the bidders' valuations. However, a difference is that now the monopolist may find it optimal to influence beliefs both *on* and *off the equilibrium path*. To illustrate, let $\underline{\theta}_B \leq \underline{\theta}_T \leq \bar{\theta}_B \leq \bar{\theta}_T$ and suppose S wants to sell to B both when $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$ and when $\boldsymbol{\theta} = (\underline{\theta}_T, \bar{\theta}_B)$. Furthermore, assume $\bar{\theta}_T$ is expected to offer a low price in the event she loses the auction without learning any information about θ_B. By inducing $\bar{\theta}_T$ to offer $\bar{\theta}_B$ instead of $\underline{\theta}_B$ *off equilibrium*, S can reduce the informational rent she must leave to the high type and hence extract more surplus from T. To change $\bar{\theta}_T$'s beliefs, S could, for example, sell to T instead of B when the two bidders report a low valuation, so that loosing the auction by announcing $\underline{\theta}_T$ becomes a perfect signal of $\theta_B = \bar{\theta}_B$. This strategy is however costly when selling to T in state $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$ is less profitable than selling to B in which case S can do better by choosing a lottery that gives the good also to B with positive probability.

Once again, the advantage of stochastic selling procedures comes from the possibility to fashion bidders' beliefs (on and off equilibrium) and at the same time implement more profitable allocations.

It is interesting to contrast this result with Zheng (2002). He shows that when it is always the winner who makes the offer in the secondary market, the optimal selling procedure is deterministic and the optimal disclosure policy is simply the announcement of the identity of the winner. In contrast, when the monopolist can not perfectly control the distribution of bargaining power in the resale game, restricting attention to deterministic mechanisms precludes the possibility to maximize revenue. Furthermore, when all bidders are expected to influence the resale price, it is in general impossible to create the desired informational linkage with the secondary market disclosing only the decision to trade with a particular buyer. In fact, even if a certain allocation rule induces the right beliefs for one bidder, it is unlikely that the same rule induces the desired beliefs also for the other bidders. When this is the case, S may gain by disclosing more information than the simple identity of the winner, such as, for example, the price paid, or more generally some statistics of the bids submitted in the auction.[21]

[21] The solution of the program of Proposition 3 for a wide range of parameters configurations, confirms this result.

4 Concluding Remarks

In primary markets where buyers anticipate the possibility to resell, the willingness to pay incorporates the surplus expected in the secondary market. The outcome in the resale game is also endogenous as it depends on the information disclosed in the primary market. Starting from these observations, this paper has suggested a tractable model to examine the intricacies associated with the design of optimal mechanisms for a monopolist who expects her buyers to resell.

A few results have been highlighted. First, in order to fashion the resale outcome, the monopolist must create an optimal informational linkage with the secondary market. This may require *stochastic allocations* as well as a *disclosure policy* richer than the simple announcement of the decision to trade with a particular buyer. Second, a monopolist benefits from resale only when she can not contract with all potential buyers. When instead all buyers participate in the primary market, resale is in general revenue-decreasing, unless the monopolist can perfectly control the distribution of bargaining power in the resale game through the allocation of the good in the primary market, such as when resale prices are always determined by secondary sellers.

Although the above results have been derived using a stylized model, we believe the main insights, as well as the methodology used to characterize the optimal mechanisms, extend to richer environments.

Finally, a last remark concerns the foundations for resale. In this paper we have assumed resale occurs as a result of (i) the impossibility for the monopolist to contract with all potential buyers, and (ii) the possibility for the bidders to correct misallocations in the primary market by trading in the secondary market. Extending the analysis to environments where resale is a consequence of changes in the bidders' valuations is likely to deliver new insights for the design of *optimal mechanisms* and represents an interesting line for future research.

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5 Appendix

Proof of Lemma 1

Given the mechanism ϕ_S, one can partition Z into three sets, Z_1, Z_2 and Z_3 such that

$$\begin{aligned} Z_1 &:= \left\{ z : t_B^T(\theta_T, z) = \bar{\theta}_B \text{ for any } \theta_T \right\}, \\ Z_2 &:= \left\{ z : t_B^T(\theta_T, z) = \underline{\theta}_B \text{ for any } \theta_T \right\}, \\ Z_3 &:= \left\{ z : t_B^T(\theta_T, z) = \bar{\theta}_B \text{ if } \theta_T = \bar{\theta}_T \text{ and } t_B^T(\theta_T, z) = \underline{\theta}_B \text{ otherwise} \right\}. \end{aligned}$$

Suppose S replaces ϕ_S with a mechanisms ϕ_S' that maps Θ_B into lotteries that assign positive measure to at most three signals, i.e. $\#_{\phi_S'} Z \leq 3$. Without loss of generality, let these three signals be labelled by z_l with $l = 1, ..., 3$. Construct ϕ_S' so that for any $\theta_B \in \Theta_B$

$$\phi_S'(1, z_l|\theta_B) = \sum_{z \in Z_l} \phi_S(1, z|\theta_B),$$

for $l = 1, ..., 3$. The mechanism ϕ_S' is payoff-equivalent to ϕ_S for all players if in the resale game that follows ϕ_S'

$$\begin{aligned} &t_B^T(\theta_T, z_1) = \bar{\theta}_B \text{ for any } \theta_T, \\ &t_B^T(\theta_T, z_2) = \underline{\theta}_B \text{ for any } \theta_T, \\ &t_B^T(\theta_T, z_3) = \bar{\theta}_B \text{ if } \theta_T = \bar{\theta}_T \text{ and } t_B^T(\theta_T, z_3) = \underline{\theta}_B \text{ otherwise.} \end{aligned}$$

This is true if given ϕ_S'

$$\begin{aligned} &\Pr(\bar{\theta}_B | x_B^S = 1, z_1) \geq \tfrac{\Delta\theta_B}{\underline{\theta}_T - \underline{\theta}_B}, \\ &\Pr(\bar{\theta}_B | x_B^S = 1, z_2) \leq \tfrac{\Delta\theta_B}{\bar{\theta}_T - \underline{\theta}_B}, \\ &\Pr(\bar{\theta}_B | x_B^S = 1, z_3) \in \left[\tfrac{\Delta\theta_B}{\bar{\theta}_T - \underline{\theta}_B}, \tfrac{\Delta\theta_B}{\underline{\theta}_T - \underline{\theta}_B} \right], \end{aligned}$$

where

$$\begin{aligned}\Pr(\bar{\theta}_B|x_B^S &= 1, z_l) = \frac{\phi_S'(1, z_l|\overline{\theta}_B)p_B}{\phi_S'(1, z_l|\overline{\theta}_B)p_B + \phi_S'(1, z_l|\underline{\theta}_B)(1-p_B)} \\ &= \frac{\sum_{z\in Z_l}\phi_S(1, z|\overline{\theta}_B)p_B}{\sum_{z\in Z_l}\left[\phi_S(1, z|\overline{\theta}_B)p_B + \phi_S(1, z|\underline{\theta}_B)(1-p_B)\right]}.\end{aligned}$$

Since for any $z \in Z_1$

$$\Pr(\overline{\theta}_B|x_B^S = 1, z) = \frac{\phi_S(1, z|\overline{\theta}_B)p_B}{\phi_S(1, z|\overline{\theta}_B)p_B + \phi_S(1, z/\underline{\theta}_B)(1-p_B)} \geq \frac{\Delta\theta_B}{\underline{\theta}_T - \underline{\theta}_B}$$

it follows that $\Pr(\overline{\theta}_B|x_B^S = 1, z_1) \geq \frac{\Delta\theta_B}{\underline{\theta}_T - \underline{\theta}_B}$. Repeating the same argument for z_2 and z_3 gives the result. ■

Proof of Lemma 2

The proof is in two steps. First, we reduce $\mathcal{P}_S$ by showing that in the optimal mechanism $(\underline{IR})$ and $(\overline{IC})$ constraints bind, which also implies that $(\overline{IR})$ is satisfied. Second, we express the reduced program for ϕ_S in terms of resale-augmented virtual valuations. Using the expressions for $U_B(\overline{\theta}_B)$ and $U_B(\underline{\theta}_B)$, the constraints $(\overline{IC})$ and $(\underline{IC})$ can be written as

$$\begin{aligned}&U_B(\underline{\theta}_B) + \textstyle\sum_{l=1}^3 \phi_S\left(1, z_l|\underline{\theta}_B\right)\left[\Delta\theta_B + \lambda_B\Delta s_B + \lambda_T\Delta r_B(z_l)\right] \leq U_B(\overline{\theta}_B) \leq \\ &\leq U_B(\underline{\theta}_B) + \textstyle\sum_{l=1}^3 \phi_S\left(1, z_l|\overline{\theta}_B\right)\left[\Delta\theta_B + \lambda_B\Delta s_B + \lambda_T\Delta r_B(z_l)\right].\end{aligned}$$

It follows that it is optimal for S to set $U_B(\underline{\theta}_B) = 0$ and

$$U_B(\overline{\theta}_B) = \sum_{l=1}^{3}\phi_S\left(1, z_l|\underline{\theta}_B\right)\left[\Delta\theta_B + \lambda_B\Delta s_B + \lambda_T\Delta r_B(z_l)\right].$$

That is, $(\overline{IC})$ and $(\underline{IR})$ are binding. Furthermore, as $\Delta\theta_B + \lambda_B\Delta s_B + \lambda_T\Delta r_B(z_l) \geq 0$ for any $l = 1, ..., 3$, $(\overline{IC})$ and $(\underline{IR})$ imply that $(\overline{IR})$ is satisfied. Substituting

$$\begin{aligned}&t_B^S(\overline{\theta}_B) = \textstyle\sum_{l=1}^3 \phi_S\left(1, z_l|\overline{\theta}_B\right)\left[\overline{\theta}_B + \lambda_B s_B(\bar{\theta}_B)\right] - \sum_{l=1}^3 \phi_S\left(1, z_l|\underline{\theta}_B\right)\left[\Delta\theta_B + \lambda_B\Delta s_B + \lambda_T\Delta r_B(z_l)\right], \\ &t_B^S(\underline{\theta}_B) = \textstyle\sum_{l=1}^3 \phi_S\left(1, z_l|\underline{\theta}_B\right)\left[\underline{\theta}_B + \lambda_B s_B(\underline{\theta}_B) + \lambda_T r_B(\underline{\theta}_B|z_l)\right]\end{aligned}$$

into $\mathcal{P}_S$ and $(\underline{IC})$ and using the expressions for the resale-augmented virtual valuations as in Definition 1, gives the result. Constraints $(1)-(3)$ guarantee that given the mechanism ϕ_S and the posterior beliefs associated with each signal z_l for $l = 1, ..., 3$, it is sequentially rational for T to follow the equilibrium strategy in the resale market. ■

Proof of Lemma 3

Consider the program for the optimal mechanism as in Lemma 2. Under A4.1, constraint (1) can be neglected, whereas constraints (2) and (3) can be written as

$$\phi_S(1,z_2|\underline{\theta}_B) \geq J(\overline{\theta}_T)\phi_S(1,z_2|\overline{\theta}_B), \quad (2)$$
$$\phi_S(1,z_3|\underline{\theta}_B) \leq J(\overline{\theta}_T)\phi_S(1,z_3|\overline{\theta}_B). \quad (3)$$

- *Favorable beliefs*: $J(\overline{\theta}_T) \geq 1$. Since $V(\underline{\theta}_B|z_3) \geq V(\underline{\theta}_B|z_2)$, and $V(\overline{\theta}_B|z_3) = V(\overline{\theta}_B|z_2)$, the optimal mechanism is $\phi_S^*\left(1,z_3|\overline{\theta}_B\right) = 1$, $\phi_S^*\left(1,z_2|\underline{\theta}_B\right) = 0$, and

$$\phi_S^*\left(1,z_3|\underline{\theta}_B\right) = \begin{cases} 1 & \text{if } V(\underline{\theta}_B|z_3) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

  Constraint $(\mathcal{M})$ is also satisfied in this case.

- *Unfavorable beliefs*: $J(\overline{\theta}_T) < 1$. The solution depends on the value of $V(\underline{\theta}_B|z_2)$. If $V(\underline{\theta}_B|z_2) < 0$, the optimal mechanism is $\phi_S^*\left(1,z_3|\overline{\theta}_B\right) = 1$, $\phi_S^*\left(1,z_2|\underline{\theta}_B\right) = 0$, and

$$\phi_S^*\left(1,z_3|\underline{\theta}_B\right) = \begin{cases} J(\overline{\theta}_T) & \text{if } V(\underline{\theta}_B|z_3) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

  Again, constraint $(\mathcal{M})$ is satisfied.

  If, instead, $V(\underline{\theta}_B|z_2) \geq 0$, then, ignoring $(\mathcal{M})$, the unconstrained solution would be $\phi_S^*\left(1,z_3|\overline{\theta}_B\right) = 1$, $\phi_S^*\left(1,z_3|\underline{\theta}_B\right) = J(\overline{\theta}_T)$, and $\phi_S^*\left(1,z_2|\underline{\theta}_B\right) = 1 - \phi_S^*\left(1,z_3|\underline{\theta}_B\right)$. However, in this case, given the price discount for the high type, $U_B(\overline{\theta}_B) = \Delta\theta_B + \lambda_B\Delta s_B - \lambda_T\phi_S^*\left(1,z_3|\underline{\theta}_B\right)p_T\Delta\theta_B$, it becomes attractive for the low type to pretend he has a high valuation and get

$$U_B(\overline{\theta}_B) - [\Delta\theta_B + \lambda_B\Delta s_B - \lambda_T p_T\Delta\theta_B] = \lambda_T p_T[1 - \phi_S^*\left(1,z_3|\underline{\theta}_B\right)]\Delta\theta_B > 0 = U_B(\underline{\theta}_B).$$

  Hence, $(\underline{IC})$, or equivalently $(\mathcal{M})$, must be binding, i.e.

$$\begin{aligned}&[\Delta\theta_B + \lambda_B\Delta s_B]\textstyle\sum_{l=2}^3 \phi_S\left(1,z_l|\underline{\theta}_B\right) - [\lambda_T p_T\Delta\theta_B]\,\phi_S\left(1,z_3|\underline{\theta}_B\right) = \\ &= [\Delta\theta_B + \lambda_B\Delta s_B]\textstyle\sum_{l=2}^3 \phi_S\left(1,z_l|\overline{\theta}_B\right) - [\lambda_T p_T\Delta\theta_B]\,\phi_S\left(1,z_3|\overline{\theta}_B\right).\end{aligned} \qquad (\mathcal{M})$$

  We proceed ignoring (2) and then show it is satisfied at the optimum. Given any $\phi_S(1,z_3|\overline{\theta}_B) \in [0,1]$, it is optimal to set $\phi_S(1,z_2|\overline{\theta}_B) = 1 - \phi_S(1,z_3|\overline{\theta}_B)$. Indeed, U_S is increasing in $\phi_S(1,z_2|\overline{\theta}_B)$ and maximizing $\phi_S(1,z_2|\overline{\theta}_B)$ relaxes $(\mathcal{M})$ and hence allows S to increase $\phi_S\left(1,z_2|\underline{\theta}_B\right)$. At the optimum, constraint (3) must also bind. If not, S could increase $\phi_S(1,z_3|\underline{\theta}_B)$ and decrease $\phi_S\left(1,z_2|\underline{\theta}_B\right)$ enhancing U_S and relaxing $(\mathcal{M})$. Using (3) and $(\mathcal{M})$, we have that U_S is increasing in $\phi_S(1,z_3|\overline{\theta}_B)$ if and only if

$$J(\overline{\theta}_T)V(\underline{\theta}_B|z_3) - \left[\frac{[\Delta\theta_B + \lambda_B\Delta s_B - \lambda_T p_T\Delta\theta_B]\,J(\overline{\theta}_T) + \lambda_T p_T\Delta\theta_B}{\Delta\theta_B + \lambda_B\Delta s_B}\right]V(\underline{\theta}_B|z_2) \geq 0,$$

or, equivalently $V(\underline{\theta}_B|z_2) \leq K\ V(\underline{\theta}_B|z_3)$, where

$$K := \frac{[\Delta\theta_B + \lambda_B \Delta s_B] J(\overline{\theta}_T)}{[\Delta\theta_B + \lambda_B \Delta s_B] J(\overline{\theta}_T) + [1 - J(\overline{\theta}_T)]\lambda_T p_T \Delta\theta_B}.$$

The optimal mechanism is then $\phi_S^*\left(1, z_3|\overline{\theta}_B\right) = 1$, $\phi_S^*\left(1, z_3|\underline{\theta}_B\right) = J(\overline{\theta}_T)$, and $\phi_S^*\left(1, z_2|\underline{\theta}_B\right) = 1 - J(\overline{\theta}_T)/K$ if $V(\underline{\theta}_B|z_2) \in [0, K\ V(\underline{\theta}_B|z_3)]$, and $\phi_S^*\left(1, z_2|\overline{\theta}_B\right) = \phi_S^*\left(1, z_2|\underline{\theta}_B\right) = 1$ if $V(\underline{\theta}_B|z_2) > K\ V(\underline{\theta}_B|z_3) > 0$. Since constraint (2) is satisfied in either case, this gives the result. ■

Proof of Proposition 2

Suppose S offers a menu of two price-lottery pairs and commits to disclose the price. The menu is such that B receives the good with certainty if he pays $\tau_H = t_B^S(\overline{\theta}_B)$ and with probability $\delta = \frac{\phi_S^*(1,z_2|\underline{\theta}_B)}{1-\phi_S^*(1,z_3|\underline{\theta}_B)}$ if he pays $\tau_L = \delta\left[\underline{\theta}_B + \lambda_B \underline{s}_B\right]$, where $\phi_S^*\left(1, z_2|\underline{\theta}_B\right)$, $\phi_S^*\left(1, z_3|\underline{\theta}_B\right)$, and $t_B^S(\overline{\theta}_B)$ are as in the direct mechanism of Lemma 3.

We want to show that it is an equilibrium for the high type to pay τ_H and for the low type to randomize over τ_H and τ_L with probability respectively equal to $J(\overline{\theta}_T)$ and $1 - J(\overline{\theta}_T)$. Given this strategy, $\overline{\theta}_T$ offers $\overline{\theta}_B$ when she observes τ_H and $\underline{\theta}_B$ when she observes τ_L, that is $z_3 = \tau_H$ and $z_2 = \tau_L$. For the low type to be indifferent between τ_H and τ_L it must be that

$$\underline{\theta}_B + \lambda_B \underline{s}_B + \lambda_T p_T \Delta\theta_B - \tau_H = \delta\left[\underline{\theta}_B + \lambda_B \underline{s}_B\right] - \tau_L. \tag{1}$$

Since $\tau_H = t_B^S(\overline{\theta}_B)$, the left hand side in (1) is also equal to the payoff $\underline{\theta}_B$ obtains from announcing $\widehat{\theta}_B = \overline{\theta}_B$ in the direct mechanism, which is equal to zero because ($\underline{IC}$) – equivalently, ($\mathcal{M}$) – and $\underline{IR}$ bind at the optimum. As a consequence, $\tau_L = \delta\left[\underline{\theta}_B + \lambda_B \underline{s}_B\right]$. The value of δ is then obtained by imposing $\delta[1 - J(\overline{\theta}_T)] = \phi_S^*\left(1, z_2|\underline{\theta}_B\right)$ which guarantees that the indirect mechanism of Proposition 2 induces the same distribution over x_B^S and Z as the direct mechanism of Lemma 3.

Next, we prove that the high type is also indifferent between τ_H and τ_L, that is $U_B(\overline{\theta}_B) = \overline{\theta}_B - \tau_H = \delta\left[\overline{\theta}_B + \lambda_B \overline{s}_B\right] - \tau_L$. Using the values of δ and τ_L, the previous equality is equivalent to

$$U_B(\overline{\theta}_B) = \left[\Delta\theta_B + \lambda_B \Delta s_B - \lambda_T p_T \Delta\theta_B\right],$$

which holds true since in the direct mechanism $\underline{IR}$ and ($\underline{IC}$) are binding which together imply that $U_B(\underline{\theta}_B) = 0 = U_B(\overline{\theta}_B) - \left[\Delta\theta_B + \lambda_B \Delta s_B - \lambda_T p_T \Delta\theta_B\right]$.

We conclude that the mechanism in Proposition 2 induces the same distribution over x_B^S and Z as the direct mechanism of Lemma 3. Furthermore, since it also gives B the same payoff, it must give S the same expected revenue. ■

Proof of Remark 1

Assume there are N potential buyers in the primary market. At the end of the auction, the winner may keep the good for himself or resell it to T in the secondary market, in which case the bargaining game is exactly as in the single-bidder case with λ_i denoting the relative bargaining power of bidder i with respect to T. Continue to assume A1-A4 hold for each bidder and let $\boldsymbol{\theta}_B := (\theta_1, \theta_2, ..., \theta_N) \in \boldsymbol{\Theta}_B := \prod_{i=1}^N \Theta_i$ denote a profile of independent private values. Following the same steps as for the single bidder case, one can show that S discloses only three signals: signal z_1 represents information such that $t_i^T(\theta_T, z_1) = \bar{\theta}_i$ for any θ_T, signal z_2 information such that $t_i^T(\theta_T, z_2) = \underline{\theta}_i$ for any θ_T, and signal z_3 information for which $t_i^T(\theta_T, z_3) = \bar{\theta}_i$ if and only if $\theta_T = \bar{\theta}_T$, where t_i^T is the resale price T offers to bidder i. Let $V(\theta_i|z_l)$ be as in Definition 1. An optimal auction in the primary market maximizes the expected sum of the bidders' resale-augmented virtual valuations,

$$\mathbb{E}_{\boldsymbol{\theta}_B}\left[\sum_{i=1}^{N}\sum_{l=1}^{3} V(\theta_i|z_l)\phi_S\left(x_i^S = 1, z_l|\boldsymbol{\theta}_B\right)\right]$$

subject to the sequential rationality constraints

$$\begin{array}{ll}
\Pr(\bar{\theta}_i|x_i^S = 1, z_1) \geq \frac{\Delta\theta_i}{\underline{\theta}_T - \underline{\theta}_i}, & (i.1)\\
\Pr(\bar{\theta}_i|x_i^S = 1, z_2) \leq \frac{\Delta\theta_i}{\bar{\theta}_T - \underline{\theta}_i}, & (i.2)\\
\Pr(\bar{\theta}_i|x_i^S = 1, z_3) \in \left[\frac{\Delta\theta_i}{\bar{\theta}_T - \underline{\theta}_i}, \frac{\Delta\theta_i}{\underline{\theta}_T - \underline{\theta}_i}\right], & (i.3)
\end{array}$$

and the monotonicity conditions – equivalently, the low-type incentive compatibility constraints –

$$\begin{array}{l}
\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left\{\phi_S\left(x_i^S = 1, z|\bar{\theta}_i, \boldsymbol{\theta}_{-i}\right)[\Delta\theta_i + \lambda_i\Delta s_i + (1-\lambda_i)\Delta r_i(z_l)]\right\} \geq \\
\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left\{\phi_S\left(x_i^S = 1, z|\underline{\theta}_B, \boldsymbol{\theta}_{-i}\right)[\Delta\theta_i + \lambda_i\Delta s_i + (1-\lambda_i)\Delta r_i(z_l)]\right\}
\end{array} \qquad (\mathcal{M}_i)$$

for $i \in \{1, 2, ..., N\}$ and $\boldsymbol{\theta}_{-i} := (\theta_1, \theta_2, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_N)$.

To prove the claim, we compare the expected revenue associated with the above program with the maximum expected revenue S could achieve in a Myerson optimal auction without resale. Recall that for any type profile $\boldsymbol{\theta}_B$, Myerson allocation rule consists in assigning the good to the bidder with the highest virtual valuation, $M(\theta_i)$, provided that $\max_i\{M(\theta_i)\} \geq 0$, and in withholding the good otherwise. The expected revenue of a Myerson optimal auction is thus $\mathbb{E}_{\boldsymbol{\theta}_B}[\max\{0,\ M(\theta_1), ..., M(\theta_N)\}]$, where $M(\bar{\theta}_i) := \bar{\theta}_i$ and $M(\underline{\theta}_i) := \underline{\theta}_i - \frac{p_i}{1-p_i}\Delta\theta_i$, for each $i = 1, ..., N$.

The proof is in two steps. The first step proves that for any $\theta_i \in \Theta_i$ and signal z_l, the resale-augmented virtual valuations are higher than the corresponding Myerson virtual valuations; that is, $V(\theta_i|z_l) \geq M(\theta_i)$ for any $l = 1, ..., 3$ and any i. This follows directly from

$$\begin{array}{c}
V(\bar{\theta}_i|z_l) = M(\bar{\theta}_i) + \lambda_i\bar{s}_i + (1-\lambda_i)\bar{r}_i(z_l)\\
V(\underline{\theta}_i|z_l) = M(\underline{\theta}_i) + \lambda_i\left[\underline{s}_i - \frac{p_i}{1-p_i}\Delta s_i\right] + (1-\lambda_i)\left[\underline{r}_i(z_l) - \frac{p_i}{1-p_i}\Delta r_i(z_l)\right]
\end{array}$$

since $\underline{s}_i \geq 0$, $\underline{r}_i(z_l) \geq 0$, $\Delta s_i \leq 0$ and $\Delta r_i(z_l) \leq 0$, for $l = 1, ..., 3$.

The second step proves that there exists a disclosure policy such that Myerson allocation rule satisfies $(\mathcal{M}_i)$ and $(i.1)-(i.3)$ for any i, and thus remains individually rational and incentive compatible also in the presence of resale. To see this, assume S discloses only the identity of the winner. Formally, conditional on i winning the auction, S sends to T always the same signal z, independently of whether i has announced a low or a high type so that $\Pr(\bar{\theta}_i|x_i^S=1,z)=\Pr(\bar{\theta}_i|x_i^S=1)$. The particular signals S sends to T depend on the posterior beliefs that are generated by the Myerson allocation rule; that is, when bidder i wins, S discloses signal $z=z_1$ if $\Pr(\bar{\theta}_i|x_i^S=1)\geq\frac{\Delta\theta_i}{\underline{\theta}_T-\underline{\theta}_i}$, $z=z_2$ if $\Pr(\bar{\theta}_i|x_i^S=1)<\frac{\Delta\theta_i}{\bar{\theta}_T-\underline{\theta}_i}$, and $z=z_3$ if $\Pr(\bar{\theta}_i|x_i^S=1)\in\left[\frac{\Delta\theta_i}{\bar{\theta}_T-\underline{\theta}_i},\frac{\Delta\theta_i}{\underline{\theta}_T-\underline{\theta}_i}\right)$. Given this disclosure policy, Myerson allocation rule trivially satisfies the constraints on the sequential rationality of T's offer in the secondary market $(i.1)-(i.3)$. Furthermore, since Myerson allocation rule is monotonic – i.e. $\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left\{\phi_S\left(x_i^S=1|\bar{\theta}_i,\boldsymbol{\theta}_{-i}\right)\right\}\geq\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left\{\phi_S\left(x_i^S=1|\underline{\theta}_i,\boldsymbol{\theta}_{-i}\right)\right\}$, constraints $(\mathcal{M}_i)$ are also satisfied for each i. It follows that Myerson allocation rule remains implementable also in the presence of resale.

Combining the two steps, we conclude that an optimal auction with resale ϕ_S^* generates a revenue

$$\mathbb{E}_{\boldsymbol{\theta}_B}\left[\sum_{i=1}^{N}\sum_{l=1}^{3}V(\theta_i|z_l)\phi_S^*\left(x_i^S=1,z_l|\boldsymbol{\theta}_B\right)\right]\geq\mathbb{E}_{\boldsymbol{\theta}_B}\left[\max\left\{0,\ M(\theta_1),...,M(\theta_N)\right\}\right],$$

which proves the result. ■

Proof of Proposition 3

To construct the reduced program for optimal auctions with inter-bidders resale, consider first how the outcome in the secondary market depends on the information disclosed in the primary market.

When bidder j wins the auction, bidder i (in the role of a resale buyer) offers a price

$$t^r(\theta_i,\widehat{\theta}_i,x_j^S=1,z^i)=\begin{cases}0 & \text{if } \theta_i<\underline{\theta}_j\\ \bar{\theta}_j & \text{if } \left[\theta_i-\underline{\theta}_j\right]\Pr(\bar{\theta}_j|x_j^S=1,z^i,\widehat{\theta}_i)\geq\Delta\theta_j,\\ \underline{\theta}_j & \text{if } \left[\theta_i-\underline{\theta}_j\right]\Pr(\bar{\theta}_j|x_j^S=1,z^i,\widehat{\theta}_i)<\Delta\theta_j.\end{cases}\tag{2}$$

The price $t^r(\theta_i,\widehat{\theta}_i,x_j^S=1,z^i)$ thus depends on bidder i's true type θ_i, as well as on his beliefs about bidder j's valuation; these in turn are a function of the allocation in the primary market, $x_j^S=1$, the information S discloses to i, z^i, and the behavior i followed in the auction, $\widehat{\theta}_i$, which is used to interpret z^i. Given a type profile $\boldsymbol{\theta}$, the surplus i obtains in the secondary market as a resale *buyer who makes the offer* is thus

$$s_i(\boldsymbol{\theta}|\widehat{\theta}_i,x_j^S=1,z^i)=\begin{cases}\theta_i-\bar{\theta}_j & \text{if } t^r(\theta_i,\widehat{\theta}_i,x_j^S=1,z^i)=\bar{\theta}_j,\\ \theta_i-\underline{\theta}_j & \text{if } t^r(\theta_i,\widehat{\theta}_i,x_j^S=1,z^i)=\underline{\theta}_j \text{ and } \theta_j=\underline{\theta}_j,\\ 0 & \text{otherwise.}\end{cases}$$

On the contrary, the surplus for bidder j in the role of a resale *seller who receives the offer* is

$$r_j(\theta_i,\underline{\theta}_j|\widehat{\theta}_i,x_j^S=1,z^i)=\begin{cases}\Delta\theta_j \text{ if } t^r(\theta_i,\widehat{\theta}_i,x_j^S=1,z^i)=\overline{\theta}_j,\\ 0 \text{ otherwise,}\end{cases}$$
$$r_j(\theta_i,\bar{\theta}_j|\widehat{\theta}_i,x_j^S=1,z^i)=0.$$

Next, consider the case where bidder i wins the auction. In this case, i (in the role of a resale seller) asks a price

$$t^r(\theta_i,\widehat{\theta}_{i,}x_i^S=1,z^i)=\begin{cases}\infty & \text{if } \theta_i>\bar{\theta}_j,\\ \bar{\theta}_j & \text{if } \left[\overline{\theta}_j-\theta_i\right]\Pr(\overline{\theta}_j|x_i^S=1,z^i,\widehat{\theta}_i)>\underline{\theta}_j-\theta_i,\\ \underline{\theta}_j & \text{if } \left[\overline{\theta}_j-\theta_i\right]\Pr(\overline{\theta}_j|x_i^S=1,z^i,\widehat{\theta}_i)\leq\underline{\theta}_j-\theta_i.\end{cases} \tag{3}$$

It follows that, given a type profile $\boldsymbol{\theta}$, the surplus i obtains in the secondary market as a resale *seller who makes the offer* is

$$s_i(\boldsymbol{\theta}|\widehat{\theta}_{i,}x_i^S=1,z^i)=\begin{cases}\bar{\theta}_j-\theta_i & \text{if } t^r(\theta_i,\widehat{\theta}_{i,}x_i^S=1,z^i)=\bar{\theta}_j \text{ and } \theta_j=\bar{\theta}_j,\\ \underline{\theta}_j-\theta_i & \text{if } t^r(\theta_i,\widehat{\theta}_{i,}x_i^S=1,z^i)=\underline{\theta}_j,\\ 0 & \text{otherwise,}\end{cases}$$

whereas the surplus for j as a *buyer who receives the offer* is

$$r_j(\theta_i,\underline{\theta}_j|\widehat{\theta}_{i,}x_i^S = 1,z^i)=0,$$
$$r_j(\theta_i,\bar{\theta}_j|\widehat{\theta}_{i,}x_i^S = 1,z^i)=\begin{cases}\Delta\theta_j \text{ if } t^r(\theta_i,\widehat{\theta}_{i,}x_i^S=1,z^i)=\underline{\theta}_j,\\ 0 \text{ otherwise.}\end{cases}$$

Using the above expressions, the monopolist's expected revenue can be written in terms of *resale-augmented virtual valuations* by substituting the expected transfers from the four binding constrains $(\overline{IC}_i)$, $(\underline{IR}_i)$, for $i=T,B$, into $U_S:=\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i=T,B}t_i^S(\theta_i)\right]$. It follows that

$$U_S=\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{h=T,B}\sum_{\mathbf{z}\in\mathbf{Z}}\left(\sum_{i=T,B}V_i(\boldsymbol{\theta}|x_h^S=1,\mathbf{z})\right)\phi_S(x_h^S=1,\mathbf{z}|\boldsymbol{\theta})\right] \tag{4}$$

where

$$V_i(\overline{\theta}_i,\theta_j|x_h^S=1,\mathbf{z}):=\ \bar{\theta}_i\mathbb{I}(h=i)+\lambda_i s_i(\overline{\theta}_i,\theta_j|x_h^S=1,z^i)+\lambda_j r_i(\overline{\theta}_i,\theta_j|x_h^S=1,z^j),$$

and

$$\begin{aligned}V_i(\underline{\theta}_i,\theta_j|x_h^S=1,\mathbf{z}):=\ &\left[\underline{\theta}_i-\tfrac{p_i}{1-p_i}\Delta\theta_i\right]\mathbb{I}(h=i)+\\ &+\lambda_i\left\{s_i(\underline{\theta}_i,\theta_j|x_h^S=1,z^i)-\tfrac{p_i\left[s_i(\bar{\theta}_i,\theta_j|,x_h^S=1,z^i)-s_i(\underline{\theta}_i,\theta_j|x_h^S=1,z^i)\right]}{1-p_i}\right\}+\\ &+\lambda_j\left\{r_i(\underline{\theta}_i,\theta_j|x_h^S=1,z^j)-\tfrac{p_i\left[r_i(\bar{\theta}_i,\theta_j|x_h^S=1,z^j)-r_i(\underline{\theta}_i,\theta_j|x_h^S=1,z^j)\right]}{1-p_i}\right\}\end{aligned}$$

Following the same steps as in the proof of Lemma 1, one can then show that S need not send more than three signals to each bidder, say z_l^i for $l = 1, ..., 3$, and $i \in \{T, B\}$. Signal z_1^i stands for information such that i prefers to ask (offer) a high price than a low price, signal z_2^i a low price than a high price, and signal z_3^i a high price for $\theta_i = \bar{\theta}_i$ and a low price otherwise. For this to be compatible with the sequential rationality constraints (2) and (3), the mechanism ϕ_S must satisfy the following constraints for any $\widehat{\theta}_i \in \Theta_i$

$$
\begin{array}{lll}
[\bar{\theta}_j - \underline{\theta}_i] \Pr(\bar{\theta}_j | x_i^S = 1, z_1^i, \widehat{\theta}_i) \geq \underline{\theta}_j - \underline{\theta}_i, & & S.1(\widehat{\theta}_i) \\
[\bar{\theta}_j - \bar{\theta}_i] \Pr(\bar{\theta}_j | x_i^S = 1, z_2^i, \widehat{\theta}_i) \leq \underline{\theta}_j - \bar{\theta}_i, & & S.2(\widehat{\theta}_i) \\
[\bar{\theta}_j - \bar{\theta}_i] \Pr(\bar{\theta}_j | x_i^S = 1, z_3^i, \widehat{\theta}_i) \geq \underline{\theta}_j - \bar{\theta}_i, & [\bar{\theta}_j - \underline{\theta}_i] \Pr(\bar{\theta}_j | x_i^S = 1, z_3^i, \widehat{\theta}_i) \leq \underline{\theta}_j - \underline{\theta}_i, & S.3(\widehat{\theta}_i) \\
[\underline{\theta}_i - \underline{\theta}_j] \Pr(\bar{\theta}_j | x_j^S = 1, z_1^i, \widehat{\theta}_i) \geq \Delta\theta_j, & & B.1(\widehat{\theta}_i) \\
[\bar{\theta}_i - \underline{\theta}_j] \Pr(\bar{\theta}_j | x_j^S = 1, z_2^i, \widehat{\theta}_i) \leq \Delta\theta_j, & & B.2(\widehat{\theta}_i) \\
[\bar{\theta}_i - \underline{\theta}_j] \Pr(\bar{\theta}_j | x_j^S = 1, z_3^i, \widehat{\theta}_i) \geq \Delta\theta_j, & [\underline{\theta}_i - \underline{\theta}_j] \Pr(\bar{\theta}_j | x_j^S = 1, z_3^i, \widehat{\theta}_i) \leq \Delta\theta_j. & B.3(\widehat{\theta}_i)
\end{array}
$$

Constraints $S.1(\widehat{\theta}_i) - S.3(\widehat{\theta}_i)$ guarantee that conditional on winning the auction by announcing $\widehat{\theta}_i$, bidder i prefers to ask a high price than a low price if $z^i = z_1^i$, a low price than a high price if $z^i = z_2^i$, and a high price for $\theta_i = \bar{\theta}_i$ and a low price for $\theta_i = \underline{\theta}_i$ if $z^i = z_3^i$. The other constraints $B.1(\widehat{\theta}_i) - B.3(\widehat{\theta}_i)$ play the same role for the case where i is a buyer in the secondary market ($x_j^S = 1$). Note that these constraints control only for i's preferences over the two prices $\bar{\theta}_j$ and $\underline{\theta}_j$. Indeed, i may actually prefer the no-trade contract when he, or she, is a seller ($x_i^S = 1$) and $\theta_i > \max \Theta_j$, or a buyer ($x_j^S = 1$) and $\theta_i < \min \Theta_j$. Also note that a constraint for signal z_l^i is relevant only if z_l^i is disclosed with positive probability to i when he announces $\widehat{\theta}_i$. Finally, some of these constraints may be impossible to satisfy for certain distributions of the bidders' valuations. When this is the case, the corresponding signals must necessarily receive zero measure in ϕ_S.

Using the expressions for the virtual valuations and the constraints on the sequential rationality for the bidders' offers in the resale game, we conclude that a revenue-maximizing mechanism for the monopolist maximizes (4) subject to ($\underline{IC}_i$), which can be rewritten as an adjusted monotonicity condition $\mathcal{M}_i$ using ($\overline{IC}_i$) and ($\underline{IR}_i$) binding , $S.1(\widehat{\theta}_i) - S.3(\widehat{\theta}_i)$, and $B.1(\widehat{\theta}_i) - B.3(\widehat{\theta}_i)$ for $i = B, T$. ■

Proof of Propositions 4 and 5

The proof for these propositions follows from the complete characterization of the optimal mechanism in the two polar cases where one of the two bidders has full bargaining power in the secondary market, i.e. $\lambda_i \in \{0, 1\}$ for $i = T, B$. Once again, it suffices to consider the case $\underline{\theta}_B \leq \underline{\theta}_T \leq \bar{\theta}_B \leq \bar{\theta}_T$.

B has all bargaining power (i.e. $\lambda_B = 1$).

As indicated in the proof of Proposition 3, the optimal mechanism ϕ_S^* maximizes the expected sum of the bidders' resale-augmented virtual valuations subject to ($\mathcal{M}_i$), $S.1(\widehat{\theta}_i) - S.3(\widehat{\theta}_i)$ and

$B.1(\widehat{\theta}_i) - B.3(\widehat{\theta}_i)$ for $i = T, B$. When it is always B who sets the price in the resale game ($\lambda_B = 1$), S does not need to disclose any information to T and thus we drop the presence of z^T in the mapping ϕ_S and eliminate the sequential rationality constraints $S.1(\widehat{\theta}_T) - S.3(\widehat{\theta}_T)$ and $B.1(\widehat{\theta}_T) - B.3(\widehat{\theta}_T)$ from the monopolist's reduced program. Furthermore, since $\underline{\theta}_B \leq \underline{\theta}_T \leq \bar{\theta}_B \leq \overline{\theta}_T$, when T wins the auction ($x_T^S = 1$), B's offer in the resale game does not depend on her beliefs about θ_T as it is always optimal for $\underline{\theta}_B$ to propose the null contract and for $\overline{\theta}_B$ to offer $\underline{\theta}_T$. Therefore, to simplify the notation, we assume no signal is disclosed to B when $x_T^S = 1$. Finally, there is no information z_2^B that can induce either type of B to ask a low price in the resale game and thus $\phi_S^*(x_B^S = 1, z_2^B|\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Substituting for the resale-augmented virtual valuations as in the proof of Proposition 3, the optimal mechanism ϕ_S^* then maximizes

$$\begin{aligned}
U_S = \;& p_T p_B \left\{\bar{\theta}_T \left[\phi_S(x_T^S = 1|\overline{\theta}_T, \overline{\theta}_B) + \phi_S(x_B^S = 1, z_1^B|\overline{\theta}_T, \overline{\theta}_B) + \phi_S(x_B^S = 1, z_3^B|\overline{\theta}_T, \overline{\theta}_B)\right]\right\} + \\
& + p_T (1 - p_B) \left\{\bar{\theta}_T \left[\phi_S(x_T^S = 1|\overline{\theta}_T, \underline{\theta}_B) + \phi_S(x_B^S = 1, z_1^B|\overline{\theta}_T, \underline{\theta}_B)\right] + \right.\\
& \left. + \left[\overline{\theta}_T - \tfrac{p_B}{1-p_B}\Delta\theta_T\right] \phi_S(x_B^S = 1, z_3^B|\overline{\theta}_T, \underline{\theta}_B)\right\} + \\
& + (1 - p_T)\, p_B \left\{\left[\overline{\theta}_B - \tfrac{p_T}{1-p_T}\Delta\theta_T\right] \phi_S(x_T^S = 1|\underline{\theta}_T, \overline{\theta}_B) + \right.\\
& \left. + \overline{\theta}_B \left[\phi_S(x_B^S = 1, z_1^B|\underline{\theta}_T, \overline{\theta}_B) + \phi_S(x_B^S = 1, z_3^B|\underline{\theta}_T, \overline{\theta}_B)\right]\right\} + \\
& + (1 - p_T)(1 - p_B) \left\{M(\underline{\theta}_B)\phi_S(x_B^S = 1, z_1^B|\underline{\theta}_T, \underline{\theta}_B) + \right.\\
& \left. + \left[M(\underline{\theta}_T) - \tfrac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T)\right] \left[\phi_S(x_T^S = 1|\underline{\theta}_T, \underline{\theta}_B) + \phi_S(x_B^S = 1, z_3^B|\underline{\theta}_T, \underline{\theta}_B)\right]\right\},
\end{aligned}$$

subject to

$$\begin{aligned}
\phi_S(x_B^S = 1, z_1^B|\underline{\theta}_T, \widehat{\theta}_B) &\leq \left[\tfrac{p_T\Delta\theta_T}{(1-p_T)(\underline{\theta}_T - \underline{\theta}_B)}\right] \phi_S(x_B^S = 1, z_1^B|\overline{\theta}_T, \widehat{\theta}_B), \qquad S.1(\widehat{\theta}_B)\\
\phi_S(x_B^S = 1, z_3^B|\underline{\theta}_T, \widehat{\theta}_B) &\geq \left[\tfrac{p_T\Delta\theta_T}{(1-p_T)(\underline{\theta}_T - \underline{\theta}_B)}\right] \phi_S(x_B^S = 1, z_3^B|\overline{\theta}_T, \widehat{\theta}_B), \qquad S.3(\widehat{\theta}_B)
\end{aligned}$$

for $\widehat{\theta}_B = \bar{\theta}_B$ and $\widehat{\theta}_B = \underline{\theta}_B$, and

$$\begin{aligned}
& p_B \left[\phi_S(x_T^S = 1|\bar{\theta}_T, \bar{\theta}_B) - \phi_S(x_T^S = 1|\underline{\theta}_T, \bar{\theta}_B)\right] + \\
& + (1 - p_B) \left[\phi_S(x_T^S = 1|\bar{\theta}_T, \underline{\theta}_B) - \phi_S(x_T^S = 1|\underline{\theta}_T, \underline{\theta}_B)\right] + \\
& + (1 - p_B) \left[\phi_S(x_B^S = 1, z_3^B|\bar{\theta}_T, \underline{\theta}_B) - \phi_S(x_B^S = 1, z_3^B|\underline{\theta}_T, \underline{\theta}_B)\right] \geq 0,
\end{aligned} \qquad (\mathcal{M}_T)$$

$$\begin{aligned}
& p_T\Delta\theta_T \left[\phi_S\left(x_B^S = 1, z_3^B|\bar{\theta}_T, \overline{\theta}_B\right) - \phi_S\left(x_B^S = 1, z_3^B|\bar{\theta}_T, \underline{\theta}_B\right)\right] + \\
& +(1 - p_T)\Delta\theta_B \left[\phi_S\left(x_B^S = 1, z_1^B|\underline{\theta}_T, \overline{\theta}_B\right) - \phi_S\left(x_B^S = 1, z_1^B|\underline{\theta}_T, \underline{\theta}_B\right)\right] + \\
& +(1 - p_T)\left(\overline{\theta}_B - \underline{\theta}_T\right) \left[\phi_S\left(x_T^S = 1|\underline{\theta}_T, \overline{\theta}_B\right) - \phi_S\left(x_T^S = 1|\underline{\theta}_T, \underline{\theta}_B\right)\right] + \\
& +(1 - p_T)\left(\overline{\theta}_B - \underline{\theta}_T\right) \left[\phi_S\left(x_B^S = 1, z_3^B|\underline{\theta}_T, \overline{\theta}_B\right) - \phi_S\left(x_B^S = 1, z_3^B|\underline{\theta}_T, \underline{\theta}_B\right)\right] \geq 0,
\end{aligned} \qquad (\mathcal{M}_B)$$

where $(\mathcal{M}_i)$ are monotonicity conditions which are obtained from the incentive compatibility constraints $(\underline{IC}_i)$ using $(\overline{IC}_i)$ and $(\underline{IR}_i)$.

1. Assume first $\max\left\{M(\underline{\theta}_T) - \frac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T); M(\underline{\theta}_B)\right\} \geq 0$.

(a) If $M(\underline{\theta}_T) - \frac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T) \leq M(\underline{\theta}_B)$, then $\frac{p_T \Delta\theta_T}{(1-p_T)(\underline{\theta}_T - \underline{\theta}_B)} \geq 1$. In this case, $\phi_S^*(x_B^S = 1, z_1^B|\boldsymbol{\theta}) = 1$ for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ maximizes U_S, and satisfies all constraints. For any θ_B, B always asks $\overline{\theta}_T$ and thus trade occurs in the secondary market if and only if T has a high valuation. In this case, the final allocation and the expected revenue coincide with that in the Myerson optimal auction if $M(\underline{\theta}_T) \leq M(\underline{\theta}_B)$. If, on the contrary, $M(\underline{\theta}_T) > M(\underline{\theta}_B)$, then in state $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$, B retains the good, contrary to what prescribed by Myerson. This in turn induces a loss of expected revenue equal to $(1-p_T)(1-p_B)\left[M(\underline{\theta}_T) - M(\underline{\theta}_B)\right]$.

(b) If $M(\underline{\theta}_T) - \frac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T) > M(\underline{\theta}_B)$, the following mechanism maximizes U_S and satisfies all constraints

$$\phi_S^*(x_T^S = 1|\overline{\theta}_T, \underline{\theta}_B) = \phi_S^*(x_T^S = 1|\underline{\theta}_T, \underline{\theta}_B) = \phi_S^*(x_T^S = 1|\overline{\theta}_T, \overline{\theta}_B) = \phi_S^*(x_B^S = 1, z_3^B|\underline{\theta}_T, \overline{\theta}_B) = 1.$$

Trade does not occur in the secondary market, the final allocation is exactly as in Myerson, but the expected revenue is just $p_T p_B \bar{\theta}_T + (1-p_T p_B)\,\underline{\theta}_T$ instead of $\mathbb{E}_{\boldsymbol{\theta}}\left[\max\{0,\ M(\theta_T),\ M(\theta_B)\}\right] = p_B[p_T\overline{\theta}_T + (1-p_T)\overline{\theta}_B] + (1-p_B)\underline{\theta}_T$.

2. Assume now $\max\left\{M(\underline{\theta}_T) - \frac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T); M(\underline{\theta}_B)\right\} < 0$. In this case, S finds it optimal to retain the good when $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$. As for the other states, the following mechanism maximizes U_S and satisfies all constraints

$$\phi_S^*(x_T^S = 1|\overline{\theta}_T, \overline{\theta}_B) = \phi_S^*(x_T^S = 1|\overline{\theta}_T, \underline{\theta}_B) = \phi_S^*(x_B^S = 1, z_3^B|\underline{\theta}_T, \overline{\theta}_B) = 1.$$

The monopolist's expected revenue is $p_T\bar{\theta}_T + (1-p_T)\,p_B\overline{\theta}_B$, trade occurs in the primary market if and only if at least one of the two bidders has a high valuation, and no offers are made in the resale game. If $M(\underline{\theta}_T) \leq 0$, the expected revenue is the same as in Myerson. On the contrary, if $M(\underline{\theta}_T) > 0 > M(\underline{\theta}_B)$, S incurs a loss equal to $(1-p_T)\,(1-p_B)\,M(\underline{\theta}_T)$.

We conclude that when $\lambda_B = 1$, the impossibility to prohibit resale results in a loss of expected revenue for the monopolist if and only if $M(\underline{\theta}_T) > \max\{0, M(\underline{\theta}_B)\}$.

T **has all bargaining power (i.e. $\lambda_T = 1$).**

In this case, S does not need to disclose any information to B. Therefore, we eliminate z^B from the mechanism ϕ_S and suppress the constraints $S.1(\widehat{\theta}_B) - S.3(\widehat{\theta}_B)$ and $B.1(\widehat{\theta}_B) - B.3(\widehat{\theta}_B)$ from the reduced program as in the proof of Proposition 3. Furthermore, since $\underline{\theta}_B \leq \underline{\theta}_T \leq \bar{\theta}_B \leq \overline{\theta}_T$, we assume no signal is disclosed to T when $x_T^S = 1$, for T always keeps the good when $\theta_T = \overline{\theta}_T$ and asks $\bar{\theta}_B$ otherwise. Finally, note that when $x_B^S = 1$, there are no signals z_1^T that can induce $\underline{\theta}_T$ to offer a high price and thus $\phi_S^*(x_B^S = 1, z_1^T|\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Using $J(\overline{\theta}_T) := \frac{p_B(\overline{\theta}_T - \overline{\theta}_B)}{(1-p_B)\Delta\theta_B}$, the program for the optimal mechanism then reduces to

$$\begin{aligned}
U_S = \;& p_T p_B \left\{ \bar{\theta}_T \left[\phi_S(x_T^S = 1|\bar{\theta}_T, \bar{\theta}_B) + \phi_S(x_B^S = 1, z_3^T|\bar{\theta}_T, \bar{\theta}_B) \right] + \bar{\theta}_B \phi_S(x_B^S = 1, z_2^T|\bar{\theta}_T, \bar{\theta}_B) \right\} + \\
& + p_T (1 - p_B) \left\{ \bar{\theta}_T \left[\phi_S(x_T^S = 1|\bar{\theta}_T, \underline{\theta}_B) + \phi_S(x_B^S = 1, z_3^T|\bar{\theta}_T, \underline{\theta}_B) \right] + \right. \\
& \left. + \left[\bar{\theta}_T - \tfrac{p_B}{1-p_B} \Delta\theta_B \right] \phi_S(x_B^S = 1, z_2^T|\bar{\theta}_T, \underline{\theta}_B) \right\} + \\
& + (1 - p_T) p_B \left\{ \left[\bar{\theta}_B - \tfrac{p_T}{1-p_T} \left(\bar{\theta}_T - \bar{\theta}_B \right) \right] \left[\phi_S(x_T^S = 1|\underline{\theta}_T, \bar{\theta}_B) + \phi_S(x_B^S = 1, z_3^T|\underline{\theta}_T, \bar{\theta}_B) \right] + \right. \\
& \left. + \bar{\theta}_B \phi_S(x_B^S = 1, z_2^T|\underline{\theta}_T, \bar{\theta}_B) \right\} + (1 - p_T)(1 - p_B) \left\{ M(\underline{\theta}_T) \phi_S(x_T^S = 1|\underline{\theta}_T, \underline{\theta}_B) + \right. \\
& + \left[M(\underline{\theta}_T) - \tfrac{p_B}{1-p_B} \Delta\theta_B \right] \phi_S(x_B^S = 1, z_2^T|\underline{\theta}_T, \underline{\theta}_B) + \\
& \left. + \left[M(\underline{\theta}_T) + \left(\tfrac{p_T}{1-p_T} - \tfrac{p_B}{1-p_B} \right) \Delta\theta_B \right] \phi_S(x_B^S = 1, z_3^T|\underline{\theta}_T, \underline{\theta}_B) \right\}
\end{aligned}$$

subject to

$$\begin{aligned}
\phi_S \left(x_B^S = 1, z_2^T|\theta_T, \underline{\theta}_B \right) &\geq J(\bar{\theta}_T) \phi_S \left(x_B^S = 1, z_2^T|\theta_T, \bar{\theta}_B \right), \qquad B.2(\widehat{\theta}_T) \\
\phi_S \left(x_B^S = 1, z_3^T|\theta_T, \underline{\theta}_B \right) &\leq J(\bar{\theta}_T) \phi_S \left(x_B^S = 1, z_3^T|\theta_T, \bar{\theta}_B \right), \qquad B.3(\widehat{\theta}_T)
\end{aligned}$$

for $\widehat{\theta}_T = \bar{\theta}_T$ and $\widehat{\theta}_T = \underline{\theta}_T$,

$$\begin{aligned}
& p_B \left(\bar{\theta}_T - \bar{\theta}_B \right) \left[\phi_S(x_T^S = 1|\bar{\theta}_T, \bar{\theta}_B) - \phi_S(x_T^S = 1|\underline{\theta}_T, \bar{\theta}_B) \right] + \\
& + p_B \left(\bar{\theta}_T - \bar{\theta}_B \right) \left[\phi_S(x_B^S = 1, z_3^T|\bar{\theta}_T, \bar{\theta}_B) - \phi_S(x_B^S = 1, z_3^T|\underline{\theta}_T, \bar{\theta}_B) \right] + \\
& + (1 - p_B) \Delta\theta_T [\phi_S(x_T^S = 1|\bar{\theta}_T, \underline{\theta}_B) - \phi_S(x_T^S = 1|\underline{\theta}_T, \underline{\theta}_B)] + \qquad (\mathcal{M}_T) \\
& + (1 - p_B) \Delta\theta_T \left[\phi_S(x_B^S = 1, z_B^T|\bar{\theta}_T, \underline{\theta}_B) - \phi_S(x_B^S = 1, z_B^T|\underline{\theta}_T, \underline{\theta}_B) \right] \\
& + (1 - p_B) (\Delta\theta_T - \Delta\theta_B) [\phi_S(x_B^S = 1, z_3^T|\bar{\theta}_T, \underline{\theta}_B) - \phi_S(x_B^S = 1, z_3^T|\underline{\theta}_T, \underline{\theta}_B)]\} \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
& p_T \left[\phi_S \left(x_B^S = 1, z_2^T|\bar{\theta}_T, \bar{\theta}_B \right) - \phi_S \left(x_B^S = 1, z_2^T|\bar{\theta}_T, \underline{\theta}_B \right) \right] + \\
& + (1 - p_T) \left[\sum_{l=2,3} \phi_S \left(x_B^S = 1, z_l^T|\underline{\theta}_T, \bar{\theta}_B \right) - \sum_{l=2,3} \phi_S \left(x_B^S = 1, z_l^T|\underline{\theta}_T, \underline{\theta}_B \right) \right] \geq 0. \qquad (\mathcal{M}_B)
\end{aligned}$$

Note that the controls $\phi_S(\cdot|\boldsymbol{\theta})$ associated with the states $\boldsymbol{\theta} = \left(\bar{\theta}_T, \bar{\theta}_B\right)$ and $\boldsymbol{\theta} = \left(\bar{\theta}_T, \underline{\theta}_B\right)$ are linked to the controls associated with the other two states $\boldsymbol{\theta} = \left(\underline{\theta}_T, \bar{\theta}_B\right)$, $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$ only through the two monotonicity conditions $(\mathcal{M}_T)$ and $(\mathcal{M}_B)$.

- Consider first $J(\bar{\theta}_T) \geq 1$.

  1. For $\boldsymbol{\theta} = (\bar{\theta}_T, \bar{\theta}_B)$ and $\boldsymbol{\theta} = (\bar{\theta}_T, \underline{\theta}_B)$, $\phi_S^* \left(x_T^S = 1|\boldsymbol{\theta} \right) = 1$ is optimal. Indeed, this maximizes U_S and since from $B.2(\bar{\theta}_T)$ $\phi_S \left(x_B^S = 1, z_2^T|\bar{\theta}_T, \bar{\theta}_B \right) < \phi_S \left(x_B^S = 1, z_2^T|\bar{\theta}_T, \underline{\theta}_B \right)$, it also helps relaxing $(\mathcal{M}_B)$. Furthermore, it guarantees $(\mathcal{M}_T)$ is always satisfied and thus can be neglected.[22]

[22] Note that $\phi_S \left(x_T^S = 1|\boldsymbol{\theta} \right) = 1$ is payoff equivalent to $\phi_S(x_B^S = 1, z_3^T|\boldsymbol{\theta}) = 1$ for $\boldsymbol{\theta} = (\bar{\theta}_T, \bar{\theta}_B)$, and $\boldsymbol{\theta} = (\bar{\theta}_T, \underline{\theta}_B)$. Nevertheless, selling to T in these two states is more effective in relaxing $(\mathcal{M}_T)$ than selling to B. This also implies that when $(\mathcal{M}_T)$ does not bind, the optimal allocation rule need not be unique.

2. Next consider the other two sates $\boldsymbol{\theta} = (\underline{\theta}_T, \overline{\theta}_B)$ and $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$. At the optimum, constraint $B.2(\underline{\theta}_T)$ is necessarily binding. If not, S can always reduce $\phi_S(x_B^S = 1, z_2^T|\underline{\theta}_T, \underline{\theta}_B)$ and increase $\phi_S(x_T^S = 1|\underline{\theta}_T, \underline{\theta}_B)$ enhancing U_S and relaxing $(\mathcal{M}_B)$. Now, assume $\phi_S(x_B^S = 1, z_2^T|\underline{\theta}_T, \overline{\theta}_B) > 0$ and suppose S reduces $\phi_S(x_B^S = 1, z_2^T|\underline{\theta}_T, \overline{\theta}_B)$ and increases $\phi_S(x_B^S = 1, z_3^T|\underline{\theta}_T, \overline{\theta}_B)$ by Δ, and at the same time, reduces $\phi_S(x_B^S = 1, z_2^T|\underline{\theta}_T, \underline{\theta}_B)$ and increases $\phi_S(x_B^S = 1, z_3^T|\underline{\theta}_T, \underline{\theta}_B)$ by $J(\overline{\theta}_T)\Delta$ so that $B.2(\underline{\theta}_T)$, $B.3(\underline{\theta}_T)$ and $(\mathcal{M}_B)$ remain satisfied. The effect on U_S is

$$(1 - p_T)\, p_B \left\{ -\frac{p_T}{1 - p_T} \left(\bar{\theta}_T - \overline{\theta}_B\right) \right\} \Delta + (1 - p_T)\,(1 - p_B) \left\{ \frac{p_T}{1 - p_T} \Delta\theta_B \right\} J(\overline{\theta}_T)\Delta = 0.$$

It follows that without loss of optimality $\phi_S^*(x_B^S = 1, z_2^T|\boldsymbol{\theta}) = 0$ for $\boldsymbol{\theta} = (\underline{\theta}_T, \overline{\theta}_B)$ and $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$.

When $\overline{\theta}_B - \frac{p_T}{1-p_T}\left(\bar{\theta}_T - \overline{\theta}_B\right) \geq 0$, $\phi_S^*\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \overline{\theta}_B\right) = 1$ is clearly optimal. In this case constraints $B.3(\underline{\theta}_T)$ and $(\mathcal{M}_B)$ are always satisfied. As for $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$, if

$$M(\underline{\theta}_T) + \left(\frac{p_T}{1 - p_T} - \frac{p_B}{1 - p_B}\right) \Delta\theta_B \geq \max\left\{0;\ M(\underline{\theta}_T)\right\},$$

then $\phi_S^*\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \underline{\theta}_B\right) = 1$ in which case the revenue is $U_S = (1-p_B)\underline{\theta}_T + p_T\Delta\theta_B + p_B\underline{\theta}_B$. If instead

$$M(\underline{\theta}_T) > \max\left\{0;\ M(\underline{\theta}_T) + \left(\frac{p_T}{1 - p_T} - \frac{p_B}{1 - p_B}\right) \Delta\theta_B\right\},$$

then $\phi_S^*\left(x_T^S = 1|\underline{\theta}_T, \underline{\theta}_B\right) = 1$ and the revenue is $(1 - p_B)\underline{\theta}_T + p_B\overline{\theta}_B$. Finally, if

$$\max\left\{M(\underline{\theta}_T) + \left(\frac{p_T}{1 - p_T} - \frac{p_B}{1 - p_B}\right) \Delta\theta_B;\ M(\underline{\theta}_T)\right\} < 0,$$

then S retains the good when $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$, that is,

$$\phi_S^*\left(x_T^S = 1|\underline{\theta}_T, \underline{\theta}_B\right) = \phi_S^*\left(\underline{\theta}_T, \underline{\theta}_B|x_B^S = 1, z_3^T\right) = 0,^{23}$$

and the revenue is $U_S = p_T\,(1 - p_B)\,\bar{\theta}_T + p_B\overline{\theta}_B$.

Next, assume $\overline{\theta}_B - \frac{p_T}{1-p_T}\left(\bar{\theta}_T - \overline{\theta}_B\right) < 0$. In this case $(\mathcal{M}_B)$ necessarily binds, i.e.

$$\phi_S^*\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \overline{\theta}_B\right) = \phi_S^*\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \underline{\theta}_B\right),$$

and hence $B.3(\underline{\theta}_T)$ is always satisfied. Furthermore, since $M(\underline{\theta}_T) \leq \overline{\theta}_B - \frac{p_T}{1-p_T}\left(\bar{\theta}_T - \overline{\theta}_B\right) < 0$, S never sells to T when the latter reports a low valuation, i.e. when $\boldsymbol{\theta} = (\underline{\theta}_T, \overline{\theta}_B)$ or $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$. At the optimum $\phi_S^*\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \overline{\theta}_B\right) = \phi_S^*\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \underline{\theta}_B\right) = 1$ if

$$p_B\left[\overline{\theta}_B - \frac{p_T}{1 - p_T}\left(\bar{\theta}_T - \overline{\theta}_B\right)\right] + (1 - p_B)\left[M(\underline{\theta}_T) + \left(\frac{p_T}{1 - p_T} - \frac{p_B}{1 - p_B}\right)\Delta\theta_B\right] \geq 0$$

and $\phi_S^*\left(x_B^S=1, z_3^T|\underline{\theta}_T,\underline{\theta}_B\right) = \phi_S^*\left(x_B^S=1, z_3^T|\underline{\theta}_T,\underline{\theta}_B\right) = 0$ otherwise. In the first case, the revenue is $U_S = (1-p_B)\underline{\theta}_T + p_T\Delta\theta_B + p_B\underline{\theta}_B$, whereas in the second $U_S = p_T\overline{\theta}_T$.

- Suppose now $J(\overline{\theta}_T) < 1$.

  In this case $(\mathcal{M}_B)$ can be neglected as it is never binding at the optimum.

  1. For $\boldsymbol{\theta} = (\overline{\theta}_T,\overline{\theta}_B)$ and $\boldsymbol{\theta} = (\overline{\theta}_T,\underline{\theta}_B)$, $\phi_S^*\left(x_T^S=1|\boldsymbol{\theta}\right)=1$ is again optimal. This also implies $(\mathcal{M}_T)$ is always satisfied.
  2. For $\boldsymbol{\theta} = (\underline{\theta}_T,\overline{\theta}_B)$ and $\boldsymbol{\theta} = (\underline{\theta}_T,\underline{\theta}_B)$, $\phi_S^*(x_B^S=1, z_2^T|\boldsymbol{\theta}) = 0$. The argument is the same as for $J(\overline{\theta}_T) \geq 1$.

     Assume first $\overline{\theta}_B - \frac{p_T}{1-p_T}\left(\overline{\theta}_T - \overline{\theta}_B\right) \geq 0$. Then $\phi_S^*(x_B^S = 1, z_3^T|\underline{\theta}_T,\overline{\theta}_B) = 1$. As for $\boldsymbol{\theta} = (\underline{\theta}_T,\underline{\theta}_B)$, if

     $$M(\underline{\theta}_T) + \left(\frac{p_T}{1-p_T} - \frac{p_B}{1-p_B}\right)\Delta\theta_B \geq \max\{0,\ M(\underline{\theta}_T)\}$$

     then constraint $B.3(\underline{\theta}_T)$ binds and thus $\phi_S^*\left(x_B^S=1, z_3^T|\underline{\theta}_T,\underline{\theta}_B\right) = J(\overline{\theta}_T)$. If in addition $M(\underline{\theta}_T)\geq 0$, then $\phi_S^*\left(x_T^S=1|\underline{\theta}_T,\underline{\theta}_B\right) = 1-J(\overline{\theta}_T)$, otherwise $\phi_S^*\left(x_T^S=1|\underline{\theta}_T,\underline{\theta}_B\right)=0$. In the former case the expected revenue is $\frac{p_B}{1-p_B}\left[\overline{\theta}_T(p_T-p_B)+(1-p_T)\overline{\theta}_B\right] + \underline{\theta}_T(1-p_B)$, whereas in the latter $(1-p_B)\underline{\theta}_T + p_T\Delta\theta_B + p_B\underline{\theta}_B$. If, on the contrary,

     $$M(\underline{\theta}_T) + \left(\frac{p_T}{1-p_T} - \frac{p_B}{1-p_B}\right)\Delta\theta_B < \max\{0,\ M(\underline{\theta}_T)\},$$

     then necessarily $\phi_S^*\left(x_B^S=1, z_3^T|\underline{\theta}_T,\underline{\theta}_B\right) = 0$. As for $\phi_S\left(x_T^S=1|\underline{\theta}_T,\underline{\theta}_B\right)$, at the optimum $\phi_S^*\left(x_T^S=1|\underline{\theta}_T,\underline{\theta}_B\right)=1$ when $M(\underline{\theta}_T)\geq 0$, whereas $\phi_S^*\left(x_T^S=1|\underline{\theta}_T,\underline{\theta}_B\right)=0$ when $M(\underline{\theta}_T)<0$, with a revenue respectively equal to $(1-p_B)\underline{\theta}_T + p_B\overline{\theta}_B$ in the first case and $(1-p_B)\underline{\theta}_T + p_T\Delta\theta_B + p_B\underline{\theta}_B$ in the second.

     Next, consider $\overline{\theta}_B - \frac{p_T}{1-p_T}\left(\overline{\theta}_T - \overline{\theta}_B\right) < 0$. At the optimum, constraint $B.3(\underline{\theta}_T)$ necessarily binds. It follows that $\phi_S^*\left(x_B^S=1, z_3^T|\underline{\theta}_T,\overline{\theta}_B\right)=1$ and $\phi_S^*\left(x_B^S=1, z_3^T|\underline{\theta}_T,\underline{\theta}_B\right) = J(\overline{\theta}_T)$ if

     $$p_B\left[\overline{\theta}_B - \frac{p_T}{1-p_T}\left(\overline{\theta}_T-\overline{\theta}_B\right)\right] + (1-p_B)J(\overline{\theta}_T)\left[M(\underline{\theta}_T) + \left(\frac{p_T}{1-p_T}-\frac{p_B}{1-p_B}\right)\Delta\theta_B\right] > 0, \tag{5}$$

     whereas $\phi_S^*\left(x_B^S=1, z_3^T|\underline{\theta}_T,\overline{\theta}_B\right) = \phi_S^*\left(x_B^S=1, z_3^T|\underline{\theta}_T,\underline{\theta}_B\right) = 0$ when (5) is reversed . In either case, S never sells to T when the latter reports a low valuation, that is, $\phi_S^*(x_T^S = 1|\boldsymbol{\theta}) = 0$ when $\boldsymbol{\theta} = (\underline{\theta}_T,\overline{\theta}_B)$ and $\boldsymbol{\theta} = (\underline{\theta}_T,\underline{\theta}_B)$. The revenue is $p_T\bar{\theta}_T + p_B(\overline{\theta}_B - p_T\overline{\theta}_T) + (1-p_T)(1-p_B)\left[M(\underline{\theta}_T) + \left(\frac{p_T}{1-p_T} - \frac{p_B}{1-p_B}\right)\Delta\theta_B\right]J(\overline{\theta}_T)$ in the former case, and $p_T\bar{\theta}_T$ in the latter.

Note that in the optimal mechanism obtained above the allocation rule is stochastic when

(I) $J(\overline{\theta}_T) < 1$,

(II) $M(\underline{\theta}_T) + \left(\frac{p_T}{1-p_T} - \frac{p_B}{1-p_B}\right) \Delta\theta_B > \max\left\{0; M(\underline{\theta}_T); -\left(\frac{p_B}{1-p_B}\right) J(\overline{\theta}_T)^{-1}\left[\overline{\theta}_B - \frac{p_T}{1-p_T}\left(\bar{\theta}_T - \overline{\theta}_B\right)\right]\right\}$.

However, since the optimal mechanism is not unique, it remains to prove that when the above two conditions hold, then the optimal allocation rule is *necessarily stochastic* in *any* revenue-maximizing mechanism. To see this, recall that constraint $B.2(\underline{\theta}_T)$ necessarily binds at the optimum. Combined with $B.3(\underline{\theta}_T)$, we have that

$$\begin{aligned}&\phi_S^*\left(x_B^S = 1, z_2^T|\underline{\theta}_T, \underline{\theta}_B\right) + \phi_S^*\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \underline{\theta}_B\right) \leq \\ &J(\overline{\theta}_T)\left[\phi_S^*\left(x_B^S = 1, z_2^T|\underline{\theta}_T, \overline{\theta}_B\right) + \phi_S^*\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \overline{\theta}_B\right)\right] < 1,\end{aligned}$$

where the last inequality follows directly from (I). At the same time, if (II) holds, then necessarily

$$\phi_S^*\left(x_B^S = 1, z_2^T|\underline{\theta}_T, \underline{\theta}_B\right) + \phi_S^*\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \underline{\theta}_B\right) > 0.$$

Indeed, suppose this is not true. Since $B.2(\underline{\theta}_T)$ necessarily binds, then $\phi_S\left(x_B^S = 1, z_2^T|\underline{\theta}_T, \overline{\theta}_B\right) = 0$. Now if $\phi_S\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \overline{\theta}_B\right) > 0$, S can set $\phi_S\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \underline{\theta}_B\right) = J(\overline{\theta}_T)\phi_S\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \overline{\theta}_B\right)$, possibly reducing $\phi_S^*\left(x_T^S = 1|\underline{\theta}_T, \underline{\theta}_B\right)$ if this is positive, still preserving constraint $B.3(\underline{\theta}_T)$. Under (II), this leads to a higher payoff for the monopolist. Furthermore, by setting $\phi_S\left(x_T^S = 1|\boldsymbol{\theta}\right) = 1$ for $\boldsymbol{\theta} = (\overline{\theta}_T, \overline{\theta}_B)$ and $\boldsymbol{\theta} = (\overline{\theta}_T, \underline{\theta}_B)$, constraints $(\mathcal{M}_T)$ and $(\mathcal{M}_B)$ are always satisfied and hence not selling to B when $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$ cannot be optimal. A similar argument holds for $\phi_S\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \overline{\theta}_B\right) = 0$, as S can always increase U_S by setting $\phi_S\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \overline{\theta}_B\right) = 1$ and $\phi_S\left(x_B^S = 1, z_3^T|\underline{\theta}_T, \underline{\theta}_B\right) = J(\overline{\theta}_T)$.

We conclude that if (I) and (II) hold, then necessarily

$$\sum_{l=2}^{3} \phi_S^*\left(x_B^S = 1, z_l^T|\underline{\theta}_T, \underline{\theta}_B\right) \in (0, 1),$$

i.e. S sells with probability positive but less than one to B when $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$, and hence it is impossible to maximize revenue through a deterministic selling procedure, which proves the claim in Proposition 5.

Finally, comparing the revenue U_S^*, with the revenue in the Myerson auction, we have that for all possible parameters configurations,

$$U_S^* < \mathbb{E}_{\boldsymbol{\theta}}\left[\max\{M(\theta_T);\; M(\theta_B); 0\}\right],$$

which along with the results for $\lambda_B = 1$ proves the claim in Proposition 4. ∎

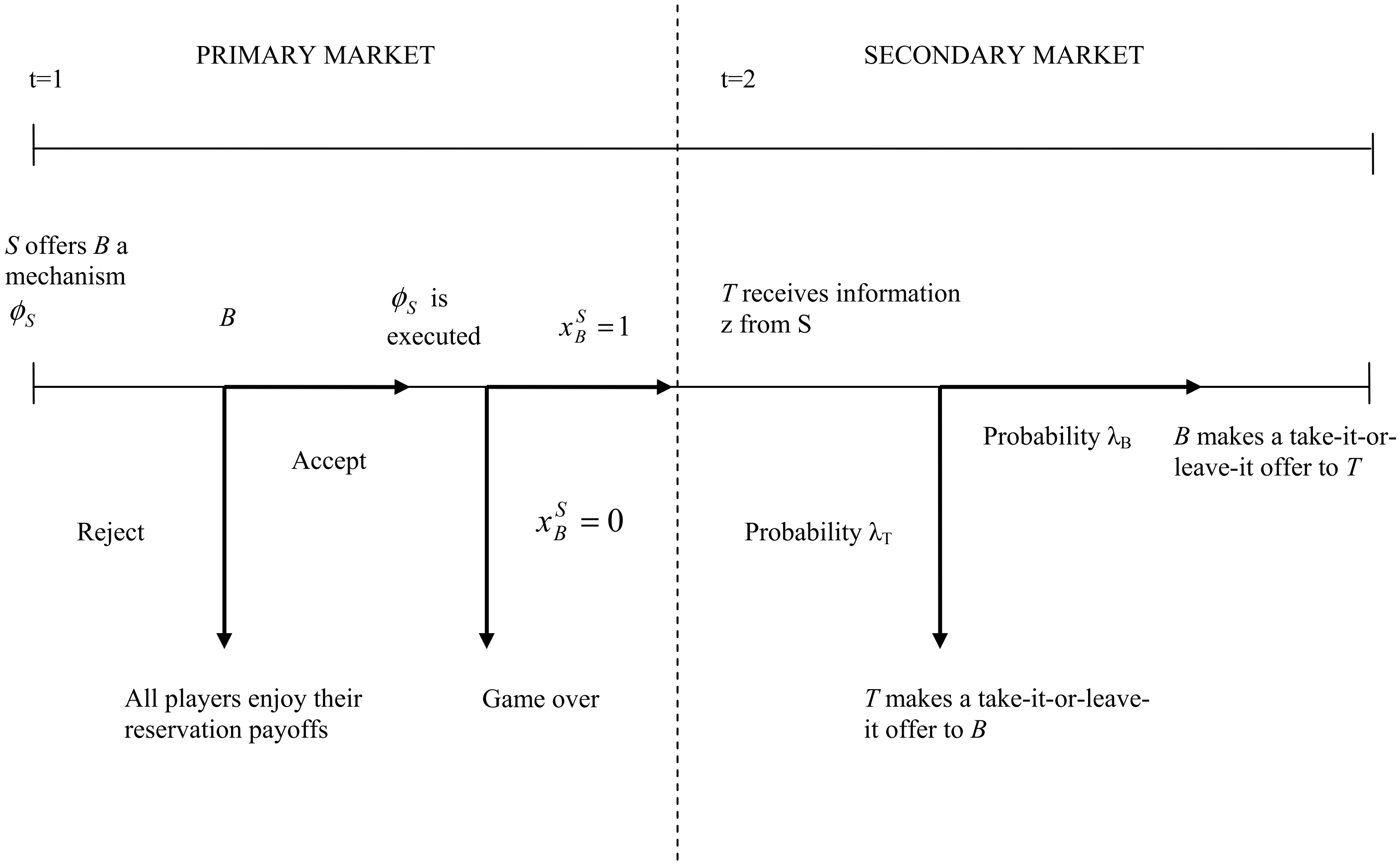

Figure 1: The trading game

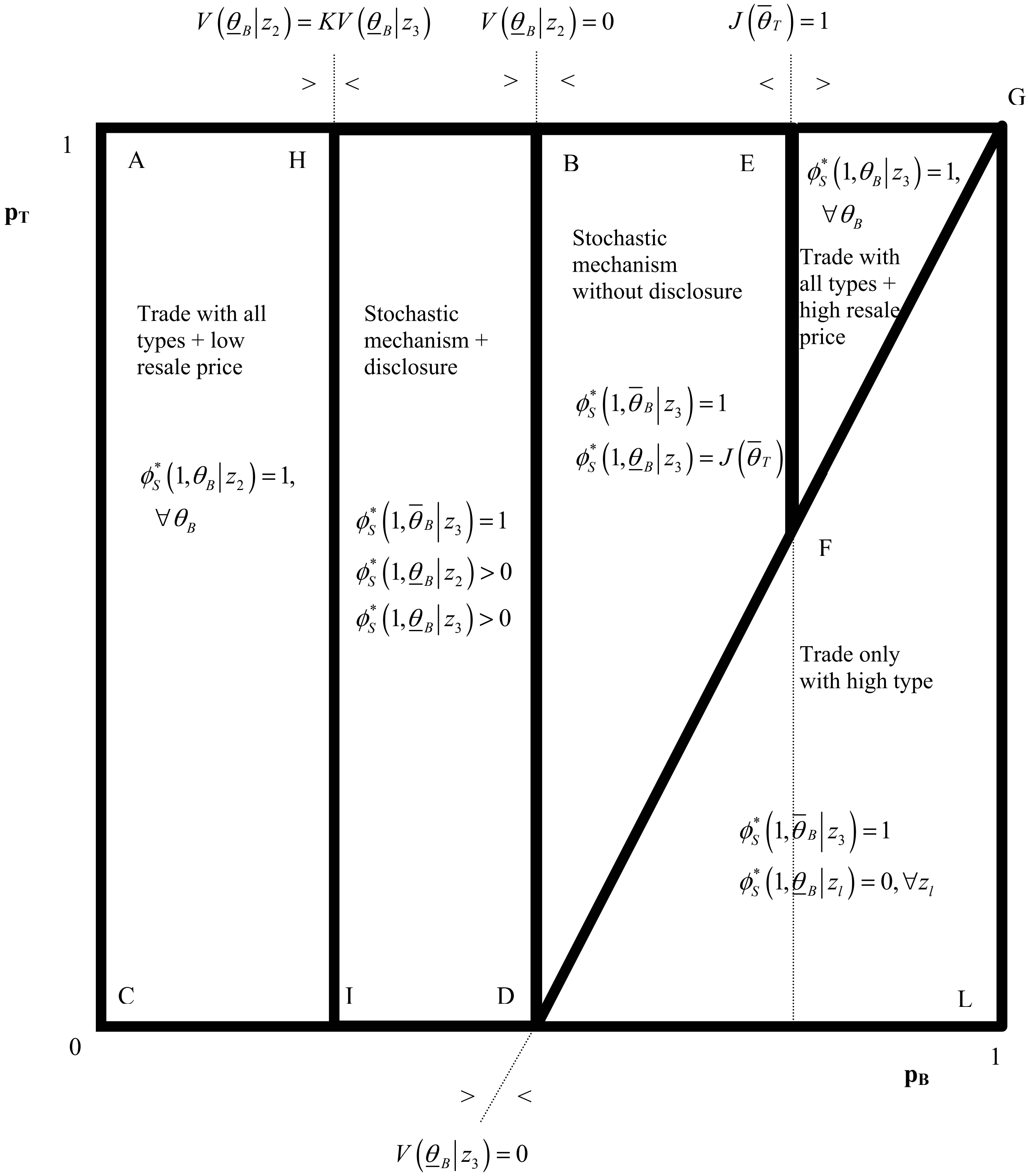

Figure 2: The Optimal Mechanism for $\lambda_T = 1$