Quitting games - An Example

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January 22, 2001

Abstract

Quitting games are n-player sequential games in which, at any stage, each player has the choice between continuing and quitting. The game ends as soon as at least one player chooses to quit; player i then receives a payoff r_S^i, which depends on the set S of players that did choose to quit. If the game never ends, the payoff to each player is zero.

In this note, we study a four-player game, where the simplest equilibrium profile is cyclic with period two.

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1 Introduction

Quitting games are I-player sequential games in which, at any stage, each player has the choice between continuing and quitting. The game ends as soon as at least one player chooses to quit; player i then receives a payoff r^i_S, which depends on the set S of players that did choose to quit. If the game never ends, the payoff to each player is 0.

In such a game, a strategy of player i is a sequence $\mathbf{x}^i = (x^i_n)_{n\geq 0}$, where x^i_n is the probability that player i continues at stage n, provided the game has not terminated before. Such a strategy is stationary if x^i_n is independent of i. We denote by $\mathbf{a}^i_n$ the action played by player i in stage n, and denote by $t = \inf\{n \geq 1, \mathbf{a}^i_n = q^i\}$ the stage in which the game terminates. Given a profile $\mathbf{x}$ of strategies, the expected payoff to player i is

$$\gamma^i(\mathbf{x}) = \mathbf{E}_{\mathbf{x}}\left[r^i_{S_t}\mathbf{1}_{t<+\infty}\right],$$

where $\mathbf{E}_{\mathbf{x}}$ stands for the expectation with respect to the probability distribution induced by $\mathbf{x}$ over the set of plays.

It is not known whether quitting games have an equilibrium payoff. Quitting games therefore form an intriguing class of stochastic games. We recall briefly existing results before presenting the contribution of this note.

In the case of two players, stationary ε-equilibria do exist. A three-player example was devised by Flesch, Thuijsman and Vrieze (1997), where ε-equilibrium strategies are more complex - they have a cyclic structure, and the length of the cycle is at least 3. However, in this example, there are equilibrium payoffs in the convex hull of the vectors $r_{\{i\}} \in \mathbf{R}^I$, $i \in I$. These payoffs can be obtained using a profile $\mathbf{x}$ that plays in any stage a perturbation of $(\mathbf{c}^i)_{i\in I}$. Therefore, it left open the possibility of finding ε-equilibrium profiles, by means of analyzing the limit behavior of stationary equilibria of discounted games, letting the discount factor go to zero. Indeed, such an analysis was provided by Solan (1999), for the more general class of three-player games with absorbing states.

The purpose of this note is to provide a four-player example, where all the ε-equilibrium payoffs involve some kind of cyclic behavior, in which the probability of quitting in any stage is bounded away from zero. The main consequence is that all the known tools for proving the existence of equilibrium payoffs in stochastic games (see, e.g., Tuijsman and Vrieze (1986), Solan (1999, 2000), Vieille (2000a,2000b)) seem likely to fail to yield any result in general I-player quitting games. In a companion paper (Solan and Vieille (2000)), we introduce new tools and provide sufficient conditions under which quitting games have an equilibrium payoffs.

2 The Example

We will study the following four player quitting game:

4

3 (top), **4** (left) — rows: **1**, columns: **2**

1	continue	$4,1,0,0$
	$1,4,0,0$	$1,1,1,1$

3 (top), **4** (right) — rows: **1**, columns: **2**

1	$0,0,4,1$	$1,1,0,1$
	$1,0,1,1$	$0,1,0,0$

3 (bottom), **4** (left) — rows: **1**, columns: **2**

1	$0,0,1,4$	$0,1,1,1$
	$1,1,1,0$	$1,0,0,0$

3 (bottom), **4** (right) — rows: **1**, columns: **2**

1	$1,1,1,1$	$0,0,1,0$
	$0,0,0,1$	$-1,-1,-1,-1$

In this game player 1 chooses a row (top row = continue), player 2 chooses a column (left column = continue), player 3 chooses either the top two matrices or the bottom two matrices, (top two matrices = continue) and player 4 chooses either the left two matrices or the right two matrices (left two matrices = continue).

Note that there are the following symmetries in the payoff function: for every 4-tuple of actions (a, b, c, d) we have:

$$\begin{aligned} v^1(a,b,c,d) &= v^2(b,a,d,c), \\ v^1(a,b,c,d) &= v^4(c,d,b,a) \qquad \text{and} \\ v^2(a,b,c,d) &= v^3(c,d,b,a), \end{aligned}$$

where $v^i(a, b, c, d)$ is the payoff to i if the action combination is (a, b, c, d) $(v^i(c^1, c^2, c^3, c^4) = 0)$.

In section 2.1 we prove that this game admits an equilibrium profile $\mathbf{y}$ that has the following structure:

$$y_n = \begin{cases} (x,1,z,1) & n \text{ odd} \\ (1,x,1,z) & n \text{ even} \end{cases}$$

where $x, z \in]0, 1[$ are independent of n; that is, at odd stages players 2 and 4 continue, while 1 and 3 quit with positive probability, whereas at even stages 1 and 3 continue, while 2 and 4 quit with positive probability.

Thus, the game admits a cyclic equilibrium with period 2.

We then prove the following:

Proposition 1 *The game does not admit a stationary 0-equilibrium.*

Proposition 2 *For ε small enough, the game does not admit an ε-equilibrium* $\mathbf{x}$ *such that* $||x_n - c|| < \varepsilon$ *for every* n.

It follows from Propositions 1 and 2 that the game does not admit a stationary ε-equilibrium, provided ε is small enough. Indeed, let us argue by contradiction, and assume that for every ε there exists a stationary ε-equilibrium x_ε. Let $x_\star$ be an accumulation point of $\{x_\varepsilon\}$ as $\varepsilon \to 0$. If $x_\star$ is terminating ($x_* \neq c$) then it is a stationary 0-equilibrium, which is ruled out by Proposition 1. Otherwise, $x_\star = c$, and then, for ε sufficiently small, there is an ε-equilibrium $\mathbf{x}$ where $\| x_n - c \| < \varepsilon$, which is ruled out by Proposition 2.

Proposition 1 is proved in section 2.2, while Proposition 2 is proved in section 2.3.

2.1 Cyclic equilibrium

We prove that the game possesses a cyclic equilibrium, where the length of the cycle is 2. At odd stages players 2 and 4 play c^2 and c^4 respectively, and players 1 and 3 continue with probability x and z respectively, both strictly less than 1. At even stages players 1 and 3 play c^1 and c^3 respectively, and players 2 and 4 continue with probability x and z respectively.

Formally, we study now profiles $\mathbf{y}$ that satisfy:

$$y_n = \begin{cases} (x, 1, z, 1) & n \text{ odd} \\ (1, x, 1, z) & n \text{ even} \end{cases}$$

where $x, z \in]0, 1[$ are independent of n.

The one-shot game played by players 1 and 3 at odd stages is

		3: z	$1-z$
1	x	γ_c^1, γ_c^3	0,1
	$1-x$	1,0	1,1

Figure 3: The game of players 1 and 3 at odd stages

In this game player 1 is the row player, player 3 is the column player, and γ_c^i is the continuation payoff of player $i = 1, 3$. The payoffs received by players 2 and 4 if termination occurs in an odd stage are given by the matrix below, in which the first coordinate of each entry is player 2's payoff, and the second coordinate is player 4's payoff.

	$0,4$
$4,0$	$1,0$

(1)

The one-shot game played by players 2 and 4 at even stages is

		4	
		z	$1-z$
2	x	γ_c^2, γ_c^4	0,1
	$1-x$	$1,0$	1,1

Figure 4: The game of players 2 and 4 at even stages

where player 2 is the row player, player 4 is the column player, and the payoffs that are received by players 1 and 3 if termination occurs are given by matrix (1). The two situations are identical (up to the continuation payoffs).

We now find necessary conditions on (x, z). First, (x, z) is a fully mixed equilibrium of the matrix game in Figure (3), so that

$$x\gamma_c^3 = 1 \text{ and } z\gamma_c^1 = 1,$$

and both players 1 and 3 receive 1 in this equilibrium.

By the symmetry of the profile, the continuation payoffs (resp. initial payoffs) of players 2 and 4 must coincide with the initial payoffs (resp. continuation payoffs) of players 1 and 3. That is, (γ_c^1, γ_c^3) is the payoff received in the matrix game (1), when the empty entry is filled with $(1, 1)$ and the row and column players play according to x and z respectively, so that

$$\begin{cases} \gamma_c^1 = xz + 4z(1-x) + (1-x)(1-z) \\ \gamma_c^3 = xz + 4z(1-x) \end{cases}$$

Set $g = \gamma_c^1$ and $h = \gamma_c^3$. Since $x = \frac{1}{h}$ and $z = \frac{1}{g}$, one gets

$$\begin{cases} g^2h = 1 + 4(g-1) + (g-1)(h-1) \\ gh^2 = 1 + 4(g-1) \end{cases}$$

which is equivalent to

$$\begin{cases} g = \frac{3}{4-h^2} \\ h \text{ root of } (h-1)(h^4 + 3h^3 - 2h^2 - 9h + 4) = 0 \end{cases} \quad (2)$$

Conversely, let (g, h) be a solution to (2) with $g, h > 1$, and define a cyclic profile by $x = \frac{1}{h}$, $z = \frac{1}{g}$. Given the above properties, in order to prove that it is an equilibrium, we need only prove that neither player 2 nor 4 can find it profitable to quit in the first stage. This is clear, since players 2 and 4 would receive at most 1 by quitting, whereas they get strictly more than 1 under the cyclic profile.

Thus, the existence of such a cyclic equilibrium is equivalent to the existence of a solution (g, h) to system (2) with $g, h > 1$. If $1 < h < 2$ then $1 < 3/(4 - h^2)$. Hence we need to assert the existence of a root in $]1, 2[$ of the polynomial

$$Q(X) = X^4 + 3X^3 - 2X^2 - 9X + 4.$$

Such a root exists since $Q(1) < 0 < Q(2)$.

2.2 No Stationary Equilibria

We check that there is no stationary equilibrium. We do it according to the number of players who play both actions with positive probability.

It is immediate to check that there is no stationary equilibrium in which at least three players play pure strategies.

We shall now verify that there is no stationary equilibrium where two players play pure stationary strategies. Indeed, assume that players 3 and 4 play pure stationary strategies. If such a case arises, players 1 and 2 are playing a 2×2 game. We will see that all the equilibria in these games are pure, and therefore they cannot generate an equilibrium in the four-player game.

Case 1: Players 3 and 4 play (q^3, q^4)
The unique equilibrium is (c^1, c^2, q^3, q^4).
Case 2: Players 3 and 4 play (c^3, q^4)
The unique equilibrium is (c^1, q^2, c^3, q^4).
Case 3: Players 3 and 4 play (q^3, c^4)— symmetric to case 2.
Case 4: Players 3 and 4 play (c^3, c^4)
There are two equilibria: (q^1, c^2, c^3, c^4) and (c^1, q^2, c^3, c^4).

We shall now see that there is no stationary equilibrium where players 2 and 4 play pure actions.

Case 1: Players 2 and 4 play (c^2, c^4)
The unique equilibrium is (q^1, c^2, q^3, c^4).
Case 2: Players 2 and 4 play (q^2, c^4)
The unique equilibrium is $(\frac{1}{2}c^1 + \frac{1}{2}q^1, q^2, \frac{1}{4}c^3 + \frac{3}{4}q^3, c^4)$. In this equilibrium player 2 receives $\frac{5}{8}$, but if he plays c^2 he gets 1.
Case 3: Players 2 and 4 play (c^2, q^4)
The unique equilibrium is (q^1, c^2, c^3, q^4).
Case 4: Players 2 and 4 play (q^2, q^4)

The unique equilibrium is (c^1, q^2, q^3, q^4).

All the other cases are symmetric to these 8 cases.

Next, we check that there is no stationary equilibrium where one player, say player 4, plays a pure strategy, and all the other players play a fully mixed strategy. We denote by (x, y, z) the fully mixed stationary equilibrium in the three-player game when player 4 plays some pure stationary strategy.

Assume first that player 4 plays q^4. Then, in order to have player 2 indifferent, we should have

$$x(1-z) = z - (1-x)(1-z)$$

which implies that $z = 1/2$. In order to have player 1 indifferent, we should have

$$(1-y)z + y(1-z) = yz - (1-y)(1-z)$$

which solves to $yz = 1/2$, and therefore $y = 1$, which is pure.

Assume now that player 4 plays c^4. First we note that $x < 1/2$, otherwise player 3 prefers to play q^3 over c^3. Next, if player 2 is indifferent between his actions, then

$$\frac{(1-x)(1+3z)}{1-xz} = x + (1-x)z$$

or equivalently,

$$(1-x)(1+2z+xz^2) = (1-xz)x.$$

Since $x < 1/2$, it follows that $1-x > x$. Therefore it follows that

$$1 + 2z + xz^2 < 1 - xz$$

or equivalently $2 + xz < -x$, which is clearly false.

2.2.1 No fully mixed stationary equilibrium

We prove now by contradiction that there is no fully mixed stationary equilibrium. Let (x^*, y^*, z^*, t^*) be such an equilibrium, where $0 < x^* < 1$ is the probability player 1 puts on c^1. Set $(a^*, b^*, c^*, d^*) = \gamma(x^*, y^*, z^*, t^*)$. Notice that $0 < a^*, b^*, c^*, d^* < 1$.

Let $0 < y, z, t < 1$. Assume that $a \in]0, 1[$ is the payoff of player 1 if quitting does not occur at the first stage. Then, by playing c^1 at stage 1, player 1 gets

$$\alpha(a; y, z, t) = yzt(a-2) - 2yz + 3zt - yt + y + z,$$

whereas by playing q^1 he gets

$$\beta(y, z, t) = t + (1-t)(y + z - 1).$$

By the equilibrium condition for player 1,

$$a^* = \beta(y^*, z^*, t^*) = \alpha(a^*; y^*, z^*, t^*).$$

Therefore, the polynomial

$$\Delta_1(y, z, t) = \alpha(\beta(y, z, t); y, z, t) - \beta(y, z, t)$$

vanishes at (y^*, z^*, t^*). For simplicity, we write

$$\Delta_1(y, z, t) = (a - 2)yzt - 2yz + 4zt + 1 - 2t,$$

with the understanding that a stands for $\beta(y, z, t)$. $\Delta_2(x, z, t), \Delta_3(x, y, t)$ and $\Delta_4(x, y, z)$ are defined in a symmetric way.

Observe that the four polynomials $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ should vanish at (x^*, y^*, z^*, t^*).

The proof goes as follows. First we prove that (x^*, y^*, z^*, t^*) is not on the diagonal of the unit four-dimensional square. We then define $D = \{y \leq z \leq t\}$ and prove that Δ_1 does not vanish on $D \cap \{z \geq \frac{1}{2}\}$ whereas Δ_4 does not vanish on $D \cap \{z \leq \frac{1}{2}\}$.

Lemma 3 *(x^*, y^*, z^*, t^*) is not on the diagonal of $[0, 1]^4$; that is, it cannot be the case that $x^* = y^* = z^* = t^*$.*

Proof. Assume to the contrary that (x, x, x, x) is a stationary equilibrium, where $0 \leq x \leq 1$. Note that $x = 1$ (everyone continues) and $x = 0$ (everyone quits) do not correspond to an equilibrium. Thus, all players play a fully mixed action at every stage.

In an equilibrium player 1 is indifferent between continuing and quitting, hence we should have

$$\frac{4x^2(1 - x) + 2x(1 - x)^2}{(1 - x)^3} = x + x^2(1 - x) - (1 - x)^3.$$

Simplifying both sides yields

$$\frac{2x + 2x^2}{1 - 2x + x^2} = -1 + 4x - 2x^2.$$

Multiplying both sides by $1 - 2x + x^2$ and rearranging the arguments yields

$$0 = 1 - 4x + 13x^2 - 8x^3 + 2x^4 = (1 - x)(1 - 3x + 10x^2) + 2x^3 + 2x^4.$$

But the polynomial on the right is positive on $]0, 1[$. ■

Without loss of generality, we assume $y^* = \min(x^*, y^*, z^*, t^*)$. We now point out several facts that will be used extensively:

1. $\frac{\partial a}{\partial t}(y,z,t) = 2 - y - z > 0$; $\frac{\partial a}{\partial y}(y,z,t) = \frac{\partial a}{\partial z}(y,z,t) = 1 - t > 0$;

2. $\frac{\partial \Delta_1}{\partial y}(y,z,t) = (a-2)zt + yzt(1-t) - 2z < 0$;

3. $\frac{\partial \Delta_1}{\partial z}(y,z,t) = (a-2)yt + yzt(1-t) - 2y + 4t$ is decreasing in y: therefore, on the region $y \leq t$, $\frac{\partial \Delta_1}{\partial z}(y,z,t) \geq \frac{\partial \Delta_1}{\partial z}(t,z,t) = (a-2)t^2 + t^2 z(1-t) + 2t > 0$.

Thus, on the region $y \leq t \leq z$,

$$\Delta_1(y,z,t) \geq \Delta_1(t,t,t) > 0.$$

Therefore, (x^*, y^*, z^*, t^*) belongs to the region $D = \{y \leq z \leq t\}$.

Lemma 4 *The polynomial Δ_1 does not vanish on $\{y \leq z \leq t\} \cap \{z \geq \frac{1}{2}\}$.*

Proof. We argue by contradiction, and denote by (y^*, z^*, t^*) a root of Δ_1. Notice that

$$y \leq \frac{1}{2} \leq z \leq t \Rightarrow \Delta_1(y,z,t) \geq \Delta_1(\frac{1}{2}, \frac{1}{2}, t) = \frac{1}{2} - \frac{t}{2} + a\frac{t}{4} > 0.$$

Thus, $y^* \geq \frac{1}{2}$.

Claim: $t^* \geq \frac{2}{3}$.

We study Δ_1 on the domain $D_1 = \{\frac{1}{2} \leq y \leq z \leq t \leq \frac{2}{3}\}$. Notice first that a is maximized at $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, where it equals $\frac{7}{9} < \frac{5}{6}$.

On D_1, $\Delta_1(y,z,t) \geq \Delta_1(z,z,t) = f(z,t) = (a-2)z^2 t - 2z^2 + 4zt + 1 - 2t$. One has $\frac{\partial f}{\partial z}(z,t) = 2zt(a-2) + 2z^2 t(1-t) - 4z + 4t$. It is easily checked that $\frac{\partial}{\partial t}(\frac{\partial f}{\partial z})(z,t) = 2z(a-2) + 2zt(2-y-z) + 2z^2(1-t) - 2z^2 t + 4$ is positive on D_1. Therefore,

$$\frac{\partial f}{\partial z}(z,t) \leq \frac{\partial f}{\partial z}(z, \frac{2}{3}) = \frac{4}{3}z(a-2) + \frac{4}{9}z^2 - 4z + \frac{8}{3}.$$

The latter quantity is maximized at $z = \frac{1}{2}$. It is then equal to $\frac{2}{3}(a-1) + \frac{1}{9}$. Since $a < \frac{5}{6}$, this is negative.

Thus, $\frac{\partial f}{\partial z} < 0$ on D_1. Therefore,

$$\Delta_1(z,z,t) \geq \Delta_1(t,t,t) > 0.$$

The claim is established.

Claim: One has $z^* < \frac{2}{3}$.

We shall prove that $\Delta_1 > 0$ on $D_2 = \{\frac{1}{2} \le y, \frac{2}{3} \le z \le t\}$. Notice first that $a \ge \frac{2}{3}$ on D_2.

Set first $D_3 = D_2 \cap \{y < \frac{2}{3}\}$. On D_3, $\Delta_1(y,z,t) \ge \Delta_1(\frac{2}{3}, \frac{2}{3}, t) = (a-2)\frac{4}{9}t + \frac{1}{9} + \frac{2}{3}t$. Now, $\forall t \ge \frac{2}{3}$, $\beta(\frac{2}{3}, \frac{2}{3}, t) \ge \frac{7}{9} \ge \frac{3}{4}$. Therefore,

$$\Delta_1(y,z,t) \ge \Delta_1(\frac{2}{3}, \frac{2}{3}, t) \ge -\frac{5}{9}t + \frac{1}{9} + \frac{2}{3}t = \frac{t+1}{9} > 0.$$

Set now $D_4 = D_2 \cap \{y \ge \frac{2}{3}\}$. On D_4, one has

$$\frac{\partial \Delta_1}{\partial t} = (a-2)yz + yzt(2-y-z) + 4z - 2 \ge (a-2)yz + 4z - 2.$$

The function $(a-2)yz + 4z - 2$ is increasing in z. Therefore, it is minimized on the diagonal $\{y = z\}$, where it is at least $-\frac{5}{4}y^2 + 4y - 2$: this minorant is minimized at $y = \frac{2}{3}$; it is then equal to $\frac{1}{9}$. Therefore, Δ_1 is increasing in t, and

$$\Delta_1(y,z,t) \ge \Delta_1(z,z,z) > 0.$$

To conclude the proof of Lemma 4, we prove that $\Delta_1 > 0$ on $D_5 = \left([\frac{1}{2}, \frac{2}{3}] \times [\frac{1}{2}, \frac{2}{3}] \times [\frac{2}{3}, 1[\right) \cap \{y \le z\}$.

On D_5, $a \ge \frac{2}{3}$, thus

$$\Delta_1(y,z,t) \ge \Delta_1(z,z,t) \ge -\frac{4}{3}z^2 t - 2z^2 + 4zt + 1 - 2t = h(z,t).$$

First,

$$h(\frac{1}{2}, t) = -\frac{t}{3} - \frac{1}{2} + 2t + 1 - 2t = \frac{1}{2} - \frac{t}{3} > \frac{1}{6}, \text{ and}$$
$$h(\frac{2}{3}, t) = -\frac{16}{27}t - \frac{8}{9} + \frac{8}{3}t + 1 - 2t = \frac{2}{27}t + \frac{1}{9} > \frac{1}{7}.$$

Now, each $z \in [\frac{1}{2}, \frac{2}{3}]$ satisfies $|z - \frac{1}{2}| \le \frac{1}{12}$, or $|z - \frac{2}{3}| \le \frac{1}{12}$. Therefore, we need only prove that $|\frac{\partial h}{\partial z}(z,t)| \le \frac{12}{7}$ on D_5. The function

$$\frac{\partial h}{\partial z}(z,t) = -\frac{8}{3}zt - 4z + 4t$$

is increasing in t and decreasing in z. Thus, it is minimal at $(\frac{2}{3}, \frac{2}{3})$, where it equals $-\frac{32}{27}$, and maximal at $(\frac{1}{2}, 1)$, where it equals $\frac{2}{3}$. ■

Lemma 5 *The polynomial Δ_4 does not vanish on $\{y \le z \le t\} \cap \{z \le \frac{1}{2}\}$.*

Proof. We argue by contradiction and denote by (x^*, y^*, z^*) a root of Δ_4. Recall that $d^* > 0$. Therefore

$$0 = \Delta_4(x^*, y^*, z^*) > -2x^*y^*z^* - 2x^*z^* + 4x^*y^* + 1 - 2y^*.$$

Hence, the polynomial $P(x, y, z) = xyz + xz - 2xy + y - \frac{1}{2}$ is positive at (x^*, y^*, z^*).

We prove now that P is negative on $D_6 = \left([0,1] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}]\right) \cap \{y \leq z\}$.

1. on $D_6 \cap \{x \leq \frac{1}{2}\}$, $\frac{\partial P}{\partial y}(x, y, z) = xz - 2x + 1 \geq 0$; thus, P is maximized at $y = \frac{1}{2}$; it is then equal to $\frac{xz}{2} + xz - x = x(\frac{3}{2}z - 1) < 0$;
2. on $D_6 \cap \{x \geq \frac{1}{2}\}$, $\frac{\partial P}{\partial z}(x, y, z) = xy + x > 0$; thus, P is maximized at $z = \frac{1}{2}$ and equals

$$Q(x, y) = y - \frac{3}{2}xy + \frac{x}{2} - \frac{1}{2}.$$

   (a) on $\{y \leq \frac{1}{3}\}$, $\frac{\partial Q}{\partial x}(x, y) = \frac{1}{2} - \frac{3}{2}y < 0$; thus, Q is maximized at $x = 1$, and then equals $y - \frac{3}{2}y < 0$;

   (b) on $\{y \geq \frac{1}{3}\}$, $\frac{\partial Q}{\partial x}(x, y) \geq 0$; thus, Q is maximized at $x = \frac{1}{2}$, and then equals $Q(\frac{1}{2}, y) = \frac{1}{4}(y - 1) < 0$.

■

2.3 Proof of Proposition 2

In this section we prove that there is no ε-equilibrium profile $\mathbf{x}$ such that $\| x_n - c \| < \varepsilon$ for every $n \in \mathbf{N}$, provided ε is sufficiently small.

We first introduce a few notations. Given a profile $\mathbf{x}$, and a stage $n \in \mathbf{N}$, we denote by $\mathbf{x}_n = (x_n, x_{n+1}, ...)$ the profile induced by $\mathbf{x}$ in the subgame starting from stage n. We let c denote the profile of actions (c^i), and by $\mathbf{c}^i$ the pure stationary strategy that plays repeatedly c^i.

Observe first that, $\mathbf{x}$ being an ε-equilibrium,

$$\mathbf{P}_{\mathbf{x}}(t < +\infty) \geq 1 - 2\varepsilon. \tag{3}$$

(Otherwise, any player i would benefit by playing according to $\mathbf{x}^i$ for many stages, before switching to q^i). Since $||x_n - c|| < \varepsilon$ for every n,

$$\mathbf{P}_{\mathbf{x}}(t < +\infty, |S_t| > 1) < 5\varepsilon. \qquad (4)$$

It follows that $\mathbf{P}_{\mathbf{x}}(t < +\infty, |S_t| = 1) \geq 1 - 7\varepsilon$, hence $\sum_{i \in \mathcal{N}} \gamma^i(\mathbf{x}) \geq 5(1-7\varepsilon) - 4 \times 5\varepsilon = 5 - 55\varepsilon$. In particular, there exists a player i such that $\gamma^i(\mathbf{x}) \geq \frac{5}{4} - \frac{55}{4}\varepsilon \geq \frac{5}{4} - 16\varepsilon$.

Define $r^i = \mathbf{P}_{\mathbf{x}}(S_t = \{i\})$.

We now show that

$$r^i \geq \frac{2}{15} - \frac{49}{15}\varepsilon \geq \frac{2}{15} - 4\varepsilon \qquad \forall i \in \mathcal{N}. \qquad (5)$$

Indeed, $\gamma^i(\mathbf{x}) \geq 1 - (\rho+1)\varepsilon \geq 1 - 2\rho\varepsilon$ for each $i \in \mathcal{N}$ (otherwise, player i can quit at stage 1 and get at least $1 - \rho\varepsilon$.) By (4) we get $r^1 + 4r^2 \geq \gamma^1(\mathbf{x}) - 5\rho\varepsilon \geq 1 - 7\rho\varepsilon$, and similarly $4r^1 + r^2 \geq 1 - 7\rho\varepsilon$. Thus, $r^1 + r^2 \geq \frac{2}{5} - \frac{14}{5}\rho\varepsilon$. In a similar way one gets $r^1 + r^2 \leq 1 - r^3 - r^4 \leq \frac{3}{5} + \frac{14}{5}\rho\varepsilon$. The set of solutions (r^1, r^2) of these equations is a triangle in the positive quadrant, and one may check that (5) holds for any such solution.

The rest of the proof goes as follows. In an equilibrium, as long as the continuation payoff of some player is more than 1, he does not quit (since by quitting he gets at most 1). In an ε-equilibrium this is no longer true, since a player may quit with small probability even when quitting yields him low payoff. We first prove that as long as the continuation payoff of some player is more than $1 + \sqrt{\varepsilon}$, the overall probability he quits cannot exceed $O(\sqrt{\varepsilon})$.

Assume w.l.o.g. that $\gamma^1(\mathbf{x}) \geq 5/4 - 16\varepsilon$, then, since $q^1 \geq 2/15 - 4\varepsilon$ it follows that for some n_1, $\gamma^1(\mathbf{x}_n) < 1 + \sqrt{\varepsilon}$. Moreover, if n_1 is the first such stage, then player 1 quits with negligible probability until stage n_1. Since the continuation payoff of 1 decreases only when 2 quits, it follows that player 2 quits with a non-negligible probability before stage n_1. Since the probability that 1 quits before stage 1 is negligible, it must be the case that the probability that 3 and 4 quit before stage n_1 is also negligible. Indeed, otherwise the continuation payoff of 2 would increase, and, as for 1, once his continuation payoff is more than $1 + \sqrt{\varepsilon}$, he would stop quitting. Thus, until stage n_1 only player 2 quits with a non-negligible probability. But that means that at stage n_1, the continuation payoff of 3 and 4 is high. However, as long as their continuation payoff is high, they do not quit, and the only way the continuation payoff of player 3 (resp. 4) can decrease is that player 4 (resp. 3) quit with non-negligible probability. That makes such an ε-equilibrium impossible.

We now formalize these ideas.

For every strategy $\mathbf{x}^i$ of player i and every $n \geq 0$ let $\tilde{\mathbf{x}}^i(n)$ be the strategy which plays c^i up to stage n, and coincides with $\mathbf{x}^i$ after stage n, and let $p^i_n = p^i_n(\mathbf{x}) = \mathbf{P}_{\mathbf{x}}(t < n, i \in S_t)$ be the probability that player i quits up to stage n, and let $p^i_\infty = \lim_{n\to\infty} p^i_n$. Note that $2/15 - 5\varepsilon \leq r^i \leq p^i_\infty \leq r^i + \varepsilon$.

Lemma 6 *Let* $\mathbf{x}$ *be a profile that satisfies (i)* $\parallel x_n - c \parallel < \varepsilon$ *and (ii)* $\gamma^i(\mathbf{x}_n) \geq 1 + \sqrt{\varepsilon}$ *for some player* i *and every* $n \leq n_0$*. Then*

$$\gamma^i(\mathbf{x}^{-i}, \tilde{\mathbf{x}}^i(n)) \geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon} p^i_n - (2N+3)\rho\varepsilon.$$

Proof. Fix a player $i \in \mathcal{N}$. We first assume that in $\mathbf{x}$ only one player quits at every stage; that is, for every $n \in \mathbf{N}$, $x^j_n \neq 1$ for at most one player j. We prove by induction that for such profile $\mathbf{x}$, for every $n \leq n_0$,

$$\gamma^i(\mathbf{x}^{-i}, \tilde{\mathbf{x}}^i(n)) \geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon} p^i_n. \tag{6}$$

Assume $n = 1$. If i continues at stage 0, then $\tilde{\mathbf{x}}^i(1) = \mathbf{x}^i$ and $p^i_1 = 0$, and (6) holds. If i quits with some positive probability at stage 0 then $p^i_1 = 1 - x^i_1$, hence

$$\gamma^i(\mathbf{x}) = p^i_1 + (1 - p^i_1)\gamma^i(\mathbf{x}^{-i}, \tilde{\mathbf{x}}^i(1)).$$

Then

$$\gamma^i(\mathbf{x}^{-i}, \tilde{\mathbf{x}}^i(1)) = \gamma^i(\mathbf{x}) + \frac{p^i_1}{1 - p^i_1}(\gamma^i(\mathbf{x}) - 1) \geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon} p^i_1,$$

where the last inequality holds by condition (ii).

Assume now that $1 < n \leq n_0$. If i continues at stage n then $\tilde{\mathbf{x}}^i(n) = \tilde{\mathbf{x}}^i(n-1)$ and $p^i_n = p^i_{n-1}$. In particular, by the induction hypothesis,

$$\gamma^i(\mathbf{x}^{-i}, \tilde{\mathbf{x}}^i(n)) = \gamma^i(\mathbf{x}^{-i}, \tilde{\mathbf{x}}^i(n-1)) \geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon} p^i_{n-1} = \gamma^i(\mathbf{x}) + \sqrt{\varepsilon} p^i_n,$$

and (6) holds.

If i quits at stage n then, applying the case $n = 1$ to the profile $\mathbf{x}_{n-1}$ we get

$$\gamma^i(\mathbf{x}^{-i}_{n-1}, \tilde{\mathbf{x}}^i(n)_{n-1}) \geq \gamma^i(\mathbf{x}^{-i}_{n-1}, \tilde{\mathbf{x}}^i(n-1)_{n-1}) + \sqrt{\varepsilon}(1 - x^i_n).$$

Using the induction hypothesis we get:

$$\begin{aligned}
\gamma^i(\mathbf{x}^{-i}, \tilde{\mathbf{x}}^i(n)) &\geq \gamma^i(\mathbf{x}^{-i}, \tilde{\mathbf{x}}^i(n-1)) + \mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}(t \geq n-1)\sqrt{\varepsilon}(1 - x^i_n) \\
&\geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}(p^i_{n-1} + \mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}(t \geq n-1)(1 - x^i_n)) \\
&\geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon} p^i_n.
\end{aligned}$$

Thus, (6) holds for every $n \leq n_0$.

Let now $\mathbf{x}$ be an arbitrary profile that satisfies (i) and (ii). We are now going to define a new profile $\mathbf{y}$ that approximates $\mathbf{x}$, and satisfies that at every stage at most one player quits with positive probability. We then apply (6) to $\mathbf{y}$ to get the desired estimate for $\mathbf{x}$.

For each i and n, define $\alpha_n^i = (1 - x_n^i) \prod_{j \neq i} x_n^j = \mathbf{P}(S_t = \{i\}, t = n \mid t \geq n) < \varepsilon$.

Define now

$$\beta_n^i = \begin{cases} \alpha_n^i & i = 1 \\ \alpha_n^i / \prod_{j<i}(1 - \beta_n^j) & i > 1 \end{cases}$$

Since $\alpha_n^i < \varepsilon$ for every i, $\beta_n^i < K\varepsilon$ for every i, for a sufficiently large K.

Finally, define for every player i a strategy $\mathbf{y}^i$ as follows:

$$y_{nN+j}^i = \begin{cases} 0 & j \neq i \\ 1 - \beta_n^i & j = i \end{cases}$$

First note that

$$\frac{\mathbf{P_y}(S_t = \{i\} \mid (n-1)N + 1 \leq t \leq nN)}{\mathbf{P_y}(S_t = \{i+1\} \mid (n-1)N + 1 \leq t \leq nN)} = \frac{\beta_n^i}{(1 - \beta_n^i)\beta_n^{i+1}} = \frac{\alpha_n^i}{\alpha_n^{i+1}}$$
$$= \frac{\mathbf{P_x}(S_t = \{i\} \mid |S_t| = 1, t = n)}{\mathbf{P_x}(S_t = \{i+1\} \mid |S_t| = 1, t = n)}$$

Hence,

$$\mathbf{P_x}(S_t = \{i\} \mid |S_t| = 1) = \mathbf{P_y}(S_t = \{i\}). \tag{7}$$

Since $\mathbf{P}(|S_t| \geq 2) < \varepsilon$, it follows that $|p_n^i(\mathbf{x}) - p_{nN}^i(\mathbf{y})| < \varepsilon$, and therefore

$$\| \gamma^i(\mathbf{x}) - \gamma^i(\mathbf{y}) \| < (N+1)\rho\varepsilon. \tag{8}$$

By applying (8) to $\mathbf{x}$ and $(\mathbf{x}^{-i}, \tilde{\mathbf{x}}^i(n))$ and using (6) we get

$$\begin{aligned} \gamma^i(\mathbf{x}^{-i}, \tilde{\mathbf{x}}^i(n)) & \geq \gamma^i(\mathbf{y}^{-i}, \tilde{\mathbf{y}}_{nN}^i) - (N+1)\rho\varepsilon \\ & \geq \gamma^i(\mathbf{y}) + \sqrt{\varepsilon} p_{nN}^i(\mathbf{y}) - (N+1)\rho\varepsilon \\ & \geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon} p_n^i(\mathbf{x}) - \sqrt{\varepsilon} \times \varepsilon - 2(N+1)\rho\varepsilon \\ & \geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon} p_n^i(\mathbf{x}) - (2N+3)\rho\varepsilon, \end{aligned}$$

as desired. ■

We define the partner $\tilde{i}$ of a player i by : $\tilde{1} = 2, \tilde{2} = 1, \tilde{3} = 4, \tilde{4} = 3$.

Lemma 7 *Let $a, b > 0$ and let $\varepsilon > 0$ be sufficiently small. Let $\mathbf{y}$ be a $b\varepsilon$-equilibrium such that $||y_n - c|| < \varepsilon$ for each n. Let $i \in \mathcal{N}$, and assume that $\gamma^i(\mathbf{y}) \geq 1 + a$. Then there exists n_1 such that (i) $\gamma^i(\mathbf{y}_{n_1}) < 1 + \sqrt{\varepsilon}$, (ii) $p_{n_1}^i \leq (b + K)\sqrt{\varepsilon}$, for some K and (iii) $a \leq 3p_{n_1}^{\tilde{i}} + 3\sqrt{\varepsilon}$.*

Proof. For convenience, assume $i = 1$. Since $p^1_\infty \geq 2/15 - 5\varepsilon$, Lemma 6 implies that there exists a stage n such that $\gamma^1(\mathbf{y}_n) < 1 + \sqrt{\varepsilon}$. Let n_1 be the first such stage. In particular, (i) holds. Observe that $\gamma^1(\mathbf{y}_{n_1-1}) \geq 1 + \sqrt{\varepsilon}$, hence by Lemma 6 $b\varepsilon \geq \sqrt{\varepsilon} p^i_{n_1-1} - (2N+3)\rho\varepsilon$, which solves to $p^i_{n_1-1} \leq b\sqrt{\varepsilon} + (2N+3)\rho\sqrt{\varepsilon}$.

Since the probability that player 1 quits in stage $n_1 - 1$ is at most ε, (ii) follows.

We now prove (iii). Since $\gamma^1(\mathbf{y}_{n_1}) < 1 + \sqrt{\varepsilon}$ one has

$$\begin{aligned}1 + a \leq \gamma^1(\mathbf{y}) &\leq p^1_{n_1} + 4p^2_{n_1} + 5\varepsilon + (1 - p^1_{n_1} - p^2_{n_1} - p^3_{n_1} - p^4_{n_1} + 5\varepsilon)\gamma^1(\mathbf{y}_{n_1}) \\ &\leq p^1_{n_1} + 4p^2_{n_1} + (1 - p^1_{n_1} - p^2_{n_1}) + 2\sqrt{\varepsilon} + 11\varepsilon \\ &\leq 1 + 3p^1_{n_1} + 2\sqrt{\varepsilon} + 11\varepsilon,\end{aligned}$$

and (iii) follows. ∎

Corollary 8 *Let $b > 0$ and $a > 3(b+1)\sqrt{\varepsilon}$. There is no $b\varepsilon$-equilibrium $\mathbf{y}$ such that :*

- $||y_n - c|| < \varepsilon$ *for each* n
- $\gamma^i(\mathbf{y}), \gamma^{\tilde{i}}(\mathbf{y}) \geq 1 + a$.

Proof. Let $\mathbf{y}$ be such a $b\varepsilon$-equilibrium. Apply Lemma 7 twice, to players i and $\tilde{i}$. Call n_1 and n_2 the corresponding two stages, and assume, w.l.o.g, $n_1 \leq n_2$. Thus, one has both $p^{\tilde{i}}_{n_1} \geq a/3 - \sqrt{\varepsilon}$, and $p^{\tilde{i}}_{n_2} \leq b\sqrt{\varepsilon}$. Moreover, $p^{\tilde{i}}_{n_1} \leq p^{\tilde{i}}_{n_2}$ since $n_1 \leq n_2$. Thus $a/3 - \sqrt{\varepsilon} \leq b\sqrt{\varepsilon}$ — a contradiction. ∎

End of proof of Proposition 2: Assume to the contrary that $\mathbf{x}$ is an ε-equilibrium with $\| x_n - c \| < \varepsilon$ for every $n \in \mathbf{N}$. We assume w.l.o.g. that $\gamma^1(\mathbf{x}) \geq 5/4 - 16\varepsilon$. We will exhibit a stage n_2 such that $\mathbf{x}_{n_2}$ is a 8ε-equilibrium, and $\gamma^3(\mathbf{x}_{n_2}), \gamma^4(\mathbf{x}_{n_2}) \geq 1 + 1/12$. By Corollary 8, we get a contradiction.

Apply Lemma 7 to $\mathbf{x}$ and $i = 1$, and denote n_1 the corresponding stage. Thus, $p^1_{n_1} \leq 2\sqrt{\varepsilon}$ and $p^2_{n_1} \geq \frac{1}{3} \times (\frac{1}{4} - 16\varepsilon) - \sqrt{\varepsilon} \geq \frac{1}{12} - 2\sqrt{\varepsilon}$. By Lemma 6, there exists a stage $N_2 < n_1$ with $\gamma^2(x_{N_2}) < 1 + \sqrt{\varepsilon}$. We set

$$n_2 = \max\{n \leq n_1, \gamma^2(\mathbf{x}_n) \leq 1 + \sqrt{\varepsilon}\}.$$

Since $p^1_{n_2} \leq p^1_{n_1} \leq 2\sqrt{\varepsilon}$, $p^1_\infty \geq \frac{1}{11}$ and $\sup_i p^i_\infty \leq 1 + 5\varepsilon$, one has $\mathbf{P}_{\mathbf{x}}(t < n_2) \leq \frac{13}{15} + 10\varepsilon \leq \frac{7}{8}$. Since $\mathbf{x}$ is an ε-equilibrium, $\mathbf{x}_{n_2}$ is a 8ε-equilibrium.

Our next goal is to prove that $p^2_{n_2} \geq \frac{1}{12} - 10\sqrt{\varepsilon}$. If $n_2 = n_1$ there is nothing to prove. Assume $n_2 < n_1$. This means that $\gamma^2(\mathbf{x}_{n_1}) > 1 + \sqrt{\varepsilon}$. Apply Lemma 6 with $\mathbf{y} = \mathbf{x}_{n_2}$ (thus $y_n = x_{n_2+n}$, for each n) and $n = n_1 - n_2$. The conclusion, rephrased in terms of $\mathbf{x}$, is that $\mathbf{P_x}(t < n_1, 2 \in S_t | t \geq n_2) \leq 8\sqrt{\varepsilon}$, hence, a fortiori, $p^2_{n_1} - p^2_{n_2} \leq 8\sqrt{\varepsilon}$. Therefore, $p^2_{n_2} \geq \frac{1}{12} - 10\sqrt{\varepsilon}$.

We use this result to prove that $\gamma^3(\mathbf{x}_{n_2}), \gamma^4(\mathbf{x}_{n_2}) \geq 1 + 1/12$.

As previously, one has

$$1 - 2\varepsilon \leq \gamma^2(\mathbf{x}) \leq 4p^1_{n_2} + p^2_{n_2} + 5\varepsilon + \left(1 - \sum_i p^i_{n_2} + 5\varepsilon\right)\gamma^2(\mathbf{x}_{n_2}). \tag{9}$$

By definition of n_2, $\gamma^2(\mathbf{x}_{n_2}) \leq 1 + \sqrt{\varepsilon}$. Since $p^1_{n_2} \leq p^1_{n_1} \leq \sqrt{\varepsilon}$, one deduces from (9) that $p^3_{n_2} + p^4_{n_2} \leq 6\sqrt{\varepsilon}$.

On the other hand,

$$1 - 2\varepsilon \leq \gamma^3(\mathbf{x}) \leq 4p^4_{n_2} + p^3_{n_2} + 5\varepsilon + \left(1 - \sum_i p^i_{n_2} + 5\varepsilon\right)\gamma^3(\mathbf{x}_{n_2}). \tag{10}$$

Since $p^2_{n_2} \geq 1/12 - 10\sqrt{\varepsilon}$, (10) yields $\gamma^3(\mathbf{x}_{n_2}) \geq 1 + \frac{1}{11} - 26\sqrt{\varepsilon} \geq 1 + 1/12$. Similarly, $\gamma^4(\mathbf{x}_{n_2}) \geq 1 + \frac{1}{12}$. Since $\mathbf{x}_{n_2}$ is a 8ε-equilibrium, we get a contradiction to Lemma 8. ■

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