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PRE-LEONTIEF FUNCTIONS AND LEAST ELEMENTS

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Introduction

In a previous paper [6] we introduced the Z-functions and showed that a nonempty feasible region defined by a continuous Z-function always contains a least element which is also a complementary solution. An iterative procedure to find the least element was also given in [6]. In this work we generalize the Z-property to introduce the pre-Leontief functions. It is then shown that these functions define feasible regions containing least elements.

Following the generalization of the linear complementarity problem presented in [2], we demonstrate a complementarity property associated with the least elements.

A modification of the algorithm presented in [6] for Z-functions, is then shown to be applicable for finding least elements of regions corresponding to pre-Leontief functions.

Definition: Let $f : R^n_+ \to R^m$ be a function from $R^n_+ = \{x \mid x \in R^n,\ x \geq 0\}$ to R^m, whose components are $f_1,\dots,f_m$. f is a pre-Leontief function on R^n_+ if for each i, $i=1,\dots,m$, there exists an integer $k(i)$, $1 \leq k(i) \leq n$, such that for all $x \in R^n_+$ and $t \geq 0$

$$f_i(x) \geq f_i(x + te_j) \ \forall\ j \neq k(i),$$

where e_j is the j^{th} unit vector in R^n.

If $m = n$ and $k(i) = i$ f is said to be a Z-function.

Note that the pre-Leontief property ensures that each component of f is nonincreasing with respect to at least $(n-1)$ of its arguments.

If f is linear and characterized by an $m \times n$ matrix then it is pre-Leontief if each of its m rows contains at most one positive element. If the matrix is square and all its off-diagonal elements are nonpositive f is a Z-function.

As indicated in [6], the Z-functions constitute a natural extension of the simple linear Leontief Interindustry Model, corresponding to n products and n industries, each with one type of output. Following the exposition of [6], we immediately observe that the pre-Leontief function can be used to describe the case where several industries may produce the same type of product.

In [6] we provided a constructive proof of the following theorem, dealing with the existence of a least element.

Theorem 1: Let $f : R^n_+ \to R^n$ be a continuous Z-function and let q be in R^n. If $X^+_q = \{x \mid f(x) + q \geq 0, x \geq 0\}$ is nonempty then it contains a least element $\bar{x}$ (i.e. $\bar{x} \in X^+_q$ and $\bar{x} \leq y$ for all $y \in X^+_q$), and $\bar{x}$ satisfies $\bar{x}'(f(\bar{x})+q) = 0$.

The proof given in [6] is based on a modification of the well known Gauss-Seidel and Jacobi iterative procedures.

Note that the main significance of the least element is that it (simultaneously) minimizes any real isotone objective function defined on X^+_q.

We now generalize Theorem 1 to pre-Leontief functions.

Existence of Least Elements

Theorem 2: Let $f : R^n_+ \to R^m$ be a continuous pre-Leontief function. Given $a \geq 0$ in R^n_+ and $q \in R^m$, if $X^+_{q,a} = \{x \mid f(x)+q \geq 0, x \geq a\}$ is nonempty it contains a least element.

Proof:

If $a \in X^+_{q,a}$ it is clearly the least element. Hence, assume that a is not contained in $X^+_{q,a}$. The continuity of f implies that $X^+_{q,a}$ is closed. Let x be the element of $X^+_{q,a}$ which is closest to a, with respect to the Euclidean norm. We show that x is the least element. Suppose that $y \in X^+_{q,a}$ and $y_j < x_j$ for some j, $1 \leq j \leq n$. Define z in R^n_+ by $z_i = \min(x_i, y_i)$ for i, i=1,...,n. To complete the proof it is then sufficient to show that $z \in X^+_{q,a}$.

We first note that $z \geq a$. Let i, i=1,...,m, and consider f_i. Using the pre-Leontief property and supposing that $z_{k(i)} = x_{k(i)}$ we get $f_i(z) + q \geq f_i(x) + q \geq 0$. (If we had assumed that $z_{k(i)} = y_{k(i)}$ we would have obtained $f_i(z) + q \geq f_i(y) + q \geq 0$). Hence $f(z) + q \geq 0$ and $z \in X^+_{q,a}$.

We note that the preceding theorem generalizes a result due to Cottle and Veinott [3], who dealt with pre-Leontief matrices, i.e. matrices for which each row contains at most one positive element.

In fact, when f(x) is linear the existence of a least element for $X^+_{q,a}$ for all $a \in R^n_+$ and $q \in R^m$, provided $X^+_{q,a}$ is not empty, implies that the matrix defining f is pre-Leontief. The following example illustrates that this is not always true when nonlinear functions are considered.

Example 1: Let $f : R^2_+ \rightarrow R^1$ be defined by

$$f(x_1,x_2) = \begin{cases} \dfrac{x_1+1}{x_2+1} & x_1 \leq x_2 \\ \dfrac{x_2+1}{x_1+1} & x_1 \geq x_2 \end{cases}$$

It is easily verified that for each $a \in R_+^2$ and scaler q $X_{q,a}^+$ contains a least element, provided it is not empty, but f is clearly not pre-Leontief.

Although the **pre-Leontief** property is not satisfied globally the following result can be interpreted as a local pre-Leontief property.

Theorem 3: Let $f : R_+^n \to R^m$ be such that for each $a \in R_+^n$ and $q \in R^n$, $X_{q,a}^+ \neq \emptyset$ implies the existence of a least element in $X_{q,a}^+$. Then for each x $x \in R_+^n$, and i, i=1,...,m there exists $k(i,x), 1 \leq k(i,x) \leq n$ such that $f_i(x+te_j) \leq f_i(x)$ for $0 \leq t$ and $j \neq k(i,x)$.

Proof:

Suppose on the contrary that for some $x \in R_+^n$ and i, i=1,...,m, there exist r and p $1 \leq r, p \leq n$ $r \neq p$ such that $f_i(x+s_o e_r) > f_i(x)$, and $f_i(x+s_1 e_p) > f_i(x)$ where $s_o > 0$, $s_1 > 0$. For any j, j=1,...,m, $j \neq i$ let $m_j = \min \{f_j(x), f_j(x+s_1 e_p), f_j(x+s_o e_r)\}$ and $m_i = \min \{f_i(x+s_o e_r), f_i(x+s_1 e_p)\}$. Consider $X_{q,x}^+$, where $q = -(m_1, m_2, \ldots, m_m)^1$. $x + s_o e_r$ and $x + s_1 e_p$ belong to $X_{q,x}^+$ which in turn implies that x is the least elemen of $X_{q,x}^+$ - a contradiction to the definition of m_i.

As shown by the next theorem, the result of Theorem 3 can be strengthened to achieve the (global) pre-Leontief property, if seperability is assumed.

Theorem 4: Let $f : R_+^n \to R^m$ be given by $f(x_1, \ldots, x_n) = f^1(x_1) + f^2(x_2) + \ldots + f^n(x_n)$ and suppose that for each $a \in R_+^n$ and $q \in R^m$, $X_{q,a}^+ \neq \emptyset$ implie the existence of a least element in $X_{q,a}^+$. Then f(x) is pre-Leontief.

Proof:

We show that for each i, i=1,...,m, there exists k(i), $1 \leq k(i) \leq n$, such that for any $j \neq k(i)$ j=1,...,n the scaler function $f_i^j(x_j)$ is nonincreasing.

Suppose on the contrary, that there exist r and p $1 \leq r, p \leq n$ $r \neq p$ such that $f_i^r(x_r)$ and $f_i^p(x_p)$ are not nonincreasing. Hence, there exist $\bar{x}_r \geq 0$, $\bar{x}_p \geq 0$, $s_o > 0$ and $s_1 > 0$ satisfying $f_i^p(\bar{x}_p+s_1) > f_i^p(\bar{x}_p)$ and $f_i^r(\bar{x}_r+s_o) > f_i^r(\bar{x}_r)$.

For any u, u=1,...,m, let $q_u = -\min[f_u((\bar{x}_p+s_1)e_p+\bar{x}_r e_r);$ $f_u(\bar{x}_p e_p + (\bar{x}_r+s_o)e_n)]$.

Then it is easily seen that $(\bar{x}_p+s_1)e_p + \bar{x}_r e_r$ as well as $\bar{x}_p e_p + (\bar{x}_r+s_o)e_r$ belong to $X^+_{q,a}$ where $q = (q_1,...,q_m)$ is defined above and $a = \bar{x}_p e_p + \bar{x}_r e_r$. This in turn implies that a is the least element of $X^+_{q,a}$ - a contradiction to the definition of q_i.

As shown in [3], if f is affine the (global) pre-Leontief property is equivalent to the following condition.

For each $q \in R^m$, $X^+_{q,o} \neq \emptyset$ implies the existence of a least element in $X^+_{q,o}$.

The next example shows that the latter condition, (i.e. when the existence of least elements for $X^+_{q,a}$ for all q and a is replaced by the existence of least elements for $X^+_{q,o}$ for all q), is not even sufficient to yield the (local) pre-Leontief property stated in Theorem 3.

Example 2: Let $f : R^2_+ \to R^1$ be defined by $f(x_1,x_2) = h(x_1) + g(x_2)$ where

$$h(x_1) = \begin{cases} x_1 & 0 \le x_1 \le 1 \\ 2-x_1 & 1 \le x_1 \le 2 \\ 0 & x_1 \ge 2 \end{cases} \quad \text{and} \quad g(x_2) = \begin{cases} 1 - x_2 & 0 \le x_2 \le 1 \\ x_2 - 1 & 1 \le x_2 \le 2 \\ 1 & x_2 \ge 2 \end{cases}$$

When some differentiability and regularity assumptions are imposed on the function f, one can derive additional necessary conditions for the existence of a least element. Specifically we will assume an arbitrary qualification for constrained optimization (see [4]).

Theorem 5: Let $f : R^n_+ \to R^m$ be a continuously differentiable function and let $\bar{x}$ be a least element of $X^+_{q,a}$. Denote $I = \{i \mid f_i(\bar{x})+q_i = 0\}$ and $J = \{i \mid \bar{x}_i = a_i\}$ and suppose that a constraint qualification is satisfied at $\bar{x}$.

Then $|I| + |J| \ge n$ and there exists an $n \times (|I| + |J|)$ nonnegative matrix A such that

$$A \cdot \begin{bmatrix} \nabla f_I(\bar{x}) \\ \hline \tilde{I}_J \end{bmatrix} = I_{n \times n} \qquad (1)$$

where $I_{n\times n}$ is the identity matrix of order n, $\nabla f_I(x)$ is the Jacobian of the functions $f_i(x)$, $i \in I$, and $\tilde{I}_J$ is the $|J| \times n$ Jacobian matrix of the functions $g_i(x) = x_i$, $i \in J$.

Proof:

It is assumed that some constraint qualification which is required by the Kuhn-Tucker conditions is satisfied. (See [4]). $\bar{x}$ being a least element of $X^+_{q,a}$ implies that for each (row) unit vector e_j in R^n

$$e_j\bar{x} \leq e_jx \qquad \forall\, x \in X^+_{q,a} \;.$$

Hence for any j, j=1,...,n, there exists (row) vectors of multipliers $u^j \geq 0$, $u^j \in R^{|I|}$ and $v^j \geq 0$, $v^j \in R^{|J|}$ such that

$$e_j - u^j \nabla f_I(\bar{x}) - v^j = 0 \quad , \quad j=1,2,\ldots,n.$$

Let A be the $n \times (|I| + |J|)$ matrix having (u^j, v^j) as its j^{th} row, then

$$A \cdot \begin{bmatrix} \nabla f_I(\bar{x}) \\ \hline \tilde{I}_J \end{bmatrix} = I_{n \times n} \;.$$

In particular we obtain that $|I| + |J| \geq n$, and the proof is complete.

A similar version of the above theorem was also proved by Bod [1].

Following Mangasarian [5], we note that (1) is equivalent to the inverse isotonicity of $\begin{bmatrix} \nabla f_I(\bar{x}) \\ \hline \tilde{I}_J \end{bmatrix}$. (A matrix B is inverse isotone if for all x $Bx \geq 0$ implies $x \geq 0$.)

As a consequence of the theorem it follows that if

$$B = \left(\frac{\partial f_i(\bar{x})}{\partial x_j} \right)_{i \in I, j \notin J}$$

, then there exists a nonnegative matrix A of order $(n-|J|) \times |I|$ such that $AB = I_{(n-|J|) \times (n-|J|)}$. The latter implies that the partial Jacobian of the binding functions, f_i, $i \in I$, with respect to the nonbinding variables, x_j, $j \notin J$ is inverse isotone.

Finally we observe that the regularity assumptions in Theorem 3 cannot be omitted as illustrated by the function $f(x_1,x_2) = -(x_1-1)^2 - (x_2-1)^2$ and $X^+_{q,a}$ where $q = 0$ and $a = (0,0)$.

An Algorithm Finding the Least Element:

Focussing on a constructive approach to find a least element we next show that the algorithm suggested in [6] for Z-functions can be modified to be applied to pre-Leontief functions. (Note that by Theorem 2 the least element can be found by solving $\{\min \sum_{i=1}^{n} x_i \mid \text{s.t. } x \in X^+_{q,a}\}$.)

Let $f : R^n_+ \to R^m$ be a pre-Leontief function and $q \in R^m$. For any i, i=1,...,n, let

$$I(i) = \{j \mid f_j \text{ is not monotonically nonincreasing in } x_i\} \quad ($$

Further, if $I(i) \neq \emptyset$ we define

$$g_i(x) = \min_{j \in I(i)} \{f_j(x) + q_j\} \quad ($$

We observe that the pre-Leontief property ensures that $I(i)$, $i=1,\dots,n$ are well defined, i.e. $i \neq j \Rightarrow I(i) \cap I(j) = \emptyset$. We also verify that $g_i(x) : R^n_+ \to R^1$ is pre-Leontief for each i such that $I(i) \neq \emptyset$. In fact, $g_i(x)$ is monotonically nonincreasing in x_j for all $j \neq i$. We define the following sets of indices

$$J = \{j \mid j \notin \cup I(i)\} \text{ and } I = \{i \mid I(i) \neq \emptyset\} \qquad (4)$$

We will further assume without loss of generality that

$$I = \{i(1), i(2), \dots, i(t)\} \text{ where } i(k) < i(k+1),\ 1 \leq k < t \qquad (5)$$

Having associated the above notation with a pre-Leontief function f, we prove the following result, which will be found useful in the application of the algorithm of [6].

Lemma 6: Let $f : R^n_+ \to R^m$ be pre-Leontief, and let $a \in R^n_+$ and $q \in R^m$. If $\bar{x}$ is a least element of $X^+_{q,a}$ then $\bar{x}_i = a_i$ for each $i \notin I$, where I is given by (4).

Proof:

$i \notin I$ implies that for each j, $j=1,\dots,m$, $f_j(x)$ is monotonically nonincreasing in x_i, i.e. for any $x \in R^n_+$ and $t \geq 0$

$f_j(x) \leq f_j(x + te_i)$. If $\bar{x}$ is the least element in $X^+_{q,a}$ then

$$0 \leq q_j + f_j(\bar{x}) \leq q_j + f_j(\bar{x} - (\bar{x}_i - a_i)e_i), \text{ for all } j,\ j=1,\dots,n,$$

which in turn proves that $\bar{x}_i = a_i$.

As a consequence of the above lemma we can set $x_i = a_i$ for all $i \notin I$, as a start in our effort to find the least element of $X^+_{q,a}$. Our next step is to introduce a Z-function which will be shown to be equivalent to $f(x)$ in the sense of finding least elements.

Let I be given by (2)-(4), and $g_i(x)$, $i \in I$ be defined by (3). For each $k=1,\ldots,t$ define

$$h_k(y_1,\ldots,y_t) = g_{i(k)} \left(\sum_{k=1}^{t} y_k e_{i(k)} + \sum_{i \notin I} a_i e_i \right) \qquad (6)$$

Note that $h = (h_1,\ldots,h_t)$ is a Z-function mapping R^t_+ to R^t, where $t \leq \min(m,n)$. We next show that for our purposes it is sufficient to concentrate on the Z-function h.

<u>Theorem 7</u>: Let $f : R^n_+ \to R^m$ be pre-Leontief. Given $a \in R^n_+$ and $q \in R^m$, define $h : R^t_+ \to R^t$ by (6), and I,J by (4).

Let

$$\tilde{X}^+_{q,a} = \{x \mid f_j(x) + q_j \geq 0,\ j \notin J \text{ and } x \geq a\} \text{ and}$$

$$Y^+_{q,a} = \{y \mid h_k(y) \geq 0,\ y_k \geq a_{i(k)},\ k=1,\ldots,t\}.$$

Then

(1) $Y^+_{q,a} = \emptyset$ if and only if $\tilde{X}^+_{q,a} = \emptyset$.

(2) If $\bar{y}$ is a least element in $Y^+_{q,a}$ then $\bar{x} = \sum_{k=1}^{t} \bar{y}_k e_{i(k)} + \sum_{i \notin I} a_i e_i$ is a least element in $\tilde{X}^+_{q,a}$. Further, if also $f_j(\bar{x}) + q_j \geq 0$ for all

$j \in J$ then $\bar{x}$ is also a least element of $X^+_{q,a}$;

otherwise $X^+_{q,a} = \emptyset$.

Proof:

(1) follows directly from the definition of h. If $y \in Y^+_{q,a}$ then it is easily verified that $x = \sum_{k=1}^{t} y_k e_{i(k)} + \sum_{i \notin I} a_i e_i \in \tilde{X}_{q,a}$. Conversely, if $x \in \tilde{X}_{q,a}$, observe that $f_j(x), j=1,\ldots,m$ is monotonically nonincreasing in x_i for all $i \notin I$. In particular for $j \notin J$

$$f_j(x) + q_j \geq 0 \text{ and } x \geq a \Rightarrow f_j\left(\sum_{k=1}^{t} x_{i(k)} e_{i(k)} + \sum_{i \notin I} a_i e_i\right) + q_j \geq 0.$$

The latter implication then yields

$$g_i\left(\sum_{k=1}^{t} x_{i(k)} e_{i(k)} + \sum_{i \notin I} a_i e_i\right) \geq 0, \quad \text{for all } i \in I.$$

Hence, $(x_{i(1)},\ldots,x_{i(t)}) \in Y^+_{q,a}$.

To prove (2), suppose that $\bar{y}$ is a least element in $Y^+_{q,a}$. Clearly, $x = \sum_{k=1}^{t} \bar{y}_k e_{i(k)} + \sum_{i \notin I} a_i e_i \in \tilde{X}^+_{q,a}$. If $\bar{x}$ was not the least element, there would exist $x^o \in \tilde{X}_{q,a}$ such that $x^o_{i(k)} < \bar{y}_k$ for some $1 \leq k \leq t$. As demonstrated above while proving (1), $(x^o_{i(1)},\ldots,x^o_{i(t)}) \in Y^+_{q,a}$, which in turn contradicts the fact that $\bar{y}$ is the least element in $Y^+_{q,a}$.

To prove the second part of (2), it will be shown that if $X^+_{q,a}$ is nonempty, then $\bar{x}$ is the least element of $X^+_{q,a}$. Suppose that x^1 is in $X^+_{q,a}$. In particular $x^1 \in \tilde{X}^+_{q,a}$ and $x^1 \geq \bar{x}$. The proof will be complete when we show that $f_j(\bar{x}) + q_j \geq 0$ for all $j \in J$. But the latter is implied by the monotonicity of f_j, $j \in J$ in all of its n arguments.

We are now ready to apply the algorithm of [6] to find the least element of $X^{+}_{q,a} = \{x \mid f(x) + q \geq 0,\ x \geq a\}$, provided $X^{+}_{q,a} \neq \emptyset$, when $f : R^n_+ \to R^m$ is a continuous pre-Leontief function.

Given a continuous Z-function $h : R^P_+ \to R^P$, and $q \in R^P$ the algorithm presented in [6] finds a least element of $X^{+}_{q,o} = \{x \mid h(x) + q \geq 0,\ x \geq 0\}$ or indicates that $X^{+}_{q,a}$ is empty.

To find the least element of $\{x \mid h(x) + q \geq 0,\ x \geq a\}$ when $a \in R^P_+$ one has to find the least element of $\{y \mid \tilde{h}(y) + q \geq 0,\ y \geq 0\}$, where $\tilde{h}(y) = h(y+a)$ and add the vector a to this least element.

Let $f : R^n_+ \to R^m$ be a continuous pre-Leontief function, $a \in R^n_+$ and $q \in R^m$. Using Theorem 7, one can apply the following procedure to verify the existence of a least element to $X^{+}_{q,a}$ and to find the element provided it exists.

<u>Step 1</u>: Define I, J and $h : R^t_+ \to R^t$ by (4) and (6) respectively.

<u>Step 2:</u> Apply the algorithm of [6] to find the least element of the set $\{y \mid h(y) \geq 0,\ y_k \geq a_{i(k)},\ k=1,\dots,t\}$. If the set is empty $X^{+}_{q,a}$ is empty: terminate. Otherwise apply Step 3, where $\bar{y}$ is the least element.

<u>Step 3</u>: Define $\bar{x} = \sum_{k=1}^{t} \bar{y}_k e_{i(k)} + \sum_{i \notin I} a_i e_i$. If $f_j(\bar{x}) + q_j \geq 0$ for all $j \in J$, $\bar{x}$ is the least element of $X^{+}_{q,a}$. Otherwise, $X^{+}_{q,a}$ is empty.

Finally we demonstrate a complementarity property associated with pre-Leontief functions. Motivated by the generalization of the linear complementarity problem due to Cottle and Dantzig [2], we refer to the generalized nonlinear complementarity problem defined as follows.

Let $F : R_+^n \to R^m$, have components $F_1,\dots,F_m$, and suppose that these components are partitioned into n sets S_j, $j=1,\dots,n$. The generalized complementarity problem is to find $x \in R^n$ such that

$$x \geq 0,\ F(x) \geq 0 \text{ and } x_j \cdot \prod_{r \in S_j} F_r(x) = 0,\ j=1,\dots,n. \tag{7}$$

Cottle and Dantzig [2] treat the linear case i.e. when f is affine and provide conditions guaranteeing the existence of a complementary solution.

It is shown in [6] that if h is a continuous Z-function and y is a least element of $\{y|h(y) + q \geq 0,\ y \geq 0\}$, then $y'(h(y)+q)=0$. Consequently we can conclude that if f is a continuous pre-Leontief function and x is a least element of $X_{q,a}^+$, then for each $i \in I$, there exists $j = j(i) \in I(i)$ such that $x_i > a_i$ implies $f_j(x) + q_j = 0$. The latter observation can be interpreted in terms of the generalized complementarity property presented above. If $f : R_+^n \to R^m$ is pre-Leontief and continuous define $S_i = I(i)$, $i=1,\dots,n$ where $I(i)$ is given by (2). Assuming for simplicity that $a = 0$ we conclude that for any $q \in R^m$, $X_{q,o}^+ \neq \emptyset$ implies the existence of a least element which is also a complementary solution to the generalized complementarity problem (7) defined by $F(x) = f(x) + q$.

References

[1] P. Bod, "On Closed Convex Sets Having a Least Element," Technical Report, 1974.

[2] R. W. Cottle and G. B. Dantzig, "A generalization of the linear complementarity problem," Journal of Combinatorial Theory, 8, 1, (1970)

[3] R. W. Cottle and A. F. Veinott, Jr., "Polyhedral sets having a least element," Mathematical Programming, 3 (1972) 238-249.

[4] F. J. Gould and J. W. Tolle, "A Necessary and Sufficient Qualification for Constrained Optimization," SIAM J. Appl. Math. 20 (1971)

[5] O. Mangasarian, "Characterization of real matrices of monotone kind," SIAM Review, 10 (1968) 439-441

[6] A. Tamir, "Minimality and Complementarity properties associated with Z-functions and M-functions," Mathematical Programming, 7 (1974).