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SYMMETRY AND DIVERSIFICATION OF INTERDEPENDENT PROSPECTS

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ABSTRACT

Dealing with optimal policy of a risk averter, characterized by a strictly concave monotonic utility function, Samuelson proved that funds should be allocated equally among prospects having a symmetric joint distribution. In this work we strengthen the above result by weakening the symmetry assumption, as well as including possible constraints on the set of possible portfolios. We also demonstrate the validity of the above result to the case where the one dimensional concave utility (of total wealth) is replaced by a multi-dimensional utility that depends on the distribution of the wealth and satisfies a weaker concavity condition.

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Introduction

In an early paper [6], P. A. Samuelson showed that given n prospects with a symmetric n-dimensional joint distribution function a risk averter's optimal strategy is to put an equal amount into each prospect. A proof of the result was also given in a recent paper by J. Hadar and W. R. Russell [3] who approached the problem using the stochastic dominance concept.

In this paper we strengthen the above result by weakening the symmetry assumption, as well as including possible constraints on the set of feasible portfolios, by allowing sets containing no continuum.

We also demonstrate the validity of the above result to the case where the one dimensional concave utility (of total wealth) is replaced by a multi-dimensional utility that depends on the distribution of the wealth and satisfies a weaker concavity condition.

We start by developing general results on a class of cyclically symmetric optimization problems and then apply it to the above portfolio selection problem.

Permutation matrices and a related class of optimization problems

In this section we present results on permutation matrices, i.e. $\{0,1\}$ - valued matrices having row and column sums equal to one.

<u>Lemma 1</u>: Let $A \subseteq R^n$ and let P be an nxn permutation matrix such that $x \in A$ implies $Px \in A$. Then $A = PA$, i.e., P maps A onto itself.

Proof: By the lemma's assumptions it follows that

$$P^k A \subseteq P^{k-1} A \subseteq A \qquad \text{for} \qquad k \geq 2,$$

where P^k is the k^{th} power of P. The lemma follows if we show that P is idempotent, i.e., that for some integer m, $P^m = I$. We recall that the permutation matrices of order n form a finite group of order n! with respect to matrix multiplication. Hence for any permutation matrix P there exists $m=m(P) \leq n!$ such that $P^m = I$.

We apply the above result to obtain the following:[1]

<u>Lemma 2</u>: Let $G:R^n \to R$ be a distribution function and $V:BxA \subseteq R^n x R^n \to R$, where A and B are subsets of R^n. Given a permutation matrix P suppose that A is symmetric with respect to P, i.e., $x \in A \Rightarrow Px \in A$, and B is symmetric with respect to P'. If $G(Px) = G(x)$ for all $x \in A$, and $V(P'\lambda, x) = V(\lambda, Px)$ for all $(\lambda, x) \in BxA$, then for any $\lambda \in B$ such that $\Phi(\lambda) = \int_A V(\lambda, x)dG(x)$ exists, $\Phi(P'\lambda)$ exists and $\Phi(P'\lambda) = \Phi(\lambda)$.

<u>Proof</u>: Let A be a Borel set. Recalling that P is a permutation matrix and applying Lemma 1 yield that $R \subseteq A$ is a rectangle if and only if $PR \subseteq A$ is a rectangle. Hence Ai is a Borel subset of A if and only if PAi is a Borel subset of A. Furthermore, $G(x) = G(Px)$ for all $x \in A$ implies that for each Borel subset $Ai \subseteq A$ $G^*(Ai) = G^*(PAi) = G^*(P^{-1}Ai)$ where G^* is the probability measure induced by the distribution G.

For each Borel subset $Ai \subseteq A$ let 1_{Ai} be the corresponding indicator function i.e. it takes on the value 1 if $x \in Ai$ and 0 otherwise.

Given the relation $V(P'\lambda, x) = V(\lambda, Px)$ for all $x \in A$ and sequences of step functions

$$\{\sum_{i=1}^{n} C_i 1_{Ai}\}_n \quad \{\sum_{i=1}^{n} C_i 1_{PAi}\}_n$$

where $Ai \subseteq A$, we observe the following

$$V(\lambda, \cdot) = \lim_n \sum_{i=1}^{n} C_i \; 1_{PAi} \iff V(P'\lambda, \cdot) = \lim_n \sum_{i=1}^{n} C_i 1_{Ai}$$

Thus,

$$\begin{aligned}
\int_A V(\lambda, x)\, dG(x) &= \lim_n \int_A \sum_{i=1}^{n} C_i 1_{PAi} = \lim_n \sum_{i=1}^{n} C_i G^*(PAi) \\
&= \lim_n \sum_{i=1}^{n} C_i G^*(Ai) = \lim_n \int_A \sum_{i=1}^{n} C_i 1_{Ai} \\
&= \int_A V(P'\lambda, x)\, dG(x).
\end{aligned}$$

and the proof is complete.

The following well known result from convexity theory is implied by the σ - additivity of the probability measure induced by the distribution function G.

<u>Lemma 3</u>: Let $G: R^n \to R$ be a joint distribution function on R^n. Given a mapping $U(\lambda,x): R^n \times R^n \to R$, suppose that for any $x \in R^n$ $U(\cdot,x)$ is concave. Let $A \subseteq R^n$ and assume further that $\psi(\lambda) = \int_A U(\lambda,x)dG(x)$ is well defined for all $\lambda \in D \subseteq R^n$. $0 \leq t \leq 1$ and $\lambda_1, \lambda_2 \in D$, then $\lambda(t) = t\lambda_1 + (1-t)\lambda_2 \in D$ implies $\psi(\lambda(t)) \geq t\,\psi(\lambda_1) + (1-t)\,\psi(\lambda_2)$.

<u>Remark</u>: If the (complete) concavity of $U(\cdot,x)$ in $\lambda = (\lambda_1,\ldots,\lambda_n)$ is replaced by a partial concavity (i.e. concavity in a subset of the variables $(\lambda_1,\ldots,\lambda_n)$) then the same property will be induced on $\psi(\lambda)$.

To present the main result we introduce the following definitions.

<u>Definition 1</u>:

$(i_1,\ldots,i_n)$ is the jth cyclic permutation of $(1,2,\ldots,n)$, if $i_{j+k} = 1+k$, $k=0,\ldots,n-j$ and $i_{j-k} = n+1-k$, $k=1,\ldots,j-1$. The $\{0,1\}$ - valued nxn matrix corresponding to the j^{th} cyclic permutation is denoted by P_j. Note that P_1 is the identity of order n.

<u>Definition 2</u>:

(1) $A \subseteq R^n$ is said to be symmetric if for any $x \in A$ and any permutation matrix P $Px \in A$.

(2) $A \subseteq R^n$ is cyclically symmetric if $x \in A$ implies $P_j\, x \in A$, $j=1,\ldots,n$, where P_j is the j^{th} cyclic permutation matrix.[2]

(3) A cyclically symmetric set $A \subseteq R^n$ is <u>cyclically convex</u> if $x \in A$ implies $\bar{x} = \frac{1}{n} \sum_{j=1}^{n} P_j x \in A$, where P_j, $j=1,\ldots,n$, is the j^{th} cyclic permutation matrix.

(4) Let $f: A \subseteq R^n \to R$, where A is symmetric (cyclically symmetric). Then f is <u>symmetric</u> (<u>cyclically symmetric</u>) if $f(x) = f(P(x))$ $(f(x) = f(P_j(x))$ for all permutation matrices P (for all cyclic permutation matrices P_j, $j=1,\ldots,n)$.

We are now ready to introduce the main result of this section.

<u>Theorem 4</u>: Given $U(\lambda,x) : R^n \times R^n \to R$, a distribution function $G(x)$: $R^n \to R$ and sets $A \subseteq R^n$, $B \subseteq R^n$, suppose the following

1. A is cyclically symmetric and B is cyclically convex.
2. G is cyclically symmetric on A. $U(\cdot,x)$ is concave for all $x \in A$.
3. $U(P'_j\lambda,x) = U(\lambda,P_j x)$ for all $(\lambda,x) \in B \times A$ and $j=1,\dots,n$, where P_j is the j^{th} cyclic permutation matrix.
4. For all $\lambda \in B$ $\psi(\lambda) = \int_A U(\lambda,x)dG(x)$ is well defined on the extended line i.e. $\psi(\lambda) \in [-\infty,\infty]$.

Then $\psi(\lambda)$ is cyclically symmetric and for each λ_1, $\lambda_2 \in B$ and t $0 \le t \le 1$, $\lambda(t) = t\lambda_1 + (1-t)\lambda_2 \in B$ implies $\psi(\lambda(t)) \ge t\,\psi(\lambda_k) + (1-t)\psi(\lambda_2)$. Moreover, $\psi(\bar{\lambda}) \ge \psi(\lambda)$ for all $\lambda \in B$, where $\bar{\lambda} \in B$ is a symmetric point all of whose components equal to $\frac{1}{n}\sum_{i=1}^{n} \lambda_i$.

<u>Proof</u>: The concavity property of $\psi(\lambda)$ is implied by Lemma 3. Using the cyclic symmetry of G we apply Lemma 2 to have $\psi(P_j'\lambda) = \psi(\lambda)$ for all $\lambda \in B$ and $j=1,\dots,n$. To obtain the cyclic symmetry of ψ we observe that $P_jP_j' = I$, i.e.,

P_j' is itself a cyclic permutation matrix. Specifically, $P_1' = P_1$ and $P_j' = P_{n+2-j}$ for $j \geq 2$. Thus, $\psi(P_j\lambda) = \psi(\lambda)$

for all $\lambda \in B$, $j=1,\ldots,n$, and ψ is cyclically symmetric.

Let $\lambda \in b$, then $\frac{1}{n} \sum_{j=1}^{n} P_j\lambda = \bar{\lambda} \in B$. All the components of $\bar{\lambda}$ are equal to $\frac{1}{n} \sum_{i=1}^{n} \lambda_i$. We then apply the concavity and the cyclic symmetry of ψ to obtain:

$$\psi(\bar{\lambda}) \geq \frac{1}{n} \sum_{j=1}^{n} \psi(P_j\lambda) = \frac{1}{n} \sum_{j=1}^{n} \psi(\lambda) = \psi(\lambda)$$

In the next section we apply the above theorem to strengthen a result of Samuelson [6].

Diversification of Interdependent Prospects

Dealing with optimal distribution of a risk averter, characterized by a strictly concave monotonic utility function, Samuelson [6] and later Hadar and Russell [3], proved the following theorem showing that funds should be allocated equally among n prospects having a symmetric joint distribution.

Theorem 5: Let $G(x_1,...,x_n)$ be the joint probability distribution of the random variables $x_1,...,x_n$. Suppose that G is symmetric and has finite means, variances and covariances. then if $U(t) : R^1 \to R^1$ is strictly concave and twice continuously differentiable the maximum of the symmetric, concave function.

$$\psi(\lambda) = \int_{R^n} U(\lambda' x)\, dG(x)$$

subject to $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$, $\lambda_i \geq 0$ is given by $\psi(\frac{1}{n},...,\frac{1}{n})$. Thus, diversification always pays.[3]

The results of the preceding section enable us to weaken the symmetry assumption of the above theorem, as well as the smoothness of the utility U. Furthermore, the convexity of the feasible set for λ can be considerably relaxed to allow finite and countable sets.

The following is our generalization:

<u>Theorem 6</u>: Let $G(x) = G(x_1,\ldots,x_n)$ be the joint probability distribution of the random variables $x_1,\ldots,x_n$. Suppose that G is cyclically symmetric and has finite means. Let U(t) be a concave function from R^1 to R^1, and let $D \subseteq R^n$ be a cyclically symmetric, cyclically convex set. If $\psi(\lambda) = \int_{R^n} U(\lambda' x)dG(x)$ exists for all λ in D then there exists a symmetric point $\bar{\lambda} \in D$, where $\bar{\lambda} = \frac{1}{n} \sum_{j=1}^{n} P_j\lambda$, and $\psi(\bar{\lambda}) \geq \psi(\lambda)$.[4]

<u>Proof</u>: The proof follows directly from Theorem 4, by verifying that all the theorem assumptions are met.

We observe that Theorem 5 is implied by Theorem 6 as D corresponds to a compact convex set. We also note that Theorem 6 yields that the equal allocation of funds among the n (cyclically symmetric) prospects is preserved as long as the set of feasible portfolios, D, is cyclically symmetric and cyclically convex. Note that D may be a finite or a countable set. (Further extensions of Theorem 6 are given in the next section.)

To demonstrate the strengthening provided by the last theorem we will illustrate that cyclic symmetry and cyclic convexity are weaker than symmetry and convexity respectively.

The function $f(u,v,w) = (u-v)(v-w)(w-u)$ is cyclically symmetric but not symmetric. We also point out that convexity of cyclically symmetric functions does not imply the complete symmetry. This is illustrated by the function

$$h(u,v,w) = (u-v)(v-w)(w-u) + u^2 + v^2 + w^2$$

in some neighborhood of the origin. Finally the set $D \subseteq R^3$ defined by the union of the convex hull of $\{(1,2,3)',(2,3,1)',(3,1,2)'\}$ and the convex hull of $\{-(1,2,3)',-(2,3,1)',-(3,1,2)'\}$ is cyclically symmetric, cyclically convex, but neither symmetric nor even connected.

Extensions and Concluding Remarks

As shown in the preceding section equal allocation of funds assures optimality when a risk averter is concerned with maximizing the expectation of the utility of his total wealth (i.e. a one dimensional utility). This result can be generalized to the case of a multi-dimensional utility that depends on the distribution of wealth and satisfies the appropriate symmetry assumption.

In particular we assume that the utility is of the form $v(\lambda_1 x_1, \ldots, \lambda_n x_n)$, where $x_1,\ldots,x_n$ denote the yields of the n prospects and $(\lambda_1,\ldots,\lambda_n)$ is a distribution vector.

If v is concave and cyclically symmetric in its n arguments then one can easily verify that the conditions of Theorem 4 with $U(\lambda, x) = v(\lambda_1 x_1,\ldots,\lambda_n x_n)$ are met and thus the extension of Theorem 6 to the case of a multidimensional concave and cyclically symmetric utility follows.

Theorem 7: Let $G(x) = G(x_1,\ldots,x_n)$ be the joint probability distribution of the random variables $x_1,\ldots,x_n$ and suppose that G is cyclically symmetric. Let $v(t_1,\ldots,t_n)$ be a real concave cyclically symmetric function from R^n to R^1 and let $D \in R^n$ be a cyclically symmetric, cyclically convex set. If $\chi(\lambda) = \int_{R^n} v(\lambda_1 x_1,\ldots,\lambda_n x_n)\, dG(x)$

exists for all $\lambda \in D$, then there exists a symmetric point $\bar{\lambda} \in D$, where $\bar{\lambda} = \frac{1}{n}\sum_{j=1}^{n} P_j\lambda$, and $\chi(\bar{\lambda}) \geq \chi(\lambda)$.

We introduce now a relaxation in the concavity assumption of the utility v to the case where the complete symmetry is satisfied by both v and G. Our generalization is based on the following lemma discussing the global optimality of the symmetric point in the hyperplane $H = \{\lambda \mid \Sigma\, \lambda_j = 1\}$.

<u>Lemma 8</u>: Let $f(\lambda_1,\ldots,\lambda_n)$ be a cyclically symmetric real function defined on the hyperplane $H = \{\lambda \mid \Sigma\, \lambda_j = 1\}$. Let f be upper semicontinuous at $t = (\frac{1}{n},\ldots,\frac{1}{n})$, (i.e. $\limsup_{x \to t} f(x) \leq f(t)$), and suppose that for each $\lambda \in H$

$$f(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n) \leq f(\frac{\lambda_1+\lambda_2}{2}, \frac{\lambda_1+\lambda_2}{2}, \lambda_3,\ldots,\lambda_n) \tag{1}$$

Then $f(\lambda) \leq f(\frac{1}{n},\ldots,\frac{1}{n})$ for all $\lambda \in H$.

<u>Proof</u>: Let $\lambda \in H$ and define the following sequence in H.

$$\lambda_i^o = \lambda_i, \quad i = 1,\ldots,n$$

For $k \geq 1$
$$\lambda_1^k = \frac{\lambda_1^{k-1} + \lambda_2^{k-1}}{2}, \quad \lambda_i^k = \lambda_{i+1}^{k-1}, \quad i=2,\ldots,n-1 \tag{2}$$

and
$$\lambda_n^k = \frac{\lambda_1^{k-1} + \lambda_2^{k-1}}{2} = \lambda_1^k.$$

We first demonstrate that $f(\lambda^k) \geq f(\lambda^{k-1})$ for $k = 1,2,\ldots,$ using (1) and the cyclic symmetry of f.

$$\begin{aligned} f(\lambda_1^{k-1}, \lambda_2^{k-1},\ldots, \lambda_n^{k-1}) &\leq f(\frac{\lambda_1^{k-1} + \lambda_2^{k-1}}{2}, \frac{\lambda_1^{k-1} + \lambda_2^{k-1}}{2}, \lambda_3^{k-1},\ldots,\lambda_n^{k-1}) \\ &= f(\frac{\lambda_1^{k-1} + \lambda_2^{k-1}}{2}, \lambda_3^{k-1},\ldots,\lambda_n^{k-1}, \frac{\lambda_1^{k-1} + \lambda_2^{k-1}}{2}) = \\ &= f(\lambda_1^k,\ldots,\lambda_n^k). \end{aligned}$$

Using the upper semicontinuity of f, the result will follow if it is first shown that $\{\lambda^k\}$ converges to the symmetric point $(\frac{1}{n},\ldots,\frac{1}{n})$ in H. From (2) it is clearly sufficient to prove that $\lambda_1^k \to \frac{1}{n}$. We observe that (2) yields the linear difference equation $\lambda_1^k = \frac{\lambda_1^{k-1} + \lambda_1^{k-(n-1)}}{2}$ which in turn implies the convergence of $\{\lambda_1^k\}$. (The characteristic polynomial, $x^{n-1} - \frac{1}{2}(x^{n-2} + 1)$, has +1 as its only positive root, while the remaining roots have moduli less than 1). The inclusion of λ^k in H and (2) yield that $\{\lambda_i^k\}$, $i=1,\ldots,n$ tend to $\frac{1}{n}$.

We note in passing that Lemma 8 extends a result due to Keilson [4], who assumes the complete symmetry and continutity of the function f.

We are now ready to introduce our extension to the (completely) symmetric case.

<u>Theorem 9</u>: Let $G(x) = G(x_1,\ldots,x_n)$ be the joint probability distribution of the random variables $x_1,\ldots,x_n$ and let $v(t_1,\ldots,t_n)$ be a continuous real function. Suppose that both G and v are symmetric and that v is concave in its first two arguments t_1 and t_2.

If $\theta(\lambda) = \int v(\lambda_1 x_1,\ldots,\lambda_n x_n)\, dG(x)$ exists for all $\lambda \in H = \{\lambda \mid \Sigma \lambda_j = 1\}$ and is upper semicontinuous on H, then the symmetric point maximizes $\theta(\lambda)$ over H, i.e.,

$$\theta(\tfrac{1}{n},\ldots,\tfrac{1}{n}) \geq \theta(\lambda), \quad \lambda \in H.$$

Proof[5]: Using the remark following Lemma 3 it follows that $\Theta(\lambda_1, \ldots, \lambda_n)$ is concave in (λ_1, λ_2). The symmetry of v and G induce the same property on Θ, (Lemma 2). Thus, $\Theta(\lambda)$ satisfies the conditions of Lemma 8 and the proof is complete.

Finally, a general comment is in order. The discussion in this paper, focusing on cyclic symmetry, can be applied to other economic situations that present symmetric solutions. We mention two such applications. The first is Samuelson's result [7], that shows that equal distribution of income among identical Benthamites will maximize the sum of social utility. A second application is the well known Modigliani-Miller theorem [5]. One of the statements of this theorem is that in the absence of default risk, the values of all firms in the same risk class are equal. Baron [1], has remarked that the risk class assumption may be weakened to the case of a symmetric joint distribution of return as considered in [6]. Thus our generalization to the cyclically symmetric case applies also.

Footnotes

1. It is assumed throughout this work that subsets of R^n are Borel sets and functions are Baire functions. The integrals are Lebesgue-Stieltjes integrals. (See [2]).

2. Since P_n generates the subgroup of cyclic permutations this definition is equivalent to $x \in A \Rightarrow P_n x \in A$.

3. We note in passing that if U is not bounded from below, then finiteness of means, variances and covariances does not imply the boundedness from below of $\psi(\lambda)$, i.e., $\psi(\lambda)$ which is bounded from above (finiteness of means), can take on the value $-\infty$. This is illustrated by the following one dimensional example.

$$\text{Set } U(x) = 1 - e^{-x} \text{ and } G(x) = \begin{cases} \dfrac{1}{2x^3} & x < -1 \\ \dfrac{1}{2} & -1 \leq x \leq 1 \\ 1 - \dfrac{1}{2x^3} & x \geq 1 \end{cases}$$

4. The concavity implies that $E(U^+) < \infty$, provided the finite means to G(x). Hence U is quasi-integrable w.r.t. G(x), i.e $\psi(\lambda)$ exists on $[-\infty, \infty]$ for all $\lambda \in D$.

5. The continuity property of $\Theta(\lambda)$ is ensured if for example we assume that $\Theta(\lambda)$ is uniformally convergent.

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