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ECONOMIC ENVIRONMENTS FOR WHICH THERE ARE PARETO SATISFACTORY MECHANISMS

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Economics has long been concerned with the possibility of decentralized economic organization. Interest attaches on the one hand to the theory of decentralized processes, and on the other to identifying those properties of economic environments which distinguish between those that can be successfully coordinated by a decentralized process and those which cannot be so organized. The broad question addressed in this paper is; "What class of economic environments can be satisfactorily (with respect to the Pareto criterion) organized by means of an economic mechanism which is informationally decentralized?" We study this question on the basis of formulations and results in [6], [7] and especially [14], where background material and fuller discussion of basic ideas may be found.

Resource allocation processes (which are abstract models of economic organization) typically involve an explicit model of communication using formal messages ([16], [14]). Informational decentralization of processes has been defined in terms of two properties. The first, involving the concept of "privacy", amounts to the requirements that all information except what is internal to an agent (e.g., his preferences) must come to him via messages; the second restricts the messages used by the process to be vectors whose dimension is bounded by a number related to the

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dimension of the commodity space [6]. In this way the information-carrying capacity of messages is constrained, and therefore the information that can be communicated is constrained. Thus, the dimension or "size" of messages is a basic component of the concept of informational decentralization. However, without further conditions on communication restricting the dimension of messages does not impose a meaningful restriction on the amount of information communicated via those messages. It seems intuitive that two dimensional messages are "bigger" than one dimensional ones in the sense that "more information" should be conveyed via a two dimensional message than a one dimensional. However, a communication process using two-dimensional messages can be replaced by one which uses one dimensional messages without sacrifice of performance. This paradoxical possibility exists because there are (continuous) functions which map a lower dimensional space onto one of higher dimension. One such map is the Peano function, which maps the unit interval continuously onto the unit square. The inverse of the Peano function can be used to encode two dimensional messages into one dimension; the Peano function can be used to recover the original two dimensional information. It is therefore necessary to impose a condition on the communication process which excludes such smuggling of information.

One such condition, given in [14], is that the message correspondence, which models communication, is locally threaded.[1/] (Another such condition given by Hurwicz is that a certain function, which plays a role analogous to the inverse of our message correspondence, should satisfy a Lipschitz condition.) These conditions, which relate to the smoothness of the

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1/ See the footnote to Definition 2.1 for the definition of "locally threaded".

communication process, have implications for the smoothness of the relation between the outcome of the process and the environment. Specifically, if the message correspondence is locally threaded, then the performance of the process is given by a continuous function from environments to outcomes. [Lemma 2.1].

We study organizations whose performance is satisfactory with respect to a criterion (usually the Pareto criterion) which can be expressed formally by means of a correspondence $\mathcal{P}$ from environments to outcomes. We consider a collection of processes using the same message space and outcome function, which we call a mechanism [Definition 2.2], and show that a mechanism whose message correspondences are locally threaded can be (e.g.) Pareto-satisfactory on a class of environments if and only if the Pareto correspondence is a union of continuous functions (a completely threaded correspondence) on that class of environments [Theorems 2.1 and 2.2].

Since Theorems 2.1 and 2.2 identify a property (complete threading) of the Pareto correspondence which is necessary and sufficient for the existence of a (decentralized) mechanism whose performance is Pareto-satisfactory, the question arises, on what class of environments does the Pareto correspondence have that property? I.e., on what class of environments is the Pareto correspondence completely threaded? Our results show that, in the presence of certain rather standard conditions, if preferences are strictly monotone, then the Pareto-utility frontier correspondence is completely threaded [Theorem 5.2]. Examples show that strict monotonicity is indispensible. If further the set of points (allocations or trades) which are Pareto equivalent to a given Pareto optimal point is a singleton, (Pointedness Assumption), then the "Contract Curve" correspondence is also completely threaded [Theorem 6.1].

The classical welfare theorems establish the Pareto-satisfactoriness of the competitive mechanism on the class of convex environments. However, it is not known whether the competitive mechanism has a locally threaded message correspondence on the full class of environments on which the welfare theorems hold. It was established in [14] that the competitive mechanism does satisfy that regularity condition on the class of pure trade environments with Cobb-Douglas utilities. Furthermore, it is clear that when the competitive equilibrium is unique (and the Walras correspondence is upper hemi-continuous), the regularity condition (local threading) is also met. The case of multiple equilibria for environments which do not satisfy the assumptions of our theorems remains open. It should be pointed out that the competitive mechanism, as it is ordinarily specified, does not meet our requirements for a mechanism in the case of multiple equilibria. We require that a particular equilibrium be selected in a continuous fashion as the environment varies. Indeed, one interpretation of our result is that the conjunction of (i) the requirement that the economic mechanism make a selection of equilibria for cases in which there are multiple equilibria, (ii) the regularity condition on communication, and (iii) Pareto-satisfactoriness of performance, restricts the allowable environments to very classical ones.

Section 2: $\mathcal{P}$-satisfactory Mechanisms

In this section we give the definitions needed to determine the structure of mechanisms which have satisfactory performance relative to an abstract criterion represented by a correspondence $\mathcal{P}$ from environments to actions. We call these $\mathcal{P}$-satisfactory mechanisms. In subsequent sections we take $\mathcal{P}$ to be the Pareto correspondence. We begin by reproducing the definition of a resource allocation process given in [14].

Definition 2.1 Let E, M and Z be topological spaces, let $\gamma:E \to M$ be a correspondence and $g:M \to Z$ a function. The pair (γ, g) is a resource allocation process on E (with message space M) if

(i) γ is locally threaded[2/]

(ii) g is continuous

(iii) g is compatible with γ, i.e., for $e \in E$ if m and m' are in $\gamma(e)$ then $g(m) = g(m')$.

The performance of a resource allocation process (γ, g) is characterized by the function $f = g \cdot \gamma : E \to Z$.

If $f:E \to Z$ is a given continuous function, we say that (γ, g) realizes f and that M is sufficient for f if $f = g \cdot \gamma$. See [14, Definition 1, p. 169].

2/ For convenience we give Definition 6 of [14, p. 173]. The correspondence $\gamma:E \to M$ is locally threaded if for each $e \in E$ there exists an open neighborhood of $e, \mathcal{O}(e) \subset E$ and a continuous function $s_e:\mathcal{O}(e) \to M$ such that $s_e(e') \in \gamma(e)$ for all $e' \in \mathcal{O}(e)$.

The structure is as yet not fully adequate for the study of Pareto satisfactory mechanisms, since the outcome of a resource allocation process is a point, while the outcomes corresponding to equilibria of a Pareto satisfactory process must cover the Pareto set.

Let $\Gamma:E \to M$ be a correspondence. We may e.g., interpret $\Gamma(e)$ to be the set of message complexes which are equilibria of a communication process in the environment e. In general Γ is not compatible with the outcome function $g:M \to Z$. Where there are multiple equilibria, different equilibria could lead to different allocations, as in the case of multiple competitive equilibria. We may consider the family of correspondences γ from E to M which lie in Γ and are compatible with g; i.e.

$$\mathbb{\Gamma} = \{\gamma \mid \gamma(e) \subset \Gamma(e) \text{ for all } e \in E \text{ and } m,m' \in \gamma(e) \text{ implies } g(m) = g(m')\}$$

Each such selection from Γ, together with the function g, determines a resource allocation process provided that condition (i) of Definition 2.1 is met, namely that the selection γ is locally threaded on E.

<u>Definition 2.2</u> The pair $(\mathbb{\Gamma},g)$ consisting of the family $\mathbb{\Gamma}$ and the function g is a <u>mechanism</u> provided (γ,g) is a resource allocation process for all $\gamma \in \mathbb{\Gamma}$.

Let $\mathcal{P}:E \to Z$ be a correspondence. We interpret $\mathcal{P}$ as the optimality criterion. Thus, for each $e \in E$, $\mathcal{P}(e) \subset Z$ is the set of points in Z which are considered optimal for e. (We do not here specify whether Z is the space of allocations, trades or other outcomes.) In the cases we will consider below $\mathcal{P}$ is the Pareto correspondence, i.e., $\mathcal{P}(e)$ is the set of Pareto optima of e.

We consider next redistributions of the initial endowment. In the conventional pure trade case, for example, such redistributions leave the Edgeworth Box and the contract curve fixed, varying only the initial endowment point in the Box. We shall in addition require redistribution to be a continuous process. We may as well think of it as a particular kind of continuous transformation of the environment by a process not part of the economic mechanism and which does not alter the optimal set $\mathcal{P}(e)$.

<u>Definition 2.3</u> Let $\mathcal{D}$ be the class of continuous functions $d:E \to E$ such that $\mathcal{P}(d(e)) = \mathcal{P}(e)$ for all $e \in E$. We call the function d a <u>redistribution rule</u>.

We may now define $\mathcal{P}$ - satisfactoriness of mechanisms.

<u>Definition 2.4</u>. A mechanism (Γ, g) is $\mathcal{P}$ - satisfactory on E if

(i) for every $\gamma \in \Gamma$, and $e \in E$, $g \cdot \gamma(e) \in \mathcal{P}(e)$.

(ii) for every $e \in E$ and $z \in \mathcal{P}(e)$, there exist $\gamma \in \Gamma$ and $d \in \mathcal{D}$ such that $g \circ \gamma \circ d(e) = z$.

Condition (i) of Definition 2.4, (non-wastefulness), requires that every outcome of the mechanism be a $\mathcal{P}$ - optimum.

Condition (ii), (unbiasedness), requires that any outcome z which is $\mathcal{P}$ - optimal for an environment e be a possible outcome of the mechanism for some choice of equilibrium, allowing for an admissible transformation of the environment e.g., a redistribution of initial endowment.[3/]

We note next that the performance of a resource allocation process is continuous.

Lemma 2.1. Let (γ, g) be a resource allocation process on E with message space M, and let $f = g \cdot \gamma : E \rightarrow Z$. Then f is continuous.

Proof. Let U be open in Z. We shall show that $f^{-1}(U)$ is open in E. Now

$f^{-1}(U) = \gamma^{-1} \cdot g^{-1}(U) = \gamma^{-1}(V)$, where V is open in M since g is continuous, and where $\gamma^{-1}(V) \equiv \{e \in E \mid \gamma(e) \subset V\}$. Let $p \in \gamma^{-1}(V)$. We shall show that $\gamma^{-1}(V)$ contains a neighborhood of p in E, and hence is open in E. Since γ is locally threaded, there exists an open set $\bar{\bar{E}}$ containing p and a continuous function $s : \bar{\bar{E}} \rightarrow M$ such that $s(e) \in \gamma(e)$ for $e \in \bar{\bar{E}}$.

3/ Hurwicz, taking $\mathcal{P}$ to be the Pareto criterion, required only that an outcome Pareto equivalent to the given alternative z be found by the mechanism [6]. Our requirement is slightly stronger if Z is the space of allocations or trades, and the same if Z is the space of utility values. The stronger requirement avoids some complications which do not seem to contribute anything essential to our understanding. The weaker form of Condition (ii) would require replacing the Pareto point z_o in theorems 2.1 and 2.2 by a Pareto equivalent point.

Since $s(p) \in V$ and V is open there exists an open neighborhood W of p in $\bar{E}$ such that $s(W) \subset V$. But this implies $W \subset \gamma^{-1}(V)$, which since $p \in W$, suffices to show that $\gamma^{-1}(V)$ is open in E. ∎

Thus, the fact that the message correspondence is locally threaded imposes a substantial degree of regularity on the performance of a resource allocation process. This has the effect of restricting the class of environments capable of being satisfactorily coordinated by means of mechanisms. Theorem 2.1 establishes necessary conditions on the correspondence $\mathcal{P}$ so that a $\mathcal{P}$ - satisfactory mechanism exists.

Theorem 2.1. Let E be a class of environments and $\mathcal{P}: E \to Z$ a correspondence. If $(\mathbb{\Gamma}, g)$ is a $\mathcal{P}$ - satisfactory mechanism on E then for each $e_o \in E$ and $z_o \in \mathcal{P}(e_o)$ there exists a (global) thread $s_o: E \to Z$ of $\mathcal{P}$ which passes through z_o at e_o.

Proof: Let $e_o \in E$ and $z_o \in \mathcal{P}(e_o)$. Since $(\mathbb{\Gamma}, g)$ is $\mathcal{P}$ - satisfactory on E, by (ii) of Definition 2.4 there exist $\gamma \in \mathbb{\Gamma}$ and $d \in \mathcal{D}$ such that $g \cdot \gamma \cdot d(e_o) = z_o$. Since γ is locally threaded, it follows from Lemma 2.1 that $f = g \cdot \gamma$ is a continuous function on E. Take $s_o = f \cdot d$. Since $d: E \to E$ is continuous, it follows that $s_o: E \to Z$ is continuous on E. By construction $s_o(e_o) = z_o$. Finally, $s_o(e) \in \mathcal{P}(e)$ for all $e \in E$, since by (i) of Definition 2.4, $g \cdot \gamma(e) \in \mathcal{P}(e)$ for all $e \in E$, and consequently $d(e) = e'$ implies $g \cdot \gamma(d(e)) \in \mathcal{P}(e') = \mathcal{P}(e)$, since it follows from Definition 2.3 that $d(e) = e'$ implies $\mathcal{P}(e) = \mathcal{P}(e')$. ∎

We next consider whether the condition that the correspondence $\mathcal{P}$ be suitably threaded is sufficient for the existence of a $\mathcal{P}$-satisfactory mechanism. It is possible to establish the converse of Theorem 2.1. However, because of our interest in informationally decentralized organization it is more interesting to state a "converse" to Theorem 2.1 with the condition that the mechanism preserve privacy. Thus, we take E to be the class of decomposable environments, i.e., $E = \prod_{i=1}^{N} E^i$, where E^i is the space of agent i's environmental characteristics (his preferences, endowment, technology, etc) and require the mechanism to preserve privacy (Definition 3 of [14].) Theorem 2.2 establishes the result that if the correspondence $\mathcal{P}$ is completely threaded, then there exists a privacy-preserving $\mathcal{P}$-satisfactory mechanism on E. The converse of Theorem 2.1 can easily be obtained as a corollary.

We introduce the class of privacy preserving processes. We reproduce from [14] the definition of a coordinate correspondence and of a privacy preserving process.

Definition 2.5 Let $E = \prod_{i=1}^{N} E^i$; we say that the correspondence $\mu: E \to M$ is a coordinate correspondence if there exist $\mu^i: E^i \to M$ such that $\mu(e) = \bigcap_{i=1} \mu^i(e)$ for all $e \in E$.

Definition 2.6. Let $E = E^1 x \ldots x E^N$. The resource allocation process (γ, g) preserves privacy if γ is a coordinate correspondence.

Definition 2.7. The mechanism (Γ, g) preserves privacy on $E = E^1 x \ldots x E^N$ if $\gamma \in \Gamma$ implies γ is a coordinate correspondence.

Lemma 2.2 Let $E = E^1 x \dots x E^N$ be a class of (decomposable) environments and let $\mathcal{P}: E \to Z$ be a correspondence on E. If for each $e_o \in E$ and $z_o \in \mathcal{P}(e_o)$ there exists a continuous function $s_o: E \to Z$ such that $s_o(e) \in \mathcal{P}(e)$ for all $e \in E$ and $s_o(e_o) = z_o$, then there exists a privacy preserving resource allocation process (γ, g) such that $g \cdot \gamma(e) \in \mathcal{P}(e)$ for all $e \in E$ and $g \cdot \gamma(e_o) = z_o$.

Proof: Take $M = E$, and for $e^i \in E^i$ and $i = 1, \dots, N$

define

$$\gamma^i(e^i) = E^1 x \dots x \{e^i\} x \dots x E^N = \{(\bar{e}^1, \dots, \bar{e}^N) \in E \mid \bar{e}^i = e^i\},$$

and let

$$\gamma(e) = \bigcap_{i=1}^{N} \gamma^i(e^i) = \{e\} \quad \text{for } e \in E.$$

Finally, let $g = s_o$.

Since γ is the identity function on E, it is a locally threaded correspondence. It follows from $g = s_o$ that $g \circ \gamma(e) \in \mathcal{P}(e)$ and $g \cdot \gamma(e_o) = z_o$. ∎

Theorem 2.2. Let $E = E^1 x \dots x E^N$ be a class of (decomposable) environments and let $\mathcal{P}: E \to Z$ be a correspondence on E. If for each $e \in E$ and $z \in \mathcal{P}(e)$ there exists a thread of $\mathcal{P}$ passing through z at e, then there exists a privacy preserving mechanism which is $\mathcal{P}$-satisfactory on E.

Proof: Let $\mathcal{S}$ be the set of continuous functions from E to Z, with the topology of pointwise convergence. Then $\mathcal{S}$ contains the (continuous) threads of the correspondence $\mathcal{P}$.

Let $M = E \times \mathscr{S}$, and let the outcome function g be given by,

$g(e,s) = s(e)$,

and note that g is continuous on $E \times \mathscr{S}$ [12, p. 218].

For $e_o \varepsilon E$ and $z_o \varepsilon \mathscr{P}(e_o)$, there exists a thread $s_o : E \to Z$ such that $s_o(e) \varepsilon \mathscr{P}(e)$ for all $e \varepsilon E$ and $s_o(e_o) = z_o$. For each $i = 1, \ldots, N$ and $e \varepsilon E$, define

$$\gamma_o^i(e^i) = E^1 x \ldots x \{e^i\} x \ldots x E^N x \{s_o\} \subset E \times \mathscr{S}$$

and

$$\gamma_o(e) = \bigcap_{i=1}^{N} \gamma_o^i(e^i) = (\{e\}, \{s_o\}).$$

Clearly, γ_o is threaded, since for fixed s_o it is a continuous function on E.

Finally,

$$g \cdot \gamma_o(e) = s_o(e) \varepsilon \mathscr{P}(e) \quad \text{for all } e \varepsilon E,$$

and

$$g \cdot \gamma_o(e_o) = s_o(e_o) = z_o, \text{ by construction.}$$

Take Γ to be the collection of all γ_o generated as s_o varies over all the threads of $\mathscr{P}$. Then (Γ, g) is the required mechanism. ∎

Theroems 2.1 and 2.2 together tell us that it is necessary and sufficient for a class of (decomposable) economies to admit of being $\mathscr{P}$-satisfactorily organized by means of a (privacy-preserving) mechanism, that the correspondence $\mathscr{P}$ be a union of threads.

Examples of Pareto-satisfactory and privacy preserving processes may be found in [6], [7] and [14] and the references cited in those papers.

Section 3: Notations and Assumptions

In what follows we shall make use of certain notations and assumptions which we list here for convenient reference.

R^{ℓ} euclidean space of ℓ-dimensions, the commodity space,

R^{ℓ}_{+} the non-negative orthant of R^{ℓ}.

$X^i \subset R^{\ell}$ the admissible consumption set of agent i, $i = 1,\dots,N$.

$\underset{i}{\succsim}$ preference preordering of X^i,

$w^i \in X^i$ initial endowment of agent i, $i = 1,\dots,N$,

Y^i set of admissible trades for agent i. In the pure trade case $Y^i = X^i - \{w^i\}$.

$E = E^1 x \dots x E^N$ the set of environments, sometimes called economies. In the pure trade case,

$E^i = \{e^i | e^i = (X^i, \Pi^i, w^i)\}$ where $\Pi^i \subset X^i x X^i$ **is** the graph of $\underset{i}{\succsim}$.

We give E^i the product topology using the topology of closed convergence for spaces of subsets and the euclidean topology for R^{ℓ}.

$X = X^1 x \dots x X^N$

$Y = Y^1 x \dots x Y^N$

$\mathcal{J}: E \to X$, where $\mathcal{J}(e) \subset X$ is the set of allocations feasible for e.

$F: E \to Y$, where $F(e) \subset Y$ is the set of trades feasible for e.

$\mathcal{G}^i(e^i, x^i) = \{\bar{x}^i \in X^i \mid \bar{x}^i \underset{i}{\succsim} x^i\}$, for $x^i \in X^i$, and

$G^i(e^i, y^i) = \{\bar{y}^i \in Y^i \mid \bar{y}^i \underset{i}{\succsim} y^i\}$ for $y^i \in Y^i$.

We understand the symbol $\underset{i}{\succsim}$ between elements $\bar{y}^i$ and y^i of Y^i to mean

$$w^i + \bar{y}^i \succsim_i w^i + y^i.$$

We shall also write $\bar{x} \succsim_i x$ for $\bar{x}, x \in X$ if and only if $\bar{x}^i \succsim_i x^i$, where

$$\bar{x} = (\bar{x}^1 \ldots \bar{x}^i \ldots \bar{x}^N) \text{ and } x = (x^1 \ldots x^i \ldots x^N).$$

Thus, $\mathscr{I}^i: E^i \times X^i \to X^i$,

and $G^i: E^i \times Y^i \to Y^i$ for $i = 1,\ldots,N$.

We write

$$\mathscr{I} = \mathscr{I}^1 \times \ldots \times \mathscr{I}^N,$$

and

$$G = G^1 \times \ldots \times G^N,$$

and hence

$$\mathscr{I}: E \times X \to X,$$

and,

$$G: E \times Y \to Y.$$

For A and B subsets of R^m, $d(A,B)$ denotes the distance between A and B in the metric topology of closed convergence. We shall denote the norm in R^m by $|\cdot|$

$\mathscr{P}: E \to X$ denotes the Pareto correspondence i.e. $\mathscr{P}(e)$ is the "contract curve" in X.

$P: E \to Y$ is the Pareto correspondence in terms of trades.

If $u(e,\cdot) = (u^1(e^1,\cdot),\ldots,u^N(e^N,\cdot))$ for $e \in E$ is a vector of utility functions on X, then

$U: E \to R^N$, where $U(e)$ is the image of $\mathscr{F}(e)$ under $u(e,\cdot)$, is the feasible utility correspondence and,

$\hat{U}: E \to R^N$, where $\hat{U}(e)$ is the image of $\mathscr{P}(e)$ under $u(e,\cdot)$, is the Pareto frontier correspondence.

Thus, for $e \in E$

$$U(e) = \{r \in R^N | r = u(e,x) \quad \text{for some} \quad x \in \mathcal{J}(e)\}$$

and

$$\hat{U}(e) = \{r \in R^N | r \in U(e), \text{ and } (r' \geqq r, r' \neq r) \text{ imply } r' \notin U(e)\}.$$

Thus $U(e)$ is the set of utility values attainable for the economy e with utility functions $u(e,\cdot)$, and $\hat{U}(e)$ is the Pareto frontier of utility values.

For a and b vectors, we write $a \geq b$ if $a_i \geqq b_i$ for all i but $a \neq b$, and $a > b$ if $a_i > b_i$ for all i.

We shall list the following assumptions. Not all assumptions will be in force simultaneously, but only as indicated. We shall, however, always assume continuity of preferences (Assumption II).

Assumption Ia. For each $i = 1,\ldots,N, X^i$ is an arc wise connected subset of R^ℓ.

Assumption Ib. For each $i = 1,\ldots,N$, $X^i = R^\ell_+$.

Assumption II. For each $i = 1,\ldots,N$, the preference relation $\underset{i}{\succsim}$ is continuous. I.e., for every $x^i \in X^i$ the sets $\{\bar{x}^i \in X^i \mid \bar{x}^i \underset{i}{\succsim} x^i\}$ and $\{\bar{x}^i \in X^i \mid x^i \underset{i}{\succsim} \bar{x}^i\}$ are closed.

Assumption IIIa. (Local non-satiation) For each $i = 1,\ldots,N$, each $x^i \in X^i$ and for every open set $\mathcal{O}^i(x^i)$ containing x^i there exists $\bar{x}^i \in \mathcal{O}^i(x^i)$ such that $\bar{x}^i \underset{i}{>} x^i$.

Assumption IIIb. (Strict monotonicity) For each $i = 1,\ldots,N$, $\bar{x}^i \geq x^i$ implies $\bar{x} \underset{i}{>} x^i$.

Assumption IV. (Strict dominance) For $e \in E$ if $x \in \mathcal{J}(e)$ and $x \notin \mathcal{P}(e)$, then there exists $\bar{x} \in \mathcal{J}(e)$ such that $\bar{x} \underset{i}{>} x$ for $i = 1,\ldots,N$.

Stated in terms of utilities, the strict dominance assumption is: For $e \in E$, if $p \in U(e)$ and $p \notin \hat{U}(e)$, then there exists $q \in U(e)$ such that $q > p$.

Assumption V. (Pointedness) For $e \in E$ and $x \in \mathcal{P}(e)$, $\mathcal{B}(e,x) \cap \mathcal{J}(e) = \{x\}$.

Assumption VIa. The correspondence $\mathcal{J}: E \to X$ is (i) upper semi-continuous, (ii) lower semi-continuous , and (iii) such that its image sets each contain more than one point.

Assumption VIb. For $e \in E$, if $x \in X$, $y \in X$, $x \in \mathcal{J}(e)$ and $\sum_{i=1}^{N} x^i = \sum_{i=1}^{N} y^i$, then $y \in \mathcal{J}(e)$.

The foregoing assumptions are clearly not all independent. We note some of the relationships among them, without proof.

(i) Ib implies Ia.

(ii) Ib and IIIb imply IIIa.

(iii) Ib, II, IIIb, and VIb imply IV.

We also note that Assumptions II and IIIa imply a weaker form of IV, namely that each non-optimal feasible point which is dominated by a point interior to the feasible set is strictly dominated.

Section 4: The Feasible Utility Correspondence and the Pareto Frontier.

In this section we study the continuity of the utility possibility set and its Pareto utility frontier.

We note first,

Lemma 4.1: If $\mathcal{J}: E \Rightarrow X$ is continuous on E (u.h.c. and l.h.c.) and $u: E \times X \Rightarrow R^N$ is continuous on $E \times X$, then $U: E \Rightarrow R^N$, $U(e) = u(e, \mathcal{J}(e))$ is a continuous correspondence on E. ([1] . p. 113). It is clear that if $\mathcal{J}$ is compact valued and u continuous, then U is compact valued.

Lemma 4.2: Let $A : X \Rightarrow R^N$ be an upper hemi-continuous correspondence on a topological space X and let $x_o \in X$. Then there exists a compact set $K \subset R^N$ and an open neighborhood $\mathcal{O}(x_o)$ of x_o such that $x \in \mathcal{O}(x_o)$ implies $A(x) \subset K$.

Proof: The upper hemi-continuity of A implies $A(x_o)$ is compact. ([1] Theorem 2, pp.110) It follows that cl $A_\varepsilon(x_o)$ is also compact, where, for $\varepsilon > 0$,
$A_\varepsilon(x_o) = \{y \in R^N \mid |y - z| < \varepsilon$ for some $z \in A(x_o)\}$.
Since $A_\varepsilon(x_o)$ is open, and A is upper hemi-continuous, there exists an open neighborhood $\mathcal{O}(x_o)$ of x_o such that $x \in \mathcal{O}(x_o)$ implies $A(x) \subset A_\varepsilon(x_o)$ ([1] p. 109). Hence $x \in \mathcal{O}(x_o)$ implies $A(x) \subset$ cl $A_\varepsilon(x_o)$. ∎

Lemma 4.3: If $\mathcal{J}: E \to R^{\ell N}$ is continuous and compact valued, (Assumption VIa), if $u: E \times R^{\ell N} \to R^N$ is continuous and satisfies Assumption IV (strict dominance) on $\mathcal{J}$, then the correspondence $\hat{U}: E \to R^N$ is upper hemi-continuous on E.

Proof: The correspondence $U : E \to R^N$ is continuous and hence upper hemi-continuous. (Lemma 4.1) Since $\hat{U}(e) \subset U(e)$ for all $e \in E$, we may write $\hat{U} = \hat{U} \cap U$. If $\hat{U}$ is a closed correspondence it follows that $\hat{U}$ is upper hemi-continuous, since the intersection of a closed correspondence with an upper hemi-continuous correspondence is upper hemi-continuous, (Theorem 7 of [1](p. 112)). We shall show that $\hat{U}$ is a closed correspondence.

Let $e_o \in E$, $e_j \to e_o$, $r_j \to r_o$ and let $r_j \in \hat{U}(e_j)$ for $j = 1,2,\ldots$. Since $\hat{U}(e_j) \subset U(e_j)$, $r_j \in \hat{U}(e_j)$ if and only if there exists $x_j \in \mathcal{J}(e_j)$ such that

1) $u(e_j,x_j) = r_j$,

and

2) $\{x \in \mathcal{J}(e_j) \mid u(e_j,x) \geq u(e_j,x_j)\} = \emptyset$

Since $\mathcal{J}(e_o)$ is compact [Assumption VIa(ii)] and $\mathcal{J}$ upper hemi-continuous [Assumption VIa(i)] by Lemma 4.2, for every $\varepsilon > 0$, there exists an integer $J(\varepsilon)$ such that $j > J(e)$ implies

$\mathcal{J}(e_j) \subset \text{cl}\, \mathcal{J}_\varepsilon(e_o)$, a compact set.

Hence, the infinite set $\{x_j \mid j > J(e)\}$ has a point of accumulation x_o, and consequently a subsequence $x_{j_k} \to x_o$. By continuity of u on $E \times R^{\ell N}$,

$$u(e_{j_k}, x_{j_k}) \to u(e_o, x_o).$$

But $u(e_{j_k}, x_{j_k}) = r_{j_k}$ and by hypothesis $r_{j_k} \to r_o$. It follows that

$u(e_o,x_o) = r_o$.

Now suppose $r_o \notin \hat{U}(e_o)$. Since $r_o \in U(e_o)$ and $r_o \notin \hat{U}(e_o)$ it follows from the strict dominance assumption applied to r_o (Assumption IV) that there exists $\bar{r} \in U(e_o)$ such that $\bar{r} > r_o$ and correspondingly $\bar{x} \in \mathcal{J}(e_o)$ such that $\bar{r} = u(e_o,\bar{x}) > u(e_o,x_o)$.

Since $\mathcal{J}$ is lower hemi-continuous (Assumption V a), it follows from $e_{j_k} \to e_o$ and $\bar{x} \in \mathcal{J}(e_o)$ that there exists $\bar{x}_{j_k} \in \mathcal{J}(e_{j_k})$ such that $\bar{x}_{j_k} \to \bar{x}$. Since u is continuous, for $\bar{\varepsilon} = |u(e_o,\bar{x}) - u(e_o,x_o)| > 0$, there exists an integer $K(\bar{\varepsilon})$ such that $k > K(\bar{\varepsilon})$ implies

$$|u(e_{j_k},\bar{x}_{j_k}) - u(e_o,\bar{x})| < \frac{\bar{\varepsilon}}{2}$$

and

$$|u(e_{j_k},x_{j_k}) - u(e_o,x_o)| < \frac{\bar{\varepsilon}}{2} \quad .$$

It follows that for $k > K(\bar{\varepsilon})$,

$$\bar{r}_{j_k} = u(e_{j_k},\bar{x}_{j_k}) > u(e_{j_k},x_{j_k}) = r_{j_k}$$

Thus, $\bar{r}_{j_k} \in U(e_{j_k})$, $\bar{r}_{j_k} > r_{j_k}$, and hence $r_{j_k} \notin \hat{U}(e_{j_k})$, contrary to construction.

Thus $r_o \notin \hat{U}(e_o)$ must be false. This establishes the upper hemi-continuity of $\hat{U}$. ∎

<u>Lemma 4.4:</u> If $U: E \Rightarrow R^N$ is continuous and compact valued on E then $\hat{U}$ is lower hemi-continuous on E.

<u>Proof:</u> Assume $e_o \in E$ and $r_o \in \hat{U}(e_o)$ and suppose $e_j \to e_o$, where $\{e_j\}$ is a convergent sequence of environments $e_j \in E$.

Since $\hat{U}(e) \subset U(e)$ for all $e \in E$, it follows that $r_o \in U(e_o)$. Since U is lower hemi-continuous on E, and since $e_j \to e_o$, it follows that there exist $r_j \in U(e_j)$ such that $r_j \to r_o$.

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* Mark Walker's suggestions have made this proof both shorter and clearer than it was before.

Let

$$\Omega(r) = \{s \in R^N \mid s \geqq r\},$$

and let $B_j = \{r \in R^N \mid r$ is maximal for $\geqq$ on $\Omega(r_j) \cap U(e_j)\}$. By hypothesis, $U(e_j)$ is compact and hence so is $U(e_j) \cap \Omega(r_j)$. Hence B_j is non-empty for all j.

Further, $B_j \subset \hat{U}(e_j)$ for all j. To see this, let $r \in B_j$ and $r' \in U(e_j)$ with $r' \neq r$ and $r' \geqq r$. Then $r' \geqq r \geqq r_j$, thus $r' \in U(e_j) \cap \Omega(r_j)$. It follows that r is not maximal on that set, which contradicts $r \in B_j$.

Given $\varepsilon > 0$, it follows from the hypothesis that U is compact-valued, that cl $U_\varepsilon(e_o)$ is compact. Since U is u.h.c. there exists an integer K such that $j > K$ implies

$$U(e_j) \subset \text{cl } U_\varepsilon(e_o).$$

Thus,

$$B_j \in U(e_j) \subset \text{cl } U_\varepsilon(e_o) \quad \text{for} \quad j > K.$$

Take $z_j \in B_j$ for $j > K$, which may be done, since $B_j \neq \emptyset$ for all j.

Compactness of cl $U_\varepsilon(e_o)$ guarantees that the infinite subset $\{z_j\}$ has at least one accumulation point, say $\bar{z}$, and that if $\bar{z}$ is the <u>only</u> accumulation point of $\{z_j\}$, then it is the limit of $\{z_j\}$. Let z_o be an accumulation point of $\{z_j\}$, and let $\{z_{jk}\}$ be a subsequence which converges to z_o. We have $z_{jk} \geqq r_{jk}$, for all $j_k > K$ (because $z_j \in B_j \subseteqq \Omega(r_j)$), and $r_{jk} \to r_o$; hence, $z_o \geqq r_o$.

But we also have $e_{jk} \to e_o$ and $z_{jk} \in U(e_{jk})$, which, since U is upper-hemi-continuous, guarantees that $z_o \in U(e_o)$. Since $r_o \in \hat{U}(e_o)$, then we must have $z_o = r_o$; that is, r_o is the only accumulation point of $\{z_j\}$, and is consequently the limit of $\{z_j\}$. Since, for $j > K$, $z_j \in B_j \subseteqq \hat{U}(e_j)$, the proof is complete. ∎

Section 5: Threads of the Pareto Frontier

In this section we give conditions sufficient for the existence of a continuous thread of the Pareto correspondence passing through an arbitrarily chosen point of the graph of that correspondence. As we found in Theorem 2.1, the existence of such threads is necessary in order that a class of economies be susceptible of being organized by a Pareto satisfactory mechanism satisfying the condition that the message correspondence is locally threaded. These conditions show that a class of economies for which the Pareto correspondence has such a thread consists of economies such that the consumption set of each agent is the non-negative orthant of the Euclidean commodity space, the production sets permit a non-zero allocation and are such that the set of attainable allocations is compact, and where preferences are strictly monotonic for each agent. Convexity of either preferences or production sets is not required. In [9] this class of economies was shown to satisfy the assumption of Openness. ([9] Lemma 5.9 and its Corollary.)

We show below in Section 6 that strict monotonicity cannot be replaced by a weaker form of monotonicity. This is not unexpected, since both local non-satiation and the strict dominance properties might then fail. Furthermore, convexity of preference, implying local non-satiation, is not sufficient to ensure that the Pareto frontier has no holes. Example 5.1 below shows this.

The property of strict dominance, which is indispensable for the Pareto frontier to be without holes, involves a relationship among the preferences of all the agents, as well as among their consumption sets (and production sets in the case of production). If we specify a class of characteristics of individuals (consumption sets, preferences, etc.) such that any N-tuple of agents drawn from that class will have the strict dominance property, then we conjecture that strict monotonicity of preference over the non-negative orthants (or consumption sets that are suitable translates of them) is essentially the largest such class. However, as Lemma 3.2 above shows, continuity of preference and local non-satiation are, when redistribution is feasible (Assumption VIb) sufficient for the strict dominance property to hold except on the set of non-optimal points dominated only by boundary points of the consumption set.

Definition 5.1 Let $e \in E$ be an economy with continuous utility indicators $u = (u^1,\ldots,u^N)$ such that for each $i = 1,\ldots,N$ $\inf\ u^i(x) \geq 0$, and let $\hat{U}(e) \subset R_+^N$ denote $u(\mathcal{P}(e))$, the Pareto set of e in the space of utility values. We say that $\hat{U}(e)$ has no holes if and only if for each non-zero vector $p \in R_+^N$, $\{tp \mid t \geqq 0\} \cap \hat{U}(e) \neq \emptyset$.

Thus, the set of Pareto utility values has no holes if every ray out of the origin in a non-negative direction intersects the Pareto set. Lemma 5.1 gives sufficient conditions , involving the no holes property, that the Pareto correspondence has a thread through every point of its graph.

Lemma 5.1: If E is a class of economies such that (i) the Pareto correspondence $\hat{U}$ is upper-hemi-continuous on E, (ii) $\hat{U}(e) \subset R_+^N$ for $e \in E$, and (iii) for each $e \in E$, $\hat{U}(e)$ has no holes, then for every $e_o \in E$, and every point $p_o \in \hat{U}(e_o)$, there exists a continuous thread $s: E \to R^N_+$ of $\hat{U}$ such that $s(e_o) = p_o$.

Proof:* Let $e_o \in E$, let $p_o \in \hat{U}(e_o)$, and for each $e \in E$, let $s(e)$ be the set $\{t\, p_o | t \geqq 0\} \cap \hat{U}(e)$, the intersection of $\hat{U}(e)$ with the ray through p_o. According to (ii) and (iii), $s(e)$ is non-empty for each $e \in E$, i.e. s is a correspondence from E to $\mathbb{R}^n_+$. Since s is the intersection of the upper-hemicontinuous correspondence $\hat{U}$ with the constant (hence continuous) correspondence $\{t\, p_o | t \geqq 0\}$, it is itself upper hemi-continuous [1], Theorem 2', p. 114). Finally, $s(e)$ obviously consists of only a single point for each e (by definition of $\hat{U}$), and hence s is a continuous function. ∎

Definition 5.2. Let E^* be the class of economies satisfying Assumptions Ib, II, IIIb, VIa, VIb. I.e., E^* is the class of economies such that

(a) for each agent $i \in (1,\dots,N)$

(i) the consumption set $X^i = R^\ell_+$

(ii) the preference relation $\underset{i}{\precsim}$ is continuous and strictly monotone,

and (b)

(iii) the correspondence $\mathcal{J}: E^* \to R^{\ell N}$ is continuous and compact valued, and such that its image sets contain more than one point

(iv) if $x \in \mathcal{J}(e)$, $x = (x^1,\dots,x^N)$, and if $y = (y^1,\dots,y^N)$ $y^i \geq 0$ for $i = 1,\dots,N$ and $\sum_{i=1}^N y^i = \sum_{i=1}^N x^i$ then $y \in \mathcal{J}(e)$,

(v) for each $e \in E^*$ there exists $x \in \mathcal{J}(e)$, $x \geq 0$.

*We owe this improved version of the proof to suggestions made by Mark Walker.

Lemma 5.2. There exists a utility representation $u^*: E^* \times R_+^{\ell N} \to R^N$ such that the correspondence

$$U^*: E^* \to R^N \text{ given by } U^*(e) = \{p \in R^N \mid u^*(e,x) = p \text{ for some } x \in \mathcal{J}(e)\}$$

is

(i) upper hemi-continuous and lower hemi-continuous

(ii) $U^*(e) \subset R_+^N$ for all $e \in E^*$

(iii) $0 \in R_+^N$ is dominated for all $e \in E^*$, (i.e., for all $e \in E^*$, 0 is not a Pareto point of $U^*(e)$).

Proof: If $e \in E^*$, under (i) and (ii) of Definition 5.2, it is well-known that there exists a vector of utility functions

$$u: E^* \times R_+^{\ell N} \to R^N$$

which is a jointly continuous representation of preferences.(Hildenbrand [11,p.798] established the existence of a utility function which is jointly continuous in preferences and commodities and which represents the class of continuous monotone preferences on R_+^ℓ, where the topology on the space of preference relations is that of closed convergence.)

Let

$$m_e^i = \min_{x \in R_+^\ell} u^i(e,x).$$

Since R_+^ℓ contains its lower bound for $\leqq$, namely, 0, and $u^i(e,\cdot)$ is strictly increasing with respect to $\leqq$, m_e^i is well-defined. For $e \in E^*$ define

$$u^*(e,\cdot) = A_u \cdot u(e,\cdot) \equiv u(e,\cdot) - (m_u^1,\ldots,m_u^N)$$

Thus,

$$u^*: E^* \times R_+^{\ell N} \to R^N.$$

Then,

$$u^{*i}(e,x^i) \geqq 0 \quad \text{for } i = 1,\ldots,N \text{ and } x^i \in R_+^\ell,$$

$$u^{*i}(e,0) = 0,$$

and u^{*i} is jointly continuous on $E^* \times R_+^\ell$, because A_u is continuous, since m_u^i is the minimum of a continuous function on a fixed set.

It follows immediately that U^* is continuous on E^*. ∎

Lemma 5.3 U^* satisfies Assumption IV on E^*.

Proof */

Let $p \in U^*(e)$ and suppose p is not a Pareto point. Then there exists $q \in U^*(e)$ such that $q \geq p$. Since p and q are in $U^*(e)$ there exists x and y in $\mathcal{J}(e)$ such that $u^*(e,x) = p$ and $u^*(e,y) = q$. Since $q \geq p \geqq 0$, there is at least one agent $i_o \in \{1,\ldots,N\}$ such that $q^{i_o} = u^{*i_o}(e,y^{i_o}) > u^{*i_o}(e,x^{i_o}) \geqq 0$. It follows from strict monotonicity of preference and from $\min_{\mathcal{J}(e)} u^*(e,x) = 0$, that there is at least one commodity $j_o \in \{1,\ldots,\ell\}$ such that $y_{j_o}^{i_o} > 0$. Since $u^{*i_o}(e,\cdot)$ is continuous and assumes the value, 0 at 0 and q^{i_o} at $y^{i_o} \geq 0$, there exists a scalar $0 < \alpha^o < 1$ such that

$$u^{*i_o}(e,\alpha^o y^{i_o}) = \tfrac{1}{2}\,(u^{*i_o}(e,y^{i_o}) + p^{i_o}) > p^{i_o}.$$

---

*/ The proof of this Lemma follows an argument in the proof of Theorem 5 in ([3], p. 25).

We may define a new allocation z, by

$$z^i = \begin{cases} y^i + \frac{1-\alpha^o}{N-1} y^{i_o} & \text{for } i \in \{1,\ldots,N\} \setminus \{i_o\} \\ \alpha^o y^{i_o} & \text{for } i = i_o \end{cases}$$

Clearly $\bar{z} \in R_+^{\ell N}$, and $\sum_{i=1}^{N} z^i = \sum_{i=1}^{N} y^i$; hence, by Assumption VI b, $z \in \mathcal{J}(e)$. Furthermore $z^i \geq y^i$ for $i \neq i_o$, which by strict monotonicity of $u^{*i}(e,\cdot)$ implies

$$u^{*i}(e,z^i) > u^{*i}(e,y^i) \geqq p^{i_o} \text{ for } i \neq i_o$$

while for $i = i_o$, $u^{*i_o}(e,z^i) = u^{*i_o}(e,\alpha^o y^{i_o}) > p^{i_o}$. Thus, it follows that

$$u^*(e,z) \in U^*(e)$$

and

$$u^*(e,z) > p \quad .$$

This concludes the proof of Lemma 5.3. ∎

Lemma 5.4 The Pareto correspondence

$$\hat{U}^*: E^* \to R_+^N \text{, given by}$$

$$\hat{U}^*(e) = \{p \in U^*(e) \mid q \in U^*(e),\ q \geqq p \ \text{ implies } \ q = p\}$$

is upper hemi-continuous on E^*.

Proof: It follows from Lemmas 5.2 and 5.3 that U^* satisfies all the hypotheses of Lemma 4.3. The conclusion follows.

Theorem 5.1.*/ If $e \in E^*$, then $\hat{U}^*(e)$ has no holes.

Proof:

For $e \in E^*$ define $\dot{U}^*(e) = \{p \in R_+^N \mid p \leqq q$ for some $q \in U^*(e)\}$ and let

$\mathcal{B}^*(e)$ denote the relative boundary of $\dot{U}^*(e)$ in R_+^N. It follows from compactness of $\mathcal{J}(e)$ and continuity of $u^*(e,\cdot)$ that $U^*(e)$ is compact and hence that $\dot{U}^*(e)$ is compact. Hence,

$$\mathcal{B}^*(e) \cap \dot{U}^*(e) = \mathcal{B}^*(e) \quad \text{for all } e \in E.$$

For each $e \in E^*$ the hypotheses of Theorem 5 ([3],p.25) are satisfied; it follows that

$$\hat{U}^*(e) = \mathcal{B}^*(e).$$

This, in turn, implies that $\hat{U}^*(e)$ has no holes. To see this, we must show that if $p \in R_+^N$, $p \neq 0$, then the half line $\{tp \mid t \geq 0\}$ has a non-empty intersection with the Pareto set $\hat{U}^*(e)$. It is clear that $\{tp \mid t \geq 0\} \cap \dot{U}^*(e) \neq \emptyset$. Since $\dot{U}^*(e)$ is compact and contains a point $q \geq 0$, there is a largest value of t, say $t = \bar{t}$, $\bar{t} > 0$, such that $\bar{t}p \in \dot{U}^*(e)$. It follows that $\bar{t}p \in \mathcal{B}^*(e)$, since $t > \bar{t}$ implies tp is exterior to $\dot{U}^*(e)$.

By the Theorem 5, of [3], $\bar{t}p \in \hat{U}^*(e)$. I.e., for each $e \in E$, $\hat{U}^*(e)$ has no holes.

*/ Chipman and Moore point out that their form of this result is implicit in T. Gorman's paper "Community Preference Fields" Econometrica 21 1953, pp. 63-80.

Theorem 5.2: If E^* is a class of economies satisfying Definition 5.2, and if $u: E^* \times R_+^{\ell N} \to R^N$ is a jointly continuous representation of preferences and $\hat{U}: E^* \to R_+^N$ the Pareto correspondence induced by u, then, for every economy $e_o \in E^*$ and every Pareto point $p_o \in \hat{U}(e_o)$ there is a continuous thread $s: E^* \to R_+^N$ which takes the value p_o at e_o.

Proof: Let A_u be the transformation which carries u to u^*, as in the proof of Lemma 5.2. Then A_u is orthogonal and non-singular. Hence A_u^{-1} is well-defined, continuous and carries Pareto points to Pareto points. I.e., if

$$q \in \hat{U}^*(e), \text{ then } A_u^{-1}(q) \in \hat{U}(e) .$$

It follows from Theorem 5.1 that for $e \in E^*$, $\hat{U}^*(e)$ has no holes. Hence Lemma 5.1 applies. We may conclude that there exists a thread $s^*: E^* \to R_+^N$ such that $s^*(e_o) = A_u \cdot (p_o)$.

Then,

$$s: E^* \to R_+^N$$

given by

$$s(e) = A_u^{-1}(s^*(e))$$

is the required function. ∎

We have assumed that the set of feasible allocations $\mathcal{J}(e)$ is compact for $e \in E^*$. Conditions on the production and consumption sets which imply that the set of attainable allocations is bounded are given in ([10] Theorem 1 p. 581). These conditions do not require convexity of either production or consumption sets. It is straightforward to show that if the aggregate production set is closed, then $\mathcal{J}(e)$ is closed.

It can be further shown [see [16] pp. 39, 40 for proofs] that conditions on individual production sets which (in the presence of others) are sufficient for boundedness of the set of feasible allocations are also sufficient to insure that the aggregate production set is closed whenever the individual production sets are closed. These conditions are stated in Lemmas 5.5 and 5.6.

<u>Lemma 5.5</u> Let $Y^1,\ldots,Y^N$ be subsets of R^ℓ and let $Y = \sum_{i=1}^{N} Y^i$. If $A\,Y \cap (-AY) = \{0\}$, then $A\,Y^1,\ldots,A\,Y^N$ are positively semi-independent.*

<u>Lemma 5.6</u> If $Y^1,\ldots,Y^N$ are closed subsets of R^ℓ and if $A\,Y \cap (-A\,Y) = \{0\}$, where $Y = \sum_{i=1}^{N} Y^i$, then Y is closed.

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* $A\,X$ denotes the asymptotic cone of the set X. N cones with vertex 0 $X^1,\ldots,X^N$, are positively semi-independent if $\Sigma\, x^i = 0$ $x^i \in X^i$ for $i = 1,\ldots,N$ implies $x^i = 0$ for $i = 1,\ldots,N$. (See [4] (1.9 m) and 1.9 n p. 22 for definitions.)

We next give an example showing that convexity of preferences, and "steepness", implying local non-satiation, are not sufficient to ensure that the Pareto frontier has no holes. Notice that in this example the utility functions are concave, but utility is not freely disposable. (Disposability of utility may be considered to be a form of monotonicity.)

Example 5.1: 2 commodities, 2 agents.

Let $\mathcal{J}(e)$(the feasible set in the space of allocations)be the Edgeworth Box

$$\{(x_1,x_2) \in R^2 \mid 0 \leq x_1 \leq 1 , 0 \leq x_2 \leq 1\}$$

so that agent 2 holds $(1 - x_1, 1 - x_2)$ if agent 1 holds (x_1,x_2).

Let $u^1(x_1,x_2) = x_1 + x_2 \qquad x_1,x_2 \geq 0$

Let $u^2(y_1,y_2) = [u^1(1 - y_1, 1 - y_2)]^{\frac{1}{2}} \qquad y_1,y_2 \geq 0$

Then, it follows that

$$u^2(y_1,y_2) = [u^1(1 - y_1,1 - y_2)]^{\frac{1}{2}} = [u^1(x_1,x_2)]^{\frac{1}{2}}$$

and hence that the image of $\mathcal{J}(e)$ is the curve $(z, z^{\frac{1}{2}})$ in R^2, where $z \in [0,1]$ and hence $z^{\frac{1}{2}} > z$.

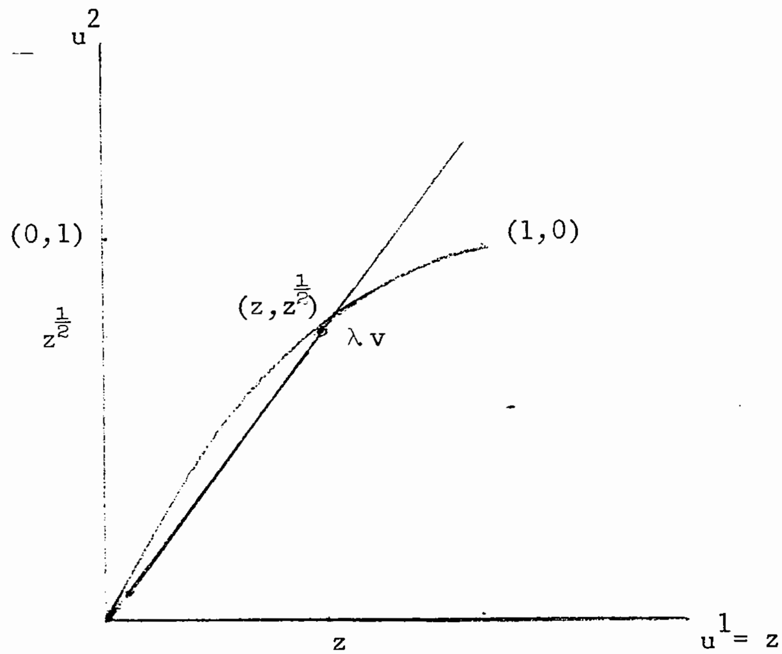

Figure 5.1

The Pareto set consists of the point (1,1). Now clearly for $v = (\frac{9}{16}, \frac{3}{4})$

$\bar{\lambda} = \max\{\lambda : \lambda v \text{ is feasible}\}$ satisfies $\bar{\lambda} = 1$. I.e.,

$\bar{\lambda} v = (\frac{9}{16}, \frac{3}{4})$ is feasible, but for $\lambda > 1$, λv is not feasible,

since for $\lambda > 1$

$$\lambda \frac{3}{4} > \lambda^{\frac{1}{2}} \frac{3}{4} = (\lambda \frac{9}{16})^{\frac{1}{2}}$$

But $(\frac{9}{16}, \frac{3}{4})$ is not a Pareto point, since

$$\frac{9}{16}, \frac{3}{4}) < (1,1).$$

Section 6: Threads of the Contract Curve

In this section we study the Pareto set in the space of allocations or of trades. We given conditions under which the utility map is a homeomorphism between the graph of the contract curve correspondence and the Pareto frontier correspondence (Lemma 6.2). This homeomorphism can be used to lift a thread of the Pareto correspondence in the utility space to a thread of the Pareto correspondence in the allocation space (Lemma 6.1).

The conditions under which the utility function is a homeomorphism between the graphs of the Pareto correspondences include the so-called "pointedness" condition, (Assumption V), namely, that the upper contour set on a Pareto point contains only a single feasible point. Example 6.1 shows that this condition is indispensible for the existence of threads, for without it there is a set of economies and a point through which there is no local thread of the Pareto correspondence, and, a fortiori, no thread.

Lemma 6.1:

Let $u: E \times X \to R^N$ be a utility function on $E \times X$. Let $\hat{u}$ denote the restriction of u to the graph $\mathcal{Q}(\mathcal{P})$ of $\mathcal{P}$. Let $\hat{u}$ be a homeomorphism of $\mathcal{Q}(\mathcal{P})$ to the graph $\mathcal{Q}(\hat{U})$ of $\hat{U}$. Let $s: E \to R^N$ be a thread of $\hat{U}$, and let Pr_x $Pr_x: E \times X \to X$ be the projection of $E \times X$ to X. Then,

$$Pr_x \circ \hat{u}^{-1} \circ s: E \to X \text{ is a thread of } \mathcal{P}.$$

Proof: The proof is immediate. ∎

Lemma 6.2: Under Assumption Ia, II, IIIa, V, and VIa(i), $\hat{u}$ is a homeomorphism of $\mathcal{Q}(\hat{\mathcal{P}})$ and $\mathcal{Q}(\hat{U})$.

Proof: Under Assumptions Ia, II and IIIa $u: E \times X \to R^N$ is jointly continuous. It follows that $\hat{u}$ is continuous on $\mathcal{Q}(\mathcal{P})$.

It follows from Assumption V (pointedness) that $\hat{u}^{-1}: \mathcal{Q}(\hat{U}) \to \mathcal{Q}(\mathcal{P})$ is single-valued. It remains only to show that $\hat{u}^{-1}$ is continuous on $\mathcal{Q}(\hat{U})$. To show this let $(e_o, p_o) \in \mathcal{Q}(\hat{U})$ and let $(e_j, p_j) \to (e_o, p_o)$, where $(e_j, p_j) \in \mathcal{Q}(\hat{U})$ for $j = 1, 2, \ldots$

Let $\hat{u}^{-1}(e_j, p_j) = (e_j, x_j)$, and $\hat{u}^{-1}(e_o, p_o) = (e_o, x_o)$. Since $e_j \to e_o$ by assumption, we need only show that $x_j \to x_o$.

Since $\mathcal{J}: E \to X$ is upper hemi-continuous (Assumption VIa(i)) it follows from Lemma 4.2 that there exists a compact set $K \subset X$ and an integer J such that $\mathcal{J}(e_j) \subset K$ for all $j > J$. Omitting the first J terms from the sequence x_j, it suffices to show that the remaining sequence converges to x_o. To establish this proposition, we shall show that every infinite subsequence of $\{x_j' : j > J\}$ converges to x_o.

Since $\{x_j\} \subset K$, it has a convergent subsequence x_{j_k} whose limit is x'. Since $\mathcal{J}$ is upper hemi-continuous, $e_{j_k} \to e_o$, $x_{j_k} \to x'$ and $x_{j_k} \in \mathcal{J}(e_{j_k})$ for all k, implies $x' \in \mathcal{J}(e_o)$. Since u is continuous, it follows that $\hat{u}(e_{j_k}, x_{j_k}) \to \hat{u}(e_o, x')$. But $\hat{u}(e_{j_k}, x_{j_k}) = p_{j_k}$, and $p_{j_k} \to p_o$, since p_{j_k} is a subsequence of p_j, and $p_j \to p_o$. It follows that $u(e_o, x') = u(r_o, x_o)$. But it follows from Assumption V that $x_o = x'$.

Consider the terms of the sequence x_j which are not in the subsequence x_{j_k}. Either there is a finite number of them in which case, $x_j \to x_o$, or there is an infinite number in which the same argument applied to the remaining infinite set of x_j's leads the conclusion that there is a convergent subsequenece with the limit x_o. Hence it follows that every infinite subsequence of $\{x_j\}$ converges to x_o and hence that $\{x_j\}$ converges to x.

This establishes the continuity of $\hat{u}^{-1}$ on $\mathcal{D}(\hat{U})$. ∎

Chipman and Moore [3, Theorems 6 and 7] have given conditions under which the "contract curve" (the set of Pareto allocations) is homeomorphic to an (N-1) simplex. These conditions include strict convexity of the preference relations, which implies that Assumption V (pointedness) is satisfied. While the pointedness assumption can be satisfied without strict convexity of preferences, it is not clear whether there is a (non-trivial) class of economies which satisfy all the assumptions of Theorem 6.1 without strictly convex preferences.

<u>Theorem 6.1</u> Under Assumption Ib, II, IIIb, V, VIa and VIb, there is for each $e \in E$ and $p \in \hat{U}(e)$

a) a thread of $\hat{U}$ $s_p: E \to R^N$, such that $s_p(e) = p$,

and

b) a corresponding thread $pr_x \circ \hat{u}^{-1} \circ s_p : E \to X$ of $\mathcal{P}$

Proof: a) follows from Theorem 5.2;

b) follows from a) and Lemmas 6.1 and 6.2.

<u>Example 6.1:</u>

We give next an example in which the Pareto correspondence in the space of allocations is not locally threaded, although the Pareto utility correspondence is threaded. We shall sketch this example rather than present it formally.

Two agents, two commodities, initial endowment (a,b)

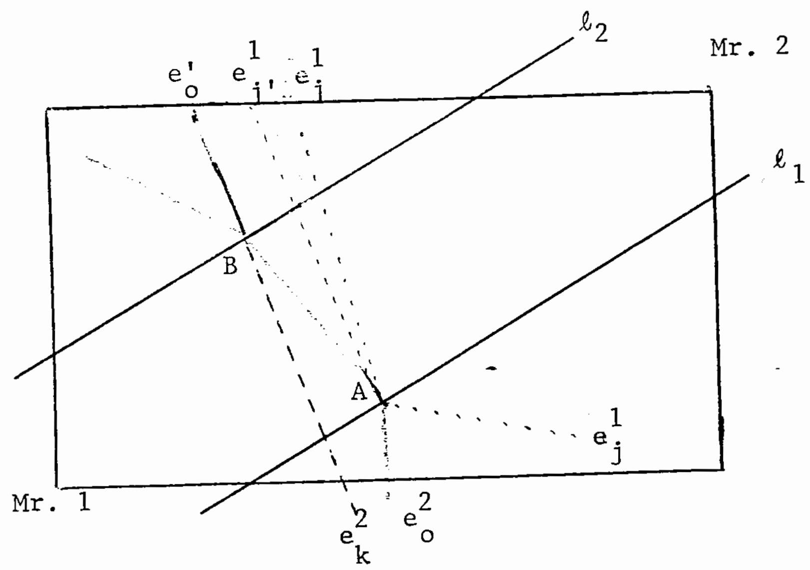

Figure 6.1

Let two lines ℓ_1 and ℓ_2 (in Figure 6.1) be given a fixed distance apart, say 1. In the environment $e_j = (e_j^1, e_j^2)$ indifference curves of agent 1 consist of parallel translates of the dotted line labelled e_j^1 in Figure 6.1, while the indifference map of agent 2 consists of parallel translates of the solid curve labeled e_0^2. For $j' > j$, the indifference j' curves of agent 1 are closer to the flat segment AB. As $j \to \infty$, $e_j^1 \to e_o^1$. Thus, for all j the Pareto set for $e_j = (e_j^1, e_o^2)$ is the line ℓ_1, while in the limit e_o, the Pareto set is the strip (closed) included between the lines ℓ_1 and ℓ_2. Similarly, for $\bar{e}_k = (e_o^1, e_k^2)$ where the indifference curves of agent 2 are parallel to the dashed curve labelled e_k^2 in Figure 6.1, the Pareto set is the line ℓ_2.

Now, every open set containing e_o contains e_j and $\bar{e}_k$ for j and k sufficiently large.

A local thread for the Pareto correspondence P would have to satisfy the following conditions.

There exists an open set U containing e_o and a continuous function $s: U \to P(U)$; (i.e. for each $e \in U$ $s(e) \in P(e)$.)

Therefore, for $e_j = (e_j^1, e_o^2) \in U$ and $\bar{e}_k = (e_o^1, \bar{e}_k^2) \in U$, we must have $s(e_j) \in \ell_1$ and $s(\bar{e}_k) \in \ell_2$. But the distance of $s(e_j)$ from $s(\bar{e}_k)$ is 1 for all j,k. Hence s is not continuous on U, for any open set U containing e_o.

The preferences in this example are continuous, convex, satisfy local non-satiation, and (can be chosen) strictly monotone. We may choose a utility function to represent these preferences say by taking the distance from the respective origins to each indifference curve measured along the diagonal of the box. Then the Pareto utility image does have a thread, although the Pareto allocation correspondence does not.

Note that, while the example shows indifference curves with kinks, smoothly differentiable curves can yield the same result, since such indifference curves can be made to turn smoothly into flat segments, with suitable tangencies along ℓ_1 and ℓ_2 respectively.

Section 7: An Economy Whose Pareto Frontier Has Holes

In Section 5 we constructed a thread of the Pareto utility correspondence by intersecting a fixed ray out of the origin with the set of feasible utility values, and taking the point of maximum distance from the origin on that ray. Under suitable conditions, including strict monotonicity of utilities, the point so obtained was shown to be a Pareto point. This property is essential to the construction of threads. We have already seen in Example 5.1 that strict monotonicity of preferences is indispensible for this construction. In that example preferences have the local non-satiation and convexity properties, but are not monotone. It is instructive to examine the borderline case in which preferences are monotone but not strictly monotone. Of course, in that case preferences may not satisfy the local non-satiation assumption. For this borderline case, we construct a class of economies with two agents and one commodity for which there is a complete description of the Pareto utility set. For this economy the feasible utility set is the line segment from (0,1) to (1,0) in $\mathbb{R}^2_+$, and the Pareto utility frontier is an arbitrary closed subset of that segment containing the end points. [See [16] pp. 47-55 for details of this construction.]

We consider the case of two consumers and one commodity. The admissible consumption set of each agent will be the non-negative real line, thus satisfying Assumption Ib, and the total endowment of the commodity will be a positive real number $w = w_e$ in the environment e. Preferences of consumers will be described by a real valued continuous function u^i, taking non-negative values on the non-negative reals. Then, the set of feasible allocations consists of all $x^1 \geq 0$, $x^2 \geq 0$ such that $x^1 + x^2 = w$. This is the usual Edgeworth Box situation.

We note that Assumptions VIa and VIb are satisfied. We may without loss of generality take $w = 1$. Then the feasible set $\mathcal{J}(e)$ consists of the line segment in $R \times R$ joining the point $(0,1)$ to $(1,0)$. Now, let $f^i: [0,1] \to R_+$ $i = 1,2$ be continuous functions and let $f(t) = (f^1(t), f^2(t))$ for $t \in [0,1]$. Let ξ be the image of $[0,1]$ in R_+^2 under f, so that ξ is a curve in R_+^2.

Now, for $x \in [0,1]$ set $u^1(x) = f^1(x)$ and $u^2(x) = f^2(1-x)$. Then for a feasible allocation (x^1,x^2), the utility image (u^1,u^2) of $(x^1,1-x^1)$ is $(u^1(x^1),u^2(1-x^1)) = (f^1(x^1),f^2(x^1))$, and hence the utility image $U(e)$ of the feasible set $\mathcal{J}(e)$ is precisely the curve ξ. Thus, in order to specify a possible utility image $U(e)$ of an economy e (with continuous utility functions) we need only specify a continuous image of the unit interval.

We then construct an economy which has as the utility image of its feasible set the line segment L from the point $(1,0)$ to $(0,1)$ in R_+^2 and whose Pareto frontier consists of the line segment from $(1,0)$ to $(0,1)$ with an open interval S deleted. This may be done by choosing u^1 and u^2 to be piecewise linear and weakly monotone, each being constant on a subinterval of $[0,1]$. The line segment parameterized by $g(t) = (t, 1-t)$ $(0 \leq t \leq 1)$ is easily seen to have as image of $(0,1)$ under (u^1,u^2), the graph shown in Figure 7.1.

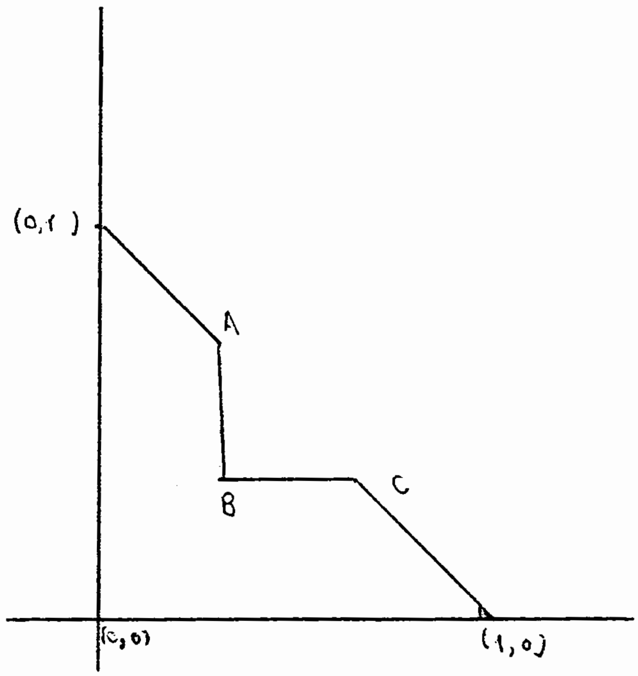

Figure 7.1

The basic idea of the construction is to use the method pictured in Figure 7.1 to eliminate an open interval from the Pareto set, defining a pair of functions on the unit interval which do the job. By composing such functions we can eliminate from the Pareto set all the open intervals of L - S. It remains only to take care of the technical details ensuring that the infinite composition of continuous functions yields a (pair of) continuous functions. We do this using the length of subintervals to be eliminated to index the functions, constructing a uniformly convergent sequence of continuous functions, thus ensuring that the limit function is continuous.

Section 8: A Class of Economies Whose Pareto Frontier Correspondence Is Not Threaded Through All Points

The example referred to in Section 7 shows that the assumption of strict monotonicity of preference is indispensable for the construction used in Lemma 5.1 to define a thread of the Pareto utility correspondence. However, failure of the construction used in our proof does not necessarily entail non-existence of a thread. It is clear, for instance, that if the utility functions in Example 5.1 are made continuous in the environments, the continuity and single-valuedness of the Pareto correspondence in that example would ensure the existence of a thread, despite the fact that the relative boundary of the feasible utility set and the Pareto frontier are not the same, and hence that the ray construction fails. The existence of threads means that for any economy e_o and any utility allocation p_o which is Pareto for e_o, there must exist a global thread s of the Pareto correspondence such that $s(e_o) = p_o$. However, we have constructed an example of a class of exchange economies with two agents and one commodity, with weakly monotone utilities which contains an economy with a Pareto point through which no thread exists. [This example is presented in detail in [16], pp. 56-74].

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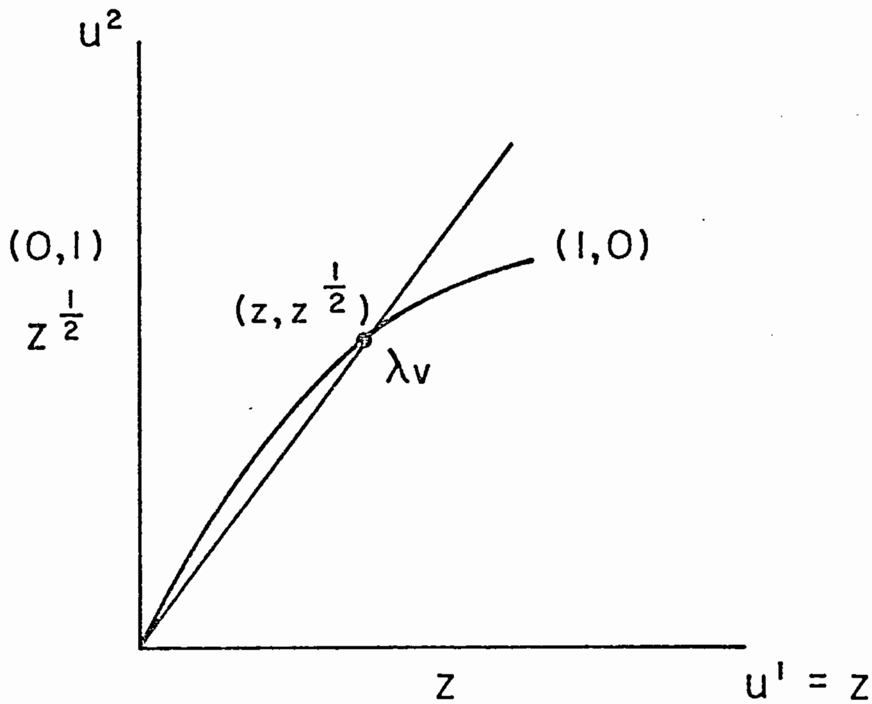

Figure 5.1

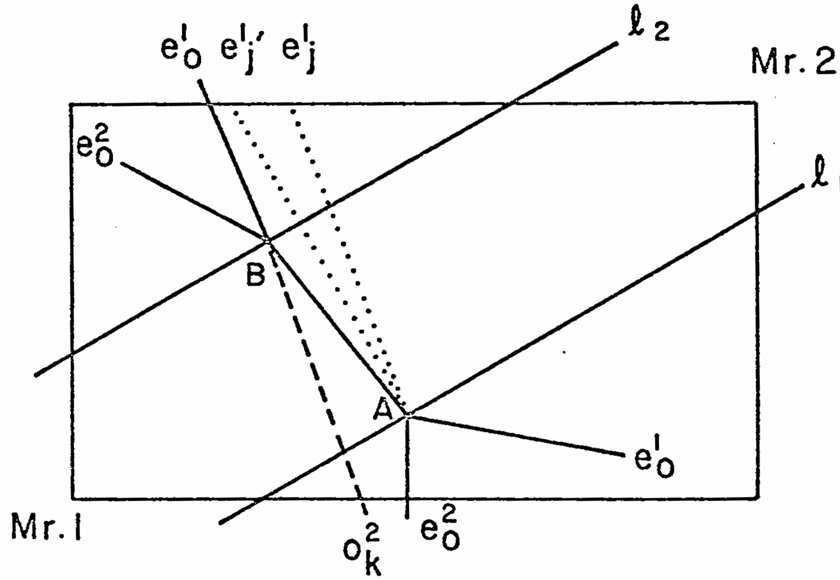

Figure 6.1