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Some Surprising Properties of Power Indices

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ABSTRACT. A troubling aspect about power indices is how the values assigned to players can depend upon the index. As shown, the problem is more severe; different indices can even generate radically different rankings; e.g., a 15-player game exists with over a *trillion* different strict power index rankings of the players. It is shown that certain indices always share the same ranking, but this assertion depends on the number of players; e.g., the Shapley and Banzhaf rankings agree with three players, but with more players they can even have opposite rankings. It is also shown how changes in certain assumptions affect the outcomes. This includes demonstrating how the rankings change when players drop out of a game.

1. INTRODUCTION

Does a voter's power in selecting the US President through the Electoral College differ from state to state? How does a voter's power vary among different multi-representative districts, or in different voting blocs in the EC, or in weighted voting schemes where the weights are determined by the number of shares a voter holds in a corporation? How does one measure the contributions of a particular player, say Michael Jordan, to the success of professional basketball, or the value added by different units in a company when assessing their respective contributions toward the design of a new product, or a division of the costs among political units for a public project? *Power indices*, which measure the contributions of each player to a game, provide a way to address these questions.

Power indices go beyond being tools for game theory to be enshrined in laws and decision procedures. They have played central roles in US Supreme Court decisions ([26]) where academics (e.g., Grofman [8]) have explained certain of the accompanying assertions which, initially, may have seemed to be ludicrous. These indices have been used to understand interactions in the Council of the European Economic Community (Brams [4]), the UN Security Council (Shapley and Shubik [23], Brams [3]), as well as the Canadian scheme for amending their Constitution (Miller [15], Straffin [24], Kilgour [13]). Other uses include analyzing the relative power of parties in a multiparty legislature, voting blocs in Congress, effects of different voting groups in GATT, WTO, UN, NATO, etc. (For a list of references, see Shapley [22].) An early, intuitive use of the concept of "power" comes from the design of the US Constitution; the "small state – large state" controversy which lead to the compromise in the adoption of two houses of Congress directly reflects the struggle to achieve parity in power for each state. Among the descriptions of the pragmatic and theoretical uses of this tool, we recommend Brams [3, 4], Grofman [8], Ordeshook [16], Shapley [21, 22], and Straffin [25].

1.1. **Power index rankings.**

An interesting but troubling fact is that different indices applied to the same game can generate different power levels for the players. (Examples are in the above references.) While the uses of power indices make it important to understand these differences, we show that this phenomenon is generic so it must be expected.But if this area is

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troubled by discrepancies in power index *values,* then it must become particularly concerned when these difference are so extreme that the power index *rankings* differ. We show that it is surprisingly easy to construct illustrating examples.

To suggest why differences in rankings of indices are of interest consider the accepted assessment that Michael Jordan is the "best professional basketball player of all times." Presumably, this comment means that a power index ranking always has Jordan top ranked. On average, then, Jordan's contribution to any team – or existing coalition – is the greatest of any player. But, could this outcome more accurately reflect the *choice of the index* rather than Jordan's worth? Namely, with the same information, would another power index rank Jordan lower? How does the choice of an index affect the rankings of other players?

These ranking differences, then, raise questions about the meaning of the indices. This becomes particularly apparent in politics where power indices frequently are used to determine the relative strengths of players – be they groups, states, or countries – in a forum. Clearly, the concern over finding a power index which adequately portrays the relative strengths of groups becomes crucial. But if indices can vary even to the extent of allowing radically different rankings, the question becomes whether any (or which) of these indices can yield an adequate measure of relative strength.

Moreover, if the rankings, or the identity of who is top-ranked, can change with the index, then the choice of an index becomes a manipulative strategy. The incentives to be strategic are obvious; if an award goes to the top-player, or political power is assigned according to values determined by indices, then the choice of an index becomes a valued strategy. If, for instance, a careful selection of power indices can persuade a group with very few votes out of the whole that their strength is greater than it appears, then these indices can be used to deter the groups from clamoring for a greater say in the political outcomes. Indeed, this reality is manifested by actual court arguments (e.g., US Supreme Court [26]) over the use and choice of indices. We indicate how our approach identifies when such strategies are possible and how to select them.

As a way to prove that a power index may more accurately reflect *which* index is being used rather than the players' actual contributions, we show that when two different indices are applied to the same game, they can have completely reversed rankings. But the game remains fixed, so both of these directly conflicting rankings cannot accurately capture the contributions of the players. Then we describe surprisinging possible differences in outcomes over all indices. We show how to determine when the associated rankings of a game ensure some level of agreement, and when we must expect the rankings to radically differ. Our geometric approach not only explains why this behavior occurs, but it displays the point spreads between different indices. Reflecting the direct interest of one of us (KKS) in political science, we characterize this ranking behavior for a wide class of voting games.

If even rankings can vary with the choice of an index, we should wonder whether the outcomes are susceptible to other parameters. Rather than an idealized academic concern, this question goes to the heart of Justice Harlan's dissenting argument in a U.S. Supreme Court case about voting power [26]. One of Harlan's concerns centered on the sensitivity of the Banzhaf index to assumptions about the game. While Grofman [8] developed a nice argument (related to the central limit theorem) to vindicate Harlan's claims, we know of no general analysis indicating the fate of other indices. Since our approach can identify assumptions which affect outcomes, we contribute to this discussion by showing, for instance, how to determine which indices are most sensitive to changes in a game, and that the Shapley value is the most resilient.

This feature where changes in the properties of a game can create a drastic change in the power structure – even to previously minor players – has other pragmatic examples. An interesting one from 1998 German elections involves the nature of the ruling coalition. Before the vote, the expectation was that the Social Democrats would form a Grand Coalition with the ruling Christian Democrats. However, the Social Democrats did better than predicted, so an alliance with the Green party, to form the "Red-Green Alliance," now became feasible. Thus, differences in the game structure created the dramatic promotion to power of a party which previously had only a minor role due to its tradition vote of about 5%.

A related topic involves the changes to rankings as players leave the game. This phenomenon can be viewed as describing how the relative importance of a player, or political party, depends upon the availability of other agents. Using the basketball example, had Scottie Pippen (another well known basketball player) quit basketball, would Jordan's relative worth dropped or increased? In politics, one can imagine how the interests of parties can change with certain unions.

While the basketball example makes it intuitively clear that settings exist where the relative importance of a player can radically change depending on the availability of another player, we also should expect some consistency. After all, we should worry about the meaning of a power index if a player is top-ranked with all n players, but always is bottom ranked whenever *any* other player drops out. We briefly describe how this consistency in rankings depends on the nature of the game and the choice of a power index.

1.2. **Axiomatic representations.** This feature where different indices support different conclusions has motivated much of the literature describing the strengths and weaknesses of the the Shapley value [20], the Shapley-Shubik value [23], the Banzhaf index [1, 2], the Deegan-Packel index [6], the Holler-Packel index [9], and so forth. To provide guidance in the selection of indices, insightful axiomatic representations have been developed (e.g., among the many references, see the above as well as Straffin [25] and Owen [17]). The intent of the axiomatic characterization is to identify what we can expect from each procedure.

While the axiomatic approach is useful, there are strong limits about what it can offer. After all, the "axiomatic characterization of a procedure" often is just a set of properties not shared by other procedures. Such a characterization does not fully identify what we can expect because a procedure most surely has many other properties which are not directly related, in any manner, to its axiomatic characterization. In fact, a procedure can have several alternative axiomatic representations which offer radically different impressions about its merits.

So, although only the Shapley-Shubik index satisfies "anonymity, null player, relative power, and the transfer axioms" and only the Banzhaf index satisfies "anonymity, null player, absolute power, and the transfer axioms," (see Dubey and Shapley [7] for definitions and a proof), this characterization does not indicate whether these indices can force conflicting orderings of the players. (They can.) If they can differ, the characterizations provide no information about how radically these rankings can differ. Can they, for instance, have opposite rankings? How about consistency; does one procedure better respect the game structure by providing rankings for subgames consistent with the full game? What about robustness of the outcomes to changes in certain assumptions; is one procedure more stable than the other?

To address these concerns, we introduce a geometric approach where the outcomes and certain properties of all power indices can be simultaneously compared. This approach allows us to explore how the outcomes from indices can differ, to understand why this is the case, and to identify new properties.

2. Recent progress

In a May, 1999, conference talk at the Université de Caen, V. Merlin introduced his results with A. Laruelle [11] about one of these themes. By combining results from references [5, 18], they showed the disturbing conclusion that a game can admit *millions upon millions of different power index rankings* by changing the power index. Their assertion uses the Calvo et al. [5] results which relate each power index of a n-player game with a particular n-candidate positional voting method. (A n-candidate positional method is defined by a voting vector $\mathbf{w}^n = (w_1, w_2, \dots, w_n)$ where $w_1 > w_n$, $w_j \geq w_{j+1}$, $\forall j = 1, \dots, n-1$. To tally a ballot, w_j points are assigned to the jth ranked candidate, $j = 1, \dots, n$.) The Calvo et al. argument transforms the voters' preferences over n-candidates into the characteristic functions for a particular n-player game.

Laruelle and Merlin used the Calvo et al. transformation to import positional voting results into game theory. (D. Haunsperger [12] did the same for statistics; she obtained unexpected conclusions for nonparametric statistics by transferring parallel results from positional voting.) In particular, Laruelle and Merlin transferred a result from Saari [18, 19] proving that there is a ten-candidate profile which defines millions of different election rankings. (About 3.2 million of them are strict rankings and the remaining 81 million rankings have at least one tie). As the voters' preferences remain fixed, these millions of outcomes are generated by varying the choice of a voting procedure. So, the Calvo et al transformation ensures there is a 10 player game with millions of different power index rankings. The game remains fixed; the millions of power index rankings reflect the different choices of a power index. Indeed, by applying this transformation to other Saari [18] conclusions, we have that with four or more players, a game can be constructed where *each player* is accorded the honor of being the most valuable – but only with certain power indices because there are other power indices which relegate the same player to a bottom ranking.

As Merlin also pointed out, their approach and results are seriously limited because the Calvo et al. transformation creates games with properties often prohibited by assumption. For instance, if ν is the characteristic value of a game, then we might require the traditional von Neumann's [27] superadditive property whereby

$$\nu(S) + \nu(T) \leq \nu(S \cup T) \quad \text{when } S \cap T = \emptyset, \tag{2.1}$$

or *monotonicity* where a player does not subtract from the value of a coalition; e.g., $v(\{a\}) \leq v(\{a, b\})$. But, this need not be the case; for many profiles the Calvo et al. transformation requires $v(\{a, b\}) < v(\{a\})$. So, do the Laruelle and Merlin results hold for games which satisfy traditional assumptions? Rather than transferring results from voting theory, we answer these and other questions by developing new direct methods.

2.1. **Power Indices.** A player's contribution can be measured by determining how his presence changes the worth of a coalition. To provide standard notation (e.g., see Lucas [14] or Ordeshook [16]), the value a game assigns to coalition S is given by the characteristic value $\nu(S)$ where, often, $0 \leq \nu(S) \leq 1$. If $\nu(S)$ is the value of the team with player j, and $\nu(S/\{j\})$ is the value without him, then it is arguable that the player's contribution is captured by the difference $[\nu(S) - \nu(S/\{j\}))]$. This leads to a standard definition of the power indices. Namely, with players $N = \{1, \dots, n\}$, the power index for player j, denoted by p_j, is defined as

$$p_j = \sum_{S \subset N, j \in S} P(S)[\nu(S) - \nu(S/\{j\})]. \tag{2.2}$$

The various power indices differ by the choice of $P(S) \geq 0$ which are normalized to satisfy

$$\sum_{S \subset N} P(S) = 1. \tag{2.3}$$

A commonly used choice requires $P(S)$ to reflect the size of S rather than the identity of the players. Here the $P(S)$ multipliers become $\lambda_i = P(S)$ for all S where $|S| = i$. By counting the number of coalitions of size $|S| = i$ with a particular player in S, Eq. 2.3 becomes

$$\sum_{i=1}^{n} \binom{n-1}{i-1} \lambda_i = 1, \quad \lambda_i \geq 0. \tag{2.4}$$

For instance, the Shapley value [20] uses $\lambda_i = 1/n\binom{n-1}{i-1}$, and the Banzhaf index [2] uses $\lambda_i = 1/2^{n-1}$. The sports example requires $\lambda_i = 0$ whenever $|S| = i$ represents a team with more members than permitted by regulations.

At times this neutrality requirement forcing $P(S)$ to depend only upon the size of a coalition makes sense; this is particularly so when issues of racial or gender equity are involved, or when the power of voting coalitions are analyzed. But the basketball example identifies settings where this neutrality assumption is inappropriate. To illustrate, adding a player from a particular position to a team where all other players already play the same position probably decreases the value of the coalition. (For instance, a soccer or ice hockey team consisting only of goalies cannot be improved by adding still another goal tender.) Here, using λ_j multipliers should raise questions about the validity of the outcome.

A political science example comes from considering the relative power of the justices on the US Supreme Court. The current alignment has three justices, Rehnquist, Scalia, and Thomas forming a conservative coalition, while Breyer, Ginsburg, Souter, and Stevens are often viewed as defining a more liberal coalition. Remaining are Justices Kennedy and O'Connor. On a divided conservative-liberal issue, it is unlikely that the conservative and liberal blocs will agree to create a majority opinion. Thus, a more accurate measure of the two main coalitions, and of Kennedy and of O'Connor, is to use multipliers defined in terms of the membership of coalitions rather than the λ_j approach. While we emphasize the Eq. 2.4 setting in our exploration of power indices, we also indicate changes which occur with the more general Eq. 2.3 setting. A safe way to characterize these difference is to assert that problems suffered by indices with the Eq. 2.4 setting become worse with the general Eq. 2.3 setting.

3. From sensitivity to trillions of outcomes

As shown next, our approach which explains why certain indices are sensitive to initial assumptions also indicates that the number of power index rankings (of the Eq. 2.4 type) grows rapidly with the number of players. For instance, we show there are 14-player games with over 80 *billion* (precisely, 80,951,270,400) different *strict* power index rankings. (By counting rankings with ties, the number escalates into the trillions.) Then, there are 15-player games with over a *trillion* different strict rankings. Our assertions about sensitivity and the number of rankings hold for a wide selection of games; e.g., they do not depend on whether they are monotonic, or super or subadditive, etc.

To develop intuition and notation, describe the matrix representation of Eq. 2.3

$$\begin{pmatrix} p_1 \\ p_2 \\ \dots \\ \dots \\ p_n \end{pmatrix} = \begin{pmatrix} \nu(\{1\}) & \dots & \sum_{|S|=j,1\in S}[\nu(S)-\nu(S/\{1\})] & \dots \\ \nu(\{2\}) & \dots & \sum_{|S|=j,2\in S}[\nu(S)-\nu(S/\{2\})] & \dots \\ \dots & \dots & & \\ \dots & \dots & & \\ \nu(\{n\}) & \dots & \sum_{|S|=j,n\in S}[\nu(S)-\nu(S/\{n\})] & \dots \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \dots \\ \lambda_j \\ \dots \\ \lambda_n \end{pmatrix}, \tag{3.1}$$

in the vector form

$$\mathbf{p} = \mathcal{M}^n(\nu)\boldsymbol{\lambda}$$

where the terms have the obvious definitions from Eq. 3.1 and $\mathcal{M}^n(\nu)$ is a $n\times n$ matrix with entries depending upon the game. If $\mathbf{P}_j$ designates the jth column of $\mathcal{M}^n$, then its kth entry represents the sum of the kth player's contributions made by joining all coalitions of size $(j-1)$. To emphasize that a power index represents the weighted contributions of each player to coalitions of all sizes, rewrite Eq. 3.1 as

$$\mathbf{p} = \sum_{j=1}^{n} \lambda_j \mathbf{P}_j. \tag{3.2}$$

Although Eq. 3.2 appears to define the convex hull spanned by $\{\mathbf{P}_i\}_{i=1}^n$, it does not because of the Eq. 2.4 limitations on the λ_j values. To correct this problem, define $\hat{\lambda}_j = \binom{n-1}{j-1}\lambda_j$ and $\hat{\mathbf{P}}_j = \mathbf{P}_j/\binom{n-1}{j-1}$ to change Eqs. 3.2, 2.3 to

$$\mathbf{p} = \sum_{j=1}^{n} \hat{\lambda}_j \hat{\mathbf{P}}_j, \quad \sum_{j=1}^{n} \hat{\lambda}_j = 1,\ \hat{\lambda}_j \geq 0. \tag{3.3}$$

Equation 3.3 requires all power index outcomes to be in the convex hull defined by the vertices $\{\hat{\mathbf{P}}_j\}_{j=1}^n$. Conversely, each point in this *power index hull* is the outcome for some power index. Thus, the power index hull and its geometry capture concomitant properties of power indices. The kth component of $\mathbf{P}_j$ defines player k's *total* contribution to all $(j-1)$-player coalitions, so the kth component of $\hat{\mathbf{P}}_j$ describes this player's *average* contribution to each coalition.

To find all power index outcomes defined by a specified game, plot the $\{\hat{\mathbf{P}}_j\}_{j=1}^n$ values. The power index hull is the convex hull defined by these vertices; the values assigned to index $\hat{\boldsymbol{\lambda}}$ define a unique geometric position within the hull; the coordinates define the index values. Indices can be described in terms of a reference point, say its center, or barycentric point. As this center point is defined by $\hat{\lambda}_i = 1/n = \binom{n-1}{i-1}\lambda_i$, it is the Shapley value. Stated in another manner, for any game *the Shapley value is the barycentric point of the associated power index hull.* (This geometric positioning dictates certain Shapley properties.) In comparison, the Banzhaf value of $\lambda_i = 1/2^{n-1}$ becomes $\hat{\lambda}_i = \binom{n-1}{i-1}/2^{n-1}$, so the Banzhaf point is closer to those $\hat{\mathbf{P}}_i$ vertices defined by mid-sized coalitions than is the Shapley point. This geometry, then, requires the Banzhaf *value* to be more strongly influenced by the contributions players make to mid-sized coalitions.

3.1. **Constructing examples.** Properties of the power index hull, and hence the properties of all power indices, are determined by the positions of the vertices $\{\hat{\mathbf{P}}_j\}_{j=1}^n$. Before describing some immediate consequences, we show how to exploit this observation. To start, notice that because the $\hat{\mathbf{P}}_j$ values are strictly determined by the ν values assigned to coalitions of size j and $(j-1)$, games can be constructed where the rankings defined by the n vectors $\{\hat{\mathbf{P}}_j\}_{j=1}^n$

all differ. For instance, it always is possible to construct a game where player j is top-ranked with $\hat{\mathbf{P}}_j$, $j = 1, \dots, n$. While this construction has the innocuous interpretation that each player has the strongest influence with coalitions of some size, it also ensures that this game *allows each player to be top-ranked with an appropriately chosen power index.* (To have player j top-ranked, choose a $\hat{\boldsymbol{\lambda}}$ which emphasizes $\hat{\lambda}_j$.)

To illustrate this construction with a three player game, the $n = 3$ version of Matrix 3.1 can be expressed as

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \nu(\{1\}) & [A_2 - A_1] + [\nu(\{1\}) - \nu(\{2,3\})] & 1 - \nu(\{2,3\}) \\ \nu(\{2\}) & [A_2 - A_1] + [\nu(\{2\}) - \nu(\{1,3\})] & 1 - \nu(\{1,3\}) \\ \nu(\{3\}) & [A_2 - A_1] + [\nu(\{3\}) - \nu(\{1,2\})] & 1 - \nu(\{1,2\}) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \tag{3.4}$$

where A_i is the sum of values assigned to i-player coalitions and where $\nu(\{1,2,3\}) = 1$. Player i is top-ranked with $\mathbf{P}_i$ if and only if the largest component of $\mathbf{P}_i$ is the ith component. Thus, the conditions to construct an appropriate game are

- that $\nu(\{1\})$ has the largest value for the one-player coalitions,
- that $[\nu(\{2\}) - \nu(\{1,3\})]$ is the algebraically largest value in column two, and
- that $\nu(\{1,2\})$ is the smallest value for two-player coalitions.

As two of these conditions involve rankings and the last one involves differences, constructing examples is easy. For instance, a superadditive game satisfying these conditions is

- $\nu(\emptyset) = 0$, $\nu(\{1,2,3\}) = 1$
- $\nu(\{1\}) = 0.2$, $\nu(\{2\}) = 0.15$, $\nu(\{3\}) = 0$,
- $\nu(\{1,2\}) = 0.35$, $\nu(\{1,3\} = 0.4$, $\nu(\{2,3\}) = 0.70$

defining $\mathbf{P}_1 = (0.2, 0.15, 0)$, $\mathbf{P}_2 = (0.6, 0.85, 0.75)$, $\mathbf{P}_3 = (0.3, 0.6, 0.65)$. The averaged vertices are $\hat{\mathbf{P}}_1 = \mathbf{P}_1$, $\hat{\mathbf{P}}_2 = \frac{1}{2}\mathbf{P}_2 = (0.3, 0.425, 0.375)$, $\hat{\mathbf{P}}_3 = \mathbf{P}_3$ with the respective power index rankings of $1 \succ 2 \succ 3$, $2 \succ 3 \succ 1$, $3 \succ 2 \succ 1$.

The barycentric Shapley value for this game is $\frac{1}{3}\sum \hat{\mathbf{P}}_j = (0.2667, 0.3917, 0.375)$ where its $2 \succ 3 \succ 1$ ranking agrees with that of $\hat{\mathbf{P}}_2$. (Corollary 1 proves that this is no accident.) The Banzhaf index of $(0.3667, 0.5333, 0.4666)$ has the same ranking (again, according to Cor. 1, this is no accident), but it assigns more significance to the second player's top standing due to the stronger influence of $\hat{\mathbf{P}}_2$ on the Banzhaf outcome. Thanks to the degrees of freedom available to design this example, it is clear that games can be constructed which satisfy other specified conditions such as, say, subadditivity, etc.

3.2. **Generic and sensitive.** This geometry is central for all of the results described here. As a sample, the first conclusion supports our earlier assertion that we must expect different indices to define different values.

Theorem 1. *For almost all choices of the characteristic values for a game, the values defined by two different indices differ. Indeed, only for those special games where all $\hat{\mathbf{P}}_j$ values agree must all indices agree.*

The proof of this assertion is immediate. As we have seen from constructing the example, changing ν values changes the vertices; generically, then, the vertices disagree. But different $\hat{\boldsymbol{\lambda}}$ choices define different positions within the hull – and, hence, different index values. The conclusion now follows. If all indices agree, then all $\{\hat{\mathbf{P}}_j\}$ vertices agree; while this setting might arise in modeling (it happens if the game is strictly additive where $\nu(S) = \sum_{j \in S} \nu(\{j\})$), it is a highly degenerate situation.

The geometry also identifies which indices are more susceptible to specified changes in assumptions. Small changes in the characteristic values of a game change the vertices. The geometry is clear; if a particular vertex moves because of changing assumptions, then those indices near that vertex are more profoundly affected. (For $n = 3$, think of this as moving one vertex of the triangular power hull; clearly, points closer to this vertex are stronger affected.) Conversely, hull point further away from any affected vertices are more minimally affected. This geometry dictates that, without imposing assumptions on which vertex is affected, the point most resilient to changes in these assumptions must be the barycentric point – the Shapley value.

Theorem 2. *With small changes in the characteristic values of a game, where these changes can affect coalitions of any size, the Shapley value is least affected by these changes.*

To indicate how this relates to Justice Harlan's comments, which are nicely captured and analyzed in Grofman [8], Harlan claims that a slight change in assumptions about voter preferences could change a voter's Banzhaf power by the astronomical multiple of 12×10^{19}. Such multiples can arise only if the original Banzhaf values are so small that even a small change in the vertices creates a large multiple change. This happens with Harlen's example of 300,000 voters in a two party election where each voter is equally likely to vote for either party. Here, each voter's minuscule Banzhaf value, derived from the power index hull with 300,000 vertices, is around $\binom{300{,}000}{150{,}000}/2^{300{,}000}$. When it is equally likely for a voter to vote either way, it is highly unlikely that a voter joining a coalition of even 200,000 will determine the election outcome by supplying a particular candidate the decisive 150,001st vote. This forces all of $\hat{\ }_j$ vertices, $j < 200,000$, to have nearly zero components. Now consider what happens when the "equally likely" assumption changes to, say, $p = 0.55$ that a voter votes for party A. This assumption makes it more likely for a voter to cast the deciding vote when joining a smaller coalition; say, the group of 200,000. Thus, this likelihood assumption changes the value of the middle valued $\hat{\mathbf{P}}_j$ vertices; changes that even if small in value, can create astronomical scale changes for indices, such as the Banzhaf value, which are strongly affected by the middle level vertices. Indices more to the center of the hull, such as the Shapley value, are not as drastically changed.

3.3. **Geometry.** The approach used to construct the above three-player example only ensures, for instance, that there is a 14-player game with 14 different power index rankings; this statement comes nowhere near satisfying our earlier assertion about a 14-player game with billions of different strict power index rankings. Yet, the proof of our more surprising conclusion involves nothing more than appropriately selecting the $\hat{\mathbf{P}}_j$ vertices. Toward our goal of verifying our conclusion, we describe the geometry of the power index hull.

Assign each R^n axis to a particular player. Because the *indifference plane* $x_i = x_j$ divides R^n into two sectors, if $\mathbf{p}$ is in the sector where $p_i > p_j$, then player i is ranked above player j. The $\binom{n}{2}$ indifference planes defined by all pairs partition R^n into *ranking regions;* the $n!$ open cones correspond to the $n!$ strict rankings of the players. Regions on an indifference plane involve at least one tied outcome. The *completely tied line* defined by the intersection of all indifference planes is where all players are ranked equal.

To visualize the geometry of the ranking regions, we reduce the dimension. Consider the positive orthant $R^n_+ = \{\mathbf{x} = (x_1, \dots, x_n) \in R^n \,|\, x_j \geq 0\}$. (As all coordinates of a R^n_+ vector are non-negative, all monotonic games are in this orthant.) In R^n_+, the simplex $Si(n) = \{\mathbf{x} \in R^n_+ \mid \sum x_j = 1\}$ intersects each ranking region. This is displayed in Fig. 1 for the special case of $n = 3$. The six small triangular regions correspond to strict rankings where the ranking is determined by the distance of a point from each vertex where "closer is better." For instance, because the $\bullet$ is in the region closest to vertex 3, next closest to vertex 1, and farthest from vertex 2, the associated ranking is $3 \succ 1 \succ 2$. The center point of the triangle, where all three indifference lines cross, corresponds to a completely tied outcome of $1 \sim 2 \sim 3$.

To visualize all power index rankings admitted by a game, project the power index hull to $Si(n)$. This is done by dividing each $\hat{\mathbf{P}}_j$ by the sum of its components to define $\hat{\mathbf{P}}_j^* \in Si(n)$. (If $\hat{\mathbf{P}}_j$ is the zero vector, its projection is the complete indifference point $\frac{1}{n}(1, \dots, 1)$.) This process defines the *normalized index value* where the sum of the values assigned to the players equals unity. Although the convex hull defined by $\{\hat{\mathbf{P}}_j^*\}_{j=1}^n$, the *normalized power index hull*, is *not* the power index hull, the ordinal rankings admitted by both hulls always agree.

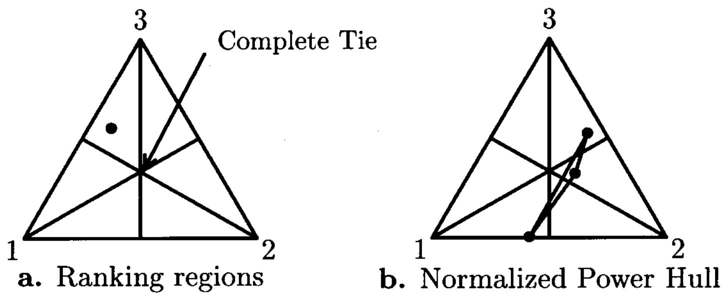

Fig. 1. Representation triangles

To illustrate with the earlier three-player game, the points

$$\hat{\mathbf{P}}_1^* = \frac{1}{0.35}\hat{\mathbf{P}}_1 = (\frac{4}{7}, \frac{3}{7}, 0), \quad \hat{\mathbf{P}}_2^* = (\frac{12}{44}, \frac{17}{44}, \frac{15}{44}), \quad \hat{\mathbf{P}}_3^* = (\frac{6}{31}, \frac{12}{31}, \frac{13}{31})$$

are plotted in Fig. 1b where $\hat{\mathbf{P}}_1^*$ is the point in the lower left region, $\hat{\mathbf{P}}_2^*$ is in the middle, and $\hat{\mathbf{P}}_3^*$ is the point in the upper right region. Earlier we showed that this game admitted three different strict rankings of the players. The normalized power index hull (defined by the three vertices $\{\hat{\mathbf{P}}_j^*\}_{j=1}^3$) intersects *four* strict ranking regions and *three* regions which involve one tie. In other words, from the geometric positioning of the vertices, we have that instead of three strict rankings, this game admits four strict rankings (the new one is $2 \succ 1 \succ 3$) and three rankings with ties ($1 \sim 2 \succ 3$, $2 \succ 1 \sim 3$, and $2 \sim 3 \succ 1$). Thus, this particular game admits *seven different power index rankings*; four are strict and three involve ties.

The geometry tells us much more. For instance, it states *for this particular game* that power indices with a stronger emphasis on individual players have the $1 \succ 2 \succ 3$ ranking, indices with emphasis on the coalition of all players have the reversed ranking, those with an emphasis on midsized coalitions have the $2 \succ 3 \succ 1$ ranking, and those with more balance between the small and midsized coalitions have the $2 \succ 1 \succ 3$ outcome. (Exact bounds on the different indices are found by using elementary algebra with the power index hull.) Moreover, the considerable freedom available in the designing this game shows that games with such multiple ranking outcomes occur with fairly high probability.

Earlier we mentioned how the differences in index outcomes converts the choice of an index into a strategic variable. The geometry shows how when such opportunities are available (i.e., when there are different outcomes) and how to identify strategic choices. For instance, the geometric argument just used to identify which indices rank each player at the top also identifies strategic choices of indices for supporters of different outcomes.

3.4. **Consequences.** The next result describes the radical variation allowed in the rankings. To motivate a technical sounding assertion, suppose we could construct a game with the completely tied point in the interior of the normalized power index hull. As the completely tied point is on the boundary of *all* ranking regions, this geometric property would require all possible rankings to occur; e.g., each of the $n!$ strict rankings would occur with some power index. Fortunately, as asserted next, such wild outcomes cannot occur.

Theorem 3. *For any $n \geq 2$, there exists an open set of choices of games where the power index ranking of the players depends upon the choice of the power index. Indeed, for each integer k satisfying $1 \leq k \leq n! - (n-1)!$, there exist games with precisely k different power index strict rankings. No game can have more than $n! - (n-1)!$ different strict power index rankings.*

For any game, the completely tied ranking never can be an interior point of the power index hull. However, for any $n \geq 4$, it is possible to construct a game where each candidate is top ranked with some power index, and then second-ranked with other power indices, ..., then bottom ranked with other power indices.

Although not everything can occur, the rankings of power indices can be so general that not much relief is provided. This is because Thm. 3 ensures there are n-player games with $n! - (n-1)!$ different power index strict rankings. (An $n = 3$ example with the maximum $3! - 2! = 4$ strict rankings is given above.) To design such a game, select ν values so that the $\hat{\mathbf{P}}_j^*$ vertices are sufficient spread among different ranking regions; while this construction designs a game where different players have success with coalitions of different sizes, it also requires the normalized power index hull to meet the largest possible number of ranking regions. In other words, the geometry defined by the $\{\hat{\mathbf{P}}_j\}$ vertices completely determines whether a game admits many, or only a few different power index rankings. This geometry also determines the differences in points assigned by different indices.

So, while no game has *all* possible rankings for power indices, the indices can include up to $(1 - \frac{1}{n})$ of all strict rankings. Thus, as n increases in value, the fraction of all rankings which can be power index rankings quickly approaches unity. The following table indicates how rapidly the number of admitted power index rankings grows into the billions and trillions.

n	Strict rankings	n	Strict Rankings
2	1	9	322,560
3	4	10	3,265,920
4	18	11	36,288,000
5	96	12	439,084,800
6	600	13	5,748,019,200
7	4,320	14	80,951,270,400
8	35,280	15	1,220,496,076,800

(3.5)

As described, constructing games with the different k values just involves the positioning of the $\hat{\mathbf{P}}_j$ vertices. For instance, if all vertices have the same ranking, then all power indices share this common ranking. If the rankings for the vertices are sufficiently diverse, then a large number of rankings are admitted. (Details are given in Sect. 5.)

The Thm. 3 restrictions depend upon using λ_j weights based on the size of the coalition rather than its members. Indeed, the weaker Eq. 2.2 formulation admits all possible rankings.

Theorem 4. *Assume that for each player, there is at least one coalition where the player adds positive value to the coalition by joining it. For any such game, the set of power indices defined by Eqs. 2.2, 2.3 include all possible (transitive, complete) rankings of the n alternatives.*

Proof. For each coalition S and player $j \in S$, let $\mathbf{V}_{j,S}$ be the vector where the jth component is $\nu(S) - \nu(S/\{j\})$ and all other components are zero. The set of all possible power indices defined by Eqs. 2.2, 2.3 is the convex hull of the set $\{\mathbf{V}_{j,S}\}$. By assumption, each R^n positive axis contains at least one of these vectors. Thus, the convex hull meets all possible ranking regions. □

3.5. **Some structure.** Suppose we want to construct a game where specified power indices have different specified rankings. As a special case, consider those $n = 3$ power indices where $\hat{\lambda}_2 = 0$. The outcome for power index $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, 0, \hat{\lambda}_3)$ is $\mathbf{p} = \hat{\lambda}_1\hat{\mathbf{P}}_1 + (1 - \hat{\lambda}_1)\hat{\mathbf{P}}_3]$. Thus, $\mathbf{p}$ is at a specified position on the line defined by the endpoints $\hat{\mathbf{P}}_1$, $\hat{\mathbf{P}}_3$. Since the locations of the endpoints $\hat{\mathbf{P}}_1$, $\hat{\mathbf{P}}_3$ are determined by the characteristic values of, respectively, single player and the difference between three and two player coalitions, it is easy to construct a game which positions this line in any desired position. In particular, since any two points on this line define two different power indices, the line can be positioned (i.e., a game can be constructed) so that each point is in a specified ranking region. By placing the line close to the completely tied center point (where all regions meet), a game can be constructed so that the rankings of the two specified indices even reverse one another. (Such a situation is in Figs. 1 and 3.)

Proposition 1. *For $n = 3$, choose two different power indices $\hat{\boldsymbol{\lambda}}^1, \hat{\boldsymbol{\lambda}}^2$ where both assign zero value to $\hat{\lambda}_2$. Choose two transitive rankings, r_1, r_2, of the three players; they need not agree in any manner. There exists a game where the power index ranking of $\hat{\boldsymbol{\lambda}}^j$ is r_j, $j = 1, 2$.*

So, games exist where the power index ranking of $\hat{\boldsymbol{\lambda}}^1 = (\frac{4}{9}, 0, \frac{5}{9})$ is $1 \succ 2 \succ 3$ even though the power index ranking of the closely related $\hat{\boldsymbol{\lambda}}^2 = (\frac{5}{9}, 0, \frac{4}{9})$ is the reversed $3 \succ 2 \succ 1$. As Prop. 1 ensures complete freedom in choosing rankings for any two indices of this $\hat{\lambda}_2 = 0$ type, we must anticipate wilder examples by using the full geometry of the power index hull. Instead, structure emerges. For motivation, notice by specifying any two columns of Matrix 3.4, the third is uniquely determined. As this relationship requires the choice of $\mathbf{P}_1$ and $\mathbf{P}_3$ to determine $\mathbf{P}_2$, we must wonder whether this dependency forces certain classes of power indices to always share the same ranking. This is the case.

To explain in terms of $n = 3$, the power index ranking is the ranking of the coordinates of $\mathbf{p} = \hat{\lambda}_1\hat{\mathbf{P}}_1 + \hat{\lambda}_2\hat{\mathbf{P}}_2 + \hat{\lambda}_3\hat{\mathbf{P}}_3$. Since common terms which appear in all coordinates of a column, such as $(A_2 - A_1)$, do not affect the ranking, the ranking is given by the coordinates of

$$\begin{pmatrix} \nu(\{1\}) \\ \nu(\{2\}) \\ \nu(\{3\}) \end{pmatrix} - \frac{2\hat{\lambda}_3 + \hat{\lambda}_2}{2\hat{\lambda}_1 + \hat{\lambda}_2} \begin{pmatrix} \nu(\{2,3\}) \\ \nu(\{1,3\}) \\ \nu(\{1,2\}) \end{pmatrix}. \tag{3.6}$$

Thus all power indices with the same $[2\hat{\lambda}_3 + \hat{\lambda}_2]/[2\hat{\lambda}_1 + \hat{\lambda}_2] = m$ value must have the same ranking as the coordinates of Eq. 3.6. These power indices, then, share the same ranking for *any* game. Call such a line of power indices an *equiranking line.* For $n = 3$, the equiranking classes of power indices are given by the two equivalent expressions

$$\hat{\lambda}_3 = \frac{m-1}{2}\hat{\lambda}_2 + m\hat{\lambda}_1, \text{ or } m = \frac{\lambda_3 + \lambda_2}{\lambda_1 + \lambda_2} \quad \forall m \geq 0. \tag{3.7}$$

Theorem 5. *For $n \geq 3$, the simplex of power indices,*

$$\mathcal{PI}^n = \{\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \dots, \hat{\lambda}_h) \mid \sum_{i=1}^{n} \hat{\lambda}_i = 1, \, \hat{\lambda}_i \geq 0\}$$

is partitioned into equiranking equivalence classes of lines. For all games, the power indices on a equiranking line have the same ranking. For an n-player game, if $\lambda_1 + \lambda_2 > 0$, the *equiranking lines are given by constant m_j values for*

$$m_j = \frac{\lambda_{j+2} + \lambda_{j+1}}{\lambda_1 + \lambda_2}, \quad j = 1, \dots, n-2 \tag{3.8}$$

where the λ_j satisfy Eq. 2.4. (If $\lambda_1 + \lambda_2 = 0$, replace the denominator by the first $\lambda_j + \lambda_{j+1}$ term which is non-zero.)

Select two power indices $\hat{\boldsymbol{\lambda}}_j$, $j = 1, 2$ from different equiranking lines and two rankings r_1, r_2. There exists a game where the power index of $\hat{\boldsymbol{\lambda}}_j$ is r_j; $j = 1, 2$.

The last conclusion, which states that rankings for indices from different equiranking lines can differ as wildly as desired, can be extended to assert that after $(n-1)$ different rankings and $(n-1)$ power indices (from different equiranking classes) are selected, there is a game where the jth power index has the jth ranking; $j = 1, \dots, n-1$. However, this result requires an extra condition on the indices (which, essentially, requires them to be linearly independent; see the comments following the proof in Sect. 5).

The equiranking lines for $n = 3$ are represented in Fig. 2. It follows from this figure (and this observation extends to all $n \geq 3$) that the common ranking shared by all indices in an equiranking class is determined by the ranking of an index from the class with some $\hat{\lambda}_j = 0$. This observation makes it easier to design games where two specified indices in the interior of the hull, $\hat{\boldsymbol{\lambda}}_j$, $j = 1, 2$, have specified rankings r_j. To illustrate with $n = 3$, rather than worrying about the actual indices, replace them with indices from their equiranking class where $\hat{\lambda}_2 = 0$. Next, as described earlier, adjust the $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_3$ positions so that the index equivalent to $\hat{\boldsymbol{\lambda}}_j$ has ranking r_j.

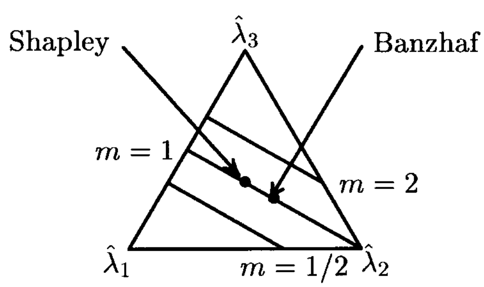

Fig. 2. Equiranking classes of power indices

An interesting consequence of Thm. 5 is that the Shapley and Banzhaf indices are in the same $n = 3$ equiranking class. This $m = 1$ equiranking class of indices, where $\hat{\lambda}_1 = \hat{\lambda}_3$, is the line connecting the midpoint of the $\hat{\lambda}_1 - \hat{\lambda}_3$ edge with the $\hat{\lambda}_2$ vertex. In turn, this means that the common ranking for these two indices is completely determined by the ranking of mid-sized coalitions as given by $\hat{\mathbf{P}}_2$; the contributions from coalitions of other sizes (given by $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_3$) influence the values of these indices, but not the ranking. It also follows from Eq. 3.8 that these two indices never are in the same class for $n \geq 4$.

Corollary 1. *For all three player games, the Shapley and Banzhaf rankings always agree. This common ranking is that of $\hat{\mathbf{P}}_2$ (or of $\hat{\mathbf{P}}_1 + \hat{\mathbf{P}}_3$). For $n \geq 4$, select any two transitive rankings r_S, r_B of the n players, they can even reverse one another. There exist games where the Shapley and Banzhaf rankings are, respectively, r_S and r_B.*

3.6. **Voting models.** As not all games allow billions of conflicting power index rankings, an interesting project is to characterize the types of games which permit some level of regularity.

The way to do this is to determine how the game structure affects the positioning of the $\hat{\mathbf{P}}_j$ vertices. When the assumptions about a game force the $\{\hat{\mathbf{P}}_j\}_{j=1}^n$ vertices to be reasonably similar, the power index rankings are closely related; when the ranking of the vertices differ (e.g., if different players make a stronger contribution over different sized coalitions), we must anticipate results of the kind described in Thm. 3. To illustrate one kind of regularity, we describe a natural class of games for political science.

A typical model measuring power in a legislative body captures when a coalition or party is decisive in certain votes. For a majority vote, for instance, one might assign $\nu(S) = 1$ if $|S| > n/2$, and $\nu(S) = 0$ if $|S| \le n/2$. To go beyond these so-called "simple games" to capture, say, when a coalition can pass legislation with a majority vote and when it can overcome a presidential veto with a two-thirds vote, we might use

$$\nu(S) = \begin{cases} 0 & \text{if } |S| \le n/2 \\ 1 & \text{if } n/2 < |S| \le 2n/3 \\ 3/2 & \text{if } 2n/3 < |S| \end{cases} \tag{3.9}$$

The following definition extends these choices.

Definition 1. *A* generalized voting game *is where the characteristic value $\nu(S)$ depends only on $|S|$ (i.e., the number of players in S), and where if $|S_1| < |S_2|$, then $\nu(S_1) \le \nu(S_2)$.*

The following result asserts that generalized voting games are spared the problems of having a large number of different power index rankings.

Theorem 6. *For a generalized voting game consisting of n voters divided into k-parties $S_1, S_2, \dots, S_k$, suppose that $|S_1| \ge |S_2| \ge \cdots \ge |S_k|$. For any choice of a power index satisfying Eq. 2.4, the power index ranking must come from $S_1 \succsim S_2 \succsim \cdots \succsim S_k$ where the different "$\succsim$" relationships are replaced with "$\succ$" or "$\sim$".*

Both technically and intuitively, this theorem captures what we expect; a party with more voters has a stronger impact. In other words, for these generalized voting games, all vertices, hence all power indices, essentially mimic the ranking given by the sizes of the coalitions. Differences among power indices primarily arise with the differing levels of significance assigned to different parties.

To provide examples where inequalities defined by party size are replaced with ranking equalities, suppose a legislative body of 100 voters is divided into three parties where $|S_1| = 36, |S_2| = 33, |S_3| = 31$ and $\nu(S) = 0$ if $|S| \le 50$, $\nu(S) = 1$ if $|S| \ge 51$. A computation proves that $\hat{\mathbf{P}}_1 = \hat{\mathbf{P}}_3 = \mathbf{0}$ while $\hat{\mathbf{P}}_2 = (1,1,1)$. Thus, all power indices have the same $1 \sim 2 \sim 3$ ranking. (Values differ as the Shapley value is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and the Banzhaf value is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.) If we include the ability to override vetos by using ν of Eq. 3.9, we have that $\hat{\mathbf{P}}_1 = \mathbf{0}$, $\hat{\mathbf{P}}_2 = (1.5, 1.25, 1.25)$, $\hat{\mathbf{P}}_3 = (0.5, 0, 0)$ where all but one power index has the $1 \succ 2 \sim 3$ ranking. The exception is $\hat{\boldsymbol{\lambda}} = (1,0,0)$ with its $1 \sim 2 \sim 3$ ranking.

The proof of Thm. 6 uses the observation that the ranking for each $\hat{\mathbf{P}}_j$ comes from $S_1 \succsim S_2 \succsim \cdots \succsim S_k$. Since all power indices are convex combinations of these vertices, the conclusion follows. To show that the vertex rankings have this property, we show that if $|S_s| > |S_t|$, then $\hat{\mathbf{P}}_j$ ranks $S_s \succsim S_t$. To do so, first consider coalitions of $j-1$ parties that do not include S_s or S_t. The generalized voting game requires S_s to provide a larger (or equal) change in the characteristic value when it joins these $j-1$ parties than would S_t. Similarly, in a coalition of j parties which includes both S_s and S_t, removing S_s rather than S_t causes a larger (or equal) change in the characteristic value of the remaining coalition. Thus, $S_s \succsim S_t$ with $\mathbf{P}_j$ (and, hence, with $\hat{\mathbf{P}}_j$). This completes the proof.

4. Dropping players

How much of the power index ranking is retained after a player is dropped? If the new game has a significantly different structure, then no relationships need be expected. Thus, answers must depend upon how the games are related. The following describes a natural setting which preserves much of the game structure when players leave.

Definition 2. *A* "standard class of games" *is where, whenever a player drops out, the resulting game changes only by ignoring all coalitions involving the missing player.*

The behavior caused by players leaving can be discouraging.

Theorem 7. *For $n \geq 3$ players, consider the set of $(n-1)$ subgames defined by players $N, \{1, 2, \dots, n-1\}, \dots, \{1, 2\}$ where each subgame is defined by a player leaving. For each subgame consisting of k players, select a ranking of the k players and a power index $\hat{\boldsymbol{\lambda}}^k$ where all $\lambda_j \neq 0$. There exists a standard class of games so that the specified power index ranking of each subgame is the specified ranking.*

This discouraging result asserts, for instance, that the power index for all n players is $1 \succ 2 \succ \dots \succ n$, but when last place n drops out, the new ranking reverses to $n-1 \succ n-2 \succ \dots \succ 1$, and when player $n-1$ drops out, the ranking is, say, the mixed $1 \succ n-2 \succ 2 \succ \dots$. In other words, no regularity whatsoever in power index rankings can be assumed when players drop out. The importance of allowing *any* index (with the one minor assumption) is to prove that the conclusion is not caused by using "incorrect power indices"; these wild changes hold for the Shapley value, or the Banzhaf index, or any other choice. This means that the problem is inherent in the structure of games and the definition of power indices.

The simple proof of this assertion is outlined next. Let r_k and $\boldsymbol{\lambda}_k$ be, respectively, the ranking and power index assigned to the subset of k players; $k = 2, \dots, n$. The problem is to define a game where the $\boldsymbol{\lambda}_k$ ranking of the set of k players is r_k. But this is easy because the two player outcome only involves assigning appropriate values to $\nu(\{1\})$, $\nu(\{2\})$, and $\nu(\{1,2\})$. For three players, we have the added variables of $\nu(\{3\}), \nu(\{2,3\}), \nu(\{1,3\}), \nu(\{1,2,3\})$ which can be used to create any desired ranking. However, in order to create the new ranking, we need to be assured that at least some of the extra variables are used; this is the purpose of the $\lambda_j \neq 0$ assumption. The same argument applies to all other values of k.

This theorem tells us, then, that consistency in the index rankings requires stronger assumptions about the game or index. But the theorem also identifies the source of the difficulty in achieving "consistency;" it is caused by the fact that new variables are used in the computation of indices for larger games. Values for these new variables – which indicate differing strengths of coalitions – can be successively selected to alter the ranking of indices in any desired manner. Thus, any assumptions added to achieve consistency must restrict the choices of values for these new variables.

To illustrate, a strong consistency holds for the standard class of games coming from generalized voting games. This is because if a political party disappears, or if a voter drops out of the legislature, then, as long as the new setting is a generalized voting game, Thm. 6 requires the power index rankings of the remaining parties to retain essentially the ranking determined by their sizes. Similarly, strictly additive games where $\nu(S) = \sum_{j \in S} \nu(\{j\})$ enjoy consistency because the ranking of each vertex agrees with the ranking of the individual players. So, a standard game structure transfers this ranking to subgames.

But even if a game and the subgames use the precise same variables, difference can occur because of the different ways the information is combined. To illustrate, consider a setting motivated by the basketball example where $\lambda_j = 0$ once j is larger than some threshold value.

Definition 3. *A threshold index for games with n players is where there is a threshold value $T < n$ so that $\lambda_j = 0$ for all $j \geq T$.*

We should expect some sort of relationship to hold among the threshold index rankings applied to a standard class of games. This is because the sum of all M player subgames, $M \geq T$, use the same variables as the n-player game. However, consistency requires using very different indices for the subgames. In particular, this consistency does not hold for the Shapley or the Banzhaf values. Actually, consistency hold only for a highly unusual class of indices.

Theorem 8. *Suppose two threshold indices $\boldsymbol{\lambda}^N$ and $\boldsymbol{\lambda}^M$, with the same threshold value T, are used with a standard class of games where $\boldsymbol{\lambda}^N = (\lambda_1^N, \ldots, \lambda_T^N)$ is used with the game of all n players while $\boldsymbol{\lambda}^M = (\lambda_1^M, \ldots, \lambda_T^M)$ is used with all M-player games where $n > M > T$. Further suppose the λ weights satisfy*

$$\binom{n-j}{M-j}\lambda_j^M = \alpha\lambda_j^N, \quad j = 1, \ldots, T \tag{4.1}$$

where α is a constant. A player who is $\boldsymbol{\lambda}^M$ top-ranked (bottom-ranked) in all subgames of size M (in which she is included) cannot be $\boldsymbol{\lambda}^N$ bottom-ranked (top-ranked) in the game involving all n players. A player who is $\boldsymbol{\lambda}^M$ top-ranked in all M-subgames is $\boldsymbol{\lambda}^N$ ranked at least as high in the n-player game as a player who is $\boldsymbol{\lambda}^M$ bottom-ranked in all M-subgames.

If $\boldsymbol{\lambda}^N$ and $\boldsymbol{\lambda}^M$ do not satisfy Eq. 4.1, then no relationship need hold; in particular, examples of these games can be created where the above conditions are violated.

5. Concluding comments

In summary, we have shown that it is surprisingly easy to create n-player games which exhibit differences in values and even in the rankings of power indices. This ease returns us to our initial concerns. For instance, not only is manipulation though the choice of indices possible, either wittingly or unwittingly, but even the meaning and some applications of these indices seems to be in need of re-examination.

As for the comments about changes in rankings of indices when players leave, an obvious challenge posed by this analysis is to find reasonably general classes of games and associated indices which permit some level of compatibility in index rankings when players drop out. We suspect that unless the games are severely restricted, the best relationships are of the type exhibited in Thm. 8 where a player who is $\boldsymbol{\lambda}^M$ top-ranked in all M-player games cannot be $\boldsymbol{\lambda}^N$ bottom-ranked in the game with all n players.

6. Proofs

The proof of several of these results is based are rewriting the matrix of Eq. 3.1 in the following manner. Leave the $\mathbf{P}_1$ column as specified. Next, define

$$A_j = \sum_{|S|=j} \nu(S), \quad A_{j,i} = \sum_{|S|=j, i \notin S} \nu(S).$$

In words, A_j is the sum of characteristic values for coalitions of size j while $A_{j,i}$ is the sum over the coalitions which do *not* include player i. With this notation, the $a_{i,j}$ entry of matrix $\mathcal{M}^n$, $2 \leq j \leq n-1$, can be expressed as

$$a_{i,j} = \sum_{|S|=j-1, i \notin S} [\nu(S \cup \{i\}) - \nu(S)] = [A_j - A_{j,i}] - A_{(j-1),i}. \tag{6.1}$$

The ith entry of the last column $\mathbf{P}_n$ is $\nu(N)-\nu(N/\{i\}) = A_n - A_{(n-1),i}$ where N is the coalition of all players. Notice that the ith entry of $\mathbf{P}_2$ can be expressed as $[A_2 - A_1] + [\nu(\{i\}) - A_{2,i}]$.

Proof of Thm. 5. When the matrix multiplication of Eq. 3.1 is carried out with matrix entries expressed as above, the ith entry is of the form

$$[(\lambda_1+\lambda_2)\nu(i) - (\lambda_2+\lambda_3)A_{2,i} - \cdots - (\lambda_n+\lambda_{n-1})A_{(n-1),i}] + \sum_{j=2}^{n}\lambda_j A_j. \tag{6.2}$$

The summation is common to all entries, so the ranking of the power index is determined by comparing the bracketed terms for different i values. Assuming $\lambda_1+\lambda_2>0$ and defining $m_j = (\lambda_{j+2}+\lambda_{j+1})/(\lambda_1+\lambda_2)$, the rankings are completely determined by the values

$$\nu(i) - m_1A_{2,i} - \cdots - m_{n-2}A_{(n-1),i} \tag{6.3}$$

over $i = 1,\dots,n$. Thus, indices with the same m_j values must have the same ranking. (If $\lambda_1+\lambda_2 = 0$, then divide by the first $\lambda_j+\lambda_{j+1}$ term which is non-zero.) This completes the proof of the first statement.

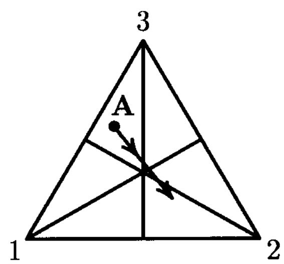

Fig. 3. Designing games

Although the second statement follows from other assertions, we outline a simple, direct proof. The most difficult setting is if the two equiranking classes differ in only one m_j value (as this reduces the number of degrees of freedom). Indeed, in this case, where the two values are $m_j^1 \neq m_j^2$, the expression 5.3 has the vector forms

$$\mathbf{A} - m_j^i\mathbf{A}_j^*, \quad i = 1,2$$

where the ith entry of $\mathbf{A}_j^*$ is $A_{j,i}$ and the ith entry of $\mathbf{A}$ is the sum of all other terms from Expression 5.3 which are common to both indices. Thus, the rankings defined by the two vector expressions depend upon the m_j^i distance $\mathbf{A}_j^*$ is moved from the common $\mathbf{A}$. Since these distances differ and since the values of the two vectors are free to be designed by selecting appropriate characteristic values, the vectors can be positioned so the two expressions have any desired ranking. This is illustrated in Fig. 3 where $\mathbf{A}$ (the base of the two arrows) and one of the outcomes (given by the first arrowhead) have the $3 \succ 1 \succ 2$ ranking, while the second outcome (given by the second arrowhead which extends in the same direction) has the reversed $2 \succ 1 \succ 3$. (The comment following the statement of the theorem is proved in much the same manner. But here we need to ensure that the indices from different ranking classes are such that the version of Expression 5.3 has $n-2$ independent directions.) $\square$

Proof of Theorem 3. First we show that the complete indifference point cannot be an interior point of the $(n-1)$-dimensional normalized power hull. If this hull has an interior point, then $\mathcal{M}^n$ has maximal rank. (If $\mathcal{M}^n$ has lower rank, then the hull is lower dimensional.) We now derive a contradiction.

To demonstrate the approach, we first prove the result for the simpler $n = 3$ setting. If the complete indifference point is an interior point, then there is some index $\boldsymbol{\lambda}$, where all $\lambda_j > 0$, so that $\mathcal{M}^3(\boldsymbol{\lambda})$ is on the line of completely tied outcomes. As this assumption requires all entries to be equal, it follows from Eq. 3.4 that

$$v(\{j\}) = \frac{\lambda_2}{\lambda_1+\lambda_2}(A_2 - A_1) - \frac{\lambda_3}{\lambda_1+\lambda_2} - \frac{\lambda_2+\lambda_3}{\lambda_1+\lambda_2}\nu(\{1,2,3\}/\{j\}) + D$$

where D is some fixed scalar. Substituting these values for $\nu(\{j\})$ into $\mathcal{M}^3$ converts each column into the form where the jth entry is $D^* + \alpha\nu(\{1,2,3\}/\{j\})$ where D^* and α are constants for that column. This means that by adding an appropriate multiple of the last column to each of the first two makes these columns with constant entries. In turn, this means that $\mathcal{M}^3$ has a lower rank, which completes our proof.

For $n \geq 4$ values, it follows from either Expression 5.2 or 5.3 that the assumption about having a completely tied interior outcome defines the expression

$$\nu(\{i\}) = \sum_{j=1}^{n-2} m_j A_{(j+1),i} + D$$

where D is a fixed value. Replace each $\nu(\{i\})$ in the first and second columns of $\mathcal{M}^n$ with this value. We now show that adding appropriate summations of the third through the nth columns to the first and to the second columns converts these two columns to columns with constant entries – as this forces $\mathcal{M}^n$ to have lower rank, the assertion would follow.

Notice from the cancellation of terms from successive columns (as given by Eq. 5.1) that the ith entry of $\sum_{k=j}^{n}(-1)^k\mathbf{P}_k$ is a constant plus $(-1)^j A_{(j-1),i}$. This means that adding appropriate combinations of these columns to the first and second columns cancels all variables. This completes the proof.

It remains to show that the different numbers of outcomes can occur. This result is based on placement of $(n-1)$ of the vertices. Since this number of vertices can be selected so they have arbitrary rankings, the results now follow from the geometric arguments concerning the placement of simplices that are developed in Saari [18]. The stronger results asserting the existence of outcomes where each alternative is ranked in different positions follows from the more delicate placement arguments of Saari [19] (which correct an oversight in [18]). □

Proof of Thm. 8. Central to this result is the fact that the term $[\nu(S \cup \{k\}) - \nu(S)]$, $k \notin S$, $|S| = j-1$, occurs once in the n-player computation of $\mathbf{P}_j$. If this term appears in computations for M-player games, it occurs only once. But, it appears in only $\binom{n-1}{M-j}$ of the $\binom{n}{M}$ M-player games. (This is because with n players, the number of times this particular coalition $S \cup \{k\}$ occurs in a M-player games is the number of ways it is possible to select the remaining $M - j$ players from $n - j$ players, or $\binom{n-j}{M-j}$.)

First assume that $\boldsymbol{\lambda}^M$ and $\boldsymbol{\lambda}^N$ have the same single non-zero term, say $\lambda_j^N = \lambda_j^M = 1$. The above computation means that the sum of the kth player's contribution over the $\binom{n}{M}$ set of M player subgames is $\binom{n-1}{M-j}$ times the $\boldsymbol{\lambda}^N$ computation for n-players. But since this multiple is the same for all entries, if a player is ranked above the average for all M-player games, she must be ranked above the average of the sum; this means she is not $\boldsymbol{\lambda}^N$ ranked below average, or at the bottom. The same analysis holds for a candidate ranked below average in each M-player game; he cannot be above average in the sum.

Using this same argument, we now prove the stronger assertion that if a player has an above (below) average $\boldsymbol{\lambda}^M$ score in all M-player games, she has an above (below) average score in the $\boldsymbol{\lambda}^N$ ranking of all n players. This statement holds for the sum of the $\binom{n}{M}$ computations for the $\boldsymbol{\lambda}^M$ indices. Therefore, the conclusion follows if this sum is used to define the $\boldsymbol{\lambda}^N$ index computation. According to the above computation of the number of times each term appears in the summation, it follows that if $\boldsymbol{\lambda}^N$ satisfies Eq. 4.1, then this condition is satisfied.

If $\boldsymbol{\lambda}^N$ does not satisfy Eq. 4.1, then the sum of the $\binom{n}{M}$ $\boldsymbol{\lambda}^M$ computations equals a term of the Eq. 4.1 form plus extra terms which are sums of multiples of various $\hat{\mathbf{P}}_j$ vertices. These extra variables can be used to create contradictory rankings for the n and M player games. $\square$

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