On the Rise and Fall of Class Societies

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Abstract

This paper develops a theoretical framework to investigate potential forces behind the rise and fall of class societies. Due to the nonconvexity of investment and credit market imperfections, only those who inherited relatively large wealth can set up firms and become employers. The equilibrium dynamics is described by the joint evolution of the wage rate, the vertical division of labor between employers and workers, and the distribution of household wealth.

For some parameter values, the model predicts the rise of class societies, where the households are permanently separated into the two classes in any steady state. The rich bourgeoisie maintain a high level of wealth due to the presence of the poor proletariat, which has no choice but to work at a wage rate strictly lower than the "fair" value of labor. For other parameter values, the model predicts the fall of class societies, where job creation by the rich employers pushes up the wage rate so much that the workers will escape from the poverty and eventually catch up with the rich. Thus, the wealth created by the rich trickles down to the poor, and, in the steady state, the inequality disappears.

The effects of self-employment are also discussed using this framework.

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1. Introduction

The division of labor between the employers and workers is one of the most contentious issues in social sciences. Some believe that the employers get rich because of cheap labor provided by the workers, who have no choice but to work for them. According to this view, the relation between the two is antagonistic. Many socialists argue that, in a capitalist society, this vertical division of labor will develop into the class structure. One class, the bourgeoisie, owns and controls the means of production and exploits the other, the proletariat, characterized by their lack of property and dependence on the sale of their labor power to the dominant class. Marx and his followers even predicted that the class struggle is an inevitable feature of capitalism and argued that the only way of realizing a classless society is the appropriation by society as a whole of the means of production. At the other end of the political spectrum, it is believed that the employers create jobs, which offer the only hope for the workers to escape the misery of poverty. According to this view, the relation between the two is mutually beneficial. Some conservatives argue that, if the market forces are allowed to operate fully, wealth generated by the rich will eventually trickle down to the poor, which will eliminate the class distinction, leading to general prosperity. Throughout the 19th century and early 20th century, the Marxist view had received wide political support among industrial workers. It seems fair to say that, by the late 20th century, the Marxist view, at least in its original form, has lost much of its intellectual appeal, as the class distinction has become less pronounced in most advanced economies. Nevertheless, there exist significant disagreements as to whether the emergence of predominantly middle class societies has been achieved by the market forces, as many conservatives argue, or by the safety net provided by social programs in the form of welfare capitalism, as many socialists argue.

Perhaps surprisingly to many social scientists outside of economics, very little formal work has been done within economics to address the issues raised above. Of course, the division of labor is a central problem of labor economics, but most formal models are concerned with the horizontal division of labor, such as occupational choices and job assignments. Some radical economists, such as Marglin (1984), have developed political economy models of classes, but they take the class structure exogenously given. In short, there have been little attempts to model an endogenous formation and/or dissolution of class societies in a capitalist economy.

This paper develops a formal framework for investigating potential forces behind the rise and fall of class societies in a systematic manner. In the model economy, there is a stationary population of inherently identical households, each of which consists of an infinite sequence of agents connected via intergenerational transfers. At any point in time, the inherited wealth is the only possible source of heterogeneity across households. The three critical assumptions are; i) setting up a firm requires a minimum level of investment, which introduces nonconvexity; ii) because of the possibility of default (and imperfect sanctions against it), any household can borrow only a limited amount to finance their investment; iii) firms need to hire labor, which means the investment is more profitable when the wage is lower. These assumptions jointly imply that the agents who inherited *relatively* large wealth become employers and those who inherited *relatively* little become workers.[2] The threshold level of inherited wealth, which divides the poor workers from the rich employers, depends on the equilibrium wage and adjusts endogenously to keep the balance between the supply of and demand for labor. The vertical division of labor excludes the relatively poor from being employers and earning as much as the relatively rich.

In this framework, the distribution of wealth in one generation affects the supply of and demand for labor, which in turn affects the wage rate and profit, and hence the distribution of wealth in the next generation. The equilibrium is thus described by the joint evolution of the wage rate, of the vertical division of labor between employers and workers, and of the distribution of household wealth. The model is simple enough to allow for a characterization of the steady states for the full set of parameter values.

Under some configurations of the parameter values, the model predicts the rise of class societies. That is to say, the household's wealth is concentrated in two points in all the steady states. In other words, the population is permanently polarized into the rich bourgeoisie and the poor proletariat. In these steady states, the proletariat possess little wealth, thereby being excluded from becoming employers. They have no choice but to work for the rich bourgeoisie at a wage rate strictly lower than the "fair" value of labor, which further contributes to their pauperization. The rich bourgeoisie maintain a high level of wealth, not only because they can finance their profitable investment, but also because they have access to the cheap labor supplied by the proletariat. Furthermore, there is a lower bound to the fraction of the households that belong to each class, and it is independent of the initial distribution of wealth. In other words, even if there was perfect equality at the beginning, there is inequality of wealth across the households in the steady state. The model thus explains how the market interactions lead to an endogenous formation of the class structure. It offers some theoretical justifications for the left-wing view that the rich employers owe their high level of wealth to a reserve army of the working class and that the class conflict is an inevitable feature of capitalism.

This is not, however, the only possible long run outcome of the model. Under different configurations of the parameter values, the model predicts the fall of class societies. That is to say, there is the unique steady state, in which all the household's wealth converges to the same level, which is high enough to allow anyone to be an employer. In this steady state, workers are paid the "fair" value of labor to make them willing to work for others; otherwise, they would prefer being employers. This outcome occurs when the demand for labor by the rich employers pushes up the wage rate so much that the workers, benefiting from a higher wage rate, eventually catch up with the rich. In other words, the job creation by the employers helps the workers escape from the poverty, thereby eliminating inequality in the long run. This case thus provides some theoretical justifications for the trickle-down economics preached by the right-wing conservatives, i.e., accumulation of the wealth by the rich is beneficial for the society as a whole, including the poor.

Demonstrating the possibility of these two alternative scenarios is important enough, providing justifications for the two opposing views of the world. More importantly, the model is simple enough that it is possible to derive the conditions for these two cases in terms of a few key parameters. The framework can thus be used as an organization principle or intuition-building device, on these highly contentious issues.

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[2]See Evans and Jovanovic (1989) for the evidence that the borrowing constraint is the major barrier for the agents with low net worth from being entrepreneurs. Holtz-Eakin, Joulfaian, and Rosen (1994) also showed that large inheritances improve the

As an application, the framework will later be extended to introduce self-employment. Self-employment provides not only the poor with an alternative to working for the rich, but also the rich with an alternative to the investment that creates jobs, which could benefit the poor. Due to this dual nature, the effects of self-employment are quite subtle and more complicated than one might imagine. Nevertheless, within the present framework, it is possible to provide a complete characterization of the steady states with self-employment.

The present study may be viewed as an addition to the literature on long run distribution of household wealth. In an important contribution, Banerjee and Newman (1993) developed a model of institutional transformation, driven by the interplay of the division of labor and the wealth distribution. Their model also features the nonconvexity of investment and the credit market imperfection, which makes the investment be wealth-constrained. One crucial difference is that, unlike in the present study, the threshold levels of wealth needed for the investment are free parameters.[3] This feature of their model leads to a plethora of steady states. In particular, they found a set of parameter values that generates the co-existence of steady states with different degrees of wealth inequality. Their main message is history dependence; the distribution of wealth and the dominant institutional form that will prevail in the long run depend on the initial distribution of wealth. If it is unequal, the economy develops a large-scale factory system, whose survival depend on the presence of a large reserve army of poor workers.[4] If it is equal, the economy becomes a nation of self-employed workers. On the other hand, the main message of the present study is *not* history dependence. What will be shown below is that, for some parameter values, the steady state distribution of wealth always displays inequality (even though

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survival probability of family-run businesses.

[3]There are other important differences. In the Banerjee-Newman model, each investment is subject to an idiosyncratic shock, which introduces social mobility even in the steady state. (Here, there is no exogenous shock, and there is no social mobility in the steady state.) Furthermore, there are two types of investment, self-employment and large-scale factory operation. (Here, there is only one, although an extension in section 4 allows for two.) While these features of their model have advantage of enabling them to tell a rich story of institutional transformation, the resulting model is so complicated that they had to restrict their analysis to a few sets of parameter values. This makes it impossible to see how the prediction of their model may change with the parameters. One major advantage of the present framework is that it is simple enough that a characterization of the steady states can be done for the full set of parameter values, and that the exact conditions for different outcomes can be derived explicitly.

[4]More precisely, what is needed for this outcome is not the inequality of the initial distribution *per se*, but the initial co-existence of the households that are so poor to be self-employed and of the households that are rich enough to invest a large-factory system.

the initial distribution may be equal) and, for other parameter values, the steady state distribution displays equality (even though the initial distribution may be unequal). Freeman (1996) is most closely related in spirit to the present study. He showed how the nonconvexity of human capital investment and the borrowing constraint lead to a permanent separation of the homogenous population into the managerial and working classes. The present study differs from his in two respects. First, his analysis is limited to the two extreme cases of the credit market; that is, the credit market is either perfect or entirely absent. Here, the borrowing limit is determined endogenously from the possibility of default. By changing the effectiveness of the penalty in the event of default, the model can be used to analyze the entire range of intermediate cases between these two extremes. Second, his model is designed to predict the formation of the class society whenever the credit market is absent. The present model is rich enough to allow for both the formation and dissolution of class societies under the credit market imperfection, and yet simple enough to allow for the characterization of the exact conditions for both. This in turn allows us to examine the effects of self-employment. Needless to say, the above comments should not be viewed as a criticism of these studies. The present study benefits immensely from theirs.[5]

The rest of the paper is organized as follows. Section 2 presents the basic model, and derives the conditions for the labor market equilibrium and for the joint evolution of the wage rate, of the division of labor and of the distribution of wealth. Section 3 offers a complete characterization of the steady states and identify the conditions for the rise and fall of class societies. The next two sections

[5]Other studies of the theory of long run distribution of household wealth include Aghion and Bolton (1997), Piketty (1997), Matsuyama (2000) and Mookerjee and Ray (2000). Mention should be made of the relation between Matsuyama (2000) and the present study. Both demonstrate the possibility of endogenous inequality (all the steady states are characterized by some inequality, even though the initial distribution may be equal) as well as the possibility of trickle-down phenomena (the steady state is characterized by equality, even though the initial distribution may be unequal.) In Matsuyama (2000), the agents interact through the supply of and demand for credit. The credit market equilibrium requires that the interest rate adjusts in such a way that the relatively poor become lenders while the relatively rich become borrowers. Here, the agents interact through the labor market, and the wage rate adjusts in such a way that the relatively poor become workers, while the relatively rich become employers. The present study not only demonstrates the robustness of the insights given in Matsuyama (2000). In addition, because of its focus on the labor market, the present framework enables us to examine the relation between the employers and workers, as well as the effects of self-employment, which was not feasible in Matsuyama (2000). It also obviates the need for imposing the credit market equilibrium, so that one can assume that the interest rate is exogenously fixed (say, the credit market may be integrated to the international capital market). It turns out that the exogeneity of the interest rate, as well as the additional structure imposed by the production, makes the characterization of the steady states much simpler here than in Matsuyama (2000). Nevertheless, an extension of the present model, presented in section 5, generates wealth dynamics somewhat similar to the model of Matsuyama (2000). Further discussion on the similarities and differences between the two will be provided in section 5.

discuss some extensions. Section 4 introduces self-employment. Section 5 allows the rich to set up multiple firms. Section 6 concludes.

2. The Model.

Time is discrete and extends to infinity. The economy produces a single numeraire good, which can either be consumed or invested. In any period the economy is populated by a unit mass of identical agents. Each agent is active for one period as a head of an infinitely-lived household (or dynasty). The only possible source of heterogeneity across households is their wealth. At the beginning of each period, the agents receive certain amounts of the numeraire good in the form of a bequest from the immediate predecessors (or parents). Let $G_t(w)$ denote the share of the households, whose agents inherited less than (but not equal to) w at the beginning of period t.

At the beginning of each period, the active agents choose their occupations as well as the allocation of their inherited wealth in order to maximize their end-of-the period wealth. (They make their consumption and inheritance decisions at the end of the period.) They have two options. First, the agents may become workers. Each worker supplies one unit of labor at the competitive wage rate, equal to v_t. When the agents become workers, they also lend their inherited wealth in the competitive credit market and earn the exogenously determined gross return equal to r per unit.[6] Thus, by becoming a worker, the agent who inherited w_t will have $v_t + rw_t$ at the end of period t.

Second, the agents may set up a firm and become employers. Setting up a firm requires F units of the numeraire good to be invested at the beginning of the period. This would enable an agent to employ n_t units of labor at the competitive wage rate, v_t, and produce $\phi(n_t)$ units of the numeraire good, which becomes available at the end of the period. The production function satisfies $\phi(n) > 0$, $\phi'(n) > 0$ and $\phi''(n) < 0$ for all $n > 0$, as well as $\phi(\infty) = \infty$ and $\phi'(\infty) = 0$. The equilibrium level of employment per firm can be expressed as a decreasing function of the wage rate, $n(v_t)$, which is defined implicitly by $\phi'(n(v_t)) \equiv v_t$ and satisfies $n(0) = \infty$. The gross profit from running a firm can be expressed as $\pi_t = \pi(v_t) \equiv \phi(n(v_t)) - v_t n(v_t) > 0$, which satisfies $\pi'(v_t) = -n(v_t) < 0$ and $\pi''(v_t) = -n'(v_t) > 0$, and $\pi(0) =$

[6]One may think that the agents can hold the financial claims issued by financial institutions that have access to the world interest rate.

$\phi(\infty) = \infty$. It is assumed that each employer can set up and manage at most one firm and that being an employer prevents the agent from earning the wage as a worker. These assumptions are, however, solely for simplicity, and will be dropped later in section 5.

When the inherited wealth falls short of the investment required (i.e., $w_t < F$), the agent needs to borrow by $b_t = F - w_t$ in the competitive credit market at the gross interest rate equal to r, in order to become an employer. If the inherited wealth exceeds the investment required, the agent can become an employer and lend by $w_t - F$ at the rate equal to r. In any case, the agent who inherited w_t will have $\pi(v_t) + r(w_t - F)$ at the end of period t by becoming an employer. This is greater than or equal to $v_t + rw_t$ (the end-of-the-period wealth if the agent becomes a worker), if and only if $\pi(v_t) - v_t \geq rF$, or

(1) $\quad v_t \leq V,$

where $V > 0$ as a unique solution to $\pi(V) - V = rF$, and may also be expressed as $V = V(rF)$, a decreasing function satisfying $V(\infty) = 0$. We shall call (1) the *profitability constraint*. If $v_t < V$, all agents prefer being employers to being workers. If $v_t = V$, they are indifferent. If $v_t > V$, then the wage is too high for the investment to be profitable; the agents are better off being workers instead of being employers. One may also call V the "fair" value of labor in the two senses of the word. First, no agent, given the choice, would be willing to work at a wage rate lower than V. Second, it is the wage rate which would equalize the net earnings of the employers and of the workers.

The credit market is competitive in the sense that both lenders and borrowers take the interest rate, r, given. It is not competitive, however, in the sense that one cannot borrow any amount at this rate. The borrowing limit exists because of the enforcement problem: the payment is made only when it is the borrower's interest to do so. More specifically, the employer, after borrowing b_t, would refuse to honor its payment obligation, rb_t, if it is greater than the cost of default, which is taken to be a fraction of the project output $\lambda\pi(v_t)$.[7] Knowing this, the lender would allow a would-be employer to borrow only up to $\lambda\pi(v_t)/r$. The parameter, $0 \leq \lambda < 1$, can be naturally taken to be the degree of the efficiency of the credit market. Note that there is no default in equilibrium. It is the possibility of default that makes

[7]A natural interpretation of the cost is that the creditor seizes a fraction λ of the gross profit in the event of default. One may also interpret that this fraction of the profit will be dissipated in the borrower's attempt to default. This makes no difference, because the default does not occur in equilibrium.

the credit market imperfect. It should also be noted that the same enforcement problem rules out the possibility that different agents may pool their wealth to overcome the borrowing constraint.[8]

Because of the borrowing constraint, the agent can set up a firm and become an employer only if

(2) $w_t \geq C(v_t) \equiv \mathrm{Max}\{0, F - \lambda\pi(v_t)/r\}$,

where $C(v_t)$ is the critical level of the household wealth needed for the agent to become employers, and, when positive, it is an increasing, concave function of the wage rate. One may also interpret $C(v_t)$ as the collateral requirement imposed by the creditors. Since $C(v) = 0$ when v is close to zero, the borrowing constraint is not binding when the wage rate is sufficiently low.[9] We shall call (2) the *borrowing constraint*.

The agents become employers if and only if both the profitability and borrowing constraints, (1) and (2), are satisfied. If one of these constraints fails, they become workers. We can now describe the labor market equilibrium. If $v_t > V$, (1) fails; it is not profitable to set up a firm. Hence, no agent would become an employer and every agent would become a worker; there would be an excess supply of labor. Thus, $v_t \leq V$ must hold in equilibrium. The agents who inherited less than $C(v_t)$ violates (2); they cannot finance their investment and have no choice but to become workers. The agents who inherited more than or equal to $C(v_t)$ can and want to become employers and hire $n(v_t)$ each, if $v_t < V$, and they are willing to do so, if $v_t = V$. Therefore, the labor market equilibrium condition is given by,

(3) $$\frac{G_t(C(v_t))}{1-G_t(C(v_t))} \leq n(v_t)\,; \qquad v_t \leq V,$$

where the first inequality can be strict either when G_t jumps at $C(v_t)$ or when $v_t = V$. Equation (3) is illustrated in Figure 1. The downward-sloping curve shows the labor demand per firm. The other curve can be interpreted as the labor supply per firm. Note that the supply curve is drawn flat at $v_t = V$, to

[8]Other studies that explore the macroeconomic implications of imperfect capital markets due to potential defaults include Banerjee and Newman (1993), Kiyotaki and Moore (1997) and Obstfeld and Rogoff (1996, Ch.6). The specification here follows Matsuyama (2000, 2001a, 2001b).

[9]This is due to the assumption, $\phi(\infty) = \infty$, which implies $\pi(0) = \infty$. Without this assumption, $\pi(0)$ would be finite and, for $rF > \lambda\pi(0)$, $C(0) > 0$, which implies that there exists a trivial steady state, where every household's wealth is zero.

capture the fact that all the agents are indifferent between being employers and being employees.[10] If $v_t < V$, all the agents prefer to be employers. As long as $G_t(C(v_t)) > 0$, however, the labor supply does not go to zero, because the borrowing constraint prevents some agents from being employers. In this range, this curve is generally upward-sloping, because a higher wage rate means a lower profit. This lowers the borrowing limit and the agents need to come up with more for the collateral. Therefore, more agents are unable to set a firm, and they have no choice but to work. The supply curve can be flat at $v_t < V$, as indicated by the dashed line. This occurs when a positive measure of the agents inherited $C(v_t)$, so that $G_t(\bullet)$ jumps at $C(v_t)$. If the labor demand curve intersects with a upward sloping part of the labor supply curve, as depicted in Figure 1, all the agents borrow up to the credit limit. If they intersect at a flat part of the labor supply curve with $v_t < V$, some agents must be credit-rationed, meaning that they cannot borrow up to the limit, even though they want to do so and they are equally qualified as others.[11] This introduces an element of chances in the dynamics of the household wealth. To deal with this situation, one needs to specify a rationing rule, which is inevitably ad-hoc. The following analysis and discussion ignores such a possibility of equilibrium credit rationing, because this situation never arises in the steady state. (Also, the steady states are independent of any rationing rule assumed.)

To summarize,

<u>Proposition 1.</u>

i) If $v_t < V$, all the rich agents, who inherited $C(v_t)$ or more, become employers. All the poor agents, who inherited less than $C(v_t)$, become workers. The poor workers earn v_t, which is lower than $\pi(v_t) - rF$, the net earning of the rich employers.

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[10]Figure 1 depicts the situation where $G_t(C(V)) < 1$. If $G_t(C(v_t)) = 1$ for some $v_t < V$, then the labor supply curve stays strictly below the line, $v_t = V$. If $G_t(C(v_t)) < 1$ for all $v_t < V$, and $G_t(C(V)) = 1$, then the labor supply curve is asymptotic to the line, $v_t = V$.

[11]While some authors use the term, "credit-rationing," whenever some credit limits exist, here it is used to describe the situation that the aggregate supply of credit falls short of the aggregate demand, so that some borrowers cannot borrow up to their credit limit. In other words, there is no credit rationing if every borrower can borrow up to its limit. In such a situation, their borrowing is constraint by their wealth, which affects the credit limit, but not because they are credit-rationed. This use of terminology is also consistent with the following definition by Freixas and Rochet (1997, Ch.5), who attributed it to Baltensperger: "some borrower's demand for credit is turned down, even if this borrower is willing to pay all the price and nonprice elements of the

ii) If $v_t = V$, some rich agents, whose inherited C(V) or more, become employers. All the poor agents, who inherited less than C(V), becomes workers. All the households receive the same level of the net earning.

To close the model, the bequest rule of the agents must be specified. To keep the matter simple, let us assume that the agent maximizes $u_t = (1-\beta)\log c_t + \beta\log w_{t+1}$, where c_t is the agent's consumption.[12] Then, each agent leaves β fraction of the end-of-the period wealth to the next generation (sibling). The wealth of each household thus changes according to the following dynamics:

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(4) \qquad w_{t+1} = \begin{cases} \beta(v_t + rw_t) & \text{if } w_t < C(v_t), \\ \beta(\pi(v_t) - rF + rw_t) & \text{if } w_t \geq C(v_t). \end{cases}
$$

We impose the restriction, $\beta < 1/r$, to ensure the existence of the steady states.

Figure 2 illustrates (4). The solid line graphs the map for the case where $v_t < V$, or equivalently, $\pi(v_t) - rF > v_t$. The map is linear and has a constant slope equal to $\beta r \in (0, 1)$, except that it jumps up at $C(v_t)$. Although all the agents want to be employers, the agents from the poor households, whose wealth falls short of $C(v_t)$, have no choice but to work for the agents from rich households. The arrows indicate the effects of a rise in the wage rate. A higher wage rate means that the terms-of-trade is more favorable for the poor worker and less favorable for the rich employer. Hence, with a high wage, the poor would have more wealth in the next period, while the rich would have less wealth in the next period (though it is still larger than the poor's.) A higher wage rate also makes the threshold higher, because the present value of the default penalty is smaller, and as a result, would-be employers need to contribute more in the form of a down payment. This suggests that a high wage rate is good for the very poor household, which cannot borrow to become an employer in any case. It is bad for the middle

[12]Banerjee and Newman (1993), Aghion and Bolton (1997), Piketty (1997), Matsuyama (2000) and many others in the literature adopted this specification, which assumes that the donor's utility depends on the amount given. The alternative specification, which assumes that the donor's utility depends on the utility of the beneficiary, not only has implications that have been rejected empirically (see, for example, Altonji, Hayashi, and Kotlikoff 1997); it would also lead to significant complications without much additional insights. The Cobb-Douglas preferences matter only to the extent that it makes the bequest a linear function of the end-of-the-period wealth, which simplifies the algebra. The homotheticity is not essential. Indeed, it is straightforward to allow for Stone-Geary preferences, so that the rich leave a large fraction of their wealth. Such an extension may be desirable to capture the point made by Kalecki, Kaldor and others, that aggregate wealth accumulation is done mostly by the capitalist.

class households, which could finance their investment at a lower wage rate (meaning the prospect of a higher profit), but not at a higher wage rate. Finally, the dashed line, $w_{t+1} = \beta(V+rw_t)$, depicts the dynamics when $v_t = V$. In this case, all the households earn V, regardless of whether they are workers or employers.

This completes the description of the model. Once a wealth distribution in period t is given, (3) determines the equilibrium wage rate, and the occupational choice of the agents. Then, from (4), one can calculate the wealth distribution in period t+1. By repeating this process, the model can be used to examine the joint evolution of the wage rate, the wealth distribution, and the division of the society between employers and workers.

3. The Steady State Analysis

Let us now look at the behavior of the economy in the long run. The steady state is associated with the limit distribution, $G_\infty(w)$, and the limit wage rate, v_∞. It is the state, which replicates itself over time, once the economy is settled in, and where all the households hold a constant level of wealth.

3A. The Classless Society: The Steady State with Wealth Equality

First, suppose that the wealth distribution is degenerate in a steady state. This is possible only when all the households earn the same net income, and hence $v_\infty = V$. From (4), this implies that the household's steady state wealth is given by the fixed point of the map, $w_{t+1} = \beta(V+rw_t)$, or

(5) $w_\infty = \beta V/(1-\beta r) \geq C(V)$.

As long as the inequality in (5) is satisfied, there exists a steady state, in which all the households maintain the same level of wealth and are rich enough to be able to become employers. Furthermore, all the households, whether they are workers or employers, earn the same net income, so that they are indifferent, and the labor market equilibrium condition, (3), is satisfied with $v_t = v_\infty = V$. (Note that $G_\infty(C(V)) = 0$, because it is the share of the household whose wealth is less than but not equal to C(V).) Therefore, (5) is the sufficient and necessary condition for the existence of a steady state, in which the wealth distribution is degenerate.

3B. The Class Society: The Steady States with Wealth Inequality

Consider now steady states with an unequal distribution of wealth. That is, some households belong to the entrepreneurial class or *bourgeoisie*; they are rich enough to become entrepreneurs and employers. The others belong to the working class or *proletariat*; they are poor and have no choice but to work. The existence of persistent inequality requires $v_\infty < V$. In such a steady state, the wealth of all the households in the bourgeoisie must converge to the fixed point of the map, $w_{t+1} = \beta(\pi(v_\infty) - rF + rw_t)$ or

(6) $w^B_\infty = B(v_\infty) \equiv \beta(\pi(v_\infty) - rF)/(1-\beta r) \geq C(v_\infty)$,

where the inequality in (6) is the condition that these households are indeed rich enough to be able to finance their investment. (B stands for "bourgeoisie".) Next, the wealth of all the households in the working class must converge to the fixed point of the map, $w_{t+1} = \beta(v_\infty + rw_t)$ or,

(7) $w^P_\infty = P(v_\infty) \equiv \beta v_\infty/(1-\beta r) < C(v_\infty)$,

where the inequality in (7) is the condition that these households are indeed too poor to be able to finance their investment, and hence has no choice but to work. (P stands for "poor" or "proletariat".) Note that the above argument also establishes that the wealth of the households is concentrated on two points in a steady state with inequality.

The labor market equilibrium condition, (3), becomes $X_\infty/(1-X_\infty) = n(v_\infty)$, where $0 < X_\infty < 1$ is the steady state fraction of the working class. Note that, for any $v_\infty < V$, this condition can be satisfied by setting

(8) $X_\infty = X(v_\infty) \equiv n(v_\infty)/(1+n(v_\infty)) \in (0, 1)$.

Therefore, the sufficient and necessary condition for the existence of a two-point steady state distribution is given by the inequalities in (6) and (7), which are reproduced as follows:

(9) $P(v_\infty) < C(v_\infty) \leq B(v_\infty)$.

3C. The Full Characterization of the Steady States.

We have just established that there are only two kinds of the steady states, and derived the conditions for their existence, (5) and (9). Figure 3a-c help to illustrate these conditions. The straight line with the slope equal to $\beta/(1-\beta r)$ depicts the steady state wealth of the proletariat, $P(v_\infty)$, while the convex, downward-sloping curve depicts that of the bourgeoisie, $B(v_\infty)$, both as functions of v_∞. The wealth gap between the two classes, $B(v_\infty) - P(v_\infty)$, shrinks as v_∞ goes up, and would disappear at $v_\infty = V$, where $P(V) = B(V) = \beta V/(1-\beta r)$. The third curve, the concave, upward-sloping one, depicts $F - \lambda\pi(v_\infty)/r$, which is equal to $C(v_\infty)$, when it is positive. There are only three generic ways in which this curve may intersect with $P(v_\infty)$ and $B(v_\infty)$. In Figure 3a, it intersects with $P(v_\infty)$ at $v^- > 0$ and with $B(v_\infty)$ at $v^+ < V$. In Figure 3b, it intersects twice with $P(v_\infty)$, first at $v^- > 0$ and then at $v^+ < V$, and it stays below $B(v_\infty)$ for all $v_\infty < V$. In Figure 3c, it never intersects with $P(v_\infty)$ nor with $B(v_\infty)$; it stays below $P(v_\infty)$ for all $v_\infty < V$.

In Figure 3a, the steady state with perfect equality does not exist, because $P(V) = B(V) = \beta V/(1-\beta r) < C(V) = F - \lambda\pi(V)/r$, which violates (5). In both Figure 3b and Figure 3c, on the other hand, (5) holds, and hence there exists a steady state, in which wealth distribution is degenerate at $w_\infty = \beta V/(1-\beta r) \geq C(V) = \text{Max}\,\{0, F - \lambda\pi(V)/r\}$.

In Figure 3a, $P(v_\infty) < C(v_\infty) \leq B(v_\infty)$ over $(v^-, v^+]$, where $0 < v^- < v^+ < V$. Thus, (9) holds for $v_\infty \in (v^-, v^+]$. This means that there is a continuum of steady states, in which the wage rate is given by $v_\infty \in (v^-, v^+]$ and the fraction of the households that belong to the working class is given by $X_\infty = X(v_\infty) \in [X(v^+), X(v^-))$, where $0 < X(v^+) < X(v^-) < 1$. All these steady states are characterized by a two-point distribution of wealth. The case of Figure 3b is similar except that (9) holds for $v_\infty \in (v^-, v^+)$ and $X_\infty \in (X(v^+), X(v^-))$. In Figure 3c, on the other hand, $C(v_\infty) < P(v_\infty) < B(v_\infty)$ for all $v_\infty < V$, which means that (9) is never satisfied. That is, there is no steady state with a two-point distribution.

Having identified the three cases to be classified, it remains to characterize the conditions for the three cases, in terms of the parameters of the model, $(\beta r, \lambda, rF) \in (0,1)^2\times(0,\infty)$, which is done in Proposition 2. To state the proposition, it is convenient to introduce some functions. For a positive constant, θ, define

(10) $\Lambda(\gamma) \equiv [\gamma - \theta V(\gamma)]/\pi(V(\gamma)) = 1 - (1+\theta)V(\gamma)/\pi(V(\gamma))$

and

(11) $\Gamma(\lambda) \equiv \lambda\phi(\theta/\lambda)$,

where $V(\gamma) > 0$ is the unique solution to $\pi(V) - V \equiv \gamma$ and satisfies $V(\infty) = 0$. The following lemma summarizes the key properties of these functions.

<u>Lemma.</u>

i) $\Lambda' > 0$, $0 < \Lambda(\gamma) < 1$ for $\gamma \in (\gamma^+, \infty)$, with $\Lambda(\gamma^+) = 0$ and $\Lambda(\infty) = 1$, where γ^+ is defined uniquely by $\gamma^+ = \theta V(\gamma^+)$ and satisfies $0 < \gamma^+ < \infty$;

ii) $\Gamma' > 0$, $\Gamma'' < 0$, and $\Gamma(0) = 0$;

iii) $\lambda > \Lambda(\Gamma(\lambda))$ for $\lambda \neq \lambda_c$ and $\lambda_c = \Lambda(\Gamma(\lambda_c))$, where $\lambda_c = \Lambda(\gamma_c) = \Gamma^{-1}(\gamma_c)$ is defined uniquely by $\theta = \lambda_c n(V(\Gamma(\lambda_c))) = \Lambda(\gamma_c)n(V(\gamma_c))$, and satisfies $0 < \lambda_c < 1$ and $\gamma^+ < \gamma_c < \infty$.

<u>Proof</u>. See the appendix.

Lemma is illustrated in Figure 4. Note that the two graphs, $\lambda = \Lambda(\gamma)$ and $\gamma = \Gamma(\lambda)$, are both increasing, and the former stays strictly above the latter except at the point of tangency, $(\lambda_c, \Gamma(\lambda_c)) = (\Lambda(\gamma_c), \gamma_c)$.

We are now ready to state the proposition.

<u>Proposition 2.</u> Let $\theta = \beta r/(1-\beta r)$. Then,

a) If $0 < \lambda < \Lambda(rF)$, (5) does not hold and (9) holds, as shown in Figure 3a. Hence, there exists a continuum of steady states, indexed by its wage rate, $v_\infty \in (v^-, v^+]$, where v^- is a unique solution to $P(v) = C(v)$ in $(0,V)$ and v^+ is a unique solution to $B(v) = C(v)$ in $(0,V)$. In these steady states, a fraction $X(v_\infty)$ of the household owns $P(v_\infty) = \beta v_\infty/(1-\beta r)$ and a fraction $1- X(v_\infty)$ of the households owns $B(v_\infty) = \beta(\pi(v_\infty) - rF)/(1-\beta r)$, where $0 < X(v^+) \leq X(v_\infty) < X(v^-) < 1$.

b) If $\lambda \geq \Lambda(rF)$ and $\Gamma(\lambda) < rF < \gamma_c$, both (5) and (9) hold, as shown in Figure 3b. Hence, there exists a continuum of steady states, indexed by its wage rate, $v_\infty \in (v^-, v^+)$, where v^- and v^+ are the two solutions to $P(v) = C(v)$ in $(0,V)$. In these steady states, a fraction $X(v_\infty)$ of the household owns $P(v_\infty) = \beta v_\infty/(1-\beta r)$ and a fraction $1- X(v_\infty)$ of the households owns $B(v_\infty) = \beta(\pi(v_\infty) -$

rF)/(1−βr), where $0 < X(v^+) < X(v_\infty) < X(v^-) < 1$. There exists also a steady state, in which $v_\infty = V$ and all the households maintain the wealth equal to $w_\infty = \beta V/(1-\beta r)$.

c) Otherwise, (5) holds and (9) does not hold, as shown in Figure 3c. Hence, there is a unique steady state, in which $v_\infty = V$ and all the households maintain the wealth equal to $w_\infty = \beta V/(1-\beta r)$.

Proof. See the appendix.

In Figure 4, Regions A, B, and C satisfy the conditions stated in Proposition 2a), 2b) and 2c), respectively. The boundary of A is given by (10), and the boundary between B and C is given by (11) for $\gamma < \gamma_c$. A higher β increases θ, which shifts both boundaries to the left and upward. Thus, as β goes up, Region A shrinks and Region C expands.

In Region A, with a combination of a large F and a small λ, the model predicts the rise of class societies. In all the steady states, there is a permanent separation between the rich bourgeoisie and the poor proletariat. The size of each class is bounded away from zero.[13] The intuition behind an endogenous formation of class societies should be easy to grasp. Because of the large investment requirement and/or the severe enforcement problem, the wage rate must become sufficiently low to make it possible for some households to be able to borrow and become employers. In order to keep the wage rate low, however, some households must stay poor, so that they are unable to borrow and forced to work. In every steady state, the rich maintain their wealth partly because the poor work for them at a low wage. And the rich's demand for labor is not strong enough to pull the poor out of the poverty. Across these steady states, the degree of inequality differs systematically. Indeed, the steady state wealth distributions can be ranked according to the Lorenz criterion. The Kuznets Ratio, the coefficient of variation, the Gini coefficient, and any other Lorenz-consistent inequality measure, all agree that there is greater inequality in a steady state with a lower wage rate. This is because a lower v_∞ implies not only that $B(v_\infty)$ is larger (i.e., the rich are richer), and $P(v_\infty)$ is smaller (i.e., the poor are

[13] The reader may wonder what would happen if the economy starts with a perfectly equal distribution of the wealth, in Region A. Suppose that the initial level of wealth is less than C(V). Then, the labor supply curve is flat below V and the equilibrium wage rate in the first period is determined in such a way that credit rationing will take place in equilibrium. The lucky households obtain credit and accumulate wealth faster than the others that are denied credit. This breaks the perfect equality. If the initial level of wealth is greater than or equal to C(V), then, the labor supply curve is flat at V. Every household earns the same net

poorer), but also that $X(v_\infty)$ is larger (i.e., a larger fraction of the households is poor). Again, the intuition behind this result is easy to grasp. The presence of a large working class keeps the wage rate low. A lower wage rate favors the rich at the expense of the poor, which increases the wealth gap. A larger demand for labor by each rich employer can be met by only when a small fraction of the households belongs to the bourgeoisie. Note that the size of the firms increases with the inequality. In a steady state with a lower wage, a smaller fraction of the households belongs to the bourgeoisie, and each of them employs a larger number of workers

In Region C, with a combination of a small F and a high λ, wealth inequality disappears in the steady state. In this case, the model predicts the fall of the class society through a trickle-down mechanism, in which the job creation by the rich, by raising the wage rate, pulls the poor out of the poverty, and the poor households will eventually catch up with the rich households. In other words, the model predicts that the class society disappears in the long run. In the steady state, some agents work for others, but they do not mind doing so because they are paid "fair" value of labor, and those who employ operate relatively at a small scale, hiring a small number of workers. In other words, the economy becomes a nation of the middle class, or *petits bourgeois*, consisting of small proprietors and well-paid employees.

Although an explicit analysis of the dynamics is beyond the scope of this paper, the transition process is not difficult to imagine.[14] Suppose that the economy starts at an underdeveloped state, where all the households are poor. There is little inequality, but some households are richer than others. Initially, the equilibrium wage rate is very low and the profit is high, so that the relatively rich households, while they may be poor in absolute terms, are able to borrow and invest. Their wealth then starts growing faster than others, magnifying inequality. In Region A, this leads to a formation of the class society. In Region C, the rich's demand for labor will drive up the wage rate so much that the working class, which benefit from a high wage rate, will be able to catch up with the rich, reducing inequality.

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income, and over time, their wealth declines until, after a finite periods, it goes below C(V), at which point, the equilibrium credit rationing occurs, and the equality is broken.

[14] An explicit analysis of the dynamics faces two major difficulties. First, the distribution of wealth is an infinite-dimensional object. Second, to analyze the dynamics for an arbitrary initial condition, one cannot avoid the possibility of equilibrium credit rationing in transition, which introduces stochastic elements into the models.

In Region B, characterized by a combination of a small rF and a small λ, both the long run scenarios are possible. Therefore, whether the economy may develop into the class society or not depends entirely on the initial wealth distribution (and possibly on the credit-rationing rule, as well.)

It might be instructive to consider the following thought experiment, which arguably traces the evolution of industrial societies. Immediately after the Industrial Revolution, λ was small and F was large, so that the economy was in Region A. Throughout much of the nineteenth century and early twentieth century, this led to a formation of the class society in industrial countries. Then the capital market and the technology gradually improve over time. With an increase in λ and/or a reduction in F, the economy eventually entered in Region C. This led to a formation of the predominantly middle-class society in the late twentieth century.

3D. Ranking of the Steady States.

In Regions A and B, there are multiple steady states, which can be indexed by their levels of the wage rate, v_∞. Since a higher v_∞ benefits the workers at the expense of the employers, the steady states are not Pareto-rankable. However, it is possible to rank them by the total surplus, $TS \equiv X_\infty v_\infty + (1-X_\infty)(\pi(v_\infty) - rF)$, or equivalently, by the aggregate wealth, $AW \equiv X_\infty P(v_\infty) + (1-X_\infty)B(v_\infty) = [\beta/(1-\beta r)]TS$. Simple algebra shows that $TS = (1-X_\infty)[n(v_\infty)v_\infty + \pi(v_\infty) - rF] = (1-X_\infty)[\phi(n(v_\infty)) - rF] = [\phi(n(v_\infty)) - rF]/(1+n(v_\infty))$. Hence, $d(TS)/dv_\infty = n'(v_\infty)[(1+n(v_\infty))\phi'(n(v_\infty)) + rF - \phi(n(v_\infty))]/(1+n(v_\infty))^2 = n'(v_\infty)[v_\infty + rF - \pi(v_\infty)]/(1+n(v_\infty))^2$, which is positive if $v_\infty < V$ and zero if $v_\infty = V$. Thus, when there are multiple steady states, the aggregate wealth and the total surplus is larger in a steady state with a higher v_∞.

This completes the analysis of the basic model. The next two sections discuss some extensions.

4. Self-Employment

In the model presented above, the relatively poor agents have no choice but to work for the relatively rich. Some readers might think that, if the poor have an alternative to working for the rich, such as self-employment, the model would not predict the rise of class societies. However, the effects of introducing self-employment are far from straightforward, because it also offers an alternative for the rich, who would otherwise invest in the job creating project, which could benefit the poor workers.

To address this issue in a formal manner, let us now suppose that the agents have a third option, self-employment. This technology requires F^S units of the numeraire good to be invested at the beginning of the period, which gives the agents π^S units of the numeraire good at the end of the period. Thus, by becoming self-employed, the agents would have $\pi^S + r(w_t - F^S)$ at the end of the period, which is greater than or equal to $v_t + rw_t$ if and only if

(12) $V^S \equiv \pi^S - rF^S \geq v_t$.

Thus, (12) is the condition under which the agents (weakly) prefer being self-employed than being an worker.

To become self-employed, the agent whose wealth, w_t, is less than the necessary investment, F^S, must borrow the difference, $w_t - F^S$. As in the case of borrowing to become an employer, the borrowing limit exists also for the agents who intend to become self-employed. They would default if the repayment obligation exceeds the cost of default, $\lambda^S\pi^S$, where $0 \leq \lambda^S < 1$. Due to this enforcement problem, the lender would allow the agent to borrow only up to $\lambda^S\pi^S/r$. Thus, the agents can become self-employed only if

(13) $w_t \geq C^S \equiv \mathrm{Max}\{0, F^S - \lambda^S\pi^S/r\}$,

where C^S may be interpreted as the collateral requirement for the investment in the self-employment technology.

Needless to say, we need to impose some restrictions on V^S and C^S, the two parameters that characterize the self-employment technology, so that this technology may provide the poor agents with a viable alternative to working for others, and yet that it would not provide the rich agents with a better option than being an employer. These restrictions are imposed by the following assumptions:

(A1) $V^S < V$;

(A2) $C^S < C(V^S)$;

(A3) $C^S \leq P(V^S)$.

The first assumption (A1) implies that, for any $v_t \leq V$, $V^S < V = \pi(V) - rF \leq \pi(v_t) - rF$, so that being an employer is preferable to self-employment. Without (A2), the self-employment technology would never affect the labor market equilibrium. (To see this, if $C(V^S) \leq C^S$, $v_t < V^S$ would imply $C(v_t) < C^S$. Thus, whenever self-employment is more desirable than working for others, any agent who is rich enough to be self-employed is rich enough to be an employer.) Finally, (A3) ensures the sustainability of self-employment. By being self-employed, the households maintain enough wealth that allows them to satisfy the borrowing constraint for being self-employed.

4A. The Labor Market Equilibrium:

Figure 5 illustrates the labor market equilibrium with self-employment, under the additional assumptions that $C^S > 0$ and that G_t has no mass point. If the self-employment technology were not available, the labor supply per firm would be equal to $G_t(C(v_t))/[1 - G_t(C(v_t))]$ for $v_t < V$, which is continuous and has no flat part (because G_t has no mass point).
Introducing self-employment does not affect the curve when $v_t > V^S$ or $v_t \leq V^0$, where $V^0 < V^S$ is defined by $C(V^0) \equiv C^S > 0$. This is because, when $v_t > V^S$, every agent is better off being a worker than being self-employed, and when $v_t < V^0$, any agent who can afford to be self-employed can also afford to be an employer, which is preferable. Self-employment thus affects the labor supply per firm solely over the interval, $(V^0, V^S]$.

Now, if $V^0 < v_t < V^S$, all the agents strictly prefer being self-employed to being a worker. With $v_t > V^0$, $C(v_t) > C^S$ holds, which means that the agents whose wealth satisfy $C^S \leq w_t < C(v_t)$ become self-employed. The agents whose wealth exceeds $C(v_t)$ become employers and those whose wealth falls short of C^S become workers. Thus, when $V^0 < v_t < V^S$, the labor supply per firm is given by $G_t(C^S)/[1 - G_t(C(v_t))]$.

If $v_t = V^S$, all the agents whose wealth satisfy $C^S \leq w_t < C(v_t)$ can be self-employed, but they are indifferent between being self-employed and being employed. Thus, the labor supply per firm can take any value between $G_t(C^S)/[1 - G_t(C(V^S))]$ and $G_t(C(V^S))/[1 - G_t(C(V^S))]$, as indicated by the flat segment at $v_t = V^S$. If the labor demand curve, $n(v_t)$, intersects this part of the labor supply curve, as shown in Figure 5, some agents who can be self-employed become workers. However, this should not be viewed as a credit rationing. They would voluntarily become workers, because their net earning is equal to $v_t = V^S$.

Of course, if G_t has one or more mass points and a positive measure of the agents have the same level of wealth, an equilibrium credit rationing may occur. In such a situation, some agents are denied the credit and unable to become self-employed, despite that they strictly prefer to be self-employed and that they may be equally qualified for the credit as some self-employed agents (for the same reason that was explained in the discussion of Figure 1). As before, however, the following analysis and discussion ignore such a possibility of equilibrium credit rationing, because it never occurs in the steady state.

4B. *Dynamics:*

The household wealth now follow

$$(14)\qquad w_{t+1} = \begin{cases} \beta(v_t + rw_t) & \text{if } w_t < C^S \\ \beta(V^S + rw_t) & \text{if } C^S \le w_t < C(v_t) \\ \beta(\pi(v_t) - rF + rw_t) & \text{if } w_t \ge C(v_t), \end{cases}$$

if $V^0 < v_t < V^S$. Otherwise, they follow Equation (4). Figure 6 illustrates (14). Now the map jumps twice, at C^S and at $C(v_t)$. If $w_t < C^S$, the agent becomes a worker, earning v_t; if $C^S \le w_t < C(v_t)$, the agent becomes self-employed, earning $V^S > v_t$; if $w_t \ge C(v_t)$, the agent becomes an employer, earning $\pi(v_t) - rF > V^S$. As before, the arrows indicate the effects of a rise in the wage rate. The effects of a higher wage rate on the workers and the employers are the same as before. As long as $v_t < V^S$, a higher wage rate does not affect the wealth dynamics of self-employed households, although more agents are forced to become self-employed because a higher wage rate increases the collateral requirement for the employer, $C(v_t)$.

4C. *The Classification of the Steady States*:

We are now ready to classify the steady states.

The One-Class Steady State without Active Self-Employment: This is the same steady state discussed in section 3A, with $v_\infty = V$. Its existence is not affected by the introduction of the self-employment technology, because $v_\infty = V > V^S$ implies that self-employment is a dominated option. Therefore, (5) remains the sufficient and necessary condition for its existence.

The Two-Class Steady States without Active Self-Employment: These are the same steady states discussed in section 3B, with $v_\infty < V$. Its existence requires that, in addition to (9),

(15) $v_\infty \leq V^0$, $v_\infty \geq V^S$, or $C^S > P(v_\infty)$,

because the poor households would switch to self-employment, if $V^0 < v_\infty < V^S$ and $P(v_\infty) \geq C^S$ would hold.

The introduction of the self-employment technology may create the following three new types of steady states, in which a positive measure of households are self-employed.

The One-Class Steady State with Active Self-Employment: In this steady state, every household is self-employed, maintaining the wealth equal to $P(V^S) \geq C^S$, and the labor market is inactive. This occurs when no one is rich enough to be an employer. To induce any self-employed agent to work for them, potential employers would need to offer a wage rate at least as high as V^S, but at such a wage rate, no one could borrow enough to be employers, that is, $P(V^S) < C(V^S)$. Thus, the condition for this steady state is given by

(16) $C(V^S) > P(V^S)$.

Even though this steady state is characterized by perfect equality, and its wealth distribution is degenerate, its steady state level of wealth, $P(V^S)$, is strictly less than $B(V) = P(V)$, the level of wealth achieved by all the households in the steady state discussed in Section 3a.

The Two Class Steady States with Active Self-Employment; In these steady states, the labor market is active with $v_\infty = V^S$. The agents are indifferent between being self-employed and being a worker. Both the self-employed and the working households own the same level of wealth, $P(V^S)$, and they are too poor to be employers, but rich enough to become self-employed; $C(V^S) > P(V^S) \geq C^S$. Some households are rich enough to become employers at $v_\infty = V^S$, thus $B(V^S) \geq C(V^S)$. The shares of the households that become self-employed, workers, and employers are given by $S_\infty \in (0,1)$,

$X(v_\infty)(1 - S_\infty)$, and $(1 - X(v_\infty))(1 - S_\infty)$, respectively, where $X(v_\infty) \in (0,1)$ was defined in (8). The sufficient and necessary conditions for these steady states are given

(17) $\quad B(V^S) \geq C(V^S) > P(V^S)$.

These steady states are also characterized by two-point wealth distributions, where a fraction, $S_\infty + X(v_\infty)(1 - S_\infty)$, of the population owns $P(V^S)$ and the rest owns $B(V^S)$.

The Three Class Steady States; In these steady states, the labor market is active with $v_\infty \in (V^0, V^S)$, and the steady state wealth distributions are concentrated at three points, $P(v_\infty)$, $P(V^S)$, and $B(v_\infty)$. The poorest are the proletariat; they are too poor to be self-employed, $P(v_\infty) < C^S$, and have no choice but to work. The richest are the bourgeoisie; they are rich enough to become employers, $B(v_\infty) \geq C(v_\infty)$. The wealth of the self-employed households converges to $P(V^S)$, which makes them too poor to be employers, but rich enough to be self-employed, $C(v_\infty) > P(V^S) \geq C^S$. Combining these conditions yields

(18) $\quad B(v_\infty) \geq C(v_\infty) > P(V^S) \geq C^S > P(v_\infty)$.

Again, the shares of the households that become self-employed, workers, and employers are given by $S_\infty \in (0,1)$, $X(v_\infty)(1-S_\infty)$, and $(1-X(v_\infty))(1-S_\infty)$, respectively.

Adding the self-employment technology may create these three new types of the steady states.[15] Nevertheless, it should be pointed out that the last two may indeed be viewed as variations of the two-class steady states discussed in section 3B. In both, the rich households enjoy the high level of wealth, because there are poor households who are willing (or forced to) work for the rich below the fair wage rate, V. The one-class steady state with active self-employment is the only new steady state, in which no household enjoys the high level of wealth by taking advantage of cheap labor supplied by the poor. It also differs from the one-class steady state with $v_\infty = V$ in that it prevents the society from developing the mutually beneficial employer-worker relation, in which all the households could enjoy the level of wealth, $B(V) = P(V)$, which is higher than $P(V^S)$, the wealth that can be achieved by the self-employment technology. In this sense, this steady state may be viewed as a poverty trap, in which there is an equalization of poverty.

4D. The Full Characterization of the Steady States

Having identified all the types of the steady states and their existence conditions, it remains to characterize them in terms of the parameters of the model.

In Regions A or B, the cases illustrated in Figures 3a) and 3b), the self-employment could eliminate some two-class steady states, while creating new ones. To conduct a systematic analysis, let us recall first the two critical values, $v^- < v^+$, defined in Propositions 2a) and 2b), which are also depicted in Figures 3a) and 3b). Recall also that V^0 was defined by $C(V^0) = C^S$. It is also useful to define V' by $P(V') = C^S$. Then, (A1)-(A3) may now be rewritten as $V^0 < V^S < V$, and $V' \le V^S$. There still remain the following six generic cases to be distinguished, depending on the values of V^S and C^S. As indicated below, each of these cases is associated with a particular ordering of v^-, v^+, V^0, V^S and V'.

Case I:	$V^S < v^-$	$\Rightarrow$	$V' < V^0 < V^S < v^- < v^+$
Case IIa:	$v^- < V^S < v^+$; $C^S < P(v^-)$	$\Rightarrow$	$V' < V^0 < v^- < V^S < v^+$
Case IIb:	$v^- < V^S < v^+$; $P(v^-) < C^S < P(V^S)$	$\Rightarrow$	$v^- < V^0 < V' < V^S < v^+$
Case IIIa:	$V^S > v^+$; $C^S < P(v^-)$	$\Rightarrow$	$V' < V^0 < v^- < v^+ < V^S$
Case IIIb:	$V^S > v^+$; $P(v^-) < C^S < P(V^+)$	$\Rightarrow$	$v^- < V^0 < V' < v^+ < V^S$
Case IIIc:	$V^S > v^+$; $P(v^+) < C^S < P(V^S)$	$\Rightarrow$	$v^- < V^0 < v^+ < V' < V^S$ in Region A
			$v^- < v^+ < V' < V^0 < V^S$ in Region B.

In each of the six generic cases, we can check to see whether it satisfies the condition for each type of the steady states, thereby making a complete list of the existing steady states.

In Region C of Figure 4, the case illustrated in Figure 3c), the introduction of the self-employment technology has no effect in the long run. The steady state is unique; in which $v_\infty = V$ and the wealth of all the households converges to $B(V) = P(V)$. Thus, as before, the model predicts the fall of class societies in this case. To see why this is the only steady state, note that all the conditions for the other types of the steady states, from (15) through (19), require that $C(v)$ exceeds $P(v)$ for some $v < V$, which is ruled out in the case of Region C, as shown in Figure 3c).

Table 1: The Steady States in the Model with Self-Employment

[15] This abundance of the steady states has some resemblance to the result obtained in Banerjee and Newman (1993).

		No Active Self-Employment		Active Self-Employment		
		One-Class	Two-Class	One-Class	Two-Class	Three-Class[16]
A	I	∅	$(v^-, v^+]$	∅	∅	∅
A	IIa	∅	$[V^S, v^+]$	V^S	V^S	∅
A	IIb	∅	$(v^-, V') \cap [V^S, v^+]$	V^S	V^S	(V'', V')
A	IIIa	∅	∅	V^S	∅	∅
A	IIIb	∅	(v^-, V')	V^S	∅	(V'', V')
A	IIIc	∅	$(v^-, v^+]$	V^S	∅	$(V'', v^+]$
B	I	V	(v^-, v^+)	∅	∅	∅
B	IIa	V	$[V^S, v^+)$	V^S	V^S	∅
B	IIb	V	$(v^-, V') \cap [V^S, v^+)$	V^S	V^S	(V'', V')
B	IIIa	V	∅	∅	∅	∅
B	IIIb	V	(v^-, V')	∅	∅	∅
B	IIIc	V	(v^-, v^+)	∅	∅	∅
C		V	∅	∅	∅	∅

Table 1 offers a complete list of the steady states in this model. For each parameter configuration and for each type of the steady states, the table entry shows the range of the steady state wage rates when they exist (or the value of the steady state wage rate when it exists uniquely), and ∅ when they do not. As shown, many of the two-class steady states discussed in section 3B, which exist in Regions A and B, survive the introduction of the self-employment technology. They continue to exist, either because self-employment is no more attractive than working for others (i.e., $v_\infty \geq V^S$ as in Cases I, IIa, and IIb), or because the workers are too poor to become self-employed (i.e., $v_\infty < V'$ or $P(v_\infty) < C^S$, as in Cases IIb, IIIb, and IIIc). Only the steady states whose wage rates satisfy $V' \leq v_\infty < V^S$ are eliminated. Even then, they are often replaced by the two-class steady states with active self-employment with $v_\infty = V^S$ (as in Cases IIa and IIb), or by the three-class steady states (as in Cases IIb, and A-IIIb, A-IIIc), both of which can be viewed as variations of the two-class steady states.

The introduction of the self-employment technology may also create an entirely new type of the steady state, where every agent is self-employed and the employer-worker relation disappears

[16] Here, V'' is defined by $C(V'') \equiv P(V^S)$. It always satisfies $V'' < v^+$ in the case of A-IIIc. In the cases of IIb and A-IIIb, $V'' < V'$ requires that C^S must be sufficiently large within the range; for C^S sufficiently close to $P(v^-)$, $V'' \geq V'$ holds and hence there is no three-class steady state.

altogether. In Cases IIa and IIb, this steady state, in which every household owns $P(V^S)$, is dominated by other steady states, including some two-class steady states with $v_\infty \geq V^S$, where the rich own $B(v_\infty)$ and the poor own $P(v_\infty) \geq P(V^S)$. In these cases, the one-class steady state with 100% self-employment should be viewed as a poverty trap, where the investment in self-employment prevents the economy from accumulating wealth, because, unlike the other investment, it does not create any job. In Cases A-IIIb, and A-IIIc, however, the one-class steady state with 100% self-employment is an improvement for the households that belong to the proletariat in the other existing steady states.

There is one case, where the introduction of the self-employment technology changes the predictions of the model drastically. That is Case IIIa, where $V^0 < v^- < v^+ < V^S$. In this case, self-employment eliminates all the two-class steady states discussed in section 3B. In Region A, the one-class steady state, where every household is self-employed, becomes the only steady state. In Region B, the one-class steady state without active self-employment is left as the only steady state. It is noteworthy that, in B-IIIa, the introduction of the self-employment technology helps to eliminate the class societies, and yet in the long run, no household remains self-employed.

Finally, in Region C, the long run prediction of the model is not at all affected by the introduction of the self-employment.

5. Constant Returns to Scale (with the Minimum Investment)

We have assumed that each agent can set up and manage at most one firm, and the amount of investment is fixed at F. This assumption implies that the employer's technology is subject not only to the minimum requirement, which implies the nonconvexity, but also to diminishing returns. Some readers might think that that this assumption of diminishing returns is responsible for the rise of class societies. One might reason that, without diminishing returns, the rich would invest more and operate many firms, until their labor demand would drive up the wage rate so much that the poor workers can catch up with the rich. In this section, we allow the employers to make variable investment with constant returns to scale, except that they must satisfy the minimum requirement for the investment, and show that the main results obtained in the basic model would carry over. Thus, what is essential is the nonconvexity of investment, not diminishing returns.

Let us go back to the model of section, and modify that model by assuming that, by investing $K_t \geq F$ units of the numeraire good and employing N_t units of labor at the beginning of period, $\Phi(N_t, K_t)$ units of the numeraire good become available at the end of period. It is assumed that Φ satisfies the standard properties of constant-returns to scale production functions for $K_t \geq F$. If $K_t < F$, $\Phi(N_t, K_t) = 0$. Let $k_t \equiv K_t/F$, $n_t \equiv N_t/k_t$, and $\phi(n_t) \equiv \Phi(n_t, F)$. Then, for $k_t \geq 1$, $Max_N\{\Phi(N, K) - vN\} = Max_n\{\phi(n) - vn\}k = \{\phi(n(v)) - vn(v)\}k = \pi(v)k$, where $n(v)$ and $\pi(v)$ are defined as before. Here, k is the scale of operation chosen by the employer, defined as the investment measured in multiples of F, and $\pi(v)$ is the equilibrium profit per unit of operation, which is independent of k, except that k must be greater than one. (One possible interpretation is that k is the number of firms (or factories) run by an agent, and the integer constraint is ignored for k greater than one.)

In the previous models, it was assumed that the employer's earning comes solely from operating a firm. In other words, one cannot be an employer and a worker at the same time. If we were to make the same assumption here, the lost wage income would be the fixed cost of being an employer, independent of the scale of operation, which introduces increasing returns to scale and leads to the nonexistence of the steady states. This is a nuisance that we want to avoid. Hence, in this section, we allow the employers to work as well. Nevertheless, we shall call only the agent who does not become an employer "a worker" and the agent who does not will be called "an employer," despite that the latter also supplies labor.

Although the technology now allows the agents to invest as much as possible, they may not be able to do so, because of the borrowing constraint. To invest by k_t, the agent who inherited w_t needs to borrow by $k_tF - w_t$. The agent would default if the payment obligation, $r(k_tF - w_t)$, exceeds the default cost, which is a fraction of the gross profit, $\lambda\pi(v_t)k_t$. Knowing this, the lender would lend only up to $\lambda\pi(v_t)k_t/r$. Therefore, to invest by k_t, the agent needs to have $w_t \geq k_tF - \lambda\pi(v_t)k_t/r$ or

(19) $w_t \geq [F - \lambda\pi(v_t)/r]k_t = C(v_t)k_t$,

where $C(v_t) \equiv F - \lambda\pi(v_t)/r$. Subject to the borrowing constraint (19), the agent chooses k_t to maximize the end-of-the-period wealth,

$$v_t + \pi(v_t)k_t - r(k_tF - w_t) = v_t + rw_t + (\pi(v_t) - rF)k_t \qquad \text{if } k_t \geq 1$$

(20)

$$v_t + rw_t \qquad \text{if } 0 \leq k_t < 1$$

The labor demand is then equal to $n(v_t)k_t$.

The wage rate adjusts to keep the balance between the labor demand and the labor supply. It is easy to see that the equilibrium wage rate satisfies $C(v_t) > 0$ and $v_t \leq V$, where V is now defined by $\pi(V) \equiv rF$. To see this, suppose $v_t > V$. Then, $\pi(v_t) < rF$ so that (20) is maximized at $k_t = 0$. All the agents prefer being workers, and not investing. Hence, the labor demand would be zero, and there is an excess supply of labor. Suppose now $C(v_t) \leq 0$, which implies $\pi(v_t) \geq rF/\lambda > rF$. Then, (20) is strictly increasing in $k_t \geq 1$, while the borrowing constraint (19) is not binding. Hence, the agents would invest by an infinite amount and the labor demand would be infinite. Therefore, the equilibrium wage rate must adjust to satisfy $0 < C(v_t) \leq C(V)$.

Note that $v_t \leq V$ may still be interpreted as the profitability constraint for becoming an employer, although the definition of V is now given by $\pi(V) \equiv rF$, not by $\pi(V) - V \equiv rF$. This is due to the change in the assumption made earlier, i.e., an agent supplies one unit of labor even as an employer. Note also that V may still be interpreted as the "fair" wage rate, because it is the wage rate that equalizes the net earnings of the employer and the worker. To make sure that the employers indeed employ more labor than they can supply themselves, it is necessary to impose the following restriction:

(A4) $\phi'(1) > V$.

This ensures that $n(v_t) \geq n(V) > 1$.

Having established that $0 < C(v_t) \leq C(V)$ in equilibrium, let us now consider the optimal investment behavior in this range. First, consider the case, $0 < C(v_t) < C(V)$, or $rF < \pi(v_t) < rF/\lambda$. Then, (20) is strictly increasing in $k_t \geq 1$, so that every agent wants to invest as much as possible. If $w_t \geq C(v_t)$, the agent invests until the borrowing constraint is binding, i.e., $k_t = w_t/C(v_t) \geq 1$. If $w_t < C(v_t)$, then the agent cannot meet the minimum requirement, so that $k_t = 0$. This can be summarized as

$$
(21)\qquad k_t = \begin{cases} w_t / C(v_t) & \text{if } w_t \geq C(v_t) \\ 0 & \text{if } w_t < C(v_t). \end{cases}
$$

Now, consider the case, $\pi(v_t) = rF$, or $v_t = V$. Then, (20) is equal to $V + rw_t$ for $k_t = 0$ and for all $k_t \geq 1$, while $0 < k_t < 1$ is strictly dominated. All the agents are hence indifferent between $k_t = 0$ and for all $k_t \geq 1$. Since the borrowing constraint is now $w_t \geq C(V)k_t$,

$$
(22)\qquad k_t \begin{cases} \in \{0, [1, w_t / C(V)]\} & \text{if } w_t \geq C(V) \\ = 0 & \text{if } w_t < C(V). \end{cases}
$$

The labor demand is equal to $n(v_t)k_t$, where k_t is given by (21) or (22). Hence, the equilibrium condition in the labor market can be given by

$$
(23)\qquad \frac{n(v_t)}{C(v_t)} \int_{C(v_t)}^{\infty} w dG_t(w) \geq 1\ ;\ \ 0 < C(v_t) \leq C(V),
$$

where the first inequality may be strict either when G_t jumps at $C(v_t)$ or when $v_t = V$. Equation (23) is illustrated in Figure 7. The aggregate labor supply is now indicated by the vertical line at one, because each agent, including the rich employer, supplies one unit of labor. The downward-sloping curve is the aggregate labor demand. Note that a higher wage rate reduces the aggregate labor demand for three reasons. First, it reduces the labor demand per unit of operation ($n(v_t)$ is decreasing in v_t). Second, it reduces the profit per unit of operation ($\pi(v_t)$ is decreasing in v_t), which tightens the borrowing constraint ($C(v_t)$ is increasing in v_t), forcing the employer to operate at a smaller scale, at $w_t/C(v_t)$. Third, with a tighter borrowing constraint, less agents are able to meet the minimum investment requirement to become employers. Note that, if G_t has a mass point at $C(v_t)$, a positive measure of the agents can meet the minimum requirement at $C(v_t)$, which causes the aggregate labor demand to jump at $C(v_t)$, as illustrated by the flat segment of the demand curve. If the vertical line intersects at the flat segment, there would be an equilibrium credit rationing. If it intersects at the downward sloping part of the labor demand curve, as indicated in Figure 7, then there is no credit rationing. As before, we ignore this possibility of equilibrium credit rationing, as it never happens in a steady state.

From (20)-(22), the dynamics of the household can be expressed by

$$
(24)\qquad w_{t+1} = \begin{cases} \beta v_t + [C(V)/C(v_t)][\pi(v_t)/\pi(V)](\beta r)w_t & \text{if } w_t \geq C(v_t) \\ \beta v_t + (\beta r)w_t & \text{if } w_t < C(v_t), \end{cases}
$$

which is illustrated by Figure 8. As in the basic model, the map (24) jumps at $C(v_t)$ when $v_t < V$. Unlike in the basic model, the slope of the map is strictly higher above $C(v_t)$ than below $C(v_t)$, when $v_t < V$. The wealth, when in the hands of the rich, earns the gross rate of return, $[C(V)/C(v_t)][\pi(v_t)/\pi(V)]r$, which is strictly greater than r. The intuition behind this is easy to grasp. When $v_t < V$, it is not only profitable to invest to become an employer. It is also profitable to invest more and operate the firm at a larger scale. Having a higher amount of wealth above $C(v_t)$ allows the rich to invest more by easing the borrowing constraint. This leverage effect allows the rich employers to earn higher returns on their wealth than the poor workers. Indeed, for a sufficiently low wage rate, the leverage effect is so strong that the slope of the map above $C(v_t)$ can be greater than one. This does not mean, however, that the rich's wealth can grow unbounded. The wealth of the rich will eventually stop growing, because a wealth accumulation by the rich will lead to a greater demand for labor, which will push up the wage rate until the slope of the map becomes less than one.[17]

As before, the arrows indicate the effects of a higher wage rate, when $v_t < V$. It raises the threshold level of wealth, increases the wealth of the worker, and reduces the gross rate of return on wealth owned by the rich. The dashed line, $w_{t+1} = \beta(V + rw_t)$, gives the dynamics of household wealth when $v_t = V$.

For a given distribution of wealth in each period, the labor market equilibrium condition (23) determines the wage rate, v_t, and then, from (24), one can obtain the wealth distribution in the next period. Thus, the equilibrium path of this economy can be solved for by applying (23) and (24) iteratively.

It is easy to see that there are at most two types of the steady states in this economy. The first type is that the steady state, characterized by an equal distribution of wealth and $v_\infty = V$. The condition for its existence is given by

$w_\infty = \beta V/(1-\beta r) \equiv P(V) \geq C(V)$,

as the labor market equilibrium condition, (23), holds because the labor demand is given by $n(V)P(V)/C(V) > 1$. The second type is a continuum of steady states, characterized by two-point distributions of wealth, and $v_\infty = V$. The rich own

$w^B_\infty = B(v_\infty) \equiv \beta v_\infty /\{1-[C(V)/C(v_\infty)][\pi(v_\infty)/\pi(V)](\beta r)\} < \infty$

and the poor owns $P(v_\infty)$. The condition that the rich can meet the minimum investment requirement and the poor cannot is given by $B(v_\infty) \geq C(v_\infty) > P(v_\infty)$. The labor market equilibrium condition, (23), is $(1-X_\infty)n(v_\infty)B(v_\infty)/C(v_\infty) = 1$, where X_∞ is the share of the poor households. Because $1 - X_\infty = C(v_\infty)/n(v_\infty)B(v_\infty) \leq 1/n(v_\infty) < 1/n(V) < 1$, one can find $X_\infty \in (0,1)$ that ensures the labor market equilibrium for each v_∞. Thus, the existence condition for this type of steady states is simply

$B(v_\infty) \geq C(v_\infty) > P(v_\infty)$.

Note that the existence condition of this type of the steady states imposes the lower bound on the steady state wage rate, which is more stringent than the restriction, $C(v_\infty) > 0$; it must be sufficiently high to satisfy $[C(V)/C(v_\infty)][\pi(v_\infty)/\pi(V)](\beta r) < 1$, which ensures that the rich's wealth would not grow unbounded. Comparing across these steady states, a lower v_∞ means a higher $B(v_\infty)$, a lower $P(v_\infty)$, and a higher X_∞, so that these steady states can be ranked according to the Lorenz criterion. A lower steady state wage rate is thus associated with greater inequality.

A lower steady state wage rate is also associated with a smaller total surplus, $TS \equiv v_\infty + (1-X_\infty)(\pi(v_\infty)-rF)k_\infty = v_\infty + (1-X_\infty)(\pi(v_\infty)-rF)B(v_\infty)/C(v_\infty)$, or equivalently, by the aggregate wealth, $AW \equiv X_\infty P(v_\infty) + (1-X_\infty)B(v_\infty) = [\beta/(1-\beta r)]TS$. Simple algebra shows that $TS = v_\infty + (\pi(v_\infty)-rF)/n(v_\infty) = [\phi(n(v_\infty))-rF]/n(v_\infty)$. Hence, $d(TS)/dv_\infty = n'(v_\infty)[n(v_\infty)\phi'(n(v_\infty)) + rF - \phi(n(v_\infty))]/(n(v_\infty))^2 = n'(v_\infty)[rF - \pi(v_\infty)]/(n(v_\infty))^2$, which is positive if $v_\infty < V$ and zero if $v_\infty = V$. Thus, when there are multiple steady states, the aggregate wealth and the total surplus is smaller in a steady state with a lower v_∞, i.e., in a steady state with greater inequality.[18]

Characterizing the condition for the co-existence of different types of steady states in the parameter space is more involved than in the previous model, because one has to go through more

[17]In this respect, this dynamics is similar to the household wealth dynamics of the model in Matsuyama (2000).

cases. Indeed, it is as cumbersome as in the model of Matsuyama (2000), which also generates wealth distribution dynamics, where the rich earns higher return on their wealth than the poor. The main source of the complication is the variability of the gross rate of return earned by the rich on their wealth. This introduces additional cases, where all the steady states are characterized by two-point distributions, and yet the share of the population that becomes the rich, while strictly positive, can be arbitrarily close to zero.[19] Nevertheless, it can be done, in a manner similar to Matsuyama (2000), and can be shown that the basic feature of the previous model is preserved. That is to say, a combination of a higher F and a smaller λ implies the rise of class societies, and a combination of a lower F and a higher λ implies the fall of class societies.

6. Concluding Remarks

This paper has presented a theoretical framework for understanding the mechanisms behind the rise and fall of class societies. The key assumptions are the nonconvexity of investment and imperfect capital markets. Under some configurations of the parameter values, the model predicts the rise of class societies, where the population is separated into the rich bourgeoisie and the poor proletariat in all the steady states, regardless of the initial distribution of wealth. In these steady states, the rich maintain a high level of wealth, partially due to the cheap labor supplied by the poor who, with their low level of wealth, have no choice but work for the rich. Under other configurations of the parameter values, the model predicts the fall of class societies. The rich's demand for labor pushes up the wage rate so much that the workers escape from the poverty, and the class distinction disappears in the long run. In this case, the wealth accumulation by the rich eventually trickles down to the poor. As an application, the framework is used to examine the effects of self-employment.

Some readers might find it unsettling that, for the case where the model predicts the rise of class societies, there is a continuum of steady states, each of which is characterized by a two-point distribution (or possibly a three-point distribution in the model with self-employment). This feature of the model is, however, a mere artifact of the simplifying assumptions that all the households are

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[18] In this respect, this model differs significantly from the model of Matsuyama (2000), whose aggregate wealth dynamics is entirely independent of the wealth distribution dynamics.

homogenous, except inherited wealth, and that there are no idiosyncratic shocks. For example, if the ability of the agent as an employer, perhaps measured by F, the minimum requirement of investment, is a random variable, there would be the unique ergodic distribution of wealth. If the model is extended to allow for such idiosyncratic shocks, however, we would have to characterize the condition under which the ergodic distribution is unimodal or bimodal, which may not be feasible analytically.

Obviously, one can think of many ways in which the models can be extended. Introducing long run growth is one. In the above models, the minimum level of investment plays a crucial role. If the economy as a whole is growing over time, perhaps due to some exogenous improvement in technology, and if the minimum level of investment is exogenously fixed, the mechanism for the formation of class societies identified in this paper loses its power in the long run. On the other hand, if the engine of growth is endogenous technological change due to investment, and if better technology requires a higher level of minimum investment, then long run growth may never eliminate the class distinction.

The theoretical framework presented above, or some variations of it, should also be useful for the policy analysis. For example, in some cases, redistributing wealth from the rich bourgeoisie to the poor proletariat, by introducing some forms of inheritance taxes, may not only help the poor to escape from the poverty, but also have the effects of increasing the aggregate wealth and the total surplus. In other cases, it may push the economy into the poverty trap, by reducing investment by the rich, which could help the poor workers by creating jobs. It is hoped that the present paper would stimulate further research on these issues.

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[19] This complication arises because B(v) may be unbounded for the range of the wage rates that satisfy the inequality C(v) > P(v).

Appendix

Proof of Lemma.

i) This follows from the fact that $\pi(v)$ are $V(\gamma)$ are positive and decreasing functions and satisfy $\pi(0) = \infty$ and $V(\infty) = 0$.

ii) By differentiation, $\Gamma'(\lambda) = \phi(\theta/\lambda) - (\theta/\lambda)\phi'(\theta/\lambda) > 0$ and $\Gamma''(\lambda) = \phi''(\theta/\lambda)(\theta^2/\lambda^3) < 0$. From L'Hospital's Rule, $\Gamma(0) = \theta \lim_{x\to\infty}[\phi(x)/x] = \theta \lim_{x\to\infty} \phi'(x) = 0$.

iii) By definition, $\pi(v) \geq \phi(n) - nv$, where the equality holds if and only if $n = n(v)$. Thus, by setting $v = V(\Gamma(\lambda))$ and $n = \theta/\lambda$, $\pi(V(\Gamma(\lambda))) \geq \phi(\theta/\lambda) - (\theta/\lambda)V(\Gamma(\lambda))$, where the equality holds if and only if $\theta/\lambda = n(V(\Gamma(\lambda)))$. This is equivalent to $\lambda \geq \Lambda(\Gamma(\lambda))$, where the equality holds if and only if $\lambda = \lambda_c$, where λ_c is defined uniquely by $\theta = \lambda_c n(V(\Gamma(\lambda_c)))$. By setting $\gamma_c = \Gamma(\lambda_c)$ and $\lambda_c = \Lambda(\gamma_c)$, this condition can be rewritten as $\theta = \Lambda(\gamma_c)n(V(\gamma_c))$. Since $\Lambda(\gamma)n(V(\gamma))$ is strictly increasing in γ with $\Lambda(\gamma^+)n(V(\gamma^+)) = 0$ and $\Lambda(\infty)n(V(\infty)) = n(0) = \infty$, $\gamma^+ < \gamma_c < \infty$, from which $0 = \Lambda(\gamma^+) < \lambda_c = \Lambda(\gamma_c) < 1 = \Lambda(\infty)$ follows.

Q.E.D.

Proof of Proposition 2.

First, note that (5) can be written to $\theta V \geq rF - \lambda\pi(V)$, which is equivalent to $\lambda \geq \Lambda(rF)$. Thus, (5) fails under the condition of Proposition 2a) and is satisfied under the conditions of Propositions 2b) and 2c). In particular, if $\lambda < \Lambda(rF)$, the three curves intersect as shown in Figure 3a. Next, let us find the condition under which $P(v_\infty) = \beta v_\infty/(1-\beta r)$ and $C(v_\infty) = F - \lambda\pi(v_\infty)/r$ are tangent below $V(rF)$. Let $z < V(rF)$ denote the point of tangency. Then, $\theta z = rF - \lambda\pi(z)$ and $\theta = -\lambda\pi'(z) = \lambda n(z)$. This implies that $rF = \lambda\pi(z) + \theta z = \lambda[\pi(z) + n(z)z] = \lambda\phi(n(z)) = \lambda\phi(\theta/\lambda) = \Gamma(\lambda)$ and $\lambda = \theta n(x) < \theta n(V(rF))$, or $rF < \Gamma(\theta n(V(rF))) = \theta\phi(n(V(rF)))/n(V(rF))$, which is equivalent to $rF < \gamma_c$. Thus, $P(v_\infty)$ and $C(v_\infty)$ are tangent below $V(rF)$ if and only if $\Gamma(\lambda) = rF < \gamma_c$. Finally, note that $rP(v_\infty) = \theta v_\infty$ is independent of rF and a higher rF moves up $rC(v_\infty) = rF - \lambda\pi(v_\infty)$. Since $P(v_\infty)$ is linear and $C(v_\infty)$ is concave, this means that, if $\Gamma(\lambda) < rF < \gamma_c$, $P(v_\infty)$ and $C(v_\infty)$ intersects twice as shown in Figure 3b, and that, if $rF \leq \Gamma(\lambda)$, $P(v_\infty) \geq C(v_\infty)$ for all $v_\infty < V$, as shown in Figure 3c.

Q.E.D.

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Figure 1: The Labor Market Equilibrium

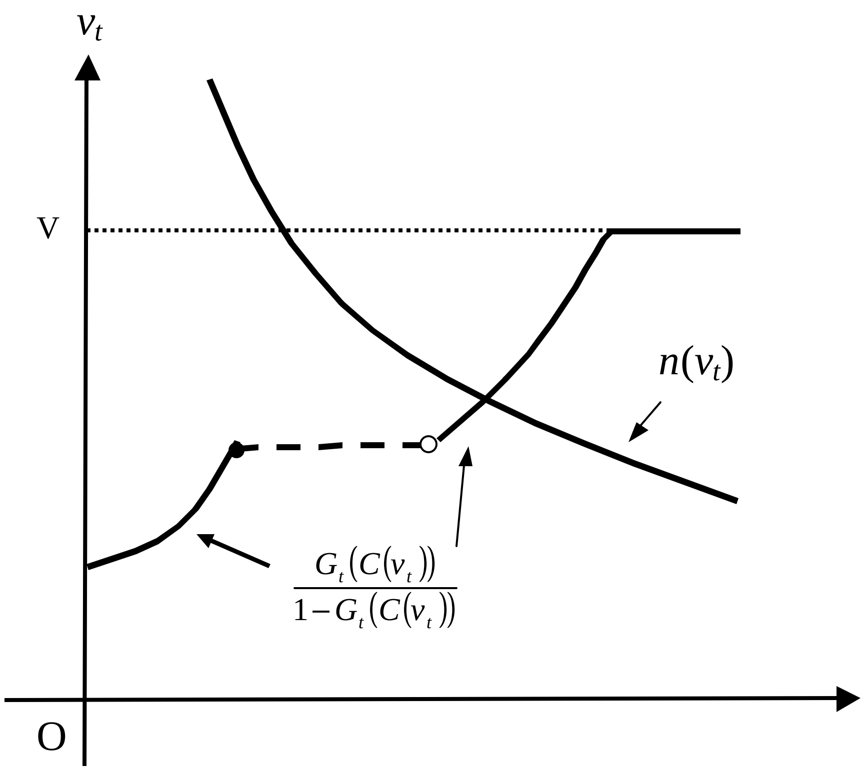

Figure 2: The Household Wealth Dynamics

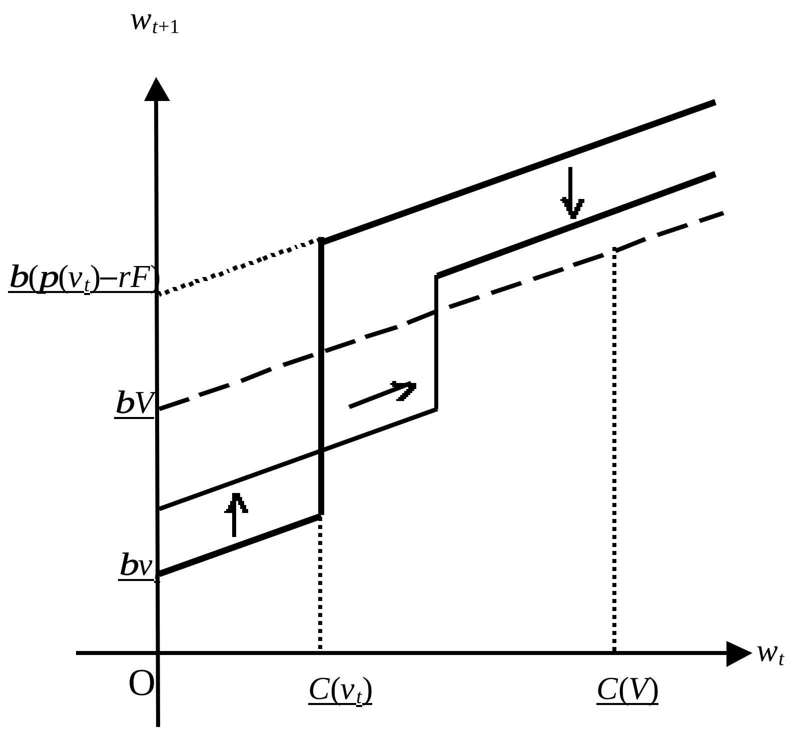

Figure 3: The Three Curves, B(v), C(v), and P(v)

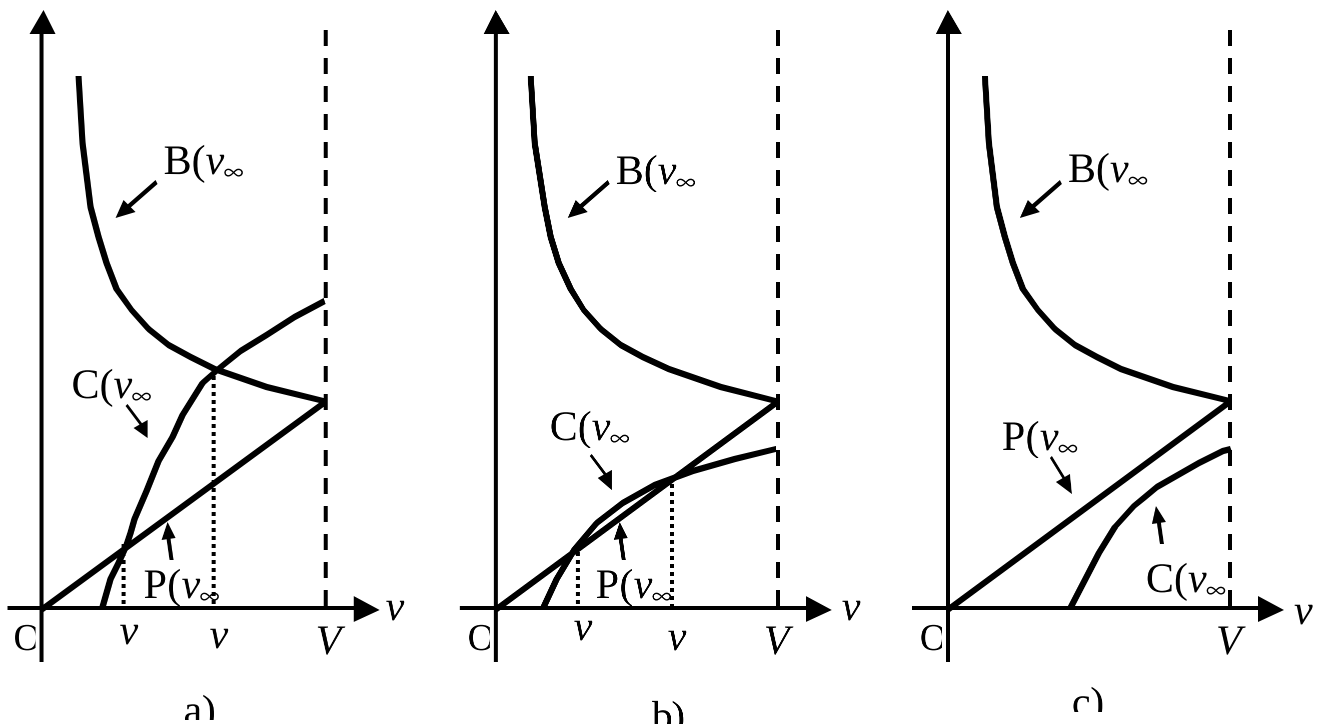

Figure 4: The Parameter Configurations

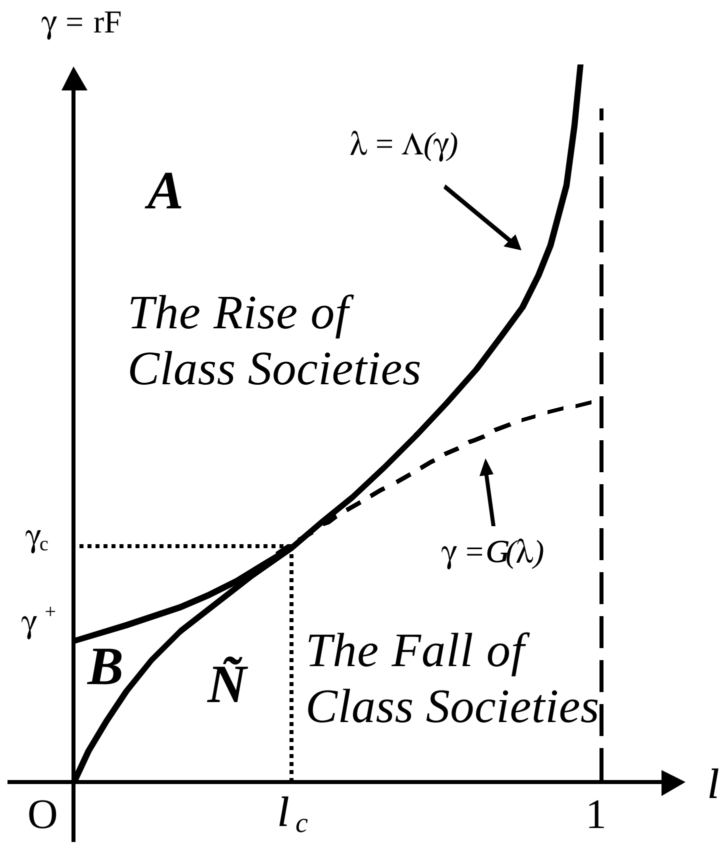

Figure 5: The Labor Market Equilibrium with Self-Employment

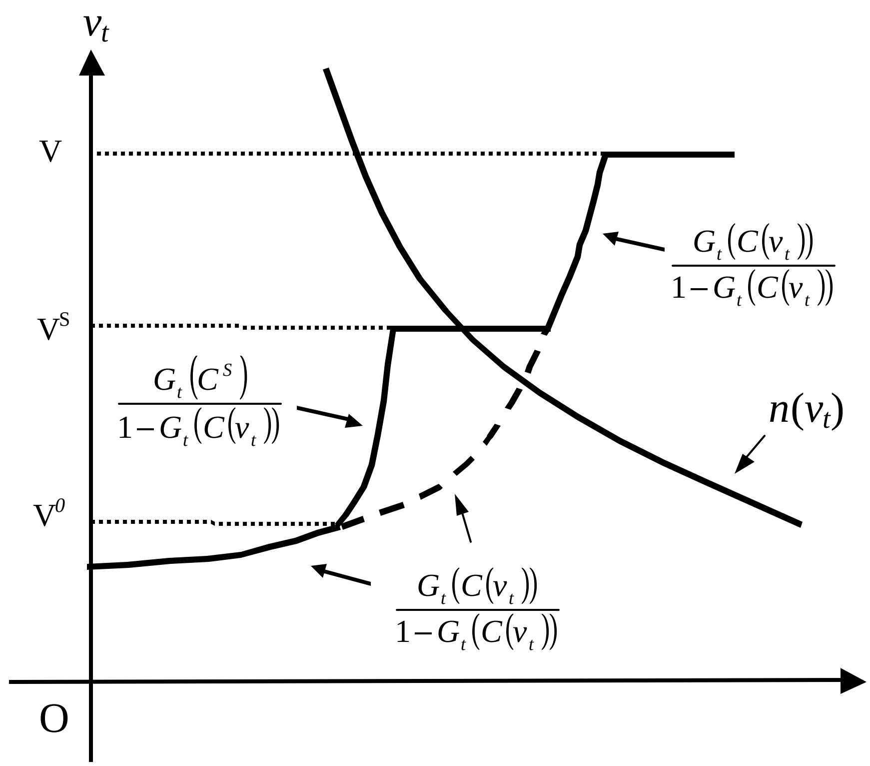

Figure 6: The Household Wealth Dynamics with Self-Employment

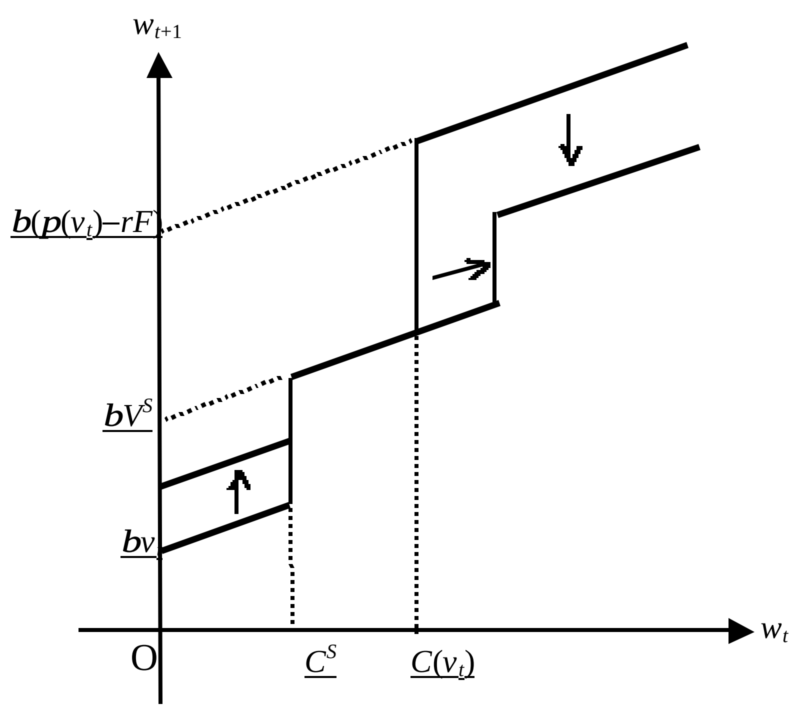

Figure 7: The Labor Market Equilibrium without Diminishing Returns

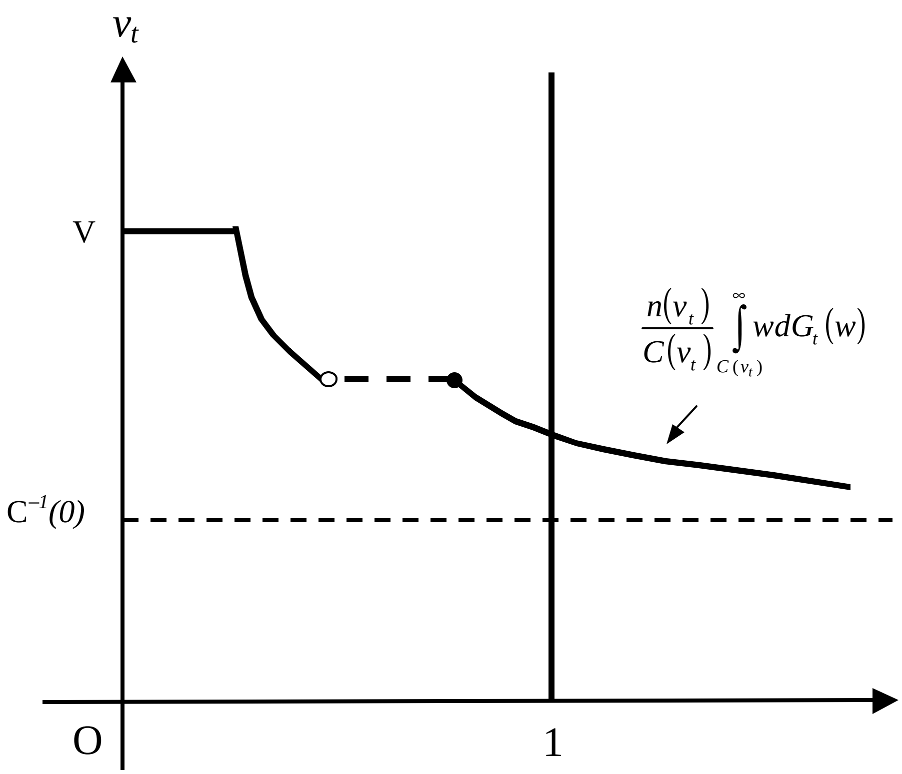

Figure 8: The Household Wealth Dynamics without Diminishing Returns