Discussion Paper No. 1262

Equilibrium Existence in Games with Incomplete Information: The Countable Case

by
Nabil Al-Najjar
and
Eilon Solan

May 1999

http://www.kellogg.nwu.edu/research/math

Equilibrium Existence in Games with Incomplete Information: The Countable Case

Nabil Al-Najjar *

and

Eilon Solan †

May 1999

Abstract

We prove an existence result for games with incomplete information with continuous type spaces under the assumption that players have atomic posteriors. This information structure is an extreme example of the failure of absolute continuity of information, hence our result complements the classical result of Milgrom and Weber (1985).

*Department of Managerial Economics and Decision Sciences, Kellogg School of Management, Northwestern University, Evanston IL 60208.<al-najjar@nwu.edu>

†Department of Managerial Economics and Decision Sciences, Kellogg School of Management, Northwestern University, Evanston IL 60208.<e-solan@nwu.edu>

1 Introduction

In their seminal paper on equilibria in games with incomplete information, Milgrom and Weber (1985, M-W) provided conditions ensuring the existence of equilibrium in incomplete information games with continuous type spaces. The interest of such games is obvious: many games in applications involve one or more player having private information about a continuous parameter, such as a bidder's valuation in an auction or a firm's cost parameter in an oligopoly setting.

The two key assumptions in M-W's theorem are that players have continuous payoffs and absolutely continuous information. While the role of payoff continuity is clear (one can easily find examples where equilibrium existence fails without it), the role of absolute continuity of information is much less obvious. First, there are no known examples in which existence fails when information is not absolutely continuous. Second, it is also easy to find simple examples where this condition is violated, and yet an equilibrium exists (see below). Finally, absolute continuity of the joint distribution does not always reflect intuitive notion that players' beliefs about their environment change continuously with their information.

This paper provides an equilibrium existence result for games where, although the marginals on players' types may be atomless, players' posteriors (conditional on their types) are atomic. In a sense, the class of games covered by our result is disjoint from that considered by M-W: every game we consider violates their absolute continuity of information, and every game with absolutely continuous information is necessarily one where players have non-atomic posteriors.[1]

[1]This is true when the marginals on players' types are atomless. The two equilibrium existence results coincide if players have countably many possible types (or more generally,

Our setup largely follows that of M-W, the only difference being the nature of the information structure. After introducing the basic setup, we state the main result and provide a sketch of the proof. The proof then follows, starting with new measurable selection results for transition probabilities, which may be of independent interest.

2 The Incomplete Information Game

We study a game of incomplete information played by a finite set of players $I = \{1, 2, \ldots, I\}$.

2.1 Strategies and Payoffs

Player i's information is represented by a type space $(T^i, \mathcal{T}^i)$, with T^i a complete separable metric space and $\mathcal{T}_i$ its Borel σ-algebra. The space of type profiles is $T = \prod_i T^i$ with the product σ-algebra $\mathcal{T}$.

Each player has a compact metric space of actions A_i. Denote action profiles by $A = \prod_i A_i$. Each player also has a payoff function $u_i : T \times A \to \mathbf{R}$.
Assumption 1: The payoff function $u_i(t, a)$ is jointly measurable on $\mathcal{T} \times A$, and continuous in A for every t.[2]

For every measurable space $(S, \mathcal{S})$, the space of probability measures over S is denoted by $\mathcal{P}(S)$.

A *(behavioral) strategy* for player i is a measurable function $\sigma_i : T_i \to \mathcal{P}(A_i)$, with $\sigma_i(t_i)$ denoting type t_i's mixed action. A *strategy profile* is a

if the marginals on their types are atomic.

[2]We do not need to require payoffs to be equicontinuous as in M-W, since, by our assumption on the information structure the strategic environment of any type is countable, and as in Proposition 3 in M-W, whenever the type spaces are countable it is sufficient to require that payoffs are continuous.

vector $\sigma = (\sigma_i)_{i \in I}$ of strategies.

2.2 Information and Equilibria

The information structure is given in terms of a joint distribution $\mu \in \mathcal{P}(T)$ on players' type profiles. Given μ, a profile σ induces a payoff $u_i(\sigma)$ for each player i:

$$u_i(\sigma) = \int_T \int_A u_i(t, a) d\sigma_1(a_1) \cdots d\sigma_I(a_I) d\mu(t).$$

Definition 2.1 *A profile σ is an* equilibrium *if for every player i and every strategy τ_i of player i*

$$u_i(\sigma) \geq u_i(\sigma_{-i}, \tau_i).$$

Thus, in equilibrium the set of types in which a player can improve his payoff has a measure 0.

2.3 Information Structure: Discussion and Examples

M-W define a game to have *absolutely continuous information* if μ is absolutely continuous with respect to the product of its marginals on players' types. They then show that if payoffs are equicontinuous and information is absolutely continuous an equilibrium in distributional strategies exists.[3]

While one can easily find examples where equilibrium existence fails without payoff continuity, the role of absolute continuity of information is much less obvious. It is easy to find simple games for which this condition is violated, yet an equilibrium exists. The game in Example 1 below (found in M-W) violates absolute continuity of information, yet an equilibrium for

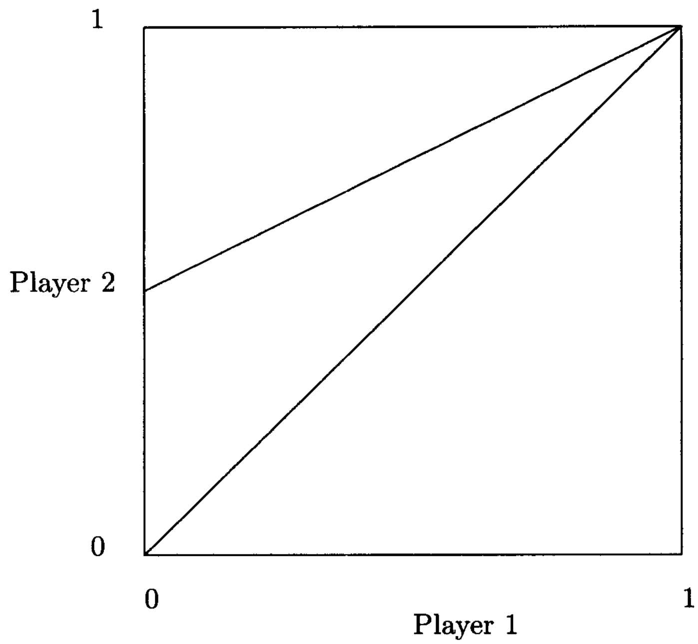
[3]They also prove the continuity of the equilibrium correspondence as a function of the information structure.

this game obviously exists:[4]

Example 1: There are two players, each with type space [0,1]. The joint distribution is uniform on the diagonal $\{t : t_1 = t_2\}$. Each player has two actions, and payoffs are those of a battle of the sexes, independently of the realized players' types. ■

A less obvious example is the following:

Example 2: Types and actions sets are as in Example 1 (payoffs are not important here). The joint distribution is one that puts a mass of 1/2 uniformly on the diagonal, and a mass of 1/2 uniformly on $\{t : t_2 = 0.5 + 0.5t_1\}$. Graphically, the information structure looks as follows.



[4]The game violates absolute continuity because the diagonal has zero mass relative to the product of the marginals, which is a uniform distribution on the square. On the other hand, any measurable mapping from the diagonal to the stage game equilibria constitutes an equilibrium.

This information structure violates absolute continuity of information, and an equilibrium (for general specification of payoffs) is not obvious. To see the problem, consider player 1 of type $t_1 = 0$. His action choice may be influenced by his belief about the actions of types 0 and 0.5 of player 2—the two types to which he assigns positive probability. But when player 2's type is 0.5, his action is affected by types 0 and 0.5 of player 1. Continuing in this manner we find that, although the payoff of player 1 of type 0 depends on the actions of only two types of his opponent, solving for this type's equilibrium behavior requires that we take into account all types of player 2 in the infinite set $\{0, 0.5, 0.75, 0.875, ...\}$, and all the types of player 1 in the infinite set $\{0.5, 0.75, 0.875, ...\}$. ■

3 Main Theorem

3.1 Statement

Our main theorem states that an equilibrium exists if the posterior of every player on the types of his opponents is atomic. While the intuitive idea underlying this assumption is straightforward, stating it precisely requires the concept of disintegration, which is more convenient to introduce later. For now we state an informal version of assumption 2; Section 4.1.3 contains the formal statement.

Assumption 2*: For every player i and almost every type t_i, his posterior on T_{-i} given t_i is purely atomic.

Theorem 1 *Every game satisfying assumptions 1 and 2 has an equilibrium.*

3.2 Sketch of the Proof

While the details of the proof are elaborate due to measurability considerations, the underlying intuition is quite simple. We illustrate it here in the context of Example 2. Since payoffs play no special role in this informal sketch, we suppress any reference to them.

In Example 2, player 1, of type t_1, say, cares (i.e. his payoff is directly affected by the actions of) two types of player 2. If t_2 is one such type, then t_2 may also care about the action taken by some other type $t_1' \neq t_1$, who in turn may care about what type $t_2' \neq t_2$ does, and so on. Clearly, the equilibrium action of t_1 must take into account all types of player 2 whose actions either directly appear in his payoff (more precisely, expected payoff, given his information), or affect him indirectly through their strategic effect on the behavior of player 2 types he cares about. Similarly, the equilibrium action should take into account all types of player 1 which affect type t_1 indirectly.

Informally, a type t_i *directly affects* his opponent's type t_j if t_i is an atom of the posterior belief of player j when his type is t_j. Let $R_1(t)$ denote the set of type profiles s such that one player type in s directly affects an opponent' type in t. Define $R_n(t)$ inductively as the set of type profiles s such that one player type in s directly affects an opponent' type in some $t \in R_{n-1}(t)$, and set $R(t) = \cup R_n(t)$.

Since the information structure is atomic, $R(t)$ is countable, and one can prove that on a set of type profiles of measure 1, R defines an equivalence relation: $s \in R(t)$ implies that $R(s) = R(t)$. Moreover, for every player i and every type t_i of player i, there exists a unique set $R(t)$ such that t_i is in the projection of $R(t)$ over T_i.

Thus, when their type profile t is randomly drawn, players find themselves in a strategic environment represented by the incomplete information game $R(t)$ with *countable* type spaces. From standard results, each such game has an equilibrium, and an equilibrium of the original game induces an equilibrium in (almost every) game $R(t)$. To prove existence of equilibrium in the oringinal game, we need to show the opposite: one can select equilibria for every game $R(t)$ that collectively form an equilibrium of the original game. To do this, we first show that the information and payoff structures of the games $R(t)$ change measurably with t. In addition, we develop new measurable selection theorems that allow us to measurably select equilibria for the games $R(t)$ that form an equilibrium for the original game.

4 Proof of the Theorem

We begin with a convenient technical tool used frequently below.

4.1 Conditional Probabilities and Disintegration

4.1.1 Disintegration of measures

The concept of disintegration of measures provides an intuitive and convenient representation of conditional distributions. See Stinchcombe (1990) for more detailed exposition and game theoretic applications. We begin with preliminary concepts.

A *transition probability* from $(X, \mathcal{X})$ to $(Y, \mathcal{Y})$ is a mapping $\mu : X \to \mathcal{P}(Y)$ such that for every $B \in \mathcal{Y}$, the real-valued function $x \mapsto \mu_x(B)$ is $\mathcal{X}$-measurable.

Let $\mathcal{A}$ be the σ-algebra generated by a measurable function $q : X \to [0,1]$ (so $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{X}$). The set $X(x) = \{\hat{x} \in X : q(x) = q(\hat{x})\}$ is called the *atom* determined by x. The idea is to think of q as defining a measurable equivalence relationship on X, in which case $X(x)$ is just the equivalence class of x.

Definition 4.1 *Let $\mu \in \mathcal{P}(X)$. A disintegration of μ relative to $\mathcal{A}$ is a transition probability $x \mapsto \mu_x$ from $(X, \mathcal{A})$ to $(X, \mathcal{X})$ such that:*

1. *For a.e. x, $\mu_x(X(x)) = 1$ (that is, the support of μ_x is the atom $X(x)$);*
2. *For every $B \in \mathcal{X}$, $\mu(B) = \int_X \mu_x(B) d\mu$*

The following theorem guarantees that in our setup a disintegration exists.

Theorem 2 *Suppose that X is a complete separable metric space, and $\mathcal{X}$ is the σ-algebra generated by the Borel sets. Then a disintegration exists relative to every σ-algebra $\mathcal{A}$ generated by a measurable function $q : \mathcal{X} \to [0,1]$.*

Proof: Dellacherie and Meyer (1978), pp. 78-79, and Stinchcombe (1993) p. 242. ■

4.1.2 Conditional Probabilities

Disintegration enables us to obtain a more powerful version of conditional probabilities.[5] Specifically, for every player i, let $\mu^i(\cdot \mid t_i)$ denote (a version of) the disintegration of μ with respect to $\mathcal{T}_i$.[6] Note that the atom $T(t)$ of a profile $t = (t_i, t_{-i})$ in this case (i.e. relative to $\mathcal{T}_i$) is simply the set $\{t_i\} \times T_{-i}$ of all profiles t' in which player i has the same type as in t.

From the definition of disintegration, we have

- for a.e. type profile t, $\mu^i(\cdot \mid t_i)$ is a probability measure on T with support $T(t) = \{t_i\} \times T_{-i}$;
- for every $B \in \mathcal{T}$, $\mu^i(B \mid t_i)$ is a version of the conditional probability of B given $\mathcal{T}_i$.

[5]Two potential problems with the usual definition of the conditional probability are: (1) in general, player i's conditional probability given his information $\mathcal{T}_i$ does not induce a probability measure on the type profiles of his opponents. That is, while it is possible, *for a given event* $A \in \mathcal{T}$, to find a version $P(A \mid \mathcal{T}_i)(t)$ of the conditional probability given this player's information $\mathcal{T}_i$, this does not guarantee that we can find versions so that, for a.e. t, $P(\cdot \mid \mathcal{T}_i)(t)$ is a probability measure on T. (2) Even if $P(A \mid \mathcal{T}_i)(t)$ defines a measure for each t, there is no guarantee that this measure is carried by $\{t_i\} \times T_{-i}$.

[6]Every complete separable metric space can be mapped via a measurable bijection into [0,1], so $\mathcal{T}_i$ can be obtained as the sub-σ-algebra generated by some function q as in the definition of disintegration.

4.1.3 Formal Statement of Assumption 2

We can now state assumption 2:

Assumption 2: For every player i and a.e. type t_i, $\mu^i(\cdot \mid t_i)$ is atomic.[7]

4.2 Selection Theorems

In this section, $(X, \mathcal{X})$ is an arbitrary measurable space, $(Y, \mathcal{Y})$ is a complete separable metric space and $\nu \in \mathcal{P}(X)$ is a measure. For every probability distribution $\nu \in \mathcal{P}(Y)$, let $\text{atom}(\nu) = \{y \in Y \mid \nu(y) > 0\}$ be the set of atoms of ν. Note that $\text{atom}(\nu)$ is always countable.

A correspondence $\phi : X \to Y$ is measurable if for every closed set $B \subset Y$, the set $\{x \in X \mid \phi(x) \cap B \neq \emptyset\}$ is measurable. We use the following facts about measurable correspondences (Himmelberg (1975)):

1. A union of countably many measurable correspondences is measurable.
2. Let $f : X \to Y$ be a measurable function, and $\phi : Y \to Z$ be a measurable correspondence. Then the correspondence $x \mapsto \phi(f(x))$ is measurable.
3. The graph of a measurable correspondence $\phi : X \to Y$ is a measurable subset of $X \times Y$ (endowed with the product topology).

Kuratowski and Ryll-Nardzewski (1965, K-RN) proved that every measurable correspondence from X to Y with closed values has a measurable selector. If the correspondence is measurable w.r.t. some sub-σ-algebra, we would like the selection to be measurable w.r.t. this sub-σ-algebra as well. We provide here a version of K-RN's theorem for simultaneous choice of selections of different correspondences.

[7] That is, there is a countable set $A \in \mathcal{T}_i$ (in fact, $A \subset T(t)$) such that $\mu^i_{t_i}(A) = 1$.

Lemma 4.2 *Let $\phi_1, \ldots, \phi_n : X \to Y$ be measurable correspondences with ν-a.s. closed values. There exist measurable functions $f_i : X \to Y$, $i = 1, \ldots, n$ such that ν-a.s. $f_i(x) \in \phi_i(x)$, and $f_i(x) = f_j(\hat{x})$ whenever $\phi_i(x) = \phi_j(\hat{x})$.*

Proof: The original proof of K-RN constructs for every correspondence ϕ a selection f such that $f(x)$ depends only on $\phi(x)$. The result follows. ■

Lemma 4.3 *Let $x \mapsto \mu_x$ be a transition probability from X to Y such that μ_x is atomic ν-a.s. Then there exists a measurable function $f : X \to Y$ such that ν-a.s.*

- $f(x) \in \mathrm{atom}(\mu_x)$.
- $\mu_x(f(x)) \geq \mu_x(\hat{x})$ *for every* $\hat{x} \in X$.
- $f(x) = f(\hat{x})$ *whenever* $\mu_x = \mu_{\hat{x}}$.

In other words, there exists a measurable selection that assigns to (almost) every x an element with maximal weight relative to μ_x.

Proof: We prove that the correspondence

$$x \mapsto H(x) = \{y \in Y \mid y \text{ has maximal weight w.r.t. } \mu_x\}$$

is measurable. Since this correspondence has finite values, we can use Lemma 4.2 with $n = 1$ to prove the desired result.

Let $(\mathcal{P}_n)$ be a sequence of increasing finite partitions of X that generates $\mathcal{X}$.

For every $n \in \mathbf{N}$ and every $P \in \mathcal{P}_n$, the real-valued function $f_n^P(x) = \mu_x(P)$ is measurable, hence so is $g_n(x) = \max_{P \in \mathcal{P}_n} f_n^P(x)$. Since for every fixed x, $(g_n(x))_n$ is a decreasing sequence of non-negative numbers, the real-valued function $g(x) = \lim_n g_n(x)$ is measurable.

Note that $g(x)$ is the weight of the maximal atom of μ_x.

Let $A \subseteq Y$ be a closed set. It is easy to verify that

$$\begin{aligned}\{x \mid \mu_x(y) \geq g(y) \text{ for some } y \in A\} &= \{x \mid \forall n\ \exists P \in \mathcal{P}_n \text{ s.t. } \mu_x(A \cap P) \geq g(x)\} \\ &= \cap_{n\in\mathbf{N}} \cup_{P\in\mathcal{P}_n} \{x \mid \mu_x(A \cap P) \geq g(x)\}\end{aligned}$$

and

$$\{x \mid \mu_x(y) \leq g(x) \text{ for all } y \in A\} = \left(\cup_{k\in\mathbf{N}}\{x \mid \mu_x(y) \geq g(x) + 1/k \text{ for some } y \in A\}\right)^c,$$

where $A^c = X \setminus A$ is the complement of A. Since $x \mapsto \mu_x$ is a transition probability and g is measurable, these sets are measurable. Finally,

$$\begin{aligned}&\{x \mid H(x) \cap A \neq \emptyset\} = \\ &\quad \{x \mid \mu_x(y) \geq g(x) \text{ for some } y \in A\} \cap \{x \mid \mu_x(y) \leq g(x) \text{ for all } y \in A\},\end{aligned}$$

hence H is measurable. ■

Corollary 4.4 *Let $x \mapsto \mu_x$ be a transition probability from X to Y such that μ_x is atomic ν-a.s. Then there exist measurable functions $f_n : X \to Y \cup \{\emptyset\}$, $n = 1, 2, \ldots$ such that ν-a.s. (i) $\cup_n\{f_n(x)\} = \text{atom}(\mu_x)$,[8] (ii) if $f_n(x) = f_m(x) \neq \emptyset$ then $m = n$ and (iii) $f_n(x) = f_n(\hat{x})$ whenever $\mu_x = \mu_{\hat{x}}$.*

Proof: By Lemma 4.3 the correspondence $x \mapsto \text{atom}(\mu_x)$ has a measurable selection f_1 such that ν-a.s. $f_1(x)$ is maximal atom of μ_x and $f^1(x) = f^1(\hat{x})$ whenever $\mu_x = \mu_{\hat{x}}$.

Define the function $\mu^1 : X \to \mathcal{P}(Y) \cup \{\emptyset\}$ as the probability distribution over $Y \setminus \{f_1(x)\}$ induced by μ_x. Note that if $\text{atom}(\mu_x)$ contains a single point, then μ^1_x is not defined, in which case we set $\mu^1_x = \emptyset$.

[8] In this union we identify the set that includes the empty set with the empty set.

μ^1 is measurable, hence as before there exists a measurable function $f_2 : X \to Y \cup \{\emptyset\}$ such that ν-a.s. whenever $\mu_x^1 \neq \emptyset$, (i)$f_2(x) \in \text{atom}(\mu_x^1)$, (ii) $f_2(x)$ is a largest atom of μ_x^1, and (iii) $f_2(x) = f_2(\hat{x})$ whenever $\mu_x^1 = \mu_{\hat{x}}^1$.

Continue this way inductively, to generate a sequence of measurable functions $f_n : X \to Y \cup \{\emptyset\}$ such that ν-a.s. whenever $A_x^n = Y \backslash \{f_1(x), \ldots, f_n(x)\}$ is not empty, $f_{n+1}(x)$ is a maximal atom of the probability distribution induced by μ_x over A_x^n. It is easy to see that (i) is satisfied, since at stage n we choose a maximal atom of μ_x^n, and if $\mu_x^n > \mu_{\hat{x}}^n$ then $\mu_x^{n+1} > \mu_{\hat{x}}^{n+1}$. (ii) is satisfied by the construction of μ^{n+1}, and (iii) is satisfied by the choice of the measurable selections f_n. ■

Corollary 4.5 *Let $x \mapsto \mu_x$ be a transition probability from X to Y such that μ_x is atomic ν-a.s. Then the set-valued function $x \mapsto$ atom(μ_x) is measurable and has a measurable graph.*

Proof: By Corollary 4.4, on a set of measure 1 $x \mapsto \text{atom}(\mu_x)$ is a union of countably many measurable functions, hence it is measurable. Since T is complete separable metric, this correspondence has a measurable graph. ■

4.3 Equivalence Classes of Types

Let $\mu^i(\cdot \mid t_i)$ be the conditional probability of μ relative to each $\mathcal{T}_i$ as defined in Section 4.1.2. For every type profile $t = (t_i)$, define

$$R_1(t) = \cup_i \text{atom}(\mu^i(\cdot \mid t_i)).$$

In words, $R_1(t) \subset \mathcal{T}$ is the set of type profiles s to which at least one player assigns positive probability when his type is the one given in t.[9]

[9]Recall that $\mu^i(\cdot \mid t_i)$ is a measure on $\mathcal{T}$ (not $\mathcal{T}_{-i}$) with the property that it puts unit mass on $\{t_i\} \times T_{-i}$.

For every $n \geq 1$, define

$$R_{n+1}(t) = \cup_{s \in R_n(t)} \cup_i \operatorname{atom}(\mu^i(\cdot \mid s_i)).$$

Finally, we set

$$R(t) = \cup_n R_n(t).$$

Example 2 (continued): In this example,

$$R_1(t_1, t_2) = \begin{cases} \{(t_1, t_1), (t_2, t_2), (t_1, 0.5 + 0.5t_1)\} & t_2 < 0.5 \\ \{(t_1, t_1), (t_2, t_2), (t_1, 0.5 + 0.5t_1), (2t_2 - 1, t_2)\} & t_2 \geq 0.5 \end{cases}$$

and $R(0, 0) = \{(0, 0), (0, 0.5), (0.5, 0, 5), (0.5, 0.75), (0.75, 0.75), \ldots\}$. ■

It is clear from the construction that:

Lemma 4.6 *If $s \in R(t)$ then $R(s) \subseteq R(t)$.*

Lemma 4.7 *$R(t)$ is countable for every t.*

Proof: $R_1(t)$ is a finite union of countable sets, hence countable. By induction on n, $R_{n+1}(t)$ is a countable union of countable sets, hence countable. Finally, $R(t)$ is a countable union of countable sets, hence countable. ■

Lemma 4.8 *The correspondences $t \mapsto R_1(t)$ and $t \mapsto R(t)$ are measurable.*

Proof: By Corollary 4.5 the correspondence $t \mapsto R_1(t)$ is measurable. By Corollary 4.4 there exist measurable functions $f_k : T \to T$, $k = 1, 2, \ldots$, such that μ-a.s. $\cup_n \{f_k(t)\} = R_1(t)$.[10] Since $R_{n+1}(t) = \cup_k R_n(f_k(t))$, it follows

[10] By Corollary 4.4, the range of the functions f_k is $T \cup \{\emptyset\}$. By redefining $f_k(t) = f_1(t)$ whenever it was equal to $\emptyset$ we get the result that we need.

by induction that $t \mapsto R_{n+1}(t)$ is measurable, and therefore $t \mapsto R(t)$ is measurable as well. ∎

Define

$$T^* = \{t = (t_i)_i \in T \mid \mu^i(t \mid t_i) > 0 \quad \forall i\}.$$

That is, T^* is the set of all type profiles $t = (t_i)_i$ such that each player, when his type is given by t_i, puts positive probability on the profile t.

Lemma 4.9 *$T^* \in \mathcal{T}$ and $\mu(T^*) = 1$.*

Proof: Define

$$S_i^* = \{t \in T \mid \mu^i(t \mid t_i) > 0\} = \text{Graph}(t \mapsto R_1(t)).$$

By Lemma 4.8 S_i^* is measurable, hence $T^* = \cap_i S_i^*$ is measurable.

If $\mu(T^*) < 1$ then, since the number of players is finite, there exist two players i and j and a subset $F \in \mathcal{T}$ such that $\mu(F) > 0$ and for every $t = (t_i)_i \in F$, $\mu^i(t \mid t_i) = 0$ while $\mu^j(t \mid t_j) > 0$. In particular, for every $t = (t_i)_i \in F$, $\mu^i(F \mid t_i) = 0 < \mu^j(F \mid t_j)$. Then

$$0 = \int_T \mu^i(F \mid t_i) d\mu(t) = \mu(F) = \int_T \mu^j(F \mid t_j) d\mu(t) > 0$$

a contradiction. ∎

As a corollary we get that R is an equivalence relation.

Corollary 4.10 *For every $t = (t_i)_i \in T^*$ and $s = (s_i)_i \in T$, if $s \in R(t)$ then $t \in R(t)$. In particular, $R(t) \subseteq T^*$.*

Proof: Clearly, R_1 is symmetric: if $s \in R_1(t)$, then for some player i, $s \in \text{atom}(\mu^i(\cdot \mid t_i))$. But then, $t_i = s_i$ (that is, player i has the same type in the two type profiles t and s), so $\text{atom}(\mu^i(\cdot \mid t_i))$ and $\text{atom}(\mu^i(\cdot \mid s_i))$

coincide. This implies $t \in \mathrm{atom}(\mu^i(\cdot \mid s_i)$, so we indeed have $t \in R_1(s)$. The proof for R follows by the definition of T^* and induction. ■

For every subset $B \subset T$, denote by $\mathrm{proj}_i(B)$ the projection of B over T_i. Clearly $\mathrm{proj}_i(R(t))$ is countable for every $t \in T$, as $R(t)$ is countable for every $t \in T$.

Lemma 4.11 *For every $s, t \in T^*$ and every $i \in I$, $R(s) = R(t)$ if and only if* $\mathrm{proj}_i(R(s)) \cap \mathrm{proj}_i(R(t)) \neq \emptyset$.

Proof: Let $s, t \in T^*$. It is clear that if $R(s) = R(t)$ then $\mathrm{proj}_i(R(s)) \cap \mathrm{proj}_i(R(t)) \neq \emptyset$.

Assume now that $\mathrm{proj}_i(R(s)) \cap \mathrm{proj}_i(R(t)) \neq \emptyset$. Then there exist $\hat{s} = (\hat{s}_i)_i \in R(s)$ and $\hat{t} = (\hat{t}_i)_i \in R(t)$ such that $\hat{s}_i = \hat{t}_i$. By the definition of T^*, $\mu^i(\hat{t} \mid \hat{s}_i), \mu^i(\hat{s} \mid \hat{t}_i) > 0$, hence $\hat{t} \in R(s)$ and $\hat{s} \in R(t)$. It follows that $R(t) \subseteq R(s)$ and $R(s) \subseteq R(t)$, as desired. ■

Example 2 (continued): One can take $T^* = \{t \mid t_1 = t_2\} \cup \{t \mid t_2 = 0.5 + 0.5t_1\}$. Note that for every $t \notin T^*$, $R(t) = R(t_1, t_1) \cup R(t_2, t_2)$. ■

We summarize the results so far in the following proposition.

Proposition 4.12 *There exists a set $T^* \in \mathcal{T}$ such that*

1. $\mu(T^*) = 1$.

2. *For every $t \in T^*$ and every $s \in R(t)$, (i) $s \in T^*$, (ii) $R(t) \subseteq T^*$ and (iii) $R(s) = R(t)$.*

3. *For every $s, t \in T^*$, $\mathrm{proj}_i(R(s)) \cap \mathrm{proj}_i(R(t)) \neq \emptyset$ if and only if $R(s) = R(t)$.*

Let T^* be the set defined by Proposition 4.12. Define a correspondence $\psi_i : T_i \rightarrow T$ as follows.

$$\psi_i(t_i) = \begin{cases} R(s) & t_i \in \text{proj}_i(R(s)) \text{ for some } s =\in T^*. \\ \emptyset & \text{Otherwise.} \end{cases}$$

By Lemma 4.11, ψ_i is well defined. The information available to each player i is his type t_i, and therefore the type profiles that indirectly affect his payoff are the type profiles in $\psi_i(t_i)$.

Corollary 4.13 *ψ_i is measurable.*

Proof: Let $f : T_i \rightarrow T \cup \emptyset$ be a measurable function that assigns to every type t_i an element in $\text{atom}(\mu^i(\cdot \mid t_i))$ whenever the latter is not empty, and is the empty set otherwise. By Lemma 4.3 such a function exists. On T^*, $\psi_i(t_i) = R(f(t_i))$ is a composition of a measurable correspondence and a measurable function. Since T is complete separable metric, it follows that ψ_i is measurable. ■

4.4 Standard Games

Definition 4.14 *A* standard game *is a game with incomplete information where*

- *The set of players is I.*
- *The type space of each player is the set of positive integers* $\mathbf{N}$.
- *The action space of player i is A_i.*

The game is parameterized by (i) an information structure, which is a (atomic) probability distribution over the space $\mathbf{N}^I$ *of type profiles and by* *(ii) measurable payoff functions $u^i : \mathbf{N}^I \times A \rightarrow \mathbf{R}$ such that for every fixed type profile $k \in \mathbf{N}^I$, $u^i(k, \cdot)$ is continuous over A.*

The space of strategy profiles in a standard game is $\Sigma = (\mathcal{P}(A)^{\mathbf{N}})^I$. Since A is compact, $\mathcal{P}(A)$ is compact in the $w^\star$-topology. Hence Σ is compact when it is equipped with the product topology.

Let $\mathcal{G}$ be the space of all standard games. We endow the space $F(A, \mathbf{R})$ of continuous real-valued functions defined over A by the supremum topology, and the space $\mathcal{G} \subset (F(A, \mathbf{R}) \times [0, 1])^{\mathbf{N}^I}$ of standard games by the product topology ($[0, 1]$ is endowed with the usual topology). Since $F(A, \mathbf{R})$ and $[0, 1]$ are complete separable metric spaces under these topologies, it follows that $\mathcal{G}$ is complete separable metric as well.

It follows from Proposition 3 and Theorem 1 in M-W that each standard game admits an equilibrium. Thus, the correspondence $E : \mathcal{G} \rightarrow \Sigma$ that assigns to each standard game G the set of Nash equilibrium strategy profiles in the game has non-empty values.

Our next goal is to show that this correspondence is upper-semi-continuous. This result does not follow from Theorem 2 in M-W, since their topology on the space $F(A, \mathbf{R})^{\mathbf{N}^I}$ is that of uniform convergence.

Lemma 4.15 *The correspondence E is upper-semi-continuous.*

Proof: Let $G^n = (\mu^n, (u_i^n)_i)$ be a sequence of standard games that converges to $G = (\mu, (u_i)_i)$. For each n let $(\sigma_i^n)_i$ be a strategy profile in G^n such that $\sigma_i^n \rightarrow \sigma_i$ for every i.

It is immediate to verify that

$$\int_{\mathbf{N}^I} \int_A u_i^n(k, a) d\sigma^n(a) d\mu^n(k) \rightarrow \int_{\mathbf{N}^I} \int_A u_i(k, a) d\sigma(a) d\mu(k)$$

which implies that a limit of a sequence of equilibrium strategy profiles in G^n is an equilibrium strategy profile in G. ■

For each player i, let $\hat{\mu}^i \in \mathcal{P}(T_i)$ be the marginal probability distribution of player i.[11]

By Corollary 4.4 and Proposition 4.12 there exist measurable functions $f_n^i : T_i \to T \cup \{\emptyset\}$, $n = 1, 2, \ldots$, that satisfy the following $\hat{\mu}^i$-a.s.:

1. $\cup_n \{f_n^i(t_i)\} = \text{atom}(\hat{\mu}^i(\cdot \mid t_i))$.
2. If $f_n^i(t_i) = f_m^i(t_i) \neq \emptyset$ then $m = n$.
3. $f_n^i(t_i) = f_n^i(\hat{t}_i)$ whenever $R(t) = R(\hat{t})$.

For every type profile t we define a corresponding standard game $G(t)$ as follows.

- The information structure is given by:

$$\mu(k_1, \ldots, k_I) = \nu_t(f_{k_1}^1(t_1), \ldots, f_{k_I}^I(t_I))$$

  whenever $f_{k_i}^i(t) \neq \emptyset$ for every $i \in I$, and 0 otherwise.

- The payoff is given by

$$u(k, a) = u(f_{k_1}^1(t_1), \ldots, f_{k_I}^I(t_I), a)$$

  whenever $f_{k_i}^i(t) \neq \emptyset$ for every $i \in I$, and 0 otherwise.

Note that if $R(s) = R(t)$ then $G(s) = G(t)$. Thus, we can write $G(R(t))$ instead of $G(t)$. By convention, $G(\emptyset)$ is an arbitrary fixed game.

4.5 The Final Proof

Proof of Theorem 1:

Recall that by Proposition 4.12 there exists a set $T^* \in \mathcal{T}$ that satisfies the following conditions

[11]That is, for every set $A \subseteq T_i$, $\hat{\mu}^i(A) = \mu(A \times T_{-i})$.

1. $\mu(T^*) = 1$.

2. For every $t \in T^*$ and every $s \in R(t)$, (i) $s \in T^*$, (ii) $R(t) \subseteq T^*$ and (iii) $R(s) = R(t)$.

3. For every $s, t \in T^*$ and every player i, $\text{proj}_i(R(s)) \cap \text{proj}_i(R(t)) \neq \emptyset$ if and only if $R(s) = R(t)$.

Define

$$\mathcal{R}^i = \{\text{proj}_i(R(t)), t \in T^*\}.$$

Each set in $\mathcal{R}^i$ is countable, and the union of all these sets has $\hat{\mu}^i$ measure 1.

Let $\mathcal{A}^i$ be the coarsest sub-σ-algebra of $\mathcal{T}_i$ for which all the sets in $\mathcal{R}^i$ are measurable. Let ν^i be a disintegration of $\hat{\mu}^i$ w.r.t. $\mathcal{A}_i$. In particular, $\nu^i(\cdot \mid t_i)$ is a measure over T_i that is carried by $\text{proj}_i(R(t_{-i}, t_i))$ whenever $(t_{-i}, t_i) \in \text{atom}(\mu^i(\cdot \mid t_i))$.

Define the correspondence $E_i^\star : t_i \mapsto E(G(\psi(t_i)))$. That is, for each type t_i we assign the set of Nash equilibrium strategy profiles in the standard game that is defined by the equivalence class that t_i belongs to.

Since the correspondence E is upper-semi-continuous, $E_i^\star$ has closed values. Since E is upper-semi-continuous, the pre-image of any closed set under E is closed. Since G is continuous, the pre-image of any closed set under G is closed as well. Since ψ is measurable, the pre-image of any closed set under ψ is measurable. Hence the pre-image of any closed set under $E_i^\star$ is measurable, and in particular $E_i^\star$ is measurable.

By Lemma 4.2, there exist measurable functions $(e_i)_i$ such that each e_i is a selection of $E_i^\star$, and $e_i(t_i) = e_i(s_i)$ whenever $\psi(t_i) = \psi(s_i)$, which is equivalent to $t_i, s_i \in \text{proj}_i(R(\hat{t}))$ for some $\hat{t} \in T$.

Moreover, $e_i(t_i) = e_j(s_j)$ whenever there is $\hat{t} \in T$ such that $t_i \in \text{proj}_i(R(\hat{t}))$ and $s_j \in \text{proj}_j(R(\hat{t}))$.

Thus, for each $t \in T^*$, $e_i(t_i)$ is an equilibrium strategy profile in the standard game induced by $R(t)$. Moreover, for every equivalence class $R(t)$ and every $s \in R(t)$, $(e_i(s_i))_i$ all refer to the same equilibrium strategy profile in the game defined by $R(t)$.

We are now ready to define the equilibrium strategy of player i in the game. The equilibrium strategy profile of player i is to play the mixed-action prescribed for him by the profile $e_i(t_i)$ when his realized type is t_i.

Since $\mu(T^*) = 1$ and whenever the realized type profile t is in T^* all the players play the same equilibrium in $R(t)$, the strategy profile induced by $(e_i)_i$ indeed forms an equilibrium. ■

References

Dellacherie, C. and P.-A. Meyer (1978): *Probabilities and Potential.* Amsterdam: North-Holland.

Himmelberg, C.J. (1975): "Measurable Relations," *Fundamenta Mathematicae*, **87**, 53-72.

Kuratowski, K. and C. Ryll-Nardzewski (1965): "A General Theorem on Selectors," *Bulletin de l'Academie Polonaise des Sciences*, **13**, 397-403.

Milgrom, P. and R. Weber (1985): "Distributional Strategies for Games with Incomplete Information," *Mathematics of Operations Research*, **10**, 619-32.

Stinchcombe, M. (1990): "Bayesian Information Topologies," *Journal of Mathematical Economics*, **19**, 233-53.