Stationary Equilibria in Asset-Pricing Models with Incomplete Markets and Collateral*

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Abstract

We consider an infinite-horizon exchange economy with incomplete markets and collateral constraints. As in the two-period model of Geanakoplos and Zame (1998) households can default on their liabilities at any time without any utility penalties or loss of reputation. Financial securities are therefore only traded if the promises associated with these securities are backed by collateral. We examine an economy with a single perishable consumption good where the only collateral available consists of productive assets. In this model competitive equilibria always exist and we show that under the assumption that all exogenous variables follow a Markov-chain there also exist stationary equilibria. These equilibria can be characterized by a mapping from the exogenous shock and the current distribution of financial wealth to prices and portfolio choices. We develop an algorithm to approximate this mapping numerically and give details on how to implement the algorithm in practice. Two computational examples demonstrate the performance of the algorithm and show some quantitative features of equilibria in models with collateral and default.

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1 Introduction

In a model with infinitely lived agents, investors' possible trading strategies have to be restricted to avoid Ponzi schemes. Levine and Zame (1996) propose a so-called 'implicit debt constraint' which ensures that in equilibrium agents' unconstrained Euler equations always hold. Unfortunately, when there are stocks in the economy which pay dividends over more than one period, when endowments and dividends are stationary, and when markets are incomplete, sequential equilibria do not always exist under this constraint[1]. Moreover, in calibrated models, an implicit debt constraint implies unrealistically high levels of individual debt – along the equilibrium path agents sometimes borrow more than 20 times their yearly income. These problems have led applied researchers to impose much tighter exogenous debt constraints as well as short-sale constraints on stocks. Without attempting to model default, however, these constraints have no economic interpretation and there is little empirical evidence in their support. This deficiency motivates the inclusion of collateral and default in these models.

Dubey et al. (2000) and Geanakoplos and Zame (1998) incorporate default and collateral into the standard GEI model. In Geanakoplos and Zame (1998), agents have to put up durable goods as collateral when they want to take short positions in financial markets. Economic agents are allowed to default on their promises but in the case of default the collateral associated with the promise is seized and distributed among the creditors. Araujo et al. (2000) extend the model to a dynamic framework with infinitely lived agents and prove equilibrium existence.

We consider a special version of the model with a single (perishable) consumption good. We assume that all exogenous variables follow a first-order Markov chain. When there are no financial securities, the only assets traded are shares in the existing firms and the model is very similar to the Lucas asset-pricing model with heterogeneous agents as in Duffie et al. (1994). The only difference is that we allow the Lucas trees' output to depend on the current owner of the tree to incorporate the possibility that the productivity of an asset depends on its owner. With financial assets, agents who sell financial securities promise payoffs at some future date. They will default on these promises if and only if the market value of the stocks held as collateral is lower than the face value of the promise.

The treatment of default in this model (as well as in Geanakoplos and Zame (1998)) is unconvincing since default 'does not matter': It does not affect households' ability to borrow in the future and it does not lead to any direct reduction in consumption at the time of default. Nevertheless, since it is analytically tractable and provides a rationale for collateral constraints, we use default to motivate collateral constraints in infinite horizon models.

The reason for assuming that all exogenous variables are stationary is that we want to approximate equilibria numerically. It is standard in modern macroeconomics to study the

[1] Magill and Quinzii (1996) show existence generically in infinite dividend sequences but their genericity conditions do not translate directly to conditions on an economy with stationary dividends and endowments.

equilibrium dynamics indirectly by using recursive methods. According to Ljungqvist and Sargent (2000) recursive methods characterize "a pair of functions: a transition function mapping the state of the model today into the state tomorrow" and a policy function "mapping the state into the other endogenous variables of the model".

Previous computational work on models with incomplete markets and heterogeneous agents (e.g. Heaton and Lucas (1996), Judd et al. (2000)) focuses on equilibria that can be described recursively using as a state the current shock together with last period's portfolio holdings. Unfortunately, for interesting specifications of the model, there are no known conditions which guarantee the existence of such equilibria (independently of collateral constraints and default – see Kubler and Schmedders (2001 a)). When the state is taken to include *all* current endogenous variables, a recursive equilibrium does exist (see Duffie et al. (1994)) but the transition function that maps state today into state tomorrow can be arbitrarily complicated and one often cannot even determine the set over which it is defined.

In this paper we develop an alternative approach to generate time series of equilibrium prices and allocations in models with incomplete markets and heterogeneous agents. We take as the endogenous state the collection of all current endogenous variables. Our stationary equilibrium is similar to a recursive equilibrium in that it also consists of a transition map and a policy map. However, in our approach the policy map is a correspondence (which may not be single valued) mapping beginning-of-period financial wealth (cash at hand) into possible current period equilibrium prices and portfolios. The transition maps the current state into next periods' endogenous states under the restriction that these have to lie in the graph of the policy map. Given the policy correspondence and a finite number of exogenous states with known Markov transition, finding next period's state from current period's state amounts to finding a solution to the system of agents' Euler equations which lies in the graph of the policy map. If the policy map is finite valued, this is generally a computationally easy task. The main problem is then to construct a policy map.

Adapting an argument from Duffie et al. (1994) one can construct a non-empty policy correspondence. This construction then implies the existence of a Markov-equilibrium. However, even when there are only two agents and the set of possible wealth distributions is one-dimensional the construction is not computationally feasible since at each iteration it requires to list all (real) solutions to a non-linear system of equations. Instead we try to approximate the correspondence by a collection of functions. It is important to emphasize that the implementation of the algorithm is not guaranteed to converge. However, in a companion paper (Kubler and Schmedders (2001 b)) we use the algorithm to investigate under which conditions collateral constraints and default increase asset price volatility in realistically calibrated models and in all of the cases considered there the algorithm finds an approximate equilibrium correspondence.

Since the dimension of the domain of the equilibrium map is independent of the number of securities traded, this setup is a significant practical improvement over existing algorithms which, due to a curse of dimensionality, cannot consider models with more than 2 assets (see Judd et al. (2000) for an overview). In particular our algorithm can be used to compute equilibria in a model where all assets are Lucas-trees and where there is no default (as in Duffie et al. (1994)).

The paper is organized as follows. We introduce the model in Section 2. In Section 3 we develop a theoretical algorithm. Section 4 describes how the algorithm is implemented and gives examples.

2 The Economic Model

We examine a model of an exchange economy which extends over an infinite time horizon and which is populated by infinitely-lived heterogeneous agents.

The Physical Economy

At each period $t \geq 0$ one of Y possible exogenous shocks $y \in \mathcal{Y} = \{1, \ldots, Y\}$ occurs. We represent the resolution of uncertainty by an event tree Σ. The root of the tree σ_0 is given by a fixed state $y_0 \in \mathcal{Y}$ in which the economy starts at time 0. Each node of the tree is characterized by a history of shocks $\sigma = (y_0 \cdots y_t)$. Each node σ has Y immediate successors, (σy), $y = 1, \ldots, Y$, and a unique predecessor σ^*. To simplify notation we sometimes refer to the root node's predecessor $\sigma_0^* = 0_-$ and include it in the event tree Σ. We collect all possible nodes which can occur at time t in a set $\Sigma_t \subset \Sigma$. The exogenous shocks follow a time-homogeneous Markov process with transition matrix π. To simplify the exposition, we assume that all elements of π are strictly positive.

At each node $\sigma \in \Sigma$ there is a single perishable consumption good. There are H agents which we collect in a set $\mathcal{H}$. At node $\sigma \in \Sigma$ agent $h's$ individual endowment in the consumption good, $e^h(\sigma)$, depends on the current shock alone, i.e. $e^h(\sigma^* y) = e^h(y)$ where $e^h : \mathcal{Y} \to I\!R_{++}$ is a time-invariant function.

In addition, the agent owns shares in physical assets (Lucas trees - these assets can be interpreted as firms, machines, land or houses). There are A different such assets which we collect in $\mathcal{A}$. At period 0 each agent h owns a share $\theta_a^h(0_-) \geq 0$ of tree a, and we normalize $\sum_{h \in \mathcal{H}} \theta_a^h(0_-) = 1$ for all $a \in \mathcal{A}$. We assume that all initial tree holdings are proportional[2], i.e. for all $h \in \mathcal{H}$ there is a $\bar{\theta}^h(0_-)$ such that $\theta_a^h(0_-) = \bar{\theta}^h(0_-)$ for all $a \in \mathcal{A}$.

At a node $\sigma = (\sigma^* y)$ let $\theta_a^h(\sigma)$ denote agent h's (end-of-period) share in tree a. From this holding he gets a dividend payment $\theta_a^h(\sigma) \cdot d_a^h(y)$ where for each agent h the dividend

[2]We need this non-standard assumption in Corollary 1 and for computational purposes in Section 4.

asset a produces when it is held by him, $d_a^h : \mathcal{Y} \to I\!R_+$ depends solely on the current state $y \in \mathcal{Y}$. If d_a^h is identical across all agents h we are in the standard Lucas-model. In order to allow for possibility of fire sales as in Kiyotaki and Moore (1997) or Shleifer and Vishny (1992) we allow the productivity of some collateralizable assets to depend of their current owner. One interpretation is that the asset is a machine whose operation requires human capital and that this human capital cannot be traded in the same way as physical capital. With this construction aggregate consumption will depend on the distribution of assets and will therefore be endogenous. We define an upper bound on aggregate consumption for each shock y

$$\bar{e}(y) = \sum_{h \in \mathcal{H}} e^h(y) + \sum_{a \in \mathcal{A}} \max_{h \in \mathcal{H}} d_a^h(y)$$

The agent maximizes a time-separable expected utility function

$$U_h(c) = E\left\{\sum_{t=0}^{\infty} \beta^t u_h(c_t)\right\}.$$

We assume that $u_h(.) : I\!R_{++} \to I\!R$ is strictly monotone, C^2, strictly concave, bounded above and unbounded from below, i.e. $u_h(x) \to -\infty$ as $x \to 0$. In order to rule out that agents are almost satiated at aggregate endowments we assume that there exists a consumption level $\hat{c}$ such that for all $h \in \mathcal{H}$,

$$u_h(\hat{c}) > \frac{1}{1-\beta} u_h(\max_y \bar{e}(y)).$$

We also assume that the discount factor $\beta \in (0,1)$ is the same for all agents, and that all agents agree on the transition matrix for the shock process.

Markets

We consider an economy with security trading in every time period. Agent h can buy $\theta_a^h(\sigma)$ shares of tree a at node σ for a price $q_a(\sigma)$. As long as $\theta_a^h \geq 0$ there is no possibility of default since no promises are made when shares of the physical assets change hands.

In addition to the physical assets there are J financial assets which we collect in a set $\mathcal{J}$. These assets are one-period securities; asset j traded at node σ promises a payoff $b_j(\sigma y) = b_j(y) > 0$ at all successor nodes (σy), $y = 1, \ldots, Y$. We assume that agents can only sell (short) a financial security if they hold shares of trees as collateral. With each financial security $j \in \mathcal{J}$ we associate a vector $k^j = (k_1^j, \ldots, k_A^j) \geq 0$ of collateral requirements with $k_a^j > 0$ for some a. If an agent sells 1 unit of security j she is required to hold k_a^j dollars worth of each asset $a = 1, \ldots, A$ as collateral. If an asset a can be used as collateral for different financial securities, the agent is required to invest k_a^j for each $j = 1, \ldots, J$. In the next period the agent can default on her promise b^j, however, in this case the buyer of the financial security gets the collateral associated with the promise.

We allow the 'margin requirement' k_a^j to be a function of the price of the financial security p_j, i.e. we allow k_a^j to vary with the current price and write

$$k_a^j(\sigma) = \bar{k}_a^j(p(\sigma))$$

for some continuous function $\bar{k}_a^j$. We therefore tie the amount of an asset which has to be held as collateral for a short position of a financial security $j \in \mathcal{J}$ at any node σ to the current price of the asset $q_a(\sigma)$ as well as possibly to the price of the financial security $p_j(\sigma)$. For the case of risk-less borrowing (i.e. the financial security being a bond), for example, it might be useful to assume that the borrower has to hold a fixed amount of collateral for each dollar borrowed (and not for each unit of the financial security). In particular, when the interest rate is high, a borrower has to hold more collateral per dollar promised next period than if the interest rate is low. In order to rule out trivial equilibria we need to assume that for all $j \in \mathcal{J}$,

$$\inf_{p \geq 0} \bar{k}_a^j(p) > 0 \text{ for at least one } a \in \mathcal{A} \tag{1}$$

To simplify notation we will suppress the dependence of k on the current price of the financial security.

Since there are no penalties for default a seller of an asset will default at a node $\sigma = (\sigma^* y)$ whenever $b_j(y) > \sum_{a \in \mathcal{A}} k_a^j \frac{q_a(\sigma)}{q_a(\sigma^*)}$. By individual rationality, the actual payoff of security j at node $\sigma = (\sigma^* y)$ is therefore always given by

$$f_j(\sigma) = \min \left\{ b_j(y), \sum_{a \in \mathcal{A}} k_a^j \frac{q_a(\sigma)}{q_a(\sigma^*)} \right\}.$$

Since the decision to default on a promise is independent of the debtor, we do not need to consider pooling of contracts as in Dubey et al. (2000) even though there might be default in equilibrium. We denote agent h's portfolio in financial assets by ϕ^h and write $p_j(\sigma)$ for the price of asset j.

Note that an agent can default on individual promises without declaring personal bankruptcy and giving up all the assets he owns. This assumption is necessary to avoid the pooling of contracts across borrowers' payments on the contract (the assumption might not be completely unrealistic, e.g. in some states in the United States, households are allowed to declare bankruptcy on their houses only).

A financial markets economy $\mathcal{E}^{fm}$ is now a collection of utility functions, endowments, assets, financial securities, and collateral requirements.

Financial Markets Equilibrium

We define a financial markets equilibrium as follows.

Definition 1 *A financial markets equilibrium for an economy $\mathcal{E}^{fm}$ is a process of portfolio holdings and consumptions $\{(\bar{\theta}^1(\sigma), \bar{\phi}^1(\sigma), \bar{c}^1(\sigma)), \ldots, (\bar{\theta}^H(\sigma), \bar{\phi}^H(\sigma), \bar{c}^H(\sigma))\}$ as well asset prices $\{(\bar{q}_1(\sigma), \ldots, \bar{q}_A(\sigma), \bar{p}_1(\sigma), \ldots, \bar{p}_J(\sigma)))\}$ for all $\sigma \in \Sigma$ satisfying the following conditions:*

(1) Markets clear:

$$\sum_{h\in\mathcal{H}} \bar{\theta}^h(\sigma) = 1 \text{ and } \sum_{h\in\mathcal{H}} \bar{\phi}^h(\sigma) = 0 \text{ for all } \sigma \in \Sigma.$$

(2) For each agent h :

$$(\bar{\theta}^h(\sigma), \bar{\phi}^h(\sigma), \bar{c}^h(\sigma)) \in \arg\max_{\theta\geq 0, \phi, c\geq 0} U_h(c) \text{ s.t. for all } \sigma = (\sigma^* y) \in \Sigma$$

$$c^h(\sigma) = e^h(y) + \phi^h(\sigma^*)\cdot f(\sigma) + \theta^h(\sigma^*)\cdot \bar{q}(\sigma) + \theta^h(\sigma)(d^h(y) - \bar{q}(\sigma)) - \phi^h(\sigma)\bar{p}(\sigma)$$

$$q_a\theta_a^h(\sigma) + \sum_{j\in\mathcal{J}:\phi_j^h(\sigma)<0} k_a^j \phi_j^h(\sigma) \geq 0, a = 1, \ldots, A$$

Note that physical assets are traded cum dividends, that is, buying a tree allows the agent to harvest the fruit in the same period.

The following observation will be important throughout the analysis of this paper. In each financial markets equilibrium, agents' optimality implies that there exists a positive lower bound on an agent's optimal consumption. For each agent h define this lower bound on consumption, $\underline{c}^h > 0$, by

$$u_h(\underline{c}^h) + \frac{\beta}{1-\beta} u_h(\max_{y\in\mathcal{Y}} \bar{e}(y)) \leq \frac{1}{1-\beta} u_h(\min_{y\in\mathcal{Y}} e^h(y)),$$

This positive lower bound for possible equilibrium consumption always exists since u_h is unbounded from below. The term on the right-hand side is a lower bound on the agent's lifetime utility if the agent has sold all her shares in trees, because she can still afford to consume her labor endowments $e^h(y)$. The second term on the left-hand side is an upper bound on the agent's utility after the first period because consumption is bounded above by aggregate endowments (see Duffie et al. (1996) for a similar argument).

Modeling default and collateral constraints

As mentioned in the introduction, there can be no doubt that our treatment of default is unrealistic: In most economic situations default leads to a loss in the borrower's reputation which has effects on his ability to borrow in the future. Furthermore, declaring personal bankruptcy results in a loss of all assets and it is rarely possible to default on some loans while keeping the collateral for others.

An alternative interpretation of the model is that all financial assets are derivative securities whose payoffs depend on the prices of the underlying physical assets.

The advantage of introducing default lies in the fact that it gives a endogenous justification for collateral constraints. Modeling default in this fashion allows us to examine socially optimal margin requirements. Following Dubey et al. (2000) we can also endogenize the margin requirements by introducing a menu of financial securities. Each security promises the same payoff but they distinguish themselves by the margin requirement k^j. In equilibrium only some of them will be traded, thereby determining the endogenous margin requirement (see Kubler and Schmedders (2001 b) for applications of the model).

2.1 Stationary Equilibria

Our objective is to compute financial markets equilibria for our economic model. It is obviously infeasible to calculate prices, asset holdings, and consumptions in an equilibrium as a function of nodes $\sigma \in \Sigma$ over the infinite event tree. Instead, we want to describe equilibria as a mapping from the beginning-of-period wealth distribution to current period equilibrium prices and portfolio holdings. From this we will be able to infer a transition mapping the current state of the economy into next period's state.

The State Space

We let the state space $\mathcal{S}$ consist of all exogenous and endogenous variables which occur in the economy at some node σ, i.e. $\mathcal{S} = \mathcal{Y} \times \mathcal{Z}$, where $\mathcal{Y}$ is the finite set of exogenous shocks and $\mathcal{Z}$ is the set of all possible endogenous variables such as (among others) prices, asset holdings, and consumption allocations. For a set $\mathcal{X}$, let $\mathcal{X}^Y$ be the Cartesian product of Y copies of $\mathcal{X}$.

It is readily apparent that in any financial markets equilibrium $q > 0$. So, we can define $\omega^h(\sigma)$ as household $h's$ fraction of financial wealth in the economy at the beginning of node $\sigma = (\sigma^* y)$, i.e.

$$\omega^h(\sigma) := \frac{\theta^h(\sigma^*) \cdot q(\sigma) + \phi^h(\sigma^*) \cdot f(\sigma)}{\sum_{a=1}^{A} q_a(\sigma)}.$$

Let $\Omega(\sigma) = (\omega^1(\sigma), \ldots, \omega^H(\sigma))$. Note that by the definition of f, in equilibrium, Ω will always lie in the $H-1$ dimensional simplex Δ^{H-1}, i.e. $\omega^h \geq 0 \ \forall h \in \mathcal{H}$ and $\sum_{h\in\mathcal{H}} \omega^h = 1$. Next, let $z(\sigma) = (\Omega(\sigma), (c^h(\sigma), \theta^h(\sigma), \phi^h(\sigma))_{h\in\mathcal{H}}, q(\sigma), p(\sigma))$ be the endogenous state at σ. We define the endogenous state space as

$$\begin{aligned}\mathcal{Z} = \{z \in \ & \Delta^{H-1} \times I\!R_+^H \times I\!R_+^{AH} \times I\!R^{HJ} \times I\!R_+^A \times I\!R_+^J : \\ & \textstyle\sum_{h\in\mathcal{H}} \theta_a^h = 1 \text{ for all } a \in \mathcal{A} \\ & \textstyle\sum_{h\in\mathcal{H}} \phi_j^h = 0 \text{ for all } j \in \mathcal{J} \\ & \textstyle q_a\theta_a^h + \sum_{\{j\in\mathcal{J}:\phi_j^h<0\}} k_a^j \phi_j^h \geq 0 \text{ for all } a \in \mathcal{A}, h \in \mathcal{H}\}.\end{aligned}$$

By definition, in any financial markets equilibrium, all endogenous variables will lie in $\mathcal{Z}$. We will also make frequent use of the set of endogenous variables without the wealth variables and denote this set by $\hat{\mathcal{Z}}$. Note that $\mathcal{Z} = \Delta^{H-1} \times \hat{\mathcal{Z}}$.

The Equilibrium Correspondence

Given an initial wealth distribution Ω and an initial shock y we collect all possible period 0 equilibrium values of the endogenous variables in a set $B_y(\Omega) \subset \hat{\mathcal{Z}}$. Formally, the equilibrium correspondence B is a mapping from $\mathcal{Y} \times \Delta^{H-1}$ to the space $\hat{\mathcal{Z}}$ where $B_y(\Omega)$ is defined to be the set of all $\hat{z} \in \hat{\mathcal{Z}}$ such that given $\theta_a^h(0_-) = \omega^h \quad \forall h \in \mathcal{H}, a \in \mathcal{A}$, and an initial shock y, there exists a financial markets equilibrium with $\hat{z}(0) = \hat{z}$.

Markov Equilibrium

We define a Markov equilibrium to be a financial markets equilibrium for which the state $s \in \mathcal{S}$ follows a Markov process. Since we rule out sunspots and the exogenous uncertainty is discrete, we can describe this process by a 'transition function' F. For each (y, Ω, b) with $b \in B_y(\Omega)$ this function assigns a value $F(y, \Omega, b) = (z_1, \ldots, z_Y) \in \mathcal{Z} \times \ldots \times \mathcal{Z}$. The function F determines the endogenous equilibrium state in a period as a function of the exogenous state in that period and the state (y, Ω, b) in the previous period, thereby justifying the notion of Markov equilibrium.

3 Existence of Equilibrium

Araujo et al. (2000) consider a model that is related to ours without long-lived assets but durable and storable goods and prove that equilibria always exist. However, since we are interested in approximating equilibria numerically an adaptation of their existence proof to our model would not help us. Instead we want to construct the equilibrium correspondence B and show that it is always nonempty. We then argue that using this correspondence, the economy can be simulated numerically: For any finite T the allocation for the first T periods of an (infinite) financial markets equilibrium can be constructed from this correspondence by solving systems of Euler equations.

In constructing B we closely follow the existence proof in Duffie et al. (1994). The basic idea of our approach is very similar to backward induction. We start with a large compact set $\mathcal{T} \subset \hat{\mathcal{Z}}$ that is large enough to ensure that for all finitely truncated economies, in equilibrium, all variables lie in $\mathcal{T}$. The fact that such a set exists is established in Lemma 2 below. The lemma also shows that for any equilibrium of the infinite economy all variables must lie in $\mathcal{T}$. Assuming that next period's equilibrium variables can be described by some correspondence $V^n : \mathcal{Y} \times \Delta^{H-1} \Rightarrow \mathcal{T}$ we define for each exogenous shock y and wealth distribution Ω, $V_y^{n+1}(\Omega)$

to be the set of all endogenous variables today that are consistent (in the sense that all agents' first-order conditions and market clearing hold) with some $((\Omega_1, \hat{z}_1), \ldots, (\Omega_Y, \hat{z}_Y))$ tomorrow for which $\hat{z}_{y'} \in V^n(\Omega)$.

We will show that with this construction $\bigcap_{n=1}^{\infty} cl(V^n(\Omega))$ is a non-empty equilibrium correspondence. In order to formalize these basic ideas we first need some additional definitions.

Expectations Correspondence

Given a state $(y, z) \in \mathcal{S}$, the 'expectations correspondence' g describes all next period states that are consistent with market clearing and agents' first-order conditions. A vector of endogenous variables $(z_1^+, \ldots, z_Y^+) \in g(s)$ if for all households $h \in \mathcal{H}$ the following conditions hold.

(a) For all $y = 1, \ldots, Y$,

$$\omega_y^{h+} = \frac{\theta^h \cdot q_y^+ + \sum_{h \in \mathcal{J}} \phi_j^h \cdot \min\left\{b_j(y), \sum_{a \in \mathcal{A}} k_a^j \frac{q_{ay}^+}{q_a}\right\}}{\sum_{a=1}^{A} q_{ay}^+}$$

and $c_y^{h+} = e^h(y) + \omega_y^{h+} \cdot \sum_{a=1}^{A} q_{ay}^+ + \theta_y^{h+} \cdot (d^h(y) - q_y^+) - \phi_y^{h+} \cdot p_y^+ \geq \underline{c}^h$

(b) There exist multipliers $\lambda^h \in I\!R^A$ and $\mu^h \in I\!R^A$ such that for all $a \in \mathcal{A}$ the following equations and inequalities hold.

$$\begin{aligned}
\mu_a^h q_a + \lambda_a^h + (d_a - q_a) u_h'(c^h) + \beta E^s\left\{q_a^+ u_h'(c^{h+})\right\} &= 0 \\
\lambda_a^h \theta_a^h &= 0 \\
\mu_a^h (q_a \theta_a^h + \sum_{j \in \mathcal{J}: \phi_j^h < 0} k_a^j \phi_j^h) &= 0 \\
\lambda_a^h &\geq 0 \\
\mu_a^h &\geq 0
\end{aligned}$$

(c) In addition, if we define $\phi_j^h(-) = \max(0, -\phi_j^h)$ and $\phi_j^h(+) = \max(0, \phi_j^h)$, there exist multipliers $\nu^h(+), \nu^h(-) \in I\!R^J$ such that for all $j \in \mathcal{J}$ the following conditions are satisfied.

$$\begin{aligned}
\sum_{a \in \mathcal{A}} \mu_a^h k_a^j - p_j u_h'(c^h) + \beta E^s\left\{f_j u_h'(c^{h+})\right\} - \nu_j^h(-) &= 0 \\
-p_j u_h'(c^h) + \beta E^s\left\{f_j u_h'(c^{h+})\right\} + \nu_j^h(+) &= 0 \\
\nu^h(+) \cdot \phi^h(+) &= 0 \\
\nu^h(-) \cdot \phi^h(-) &= 0 \\
\nu^h(+) &\geq 0 \\
\nu^h(-) &\geq 0
\end{aligned}$$

Conditions (a) follow from the definition of ω and the budget constraint. The conditions (b) are part of the standard first-order conditions with respect to θ_a^h. Finally, conditions (c) consist of the Kuhn-Tucker conditions with respect to ϕ_j^h, where we treat the choices of long and short positions in the asset separately (i.e. the agent chooses both $\phi_j^h(-)$ and $\phi_j^h(+)$ and therefore has two first-order conditions for each financial asset). Note that the constraints on the tree holdings are already included in the definition of the endogenous state space $\mathcal{Z}$ and that the sign restrictions on the two portfolio variables for each financial asset are satisfied by definition. Also note that there is a redundancy in the Kuhn-Tucker conditions in (b) – if the collateral constraint is satisfied the short-sale constraint on the collateralizable asset will hold automatically. For pedagogical reasons we keep the redundant conditions in the list of conditions.

The following lemma assures a property of the expectations correspondence g that is important for the existence proof of a financial markets equilibrium. We give the proof of this lemma in the Appendix.

Lemma 1 *The graph of g is a closed subset of the closed set $\mathcal{S} \times \mathcal{Z}^Y$.*

Using the expectations correspondence we can now formalize the details of the backward induction approach for our model.

Constructing an Equilibrium Correspondence

Given a compact set $\mathcal{T} \subset \hat{\mathcal{Z}}$ and a correspondence $V : \mathcal{Y} \times \Delta^{H-1} \Rightarrow \mathcal{T}$, define an operator $G_{\mathcal{T}}$ which maps the correspondence $V : \mathcal{Y} \times \Delta^{H-1} \Rightarrow \mathcal{T}$ to a new correspondence $W : \mathcal{Y} \times \Delta^{H-1} \Rightarrow \mathcal{T}$ as follows.

$W = G_{\mathcal{T}}(V)$ if for all $\Omega \in \Delta^{H-1}$,

$$W_y(\Omega) = \{(c,\theta,\phi,q,p) \in \mathcal{T} : \quad \exists\ (z_1,\ldots,z_Y) \in g(y,\Omega,c,\theta,\phi,q,p) \text{ which satisfy for all } y' \in \mathcal{Y}$$
$$z_{y'} = (\Omega_{y'}, c_{y'}, \theta_{y'}, \phi_{y'}, q_{y'}, p_{y'}) \text{ with } (c_{y'}, \theta_{y'}, \phi_{y'}, q_{y'}, p_{y'}) \in V_{y'}(\Omega_{y'}).\}$$

In order to construct the equilibrium correspondence we first need a suitable set $\mathcal{T}$ which is large enough to contain all endogenous variables. We prove the following lemma in the Appendix.

Lemma 2 *For all $T > 2$ there exists a financial markets equilibrium for the truncated economy in which values of all prices, asset holdings, and consumption allocations lie in a compact set $\mathcal{T}^* \subset \hat{\mathcal{Z}}$.*

Moreover in any financial market equilibrium for the infinite economy the endogenous variables will lie in this set.

Define V^0 by $V_y^0(\Omega) = \mathcal{T}^*$ for all $\Omega \in \Delta^{H-1}$ and all $y \in \mathcal{Y}$. Given a correspondence V^n, $n = 0, 1, \ldots$ define recursively $V^{n+1} = G_{\mathcal{T}^*}(V^n)$. Finally let

$$V^*(\Omega) := \bigcap_{n=1}^{\infty} cl(V^n(\Omega)) \tag{2}$$

We can now state our main theorem.

Theorem 1 *The correspondence V^* is non-empty and an equilibrium correspondence as defined in Section 2 above.*

Proof of Theorem 1. We first prove that V^* is non-empty. Since the first-order conditions are necessary for optimality, Lemma 2 implies that for all n, the correspondence V^n is non-empty; for each initial wealth distribution there exists an n-horizon financial markets equilibrium for which all endogenous variables lie in $\mathcal{T}^*$.

By definition, $V^1(\Omega) \subset V^0(\Omega)$ for all $\Omega \in \Delta^{H-1}$. But if $V^n(\Omega) \subset V^{n-1}(\Omega)$, the definition of G implies directly that $V^{n+1}(\Omega) \subset V^n(\Omega)$. Since the infinite intersection of nested, closed and nonempty sets is itself nonempty, the set $\bigcap_{n=1}^{\infty} cl(V^n(\Omega))$ must be nonempty for all $\Omega \in \Delta^{H-1}$.

It is now straightforward to verify that $V_y^*(\Omega)$ describes a financial markets equilibrium. Since by Lemma 1 the expectations correspondence has a closed graph, there will exist an infinite sequence of variables lying in the graph of V^* which satisfy the expectations correspondence.

Therefore it only remains to be established that the conditions (b) and (c) are necessary and sufficient for agents' optimality. Necessity is standard. The proof of sufficiency follows directly from Duffie et al. (1994), Proposition 3.2. The presence of short-lived assets does not provide any additional technical difficulties because agents' positions in these assets are uniformly bounded in equilibrium through the collateral constraint. □

By construction of the equilibrium correspondence, the proof of the theorem and the assumption that agents' initial endowments in assets are proportional imply the following corollary.

Corollary 1 *There exists a Markov equilibrium.*

4 Computation of Markov Equilibria

The analysis of the previous section suggests that one possible algorithm for the approximation of an equilibrium correspondence would be to recursively approximate V^n until

$$\sup_{\Omega \in \Delta^{H-1}} \left[\sup_{x^{n+1} \in V^{n+1}(\Omega), x^n \in V^n(\Omega)} \|x^n - x^{n+1}\| \right] < \epsilon$$

for some small $\epsilon > 0$. Unfortunately, implementing this strategy computationally is impossible. The recursively constructed correspondences V^n are in general not convex-valued; therefore, they cannot be approximated efficiently. Furthermore, applying the operator $G_{\mathcal{T}}$ involves finding all solutions to a complicated system of nonlinear equation. This task is known to be impossible in general and intractable for the special case of polynomial equations. But even if we were able to approximate the equilibrium correspondence, it would be difficult to simulate time series of endogenous variables if we do not know the transition behavior. In summary, we need to develop a different approach to computing equilibria.

4.1 The Policy Correspondence

In some cases there may exist Markov equilibria whose support is only a subset of the graph of the equilibrium correspondence and it may not be necessary to approximate the entire equilibrium correspondence.

Instead one is interested in a 'policy correspondence', a non-empty valued mapping $P : \mathcal{Y} \times \Delta^{H-1} \Rightarrow \hat{\mathcal{Z}}$ such that $P(y, \Omega) \subset B_y(\Omega)$ for all $\Omega \in \Delta^{H-1}, y \in \mathcal{Y}$, and such that there exists a Markov equilibrium whose support lies in the graph of P. Given such a policy correspondence P, we can generate time series of endogenous and exogenous variables as follows. For $(y, \Omega, \tilde{z})$ with $\tilde{z} \in P_y(\Omega)$ find $s^+ = \left((\Omega_1^+, \tilde{z}_1^+), \ldots, (\Omega_Y^+, \tilde{z}_Y^+)\right)$ such that for each y',

$$\tilde{z}_{y'}^+ \in P_{y'}(\Omega_{y'}^+) \text{ and } s^+ \in g(y, \Omega, \tilde{z}). \tag{3}$$

Note that, trivially by definition, the equilibrium correspondence B itself is a policy correspondence. But more importantly, there may exist Markov equilibria that are represented by a policy correspondence P with the property that $P(y, \Omega)$ is a strict subset of $B_y(\Omega)$. In particular in cases where there exist several equilibria, a policy correspondences could be a single-valued selection of the equilibrium correspondence.

A priori we have, of course, no indication that such "minimal" single-valued policy correspondences do exist. But there is very surprising computational evidence that for many parameterized examples such policy functions appear to exist. This observation is very important for the development and implementation of a robust and fast algorithm. Before we describe our algorithm we first discuss the existence issue of policy functions.

'Recursive' Equilibria and Policy Functions

When the equilibrium correspondence is single-valued then there clearly exists a policy function. In that special case there would exist a unique financial markets equilibrium. Unfortunately there are no known conditions ensuring uniqueness of infinite-horizon equilibria when markets are incomplete. Although uniqueness is not a necessary condition for the existence of a single-valued policy correspondence, it is also not known if there are weaker conditions

which ensure the existence of a single-valued selection of the equilibrium correspondence which describes a financial markets equilibrium.

The issue is closely related to the existence of 'recursive' equilibria for which the endogenous state space consists of agents' previous period's portfolio positions. This notion of equilibrium is typically used in applications, see for example Heaton and Lucas (1996) and Judd et al. (2000). There are also no known conditions for the existence of such recursive equilibria (see Kubler and Schmedders (2001 a)) even though their existence is routinely assumed in applied work.

However, there are a few important differences between recursive equilibria and the Markov equilibria that can be described by a single-valued policy correspondence. To understand these differences consider the simplest case of a model without financial securities (and therefore no collateral constraints and no default). In this case a recursive equilibrium is described by a function mapping $\left(\Delta^{H-1}\right)^A \times \mathcal{Y}$ to current-period prices and portfolios. If in addition to single-valuedness we also assume continuity the existence of a recursive equilibrium implies the existence of single-valued (continuous) policy correspondence but not vice versa. In this situation our concept is therefore more general.

If the policy correspondence is single valued and *continuous* a recursive equilibrium, if it exists, can be constructed as follows. For all $\Theta = (\theta^h)_{h\in\mathcal{H}} \in \left(\Delta^{H-1}\right)^A$, the function $\Gamma_\Theta : \Delta^{H-1} \to \Delta^{H-1}$ maps a wealth distribution Ω to a wealth vector with elements $\frac{\theta^h \cdot q(\Omega)}{\sum_{a=1}^A q_a(\Omega)}$. This function is continuous and therefore always has a fixed-point. The policy function therefore describes a recursive equilibrium.

However, there is no guarantee that a recursive equilibrium does exist: For a given $\Theta = (\theta^h)_{h\in\mathcal{H}}$ there may be several fixed-points, that is, there may be several different pairs of asset prices $q(\Omega)$ and wealth distribution Ω in the policy correspondence. In that situation a recursive equilibrium may not exist because the portfolio variables do not determine a unique equilibrium behavior (e.g., asset prices). Intuitively, the fact that a recursive equilibrium does not use any information about last period's prices or consumption leads to non-existence. In contrast, the fact that Equation (3) and the expectations correspondence are using more information from the previous period is then crucial to establish existence of a Markov equilibrium.

On the other hand, if there does exist a recursive equilibrium with *continuous* pricing and policy functions, a single-valued policy correspondence must exist. Clearly, for each Θ, y, and $q = q(\Theta)$ the wealth share ω^h is simply given by $\frac{\theta^h \cdot q(\Theta)}{\sum_{a=1}^A q_a(\Theta)}$. If the resulting policy correspondence is not single valued and if for some initial wealth distribution $\Omega \in \Delta^{H-1}$ there exist several equilibria then there will also exist several equilibria for the initial distribution of the tree $\theta_a^h = \omega^h$ for all $a \in \mathcal{A}$. But if a continuous recursive equilibrium exists, there exists a continuous selection over $\left(\Delta^{H-1}\right)^A$ and there must also exist a selection over Δ^{H-1}.

Evidently, without continuity or with financial assets, the existence of a recursive equilib-

rium is neither necessary nor sufficient for the existence of a single-valued policy correspondence in our model: This correspondence is defined on the endogenous shares of financial wealth but not on predetermined previous period's variables. Moreover, previous period's prices affect the payoffs of financial securities.

4.2 Implementation of the Algorithm

The Algorithm to Approximate Policy Functions

While there is no general existence proof for a policy *function*, the construction of our algorithm is based on the assumption that a policy function does exist. In the remainder of the paper we denote policy functions by $\rho : \mathcal{Y} \times \Delta^{H-1} \to \mathcal{T}^*$.

We compute a policy function via an iterative algorithm. First, as a starting point we choose a continuous function $\rho^0 : \mathcal{Y} \times \Delta^{H-1} \to \mathcal{T}^*$. Secondly, in each iteration of the algorithm, given functions ρ^n, we construct $\rho_y^{n+1}(\Omega)$ for all $y \in \mathcal{Y}, \Omega \in \Delta^{H-1}$ by solving for prices and optimal choices in a period that satisfy the expectation correspondence g for a given shock y and wealth distribution Ω. Intuitively, we assume that the values of the endogenous variables in the next period as a function of agents' wealth is determined by the function ρ^n. We then proceed one period backwards and determine ρ^{n+1} by solving for an equilibrium in the current period. In the language from Section 3, we search for a single-valued selection of $G_{\mathcal{T}^*}(\rho^n)$.

The algorithm terminates if for some prespecified $\epsilon > 0$,

$$\sup_{y,\Omega} \|\rho^{n+1} - \rho^n\| < \epsilon.$$

We then have found an approximate policy function $\rho^* = \rho^{n+1}$ describing a Markov equilibrium.

If there does not exist a policy function, then we would expect the algorithm to fail to terminate. There are two main reasons for such a failure. First, $G_{\mathcal{T}^*}\rho^n$ could be empty for some y, Ω. Secondly, even if for all n the correspondence $G_{\mathcal{T}^*}\rho^n$ is non-empty, it is not guaranteed that ρ^n converges as $n \to \infty$. In our practical computational experience neither of these two cases has occurred. The implemented algorithm has always produced a continuous (approximate) policy function.

Outline of Main Steps

The actual implementation of the algorithm on a computer involves many tedious details, many of which are common in the literature on finding equilibrium functions for infinite-horizon problems. We do not discuss them in detail here and instead refer to the survey by Judd et al. (2000). Instead, we focus on the aspects of the algorithm that are new to the literature. The most important innovations in our algorithm are necessary due to the fact that the wealth distribution is used as a sufficient endogenous state variable.

We restrict attention to economies with $H = 2$ agents. In this case the wealth distribution is represented by a single number in the unit interval and so the equilibrium policy function ρ is a function of the discrete exogenous state variable y and the single continuous wealth variable ω^1 only.

We approximate the equilibrium functions by cubic splines (i.e. twice differentiable piecewise cubic functions). In order to handle non-differentiabilities in the policy functions we use 300 collocation points. We solve for the spline coefficients using a time iteration collocation algorithm similar to the one described in detail in Judd et al. (2000).

Given a wealth distribution, $\Omega = (\omega^1, 1 - \omega^1)$, the current shock y today and given the spline coefficients of an approximate map from tomorrow's wealth distribution into tomorrow's prices and portfolio holdings, ξ, one needs to solve for prices and optimal choices today which satisfy the expectation correspondence. In order to make this computationally tractable we formulate the problem as a system of non-linear equations

$$F(q, p, (\theta^h, \phi^h)_{h=1,2}; \omega^1, y, \xi) = 0 \tag{4}$$

The Kuhn-Tucker inequalities in the agents' first-order conditions can easily be replaced by equalities via a change of variable (see Garcia and Zangwill (1981)). The main problem one faces is that in order to infer tomorrow's wealth distribution (which is needed to obtain the (approximate) optimal choices and prices tomorrow from the guess of tomorrow's equilibrium map one needs to solve a nonlinear system of equations. Formally, given choices $(\theta^h, \phi^h)_{h=1,2}$ today, as well as approximate equilibrium price functions $\hat{q}(\Omega; \xi)$, tomorrows wealth distribution Ω^+ must solve

$$\omega^{1+}(y) := \frac{\theta^1 \cdot \rho_q^n(y, \omega^{1+}; \xi) + \phi^1 \cdot f(y, \omega^{1+})}{\sum_{a=1}^{A} \rho_{q_a}(y, \omega^{1+}; \xi)}. \tag{5}$$

We can solve this single equation in the one unknown ω^{1+} using a standard Newton solver. Notice that via Equation 5 we have tomorrow's optimal choices implicitly defined as functions of today's portfolio choices, and so Equation 4 is well defined.

The subproblem of solving Equation 5 is an innovation that is necessary due to the use of the state variable wealth. The novel issue in our set-up is that, contrary to the literature of computing recursive equilibria, the agents' choice (portfolio) variables are not state variables. Agents' portfolio decision today do not immediately yield tomorrow's wealth distribution.

At the beginning of iteration n some spline coefficients ξ^n for the policy function iterate ρ^n are given. For each shock $y = 1, \ldots, Y$, and on a grid of $\omega^1 \in [0, 1]$ the algorithm solves the nonlinear system of Equations 4 for prices and choices $(q, p, (\theta^h, \phi^h)_{h \in \mathcal{H}})$. The resulting functions $(q, p, (\theta^h, \phi^h)_{h \in \mathcal{H}})(y, \omega^1)$ are then approximated by splines with new coefficients ξ^{n+1} and the next iteration begins.

The algorithm terminates if $\|\xi^{n+1} - \xi^n\| < \epsilon$ for some small (prespecified) $\epsilon > 0$. We then examine the quality of the approximate solution by computing the maximum relative error of

Equations 4 using the approximated equilibrium functions. In the examples presented below these maximum errors were never above 10^{-5}.

4.3 Examples

The purpose of the following example is to illustrate the output of the algorithm, that is, the policy functions describing the equilibrium portfolios and prices as a function of the state variable wealth. In addition, we show simulated time series of simulations of the model based on the computed policy functions. We use the simple example to illustrate how *high* margin requirements can lead to excess volatility in asset prices and that there are cases where low requirements lead to substantial default but low volatility.

Consider the following simple specification of an economy. There are $Y = 4$ possible exogenous shocks in each period. There is a single physical asset with shock- and agent-dependent payoffs, which, for simplicity, we just call stock. The agents can use the stock for collateralized borrowing, that is, there is a single one-period bond with payoffs $b_1(y) = 1$ for all $y \in \mathcal{Y}$. The two agents have identical log-utility with a discount factor of $\beta = 0.96$. The following table summarizes individual endowments and the dividends of the stock.

y	1	2	3	4
e^1	3	3	7	7
d_1^1	0.6	1.2	0.6	1.2
e^2	7	7	3	3
d_1^2	1	2	1	2

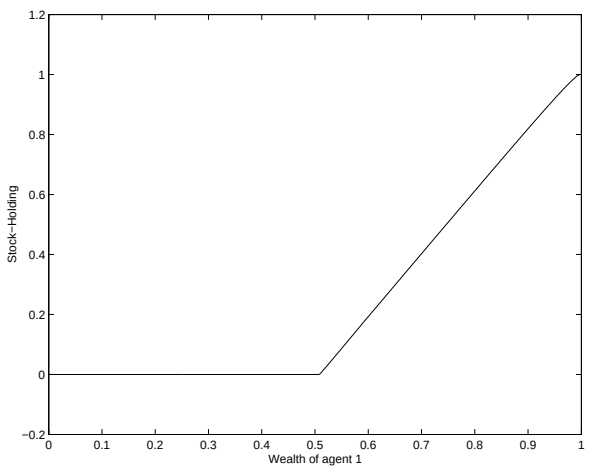

Figure 1a. Stock holding of agent 1

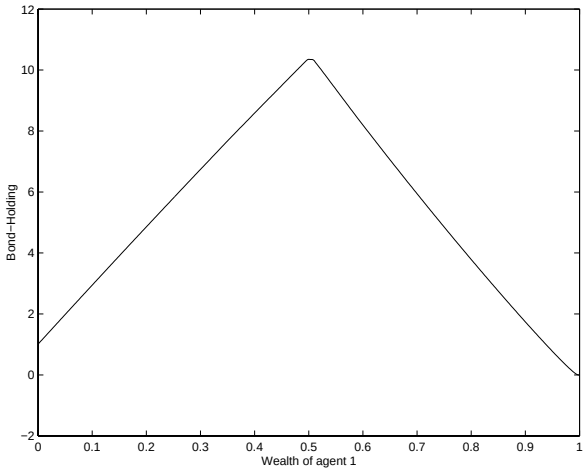

Figure 1b. Bond holding of agent 1

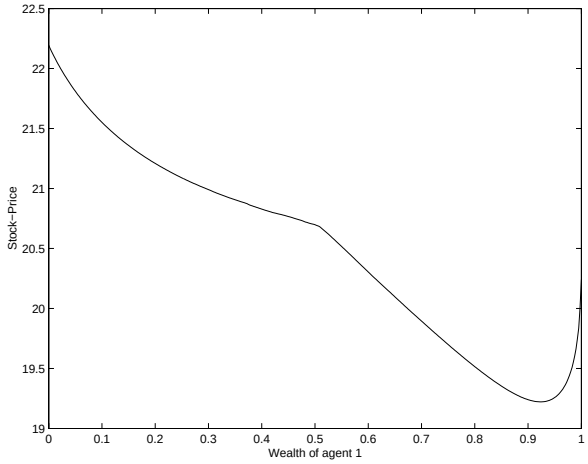

Figure 2a. Price of Stock

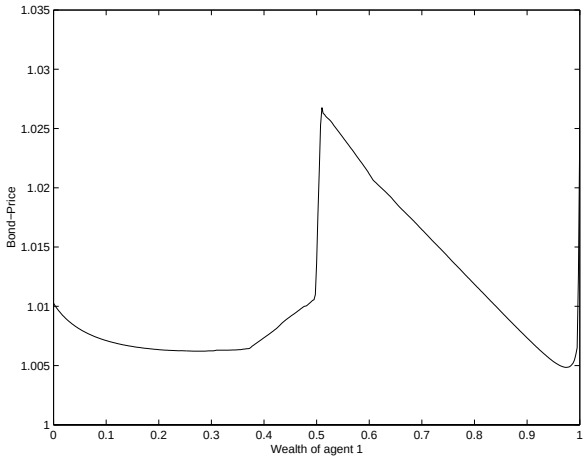

Figure 2b. Price of Bond

The shocks to individual endowments are persistent (the probability to stay in the same income state is 0.8) while shocks to dividends are i.i.d. and independent of the shocks to endowments. Note that the output of the stock is substantially higher if it is owned by agent 2.

We consider two different specifications for the collateral requirement, $\kappa_1 = 1$ and $\kappa_2 = 2$. For $\kappa = 2$ there is never default in equilibrium. Figures 1a and 1b depict the policy portfolio functions, mapping the wealth distribution and shock $y = 2$ into the asset holdings of agent 1. Figures 2a and 2b show the corresponding policy price functions.

Agent 2's human capital allows her to obtain a higher output of the stock than agent 1, and so she tries to hold the entire stock whenever that is affordable for her. However, when her wealth becomes sufficiently low, the collateral constraint forces her to sell off part of the stock. When this sell-off occurs, then the price of the stock decreases, since it is now held by the unproductive agent yielding lower dividends. The price of the bond peaks up, when the collateral constraint starts binding.

The policy functions themselves are of limited interest for applied work since they do not directly say anything about equilibrium prices and allocations along a path of the event tree. In order to obtain this information, we simulate the economy as described in Section 4.1, Equation (3). Figures 3a and 3b show agent 1's stock holdings and the price of the stock, respectively, along one such simulation. The figures show clearly how the collateral constraint can lead to a low stock price. When the collateral constraint starts binding and agent 2 has to sell of part of the stock, the price of the stock decreases (because only agent 1 is there to buy it) which results in 'fire-sales' and a further decrease of the price of the stock. The figures are consistent with the story told by Kiyotaki and Moore (1997).

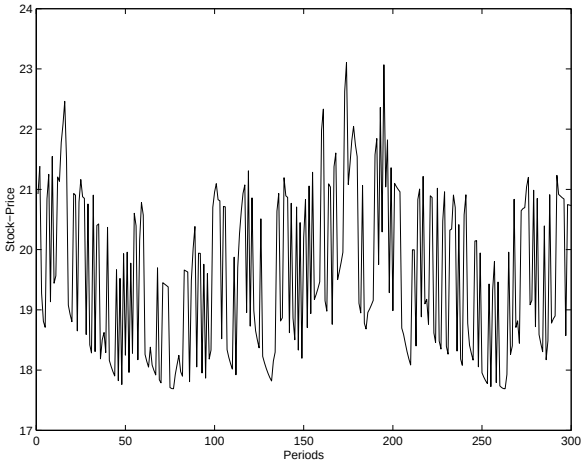

Figure 3a. Simulated stock price

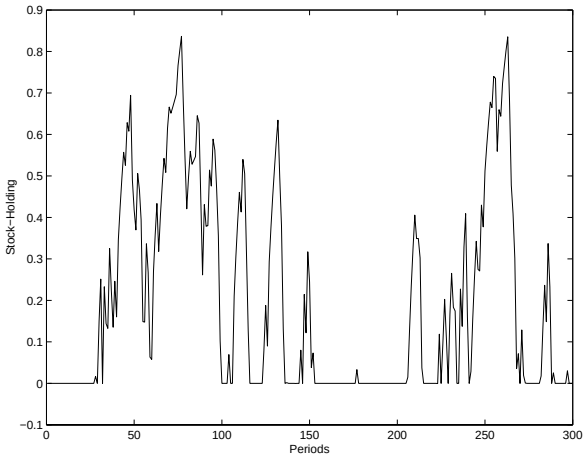

Figure 3b. Simulated stock holding

However, while the differences in productivity between agent 1 and agent 2 are substantial, the stock price effect of the collateral constraint is quantitatively quite small. Furthermore, these effects disappear when the margin requirement is lowered. In the extreme case with

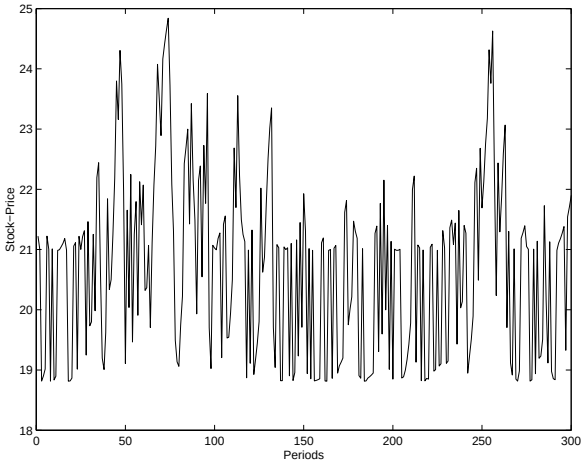

Figure 4a. Simulated stock price for $\kappa = 1$

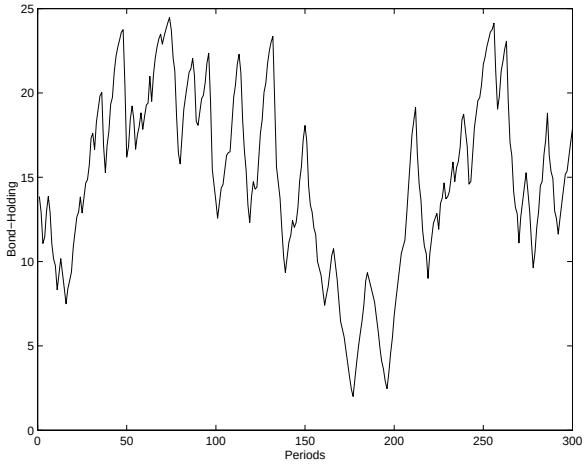

Figure 4b. Simulated bond holding for $\kappa = 1$

$\kappa = 1$, there is default in 35 percent of the cases. But the productive agent always holds the stock and fire-sales never occur! Figures 4a and 4b show stock prices and bond holdings of agent 1 for the same sequence of exogenous shocks. Since agent 2 always holds the entire stock the price never falls below 18.8. Instead of selling the stock, the agent can smooth his consumption sufficiently by trading in the bond.

5 Conclusion

In this paper we formulate an asset pricing model with infinitely lived heterogeneous agents, collateral constraints and incomplete markets. We prove the existence of a Markov equilibrium and show how one can attempt to describe this equilibrium numerically. For the special case where there exists a single-valued policy correspondence we develop an algorithm to compute equilibrium. The practical advantage of our approach is that we do not face a 'curse of dimensionality' in the number of securities.

There is a related, applied literature in macroeconomics which considers models with collateral constraints. Fernandez and Krueger (2000) consider a finite horizon model very similar to ours but without aggregate uncertainty. In their model, margin requirements are endogenous and set to ensure no default. Lustig (2000) considers a model with infinite time-horizon and possibly aggregate uncertainty, but he assumes that a complete set of Arrow securities is traded each period. In his model default is never optimal and margin requirements ensure that it never happens in equilibrium. This literature relates collateral constraints to debt constraints of the Kehoe and Levine (1993) type.

Appendix 1: Proofs

Proof of Lemma 1. Consider any sequence $(s^n, z_1^n, \ldots, z_Y^n)$ in the graph of the g. If $s^n \to \bar{s}$ and $(z_1^n, \ldots, z_Y^n) \to (\bar{z}_1, \ldots, \bar{z}_Y)$ then $(\bar{z}_1, \ldots, \bar{z}_Y)$ will lie in $\mathcal{Z}$ because $\mathcal{Z}$ is closed by definition. Furthermore the limit will satisfy all conditions (a) – (c) because they are all equations or weak inequalities and are satisfied by any point in the sequence.□

Proof of Lemma 2. We extend the existence proof of Radner (1972) to our model with collateral and default in order to show that for every finite T there exists a financial markets equilibrium, see Geanakoplos and Zame (1998) for a similar argument. Denote the set of all nodes over periods $0, 1, \ldots, T$ of the truncated event tree by $\Sigma^T = \cup_{t=0}^T \Sigma_t$. Recall that there exists a $\hat{c}$ such that $u_h(\hat{c}^h) > \frac{1}{1-\beta} u_h(\max_y \bar{e}(y))$. We define $\bar{c} = 2\hat{c}$.

In order to establish upper-hemicontinuity of the best response correspondences we add the following two sets of constraints to each agent's portfolio optimization problem for all $\sigma \in \Sigma^T$.

$$\begin{aligned} \frac{1}{2}\underline{c}^h \leq \quad & c^h(\sigma) \quad \leq \bar{c} \\ & \theta_a^h(\sigma) \quad \leq 2 \end{aligned}$$

Define bounds on the prices for trees.

$$\underline{q}_a = \min_{y \in \mathcal{Y}} \{\max_{h \in \mathcal{H}} d_a^h(y)\} \quad \text{and} \quad \hat{q}_a = H\bar{c}$$

Since it will be shown that in equilibrium it will always be true that $\underline{q}_a \leq q_a(\sigma) \leq \hat{q}_a$ we can use these prices to obtain bounds on the agents' holdings of financial assets in equilibrium. The upper bounds on $\theta_a^h(\sigma)$ and a price $q_a(\sigma)$ imply that the collateral constraint yields a lower bound $\underline{\phi}_j$ on the asset holding $\phi_j^h(\sigma)$ for all agents. Now we can also add the following constraint to an agent's optimization problem.

$$2\underline{\phi}_j \leq \phi_j^h(\sigma) \leq -2H\underline{\phi}_j$$

For the purpose of showing equilibrium existence we change our price normalization. Instead of setting the price of the consumption good at every node σ to 1, we define the good price to be $\rho(\sigma)$ and the price vector $(\rho(\sigma), q(\sigma), p(\sigma))$ to be an element of the unit simplex Δ of dimension $(1 + A + J) - 1$. Agent h's budget equation at node σ is then written as follows.

$$\rho(\sigma)c^h(\sigma) = \rho(\sigma)e^h(y)+\rho(\sigma)\phi^h(\sigma^*)\cdot f(\sigma)+\theta^h(\sigma^*)\cdot q(\sigma)+\rho(\sigma)\theta^h(\sigma)\cdot d^h(y)-\theta^h(\sigma)\cdot q(\sigma)-\phi^h(\sigma)\cdot p(\sigma)$$

Note that summing over all agents yields the equation

$$\begin{aligned}
& \rho(\sigma)\left(\sum_{h=1}^{H} c^h(\sigma) - e^h(y) - \theta^h(\sigma)\cdot d^h(y)\right) \\
- \quad & \rho(\sigma)\left(\sum_{h=1}^{H} \phi^h(\sigma^*)\right)\cdot f(\sigma) \\
+ \quad & q(\sigma)\left(\sum_{h=1}^{H} \theta^h(\sigma) - \theta^h(\sigma^*)\right) \\
+ \quad & p(\sigma)\cdot\left(\sum_{h=1}^{H} \phi^h(\sigma)\right) \\
= \quad & 0
\end{aligned} \tag{6}$$

The augmented budget set of an agent is a compact-valued, convex-valued, and continuous correspondence of the asset price vectors q and p and the good price ρ. Therefore, the agent's demand correspondence for the consumption good and the assets is non-empty, upper-hemicontinuous, compact-valued, and convex-valued.

Now define the total excess good and asset demand correspondence

$$D(\rho, q, p) = (D_c(\rho, q, p), D_\theta(\rho, q, p), D_\phi(\rho, q, p)) = (\sum_{h\in\mathcal{H}} c^h - e^h - \theta^h\cdot d^h, \sum_{h\in\mathcal{H}} \theta^h - 1, \sum_{h\in\mathcal{H}} \phi^h).$$

Moreover, for $\delta = (\delta_c, \delta_\theta, \delta_\phi) \in D(\rho, q, p)$ and $\sigma \in \Sigma^T$ define the optimization problem

$$(OP(\sigma)) \quad \max_{\rho', q', p'} \rho'\delta_c + q'\cdot\delta_\theta + p'\cdot\delta_\phi \text{ s.t. } (\rho', q', p') \in \Delta.$$

The set of optimal (ρ', q', p') for this optimization problem is a nonempty, compact-valued, and upper-hemicontinuous correspondence. We denote this correspondence by $PP_\sigma(\delta)$. The product of these correspondences for all $\sigma \in \Sigma^T$ will be denoted by just $PP(\Sigma^T)$.

The correspondence $PP(\Sigma^T) \times D(\rho, q, p)$ maps $(\rho, q, p, \delta) \in \Delta^{|\Sigma^T|} \times I\!R^{(1+A+J)|\Sigma^T|}$ to a subset of $\Delta^{|\Sigma^T|} \times I\!R^{(1+A+J)|\Sigma^T|}$. This correspondence is nonempty, upper-hemicontinuous, compact-valued, and convex-valued. Kakutani's fixed-point theorem guarantees that this correspondence has a fixed point $(\rho^*, q^*, p^*, \delta^*)$. We show next that $\delta^* = 0$ and $(\rho^*, q^*, p^*) > 0$. The proof is by contradiction.

Suppose there is positive excess demand in some market at time $t = 0$. Suppose the largest excess demand is in the good market. Then the optimal solution for (OP) would be to set $\rho = 1$ and all other prices to 0, resulting in a positive optimal objective function value. But this outcome contradicts equation 6. If the largest excess demand is in one of the asset markets the argument is similar due to the assumption that $\sum_h \theta^h(0^*) = 1$ and $\sum_h \phi^h(0^*) = 0$.

Next, suppose there is negative excess demand in some market at time $t = 0$. Suppose the smallest (that is, most negative) excess demand is in the good market. Clearly it is optimal to set $\rho(0) = 0$. Then all agents want to consume $\bar{c}$ resulting in a large positive excess demand, which is a contradiction. Hence, the good price is positive and the excess good demand equal to zero. Similar arguments hold for all trees $a \in \mathcal{A}$. Since tree prices q are positive, the payoffs of the financial assets are nonzero, and so a similar argument holds for the financial assets as well.

In summary, $\delta^*(0) = 0$ and all prices are positive. By induction it quickly follows that $\delta^*(\sigma) = 0$ and $(\rho(\sigma), q(\sigma), p(\sigma)) > 0$ for all $\sigma \in \Sigma^T$.

Next we prove that all prices stay bounded away from 0 by showing that the quotients $\frac{q^*(\sigma)}{\rho^*(\sigma)}$ and $\frac{p^*(\sigma)}{\rho^*(\sigma)}$ stay within certain positive bounds. The best way to do this is to revert to the original normalization of prices, that is, set $\rho(\sigma) = 1$ for all $\sigma \in \Sigma^T$. Now the task is to show that the asset prices always lie in compact sets not containing zero.

We start by specifying bounds on asset prices. Bounds on tree prices are as above, bounds on the prices of financial assets are as follows.

$$\underline{p}_j = \min_{h \in \mathcal{H}, a \in \mathcal{A}, y \in \mathcal{Y}} \frac{u_h'(\underline{c}^h)}{u_h'(\bar{c})} \min\{b_j(y), \frac{k_a^j \underline{q}_a}{\hat{q}_a}\} \quad \text{and} \quad \hat{p} = \max_{h \in \mathcal{H}, j \in \mathcal{J}, y \in \mathcal{Y}} \frac{u_h'(\underline{c}^h)}{u_h'(\bar{c})} b_j(y).$$

We argue that at all nodes σ it is true that $\underline{q}_a \leq q_a(\sigma) \leq \hat{q}_a$. In equilibrium, at a node $\sigma = (\sigma^*, y)$ the price of tree a cannot be below the highest dividend d_a^h some agent h can obtain from owning the tree. Otherwise she would have an arbitrage opportunity. Suppose that in equilibrium the asset price of tree a exceeds $\hat{q}_a$. Then an agent holding at least $\frac{1}{H}$ units could sell some portion of the tree resulting in a current-period utility of $u(\hat{c}) > \frac{1}{1-\beta} u(\bar{e})$.

Hence, her behavior would not have been optimal before, which is a contradiction.

The first-order conditions with respect to ϕ_j imply directly that $\underline{p}_j \leq p_j(\sigma) \leq \hat{p}$ for all $\sigma \in \Sigma^T$.

It remains to prove that the additional constraints in the augmented utility maximization problems do not affect the optimal solution. First, they cannot be binding in equilibrium by construction. Second, the utility optimization problems have objective functions that are strictly concave in the consumption variables and constant (and so trivially concave) in the asset variables. □

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