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"A Theory of the Firm with Non-Binding Employment Contracts"

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A THEORY OF THE FIRM WITH NON-BINDING EMPLOYMENT CONTRACTS

by

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Abstract

The purpose of this paper is to develop the theory of the firm to get better understanding of situations in which individual employees enjoy some bargaining power in their relations with the firm, and in which the terms of employment are determined and adjusted through individual contracting and recontracting with the firm. The main elements of the situations studied here are that the employees are not organized, that the employment contracts are non-binding or at least not for very long, and that the firm has opportunities to replace employees.

The paper develops analyzes a <u>dynamic</u> model in which the processes of contracting and re-contracting between the firm and its employees are intertwined with the dynamic evolution of the firm's workforce. The analysis of the model is somewhat complicated because the employment level is a non-degenerate state variable that evolves over time and is affected by past decisions.

The main analytical results characterize certain important equilibria: the profit maximizing, profit minimizing and stationary equilibria. The unique stationary equilibrium is markedly inefficient: it exhibits inefficient over-employment and the steady state wages coincide with the workers' reservation wage. It confirms earlier results derived by Stole and Zwiebel (1996a,b) in the context of a static model and shows that they are very robust even when the firm has nearly frictionless hiring opportunities. In contrast, the profit maximizing equilibrium captures a very different pattern. The outcome is nearly efficient and the wage exhibits a mark-up over the reservation wage. The path of the wages exhibits an interesting behavior--it declines sharply when it reaches its steady state level.

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1. Introduction

The purpose of this paper is to develop the theory of the firm to get better understanding of situations in which individual employees enjoy some bargaining power in their relations with the firm, and in which the terms of employment are determined and adjusted through a process of individual contracting and recontracting with the firm.

Obviously, the general situation addressed here is widespread. In many or even most firms, the terms of employment of certain key employees are determined in direct bargaining between the employee and the firm. But, more importantly, there are many firms in which relations of this form encompass a substantial subset of a firm's workforce. The latter category includes firms that employ highly skilled labor such as high technology firms, law-firms, universities, and perhaps even small firms that employ less skilled labor. These employment relations are clearly distinct from those described by the standard wage setting firm's model or the unionized firm's model.

Despite the obvious relevance of employment relations of this form, they have not been discussed extensively in the theoretical literature. The only analysis that I am aware of is presented by Stole and Zwiebel (1996a,b)[1] (henceforth referred to as SZ). They develop a model of a firm that contracts with each of its employees through separate bilateral negotiations. Their analysis takes into account the intricate interdependencies among the simultaneous bilateral relationships of the different individuals with the firm. They derive a variety of insights that demonstrate the importance of this approach. The central insights are perhaps that there is substantial over-employment (in the sense that the value of the marginal product of labor is well below the opportunity cost of labor), and that, despite the bargaining, the wage is driven down to its competitive level. In some sense the firm is maintaining a "reserve army" of inefficiently employed to reduce the bargaining power of their colleagues. Perhaps the main limitation of their theory is that the hiring decision of the firm is made once at the beginning, so in particular workers who might leave in disagreement cannot be replaced. This feature might have significant consequences in a bargaining situation in which the threats of quits and terminations play a major role.

The departure point of the present paper is the recognition that this theory should be completed to include considerations of employee substitution. The main concern is not about the imperfect realism of the model itself, but rather regarding the extent to which this theory is valid in an environment with an access to a labor market, which is the sort of environment we are normally interested in. The model of this paper expands SZ's basic framework to incorporate this missing element. This <u>necessitates</u> the construction of a <u>dynamic</u> model in which the processes of contracting and re-contracting between the firm and its employees are intertwined with the dynamic evolution of the firm's workforce.

In practice, the employment relations we want to study appear in a wide variety of forms, and the theory discussed here will naturally have to abstract of many of the features characterizing realistic situations. It will focus on situations with the following features. The contracts between the firm and its employees are <u>non-binding</u> or at least not binding for very long, in the sense that they can be repeatedly renegotiated to adjust to changing situations; the employees are <u>"replaceable"</u> but their replacement is not frictionless in the sense that it requires time and effort. The assumption that contracts are non-binding is obviously an abstraction, but it is not as strong as it might seem at first (it is probably more appealing than the alternative extreme of fully binding contracts). First, the degree to which employees could be bound by contracts is greatly limited both by law and by custom. Second, since only a few of the many dimensions of employment terms are contracted upon formally, the degree to which employers can be bound by long term contracts is also rather limited[2].

We present what is perhaps the simplest model that still captures the essence of this interaction. The focus of this paper is not on the details of the negotiations, but rather on the significant consequences of these circumstances for employment and output, and the model is constructed with this in mind. The model features a firm which uses labor as its only variable input. Two processes take place over time. First, a stochastic process brings new potential employees to the firm and, second, the wages of the employees are constantly renegotiated. In the absence of binding contracts, the firm and a worker can only agree on the worker's current wage rate. But in so doing they take into account the expected value of their continued relations. The bargained wage splits equally the surplus of this value over the value of the disagreement outcomes. The disagreement value for the worker is the value of his outside opportunities; the disagreement value for the firm is the continuation value of the hiring and renegotiation processes without the quitting worker. The behavior of the firm is described by an employment policy which specifies its employment decisions as a function of the history of the process. An equilibrium employment policy is such that the firm does not wish to deviate after any history. This model has multiple equilibria. We characterize the equilibria that maximize and minimize the firm's profit (for any initial pool of employees) and the (unique) stationary equilibrium.

These equilibria crystalize different patterns of interaction that might arise in this situation. The stationary equilibrium captures the behavior of a "mean" firm which keeps its wages low by excessive employment. This equilibrium reaches a steady state that coincides with the equilibrium outcome in SZ, so it confirms the theory of SZ fully. Furthermore, this equilibrium is very robust and its steady state outcome is independent of how fast the firm can replace departing employees. The maximum profit equilibrium captures the behavior of a "nice" firm which does not attempt to depress wages through excessive hiring. The employment level ends up being nearly efficient or somewhat lower than the efficient level. In this equilibrium, starting from a low initial pool of employees, the behavior of the wage as a function of employment fits into <u>two distinct phases</u>. In the build-up phase in which the employment level is short of its steady state level, the wage is relatively high. In the terminal phase, after employment reached its steady state level, the wage is distinctly lower (in the continuous labor limit of the model, the wage function is discontinuous at this point). This wage profile owes to the absence of binding wage contracts. Earlier hires have a stronger position, but since they anticipate their wages to be renegotiated downwards as more workers are hired, they have to receive a relatively higher wage initially. The wage paid in the terminal phase, after the steady state employment level was reached, exhibits a <u>mark-up</u> over the workers' reservation wage which represents their alternative opportunities. Since the employment level is determined in the model, this observation is not a straightforward consequence of the fact that wages are determined in bargaining and indeed it does not appear in the SZ model. The significance of this insight is in pointing out another source of wage mark-up to the existing list that includes efficiency wage and unionized labor. Finally, unlike the stationary equilibrium, the profit maximizing one is affected importantly by how easy it is for the firm to replace quitting employees. When the firm can do it quickly, the steady state outcome reached by this equilibrium coincides with the efficient equilibrium outcome of the standard model of a firm facing an infinitely elastic labor supply curve at the workers' reservation wage.

The insights reported above can be traced clearly to the fundamental ingredients of the situation we study. They are therefore quite robust. Their robustness is also illustrated by the two variations on the model that are briefly outlined[3] in Section 7.

This paper should be probably classified as a contribution to "applied theory." Namely, the focus is on modeling an economic situation, rather than on the introduction of new methodological advances. At the same time, it should be noted that the model has essentially the structure of a stochastic game with a non-degenerate state variable. This family of games is not as well mapped as, say, its sub-family of repeated games, and, consequently, the equilibrium analysis here is not an immediate corollary of existing work. So, the methodological contribution, if any, lies in the provision of an interpretable and somewhat rich example with such structure, which is sufficiently tractable to produce sharp characterization results.

The plan of the paper is as follows. Section 2 presents the basic model. Section 3 contains preliminary analysis and introduces a certain class of simple equilibria. Section 4 presents the main equilibrium analysis: characterization of the maximum profit, minimum profit and stationary equilibria. Section 5 derives the limit equilibrium outcomes for the case in which workers are negligible relative to the size of the firm. Section 6 discusses the insights emerging from the equilibrium analysis. Section 7 reviews briefly two extensions of the basic model. Section 8 contains a few remarks on the modelling. Section 9 brings concluding remarks.

2. The Model

This model considers a firm that uses labor as its only variable input and workers who are identical in their preferences and productivity. The events in the model take place over time. The time dimension is discrete and denoted by $t=1,2,\cdots$. The significance of a period is captured by a common discount factor $\delta\in(0,1)$. Following are the main ingredients of the model and the solution concept.

Technology: The total present value of the output generated by perpetual employment of n workers is $F(n)$; the value of the output at any period in which the firm employs n workers is $f(n)=(1-\delta)F(n)$. F and hence f are increasing and concave. The analysis below will be conducted only in terms of f, but we introduce the dependency of f on δ in this manner to make sure that, when δ is varied and the length of the period changes, the production varies with it appropriately.

Payoffs: A worker's von-Neumann-Morgenstern utility, evaluated in the beginning of period t, of being employed (by this firm) in periods $t,\ldots,T$ at wages $w_t,\ldots,w_T$ and not after that is

$$\sum_{s=t}^{T}\delta^{s-1}w_s+\delta^{T-t+1}W_U \ ,$$

where W_U is a worker's (exogenously given) utility of not being employed by this firm. We shall assume that, once a worker is separated from the firm, he will not be employed by it again, so that only employment profiles of the above form are relevant.

The firm's profit given a profile $\{n_t, e_t\}_{t=1,\ldots,\infty}$ of employment levels n_t and expenditures e_t is

$$\sum_{t=1}^{\infty} \delta^{t-1}[f(n_t)-e_t]$$

The magnitude $(1-\delta)W_U$ will be referred to as the reservation wage of the workers, since the worker's utility of being employed in perpetuity at this wage, $w/(1-\delta)$, is just equal to the utility of being unemployed, W_U. To assure that the model is not degenerate, it is assumed that $f(1)>(1-\delta)W_U$.

The evolution of the workforce: At the beginning of period t, the firm faces a pool of m_t potential employees consisting of the n_{t-1} who were employed in period t-1 plus at most one new prospective employee who may arrive through an exogenous arrival process. The per-period probability of a new arrival is α, up to a large upper bound M,

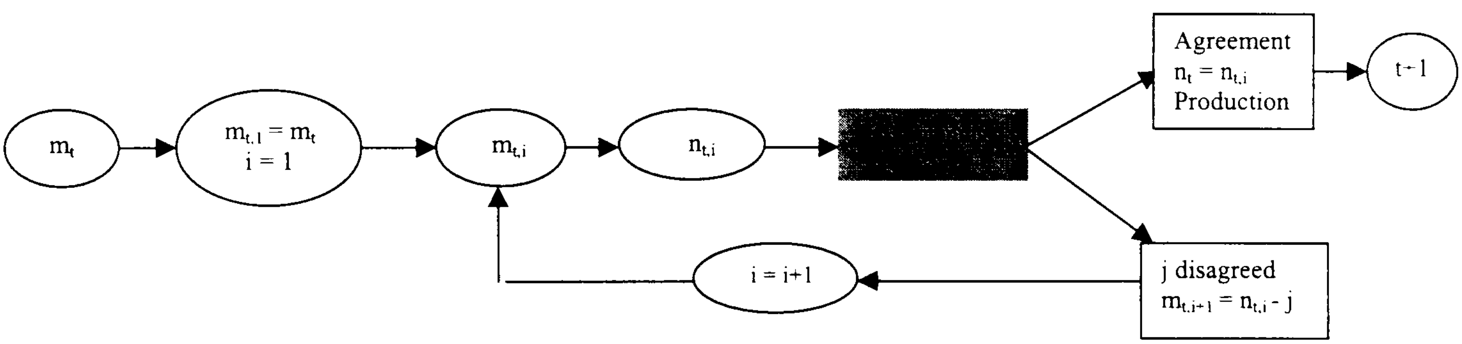
$$\text{Prob}(m_t|n_{t-1}<M) = \begin{cases} \alpha & \text{if } m_t=n_{t-1}+1 \\ 1-\alpha & \text{if } m_t=n_{t-1} \end{cases}$$

If $n_{t-1}=M$, then $m_t=n_{t-1}$. M is assumed very large[4], in particular, for some m well below M, $f(m)<m(1-\delta)W_U$.

The speed of arrival depends on the probability α as well as on the "length of a period." Therefore, the speed of arrival will be measured here by the ratio $\alpha/(1-\delta)$.

The flow of events: Roughly speaking, the events within any given period t proceed as follows. Out of the m_t potential employees, the firm chooses those it actually wants to employ in period t. Then the wages are determined in bilateral bargaining between each worker and the firm. Finally, production takes place, wages are paid and the period ends. The bargaining component is not modelled as a non-cooperative game, but is rather left in a black box which produces the bargaining outcome according to a rule that will be specified later.

The above description is incomplete, since in order to describe the bargaining outcome, we have to introduce the possibility of disagreement. This requires the following more elaborate description of the "off the path" developments.



As before, out of the $m_t=m_{t,1}$ potential employees, the firm picks $n_{t,1}$ at the start. If the bargaining phase is concluded with no disagreement, the firm proceeds to the production phase with $n_t=n_{t,1}$ employees. If the bargaining results in disagreement with some j employees, these j employees depart and the firm is facing $m_{t,2} = n_{t,1}-j$ employees whose terms have to be renegotiated. Before entering again the bargaining phase, the firm may further adjust this pool to $n_{t,2} \leq m_{t,2}$ and so on.

Notice that an employment decision is always followed by bargaining (as opposed to the alternative of first negotiating the wages with the pool of potential employees and then deciding on employment). This is an important assumption which captures the essence of what we mean by non-binding contracts. It is discussed further in Section 8 below.

__History__: A history records the evolution of the workforce including all the information described in the diagram. The component added to history at period t is of the form $(m_{t,1},n_{t,1},\ldots,m_{t,k_t},n_{t,k_t})$, where $m_{t,i}\geq n_{t,i}>m_{t,i+1}$ so that the number of rounds, k_t, satisfies $1\leq k_t\leq m_{t,1}$. A possible history in period t is a sequence of the form

$h^- = (m_{1,1},n_{1,1},\ldots,m_{1,k_1},n_{1,k_1}),\ldots,(m_{t,1},n_{t,1},\ldots n_{t,i-1},m_{t,i})$, or

$h^+ = (m_{1,1},n_{1,1},\ldots,m_{1,k_1},n_{1,k_1}),\ldots,(m_{t,1},n_{t,1},\ldots,m_{t,i},n_{t,i})$

where in addition to the above properties, for $t\geq 2$, $m_{t1} = n_{t-1,k_{t-1}}$ or $n_{t-1,k_{t-1}}+1$. We will continue to use m_t and n_t to refer to the initial number, $m_{t,1}$, and the terminal number, n_{t,k_t}. Thus, a history of type h^- precedes an employment decision; a history of type h^+ precedes the bargaining phase. Let H^-, H^+ denote the sets of histories of the two types and let $H=H^-\cup H^+$. Let $\ell(h)$ denote the last number in the sequence $h\in H$.

Observe that the history records only employment levels. In particular, it does not record wages. This is supposed to capture a situation in which changes in employment are publicly observable, but wages are not. We will return to discuss this point in Section 8 below.

__Policy (employment policy)__: A policy is a function ν: $H^- \to \{0,\ldots,M,C\}$. $\nu(h) \in \{0,\ldots,M\}$ means that the firm hires $\nu(h)$ workers and hence $\nu(h)\leq\ell(h)$; $\nu(h) = C$ means that the firm closes down and ceases to operate.

The policy is the counterpart of a strategy in a non-cooperative game. The distinction between $\nu(h)=0$ and shutdown is that in the former case the firm can continue hiring.

It is assumed that, when an employment policy is implemented, the following rules apply: (i) **Precedence of continuing employees**: if at the beginning of t the firm retains fewer than the m_t potential employees, the n_{t-1} continuing employees have precedence over a new arrival. (ii) **Equal retention probabilities**: If $n_{t,i}<m_{t,i}$, the retention probabilities are $n_{t,i}/m_{t,i}$ if $i>1$ or $t=i=1$; $n_{t,1}/n_{t-1,k_{t-1}}$ if $t>1$ and $i=1$.

Property, (i), is a simple way to build into the model a plausible feature, without imposing additional structure such as training costs that would rationalize this assumption. Its removal would not change the analysis qualitatively. Property (ii) just gives further content to the symmetry of this model with respect to the workforce.

__Wage function__: A function $w\colon H^+ \to R$.

$w(h)$ describes the wage that arises in the bargaining phase after the history h. At the moment no restrictions are placed on the function w, but we will soon impose on w a condition motivated by the underlying bargaining.

__Value functions__: Given a policy-wage function pair (ν,w) and $h\in H^+$, let:

$W(h;\nu,w)$ = An employee's expected utility after the history h.

$\Pi(h;\nu,w)$ = Expected profit after h.

$$(2.1)\quad W(h;\nu,w) = w(h) + \delta\alpha\{\min[1,\nu(h,\ell(h)+1)/\ell(h)]W[h,\ell(h)+1,\nu(h,\ell(h)+1);\nu,w]+ (1-\min[1,\nu(h,\ell(h)+1)/\ell(h)])W_U\} + \delta(1-\alpha)\{\min[1,\nu(h,\ell(h))/\ell(h)]W[h,\ell(h),\nu(h,\ell(h));\nu,w] + (1-\min[1,\nu(h,\ell(h))/\ell(h)])W_U\}$$

If $\ell(h)=C$, then $\Pi(h;\nu,w)=0$. If $\ell(h)\neq C$, then

$$(2.2)\quad \Pi(h;\nu,w) = f(\ell(h)) - \ell(h)w(h) + \delta\{\alpha\Pi[h,\ell(h)+1,\nu(h,\ell(h)+1);\nu,w] + (1-\alpha)\Pi[h,\ell(h),\nu(h,\ell(h));\nu,w]\}$$

Thus, $W(h;\nu,w)$ consists of the current wage, $w(h)$, plus the discounted value of continuation, which is either the value of W in the next period or W_U, depending on whether the worker is retained. Both the future value of W and the retention probability may depend on whether a new worker will arrive, and this explains the separate terms multiplied by $\delta\alpha$ and $\delta(1-\alpha)$ on the RHS of (2.1). Equation (2.2) is explained similarly.

The purpose of the following condition is to introduce into the model the idea that the wage is determined in bargaining.

__Equal split bargaining condition__:

Given ν, the wage function w satisfies __equal split__ if for all $h\in H^+$

$$(2.3)\quad W(h;\nu,w) - W_U = \max\{\Pi(h;\nu,w) - \Pi(h,\ell(h)-1,\nu(h,\ell(h)-1);\nu,w),0\}$$

When $\Pi(h;\nu,w) \geq \Pi(h,\ell(h)-1,\nu(h,\ell(h)-1)$, this condition says that the wage is determined so as to equate the worker's utility gain from continued employment (the LHS) to the firm's gain from retaining this worker (the RHS). But, if for some reason, the firm insists on unprofitably keeping workers even though $\Pi(h;\nu,w) < \Pi(h,\ell(h)-1,\nu(h,\ell(h)-1)$, the workers cannot be bargained to below their reservation values, so in such a case the wage will only fall down to equate $W(h;\nu,w)$ to W_U. As was mentioned before, the bargaining phase is not modelled explicitly, and this condition only characterizes its outcome. We shall return to discuss this issue in Section 8 below.

Equilibrium: The pair (ν,w) is an equilibrium if for all $h\in H^-$ and $j\leq\ell(h)$

(2.4) $$\nu(h) \in \mathrm{Argmax}_{n\in\{0,\ldots,\ell(h),C\}}\Pi(h,n;\nu,w)$$

(2.5) $$\Pi(h,\nu(h);\nu,w) \geq \Pi(h,\nu(h),j,\nu(h,j);\nu,w)$$

where the functions Π, W satisfy (2.1-3).

Let $\mathscr{E}$ denote the set of all equilibria. Condition (2.4) requires that after any history, h, it is in the best interest of the firm to choose employment level (or shutdown) $\nu(h)$, given the common expectation that future employment and wages will continue to be governed by ν and w. (2.5) requires that, after an employment decision prescribed by the equilibrium policy, it is more profitable for the firm to have the expected agreement with these employees than to have disagreement with some of them. Condition (2.5) is based on the premise that, in the unmodelled bargaining phase, the firm can force disagreement with any number of its employees.

Since the workers do not take any action in the above model, it might be supposed that the modeled situation is a decision problem rather than a strategic equilibrium problem. But, in fact, this a genuine equilibrium problem. The expectations of the workers regarding the future determine $W(h;\nu,w)$ which enters the bargaining condition (2.3) and in turn affects the wages. The workers' missing moves are hidden in the bargaining black-box.

3. Preliminary Analysis

The equilibrium analysis is developed in three steps. The first is characterization of the wage and profit functions arising for a class of simple employment policies. The second step presents the equilibria sustainable by the policies of this class. The third step characterizes the extremal equilibria in the class of all policies.

3.A. Wage and profit with simple employment policies

A stationary N policy is ν such that $\nu(h) = \mathrm{Min}[\ell(h),N]$, for all $h\in H^-$.

That is, when following such policy, the firm always aims at reaching N in the fastest way, regardless of the history. Given a stationary N policy ν and a wage function w such that w(h) depends only on $\ell(h)$ and N, i.e., $w(h)=w(\ell(h);N)$, we also have $W(h;\nu,w)=W(\ell(h);N)$ and $\Pi(h;\nu,w)=\Pi(\ell(h);N)$. For $h\in H^+$ s.t. $\ell(h)=n\leq N$, system (2.1-3) can be rewritten as follows.

$$(3.1)\quad W(n;N) = w(n;N) + \delta[\alpha W(\min\{n+1,N\};N) + (1-\alpha)W(n;N)],\quad 1\leq n\leq N$$

$$(3.2)\quad \Pi(n;N) = f(n)-w(n;N)n + \delta[\alpha\Pi(\min\{n+1,N\};N)+(1-\alpha)\Pi(n;N)],\quad 0\leq n\leq N$$

$$(3.3)\quad W(n;N) - W_U = \mathrm{Max}[\Pi(n;N) - \Pi(n-1;N),0]\quad 1\leq n\leq N$$

A solution to system (3.1-3) is an assignment of values to $w(n;N)$, $W(n;N)$, $1\leq n\leq N$, and to $\Pi(n;N)$, $0\leq n\leq N$, which satisfy the system. A solution is feasible if it satisfies $\Pi(n;N)\geq\Pi(n-1;N)\geq 0$, for $1\leq n\leq N$. If a feasible solution exists, the associated N is also called feasible. Notice that feasibility assures that the firm indeed wants to follow the prescription of the stationary N policy at levels $n\leq N$.

Let $\Delta f(n) = f(n) - f(n-1)$, the value of the marginal product. Let

$$\Psi(n) = \frac{2}{n(n+1)}\sum_{i=1}^{n} i\Delta f(i) .$$

Since $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$ the term $\frac{2}{n(n+1)}\sum_{i=1}^{n} i\Delta f(i)$ is a weighted sum of the marginal products with higher weights assigned to the marginal products of later units[5]. Since f is concave, Ψ is a decreasing function of n. The function Ψ will play an important role in the model. In a static version of this model in which the firm hires n workers once and for all (the SZ model), $\Psi(n)$ captures the gross surplus (gross of the worker's opportunity cost $(1-\delta)W_U$) to be shared by the firm and a worker. While the direct contribution of a worker is $\Delta f(n)$, there is also an indirect contribution in depressing the wages of the others, which explains the appearance of the infra marginal productivities (if the worker quits, the remaining workers will renegotiate the wage to reflect their increased contribution).

Let N^0 denote the maximal n such that $\Psi(n)\geq(1-\delta)W_U$. Since $f(1)>(1-\delta)W_U$, such a positive N^0 exists. Assume that $\Psi(N^0)$ is <u>exactly</u> equal to $(1-\delta)W_U$. This assumption will simplify the presentation of later arguments by avoiding "integer problems."

<u>Proposition 1</u>: (i) N is feasible iff $N\leq N^0$

(ii) For $N\leq N^0$, the unique feasible solution to (3.1-3) is

$$(3.4)\quad w(n;N) = \begin{cases} [\Psi(n)+(1-\delta)W_U]/2 & n<N \\ [(N+1)(1-\delta)\Psi(N)+[(N+1)(1-\delta)+2\delta\alpha](1-\delta)W_U]/2[(N+1)(1-\delta)+\delta\alpha] & n=N \end{cases}$$

The proof (and all subsequent proofs in this section) are relegated to Appendix 1. Both at n<N and at n=N, the wage w(n;N) is a weighted average of $\Psi(n)$ and the reservation wage $(1-\delta)W_U$, with a higher weight on $(1-\delta)W_U$ when n=N. Since $\Psi(n)$ is monotonically decreasing in n, so is w(n;N). Due to the change in weights at n=N, the downward jump of w(n;N) is larger than it would be if w(n;N) were governed by the same formula at n<N and n=N.

Since $\Pi(N;N)=[f(N)-w(N;N)]/(1-\delta)$, (3.4) also yields an explicit expression for $\Pi(N;N)$. Let $\pi(n)$ denote the profit flow accruing to a firm who employs n workers at the workers' reservation wage $(1-\delta)W_U$,

$$\pi(n) = f(n) - n(1-\delta)W_U.$$

In other words, π is the profit flow in the standard benchmark case of a <u>wage setting</u> or a <u>neo-classical</u> firm.

$$(3.5)\quad \Pi(N;N)=\frac{1}{1-\delta}\left\{\frac{\delta\alpha}{(N+1)(1-\delta)+\delta\alpha}\pi(N)+\frac{(N+1)(1-\delta)}{(N+1)(1-\delta)+\delta\alpha}\left[\frac{1}{N+1}\sum_{i=1}^{N}\pi(i)\right]\right\}$$

Thus, $(1-\delta)\Pi(N;N)$ -- the flow equivalent of $\Pi(N;N)$ -- is a weighted average of $\pi(N)$ and the arithmetic average of $\pi(i)$ for $i=0,\ldots N$.

Figure should be placed about here

The figure depicts the relationship between $(1-\delta)\Pi(n;n)$, $\pi(n)$ and $\frac{1}{n+1}\sum_{i=1}^{n}\pi(i)$. For convenience we draw continuous curves although these are actually discrete points. The concavity of $\pi(n)$ assures that the two solid curves are as shown. $\pi(n)$ is maximized at one or two adjacent n's. Without any loss we assume that it is maximized at a single point: $N^s = \text{Argmax}\pi(n)$. Clearly, $N^S<N^0$ since N^S is the largest n such that $\Delta f(n)\geq(1-\delta)W_U$. N^* denotes the smallest maximizer of $\Pi(0;n)$. Since for $n<N$, $w(n;N)$ is independent of N, N^* also maximizes $\Pi(n;N)$, for any $n\leq N^*$. The following proposition confirms the features shown in the figure.

<u>Proposition 2</u>: (i) The curves π, $(1-\delta)\Pi(n;n)$ and $\frac{1}{n+1}\sum_{i=0}^{n}\pi(i)$ intersect at N^0, where the latter has its unique maximum. (ii) $\text{Argmax}\Pi(n,n) = N^{**}$ or $\{N^{**},N^{**}+1\}$. (iii) $N^*\leq N^S\leq N^{**}<N^0$.

In fact, it is easy to see that, when the contribution of the individual worker is not too large relative to the total production at N^s, then the above inequalities are strict, $N^*<N^S<N^{**}$. Notice that the observation that the maximizer of $\Pi(0;n)$ is strictly smaller that the maximizer of $\Pi(n;n)$ is somewhat surprising. It owes to a certain dynamic effect that will be discussed later.

3.B. Equilibria in N policies

Section 4 below brings comprehensive equilibrium analysis. Before turning to it, we present here special equilibria in which the behavior is of the simple variety described above (i.e., there is a level N that the firm plans and everybody expects it to reach and retain thereafter), and the wages and profits along the paths of these equilibria are as summarized by (3.4-5). The purpose of this part is both to give some idea on how the model works and to introduce equilibria that play an important role in the subsequent analysis.

Recall the stationary N policies defined above. A stationary N equilibrium is a regular equilibrium in which the policy happens to be a stationary N policy.

Proposition 3: There exists a unique stationary N equilibrium. In it $N=N^0$.

The stationary N^0 equilibrium is sustained by the following considerations. At $n<N^0$, the hiring of an additional worker is profitable, since the worker's combined contribution to production, $\Delta f(n+1)$, and to the reduction of the total wage bill (recall the declining wage profile between n and N^0), exceeds the firm's cost of employing this worker. At $n>N^0$, the workers have to be paid at least their reservation wage $(1-\delta)W_U$, since the firm is expected to return to N^0 where the workers' continuation utility will be W_U. This means that hiring beyond N^0 does not reduce the total wage bill, so it must be unprofitable as $\Delta f(n)<(1-\delta)W_U\leq w(n;N)$.

Consider next a somewhat broader class of policies. Let $n_{max}(h)$ denote the maximal employment decision (even numbered coordinate) in the history h.

Definition: A quasi-stationary N policy is ν such that, for any h with $n_{max}(h) \leq N$, $\nu(h) = \min[\ell(h), N]$.

That is, a quasi-stationary N policy is like a stationary N policy as long as N was not exceeded. Let $\hat{N}$ be the minimal n such that $\Pi(n;n) \geq \Pi(N^0;N^0)$.

Proposition 4: For each $N \in [\hat{N}, N^0]$ there exists a quasi-stationary N equilibrium.

The quasi-stationary N equilibria with $N<N^0$ are sustained by trigger policies. If the firm deviates and exceeds the original N, a punishment phase is triggered in which the stationary N^0 equilibrium is played. Consider an equilibrium with steady state employment level $N<N^0$. At $n<N$, continued hiring is profitable, just as was explained following Proposition 3 above. However, if at $n=N+1$, the firm was still expected to return to N, the argument given for the N^0 equilibrium would not work, since here $W(N;N)>W_U$ and the firm would achieve a reduction of the wage bill by such additional hiring. Instead, such additional hiring is deterred by triggering the less profitable N^0 equilibrium.

Recall that the speed of arrival of new employees is measured by the ratio $\alpha/(1-\delta)$. Notice, that the levels N^s and N^0 are independent of $\alpha/(1-\delta)$, but $\hat{N}$ depends on it. When $\alpha/(1-\delta)$ is sufficiently large, so that arrival is fast, $\hat{N} < N^*$. When $\alpha/(1-\delta)$ is relatively small, so that arrival is slow, $\hat{N}$ is close to N^0 and hence $N^s<\hat{N}$. In such a case N^s and N^* are not supported in this manner by triggering the N^0 equilibrium.

4. Further equilibrium analysis (all equilibria)

Obviously, this model has other equilibria outside the class discussed above. For example, a variety of complicated patterns can be sustained by using the N^0 equilibrium as a punishment phase. We now look at the set of all equilibria--not only those supportable by quasi-stationary policies. Specifically, we characterize the profit maximizing equilibria, the profit minimizing equilibria and the stationary equilibria. All the proofs for this section are collected in Appendix 2, but an intuitive exposition of the ideas of the main proofs are provided in 4.D below.

4.A. Preliminaries

Since the following analysis considers histories in more detail, it will be useful to review some relevant notation. First, recall that components of a history that refer to different periods are separated by brackets. Thus, $h=(8,8,7)$ refers only to the beginning of the first period, before the final employment for that period was determined: the initial number of potential employees was 8, the firm decided to hire all 8, but one of them dropped out through disagreement. In comparison, $h=[(8,8),8]$ covers the first period and the beginning of the second: the initial number was 8, who were also employed for the first period, and there was no new arrival in the second period. Both of these histories belong to H^- so both will be followed by an employment decision of the firm.

Second, let $(\nu,w)|_h$ denote the continuation prescribed by policy-wage pair (ν,w) after the history h in the obvious way.

Third, given (ν,w) and a history $h\in H^-$, let $\Pi^{\nu,w}(h)\equiv\Pi(h,\nu(h);\nu,w)$ and $W^{\nu,w}(h)\equiv W(h,\nu(h);\nu,w)$. Notice that argument of $\Pi^{\nu,w}$ is a history in H^-, while the argument of $\Pi(\cdot;\nu,w)$ is a history in h^+. This notation will help cut down the length of histories we have to actually write.

Next consider the set of equilibrium payoffs. Let V(m) denote this set, given an initial pool of m potential workers,

$$V(m) = \{(\Pi^{\nu,w}(m), W^{\nu,w}(m)) : (\nu,w)\in\mathscr{E}\}.$$

The next result is obviously needed before proceeding to characterization.

<u>Proposition 5</u>: For any m, V(m) is compact.

Thus, in particular, there exist equilibria which maximize (minimize) the expected profit, Π, and equilibria which maximize (minimize) employees' expected utility, W.

<u>4.B. Maximum profit equilibria</u>

Theorem I is the main result of this section. Its main messages regarding the maximum profit equilibrium are: when the arrival of new employees is sufficiently fast, this equilibrium reaches a steady state level in $[N^*,N^s]$, and for an initial $m\le N^*$, it coincides with the quasi-stationary N^* equilibrium; when the arrival is slow, this equilibrium essentially coincides with the stationary N^0 equilibrium.

Let $(\bar{\nu},\bar{w})$ denote a maximum profit equilibrium from among <u>all</u> equilibria. That is, for all m, $(\bar{\nu},\bar{w})\in \text{Argmax}\{\Pi^{\nu,w}(m) : (\nu,w)\in\mathscr{E}\}$. When there are multiple such equilibria, we will select one for which the initial employment decision, $\nu(m)$, is maximal, for each m. Also let $\bar{\Pi}(h)\equiv\Pi^{\bar{\nu},\bar{w}}(h)\equiv\Pi(h;\bar{\nu},\bar{w})$ and $\bar{W}(h)\equiv W^{\bar{\nu},\bar{w}}(h)\equiv W(h;\bar{\nu},\bar{w})$. Analogously, let $\underline{\Pi}(h)$ denote the minimum equilibrium profit after h, i.e., $\underline{\Pi}(h) = \text{Min}\{\Pi^{\nu,w}(h) : (\nu,w)\in\mathscr{E}\}$.

<u>Theorem I</u>: (i) <u>Equilibrium employment and profit</u>:

(a) There exists $\bar{N} \geq N^s$ such that, for $m \leq \bar{N}$, $\bar{\nu}(m) = m$ and

$$\bar{\Pi}(m) = \text{Max}_{j \leq m, k \leq m+1} \{ \pi(m) + m\bar{\Pi}(m-1) + \delta(1-\alpha)[\bar{\Pi}(j) + j[\bar{W}(j) - W_U]] + \delta\alpha[\bar{\Pi}(k) + \min(k,m)[\bar{W} - W_U]] \} / (m+1)$$

$$\text{s.t. } \bar{\Pi}(j) \geq \underline{\Pi}(m) \text{ and } \bar{\Pi}(k) \geq \underline{\Pi}(m+1).$$

For $m > \bar{N}$, $\bar{\nu}(m) = \bar{N}$ and $\bar{\Pi}(m) = \bar{\Pi}(\bar{N})$.

(b) For any $h \in H^+$ on the path of $(\bar{\nu}, \bar{w})$, $(\bar{\nu}, \bar{w})|_h = (\bar{\nu}, \bar{w})|_{(\ell(h), \ell(h))}$

(ii) <u>If the rate of new arrivals $\alpha/(1-\delta)$ is sufficiently large, then</u>:

(a) For $m \leq N^*$, $(\bar{\nu}, \bar{w})$ coincides with the quasi-stationary N^* equilibrium.

(b) For $m \in (N^*, N^s]$, $\bar{\nu}[(m,m),m] = m$, $\bar{\nu}[(m,m),m+1] \in \{m, m+1\}$, $\bar{\nu}[(N^s,N^s),N^s+1] = N^s$ (i.e., employment reaches a steady state level in $[m, N^s]$ and remains there).

(c) For $m \in (N^s, \bar{N}]$, $\bar{\nu}(m) = m$, $\bar{\nu}[(m,m),m] = \bar{\nu}[(m,m),m+1] = N^s$ (i.e., m are hired for one period; thereafter N^s are retained and $(\bar{\nu}, \bar{w})$ coincides with $(\bar{\nu}, \bar{w})|_{N^s}$).

(iii) <u>If the rate of new arrivals $\alpha/(1-\delta)$ is sufficiently small, then</u>:
For $m \leq N^0 - 1$, $(\bar{\nu}, \bar{w})$ coincides with the quasi-stationary N^0-1 equilibrium. For $m > N^0 - 1$, $(\bar{\nu}, \bar{w})$ coincides with it after at most one period, as in (ii)-(b),(c).

Theorem I-(i) gives the general structure of the profit maximizing equilibrium. It prescribes hiring the initial pool only if it does not exceed $\bar{N}$. It then continues to a profit maximizing equilibrium that maximizes the total net surplus (of the firm and its initial set of employees). Part (i)-(b) says that this equilibrium has an almost Markovian property in the sense that, from any employment level reached on the path, $(\bar{\nu}, \bar{w})$ restarts again, i.e., for $h \in H^+$, $(\bar{\nu}, \bar{w})|_h = (\bar{\nu}, \bar{w})|_{(\ell(h), \ell(h))}$. Notice, however, that we <u>cannot</u> say that $(\bar{\nu}, \bar{w})$ restarts after any history on the path. For $h' \in H^-$ on the path, the correct statement of the "almost" Markovian property is that the continuation equilibrium depends on the current pool and last period's employment: if $h=(m_1,n_1),\ldots,(m_{t-1},n_{t-1}),m_t$, then $(\bar{\nu},\bar{w})|_h = (\bar{\nu},\bar{w})|_{(m_{t-1},n_{t-1}),m_t)}$.

Part (ii) reports what is perhaps the central message of this theorem. When the arrival of new employees is sufficiently fast, the steady state employment level will settle in $[N^*,N^s]$. If the size of the initial pool is below N^*, the maximum profit is achieved by the quasi-stationary N^* equilibrium, i.e., for $h \in H^-$ on the path, $\bar{\nu}(h)=\min[\ell(h),N^*]$ and $\bar{\Pi}(m)=\Pi(m;N^*)$. Starting from an initial m in $(N^*,N^s]$, employment will remain at m or climb steadily to another steady state level in $[m,N^s]$. In this range, the equilibrium policies are the same as quasi-stationary N policies on the path, but are somewhat different off the path. If, for example, "off the path" of the equilibrium starting with $m=N^*+1$, employment fell to N^* due to a disagreement, the equilibrium policy will entail staying at N^* rather than returning to N^*+1, as an N policy would prescribe.

Starting from initial m's above N^s but not too large, the firm hires all m employees for one period, in the following period it cuts its workforce down to the efficient level N^s and thereafter remains there. The intuition is that, after settling at N^s, the retained workers are paid strictly more than their reservation wage. The firm extracts some of this surplus by employing everybody at relatively low wages (possibly below their reservation wage). The workers agree to such low wages for the chance of being retained after the subsequent reduction in the workforce. The case of initial m's above $\bar{N}$ is explained in the same way except that the initial number is so large that hiring all of them for one period is not profitable. The notion of hiring workers for a single period and then firing them is of course extreme. If the model was modified to include significant training costs or other restrictions, the firm would go directly to N^s without this intermediate step.

Part (iii) claims that, when the arrival of new employees is sufficiently slow, the maximum equilibrium essentially coincides with the quasi-stationary N^0-1 equilibrium. If a single worker is not too significant, this equilibrium is near the stationary N0 equilibrium. Together with Theorem II-(iii) below this implies that the entire equilibrium set collapses around the N^0 equilibrium.

4.C. Minimum profit equilibria

The minimum profit equilibrium is also Pareto inferior to other equilibria and generally seems as a less convincing prediction for behavior. For this reason, Theorem II below is somewhat less central for the substantive messages of this paper. It is presented mainly to complete the equilibrium analysis. So, a reader who is less interested in the equilibrium analysis itself, can skip this part without losing much in terms of the substantive discussion.

The analysis here requires another preliminary technical point. In a couple of points in the proof of Theorem II, the argument requires convexity of V(m). The problem is that V(m) need not be convex. The points for which the convexity is needed are relatively minor, so one possibility would be to just drop them. Alternatively, V(m) can be convexified by assuming that everybody observes the realization of a public randomizing device at the beginning of each round. The introduction of public signals involves no conceptual or technical difficulty, but it would complicate the notation. We will therefore adopt the following approach. The proof of Theorem II will be conducted under the assumption that V(m) is convex and we will indicate clearly where this property is used. A remark following the proof (in Appendix 2) takes up this point and explains how the randomization can be formally added, why all the arguments remain in tact, and what might be lost without it.

Let $(\underline{\nu},\underline{w})$ denote a minimum profit equilibrium from among <u>all</u> equilibria. That is, for all m, $(\underline{\nu},\underline{w}) \in \text{Argmin}\{\Pi^{\nu,w}(m): (\nu,w)\in\mathscr{E}\}$. When there are multiple such equilibria, we will select one for which the initial employment decision, $\nu(m)$, is maximal, for each m. Also, let $\underline{\Pi}(h)\equiv\Pi^{\underline{\nu},\underline{w}}(h)\equiv\Pi(h,\nu(h);\underline{\nu},\underline{w})$ and $\underline{W}(h)\equiv W^{\underline{\nu},\underline{w}}(h)\equiv W(h,\nu(h);\underline{\nu},\underline{w})$.

<u>**Theorem II**</u>: (i) <u>Equilibrium employment and profit</u>: There exists $\underline{N}\geq N^0$ such that, for $m<\underline{N}$, $\underline{\nu}(m)=m$, $\underline{\Pi}(m)$ is strictly increasing in m and

$$\underline{\Pi}(m) = f(m) - m\underline{w}(m,m) + \delta[(1-\alpha)\underline{\Pi}(m) + \alpha\underline{\Pi}(m+1)].$$

For $m\geq\underline{N}$, $\underline{\Pi}(m)=\underline{\Pi}(\underline{N})$.

(ii) <u>If the rate of new arrivals $\alpha/(1-\delta)$ is sufficiently large, then</u>: For $m\leq\underline{N}$, $\underline{\nu}[(m,m),m]=\underline{\nu}[(m,m),m+1]=0$. $\underline{\Pi}$ satisfies the following system:

$$\underline{\Pi}(m) = \{\pi(m) + m\underline{\Pi}(m-1) + \delta(1-\alpha)\underline{\Pi}(m) + \delta\alpha\underline{\Pi}(m+1)\}/(m+1).$$

(iii) <u>If the rate of new arrivals $\alpha/(1-\delta)$ is sufficiently small, then</u>: $(\underline{\nu},\underline{w})$ coincides with the stationary N^0 equilibrium.

Thus, when the initial pool of employees is smaller than $\underline{N}$, all are hired for the first period. When the arrival of new employees is fast, all employees are fired after the first period. Then the continuation from zero is with an equilibrium (ν,w) such that $\Pi^{\nu,w}(0)=\underline{\Pi}(m)$ or $\underline{\Pi}(m+1)$, depending on whether or not there was new arrival in the beginning of the second period. That is, the equilibrium $(\nu,w)|_0$ is more profitable than $(\underline{\nu},\underline{w})|_0$. The expected dismissal of all workers hardens their bargaining position in the first period by removing any expected future gains. This results in relatively high wages which minimize the profit.

Notice that the difference equations in (ii) allow to solve completely for the equilibrium profits and the number $\underline{N}$. This is done by solving this system for $m=0,\ldots N$, under the assumption $\underline{\Pi}(N)=\underline{\Pi}(N+1)$. Denoting this solution by $\Pi_N(m)$, $m=0,\ldots,N$, we have that $\underline{N}$ is the first N for which $\Pi_N(N)>\Pi_{N+1}(N+1)$, and $\underline{\Pi}(m)=\Pi_N(m)$ for all $m=0,\ldots N$.

Like in other models of dynamic games, the minimum profit equilibria have poor welfare properties, and do not seem to offer a convincing prediction for behavior.

4.D. Intuitive discussion

The precise intuition behind Theorems I and II is embodied in the proofs. The rough intuition relies on the positive relation between the firm's profit and the total surplus implied by the bargaining condition. Specifically, let $S_m(\nu,w)$ denote the expected total net payoff of the firm and its employees, evaluated at the beginning of a period, before the uncertainty about the new arrival was resolved, when the preceding period had m employees and when the continuation is expected to follow the equilibrium (ν,w) played from its start. That is,

$$S_m(\nu,w) = (1-\alpha)\{\Pi^{\nu,w}(m)+\nu(m)[W^{\nu,w}(m)-W_U]\} + \alpha\{\Pi^{\nu,w}(m+1)+\min[m,\nu(m+1)][W^{\nu,w}(m+1)]-W_U]\}$$

The appearance of $\min[m,\nu(m+1)]$ reflects the fact that, in the event of new arrival in the beginning of the next period, only the payoff of the original m employees is taken into account. Consider now an equilibrium (ν,w) such that $\nu(m)=m$. Using condition (2.3) to substitute out $w(m,m)$ in

$$\Pi^{\nu,w}(m) = f(m) - w(m,m)m + \delta\{(1-\alpha)\Pi^{\nu,w}[(m,m),m] + \alpha\Pi^{\nu,w}[(m,m),m+1]\},$$

we get the following key observation

$$(4.1) \qquad \Pi^{\nu,w}(m) = \{\pi(m) + m\Pi^{\nu,w}(m,m,m-1) + \delta S_m[(\nu,w)|_{(m,m)}]\}/(m+1),$$

where $\pi(m) \equiv f(m)-(1-\delta)W_U$ is the increment to the net total surplus associated with employment m and $(\nu,w)|_{(m,m)}$ is the continuation of (ν,w) after the history (m,m). Expression (4.1) is interpreted as follows. The surplus available for the bargaining, $\pi(m) + \delta S_m[(\nu,w)|_{(m,m)}] - \Pi^{\nu,w}(m,m,m-1)$, is split equally among the firm and the m workers. Hence the firm's payoff is 1/(m+1) of that plus its disagreement payoff, $\Pi^{\nu,w}(m,m,m-1)$, which amounts to the RHS of (4.1).

Now, the RHS of (4.1) is maximized when $\Pi^{\nu,w}(m,m,m-1)$ and $S_m[(\nu,w)|_{(m,m)}]$ are maximized. Clearly $\Pi^{\nu,w}(m,m,m-1)$ is maximized when the continuation after the history (m,m,m-1) coincides with $(\bar{\nu},\bar{w})|_{m-1}$ which gives the maximum possible profit starting from m-1. The total net future payoff, $S_m[(\nu,w)|_{(m,m)}]$, is bounded from above by $\pi(N^s)/(1-\delta)$, which is the maximum possible total payoff generated by this firm. This is achieved by an equilibrium that starts and remains perpetually at N^s. Therefore, for $m \geq N^s$ the profit maximizing equilibrium continues to N^s after one period. Starting from $m<N^s$ the equilibrium employment proceeds upwards in the direction of N^s for the same reason. In these cases, the equilibrium employment may not reach N^s owing to additional short run considerations that will be discussed later.

The analysis of the minimum profit equilibrium is understood analogously. $\Pi^{\nu,w}(m,m,m-1)$ is minimized of course by the continuation with $(\underline{\nu},\underline{w})|_{m-1}$, and $S_m[(\nu,w)|_{(m,m)}]$ is minimized by a continuation that reduces employment to 0 when this is possible in equilibrium (i.e., when there are equilibria whose total profit starting at employment 0 exceeds $\underline{\Pi}(m)$).

3.E. Features of the main equilibria

The N^0 equilibrium

In the model of SZ, on which the present model builds, there is once and for all hiring decision in the beginning. In the unique equilibrium of that model the employment level is N^0. My initial conjecture was that this outcome would not be supported as an equilibrium in the presence of substitution possibilities. As it turns out, the N^0 equilibrium has a very robust existence in the model. Following are some of its main properties.

(i) Stationarity: A stationary equilibrium (ν,w) is such that $\nu(h) = \nu(\ell(h))$, for all $h\in H^-$ and $w(h) = \nu(\ell(h))$, for all $h\in H^+$.

Proposition 6: The N^0 equilibrium is the unique stationary equilibrium.

This result strengthens the result of Proposition 3 which establishes N^0 as the only equilibrium in a stationary N policy.

(ii) Almost uniqueness for slow arrival rate: When the speed of arrival of new employees, $\alpha/(1-\delta)$, is small, the equilibrium set collapses around the N^0 equilibrium. The minimum profit equilibrium coincides with the N^0 equilibrium and the maximum profit equilibrium coincides with the quasi-stationary N^0-1 equilibrium. The latter equilibrium is near the former when N^0 is not too small.

(iii) Independence of $\alpha/(1-\delta)$: The wages along the path as well as the long run outcome (after the level N^0 was attained) are independent of α: N^0 itself as well as $W(N^0;N^0)$ and $\Pi(N^0;N^0)$ do not depend on $\alpha/(1-\delta)$. To see this recall that N^0 is the solution to $\Psi(n)=(1-\delta)W_U$, which is obviously independent of α, and $w(N^0;N^0)=(1-\delta)W_U$, $W(N^0;N^0)=W_U$ and $\Pi(N^0;N^0)=\pi(N^0)/(1-\delta)$ are also independent of α. Since $f(n)=(1-\delta)F(n)$, it follows that N^0 and the long run outcome,

$W(N^0;N^0)$ and $\Pi(N^0;N^0)$, are also independent of $(1-\delta)$.

__(iv) Inefficiency__: The N^0 equilibrium clearly does not maximize the total net surplus. This follows immediately from $N^0>N^s$ which uniquely maximizes the surplus. The N^0 equilibrium is also Pareto dominated by other equilibria. For example, the quasi-stationary N^0-1 equilibrium clearly yields higher profit for the firm and to any initial pool of employees. Since $W(N^0;N^0)=W_U$, so that workers are just indifferent to being employed, even for an initial pool $m\geq N^0$, no worker loses in the N^0-1 equilibrium relative to the N^0 equilibrium.

The N^* and N^s equilibria

These equilibria maximize the firm's profit starting from employment levels $m\leq N^*$ and $m\geq N^s$ respectively. The N^* equilibrium belongs to the family of quasi-stationary N equilibria examined in Section 3 but the N^s equilibrium does not.

The N^s equilibrium obviously generates the maximum possible total surplus, while the long run outcome of the N^* equilibrium is inefficient (since $N^*<N^s$). The reason that, despite the close relationship between profit and total surplus in this model, the N^* equilibrium is inefficient will be explained in Section 6 below. But, it is worth noting that, if arrival of new employees is fast, i.e., $\alpha/(1-\delta)$ is large, then N^* is near N^s.

When the arrival of new employees is slow, the levels N^* and N^s cannot be sustained as a steady state by any equilibrium.

5. Limit versions of the outcomes

Before turning to discuss the equilibrium results further, we derive two limit versions of the outcomes arising with N policies. The first version corresponds to the case in which the length of the time period is relatively short, so that decisions can be updated relatively quickly. The second captures a situation in which each worker is "small" relative to the size of the firm. The substantive insight emerging from the latter is that, when the individual worker is of negligible size, the outcomes remain essentially unchanged, using an appropriate sense of equivalence. This version also facilitates occasionally sharper exposition of results. A reader who will skip this section and go directly to the following discussion should not lose much in terms of the understanding.

<u>**The continuous time limit**</u>

Let the length of a time period be denoted by z. The parameters of the model now take the form

$$\delta(z) = e^{-rz} \quad \text{and} \quad \alpha(z) = 1-e^{-\alpha z},$$

so that r is the discount rate and α is the arrival rate (rather than a probability).

In the limit as $z \to 0$, system (3.1-3) becomes:

(5.1) $\Pi(n;N) = [f(n) - nw(n;N) + \alpha\Pi(\min\{n+1,N\};N)]/(r+\alpha)$ $\quad 0 \le n \le N$

(5.2) $W(n;N) = [w(n;N) + \alpha W(\min\{n+1,N\};N)]/(r+\alpha)$ $\quad 1 \le n \le N$

(5.3) $W(n;N) - W_U = \text{Max}[\Pi(n;N) - \Pi(n-1;N), 0]$ $\quad 1 \le n \le N$

The limit the versions of w(n;N), Π(n;N) and W(n;N) can be obtained either by solving (5.1-3) or by taking directly the limits of (3.4-5).

(5.4)
$$w(n;N) = \begin{cases} [\Psi(n)+rW_U]/2 & n<N \\ [(N+1)r\Psi(n)+[(N+1)r+2\alpha]rW_U]/[2(N+1)r+2\alpha] & n=N \end{cases}$$

(5.5)
$$\Pi(N,N) = \frac{1}{r}\{\frac{\alpha}{(N+1)r+\alpha}\pi(N) + \frac{r(N+1)}{(N+1)r+\alpha}[\frac{1}{N+1}\sum_{i=o}^{N}\pi(i)]\}$$

The continuous labor limit

Consider the continuous time version summarized in (5.1-5) above. In an ε-version of the model, the size of the individual worker is ε. This is in the sense that, in terms of the contribution to production and the rate of arrival, a batch of $1/\varepsilon$ workers (ε is restricted to be an inverse of a natural number), is equivalent to a single worker in the original version, as if each worker was split into $1/\varepsilon$ smaller ones. The data of an ε-version are $f^{\varepsilon}(n)=f(n\varepsilon)$, $\alpha^{\varepsilon}=\alpha/\varepsilon$ and $W_U^{\varepsilon}=W_U\varepsilon$, so that for $\varepsilon=1$ it coincides with the original model. The term labor unit will be used to describe one worker in the original version of the model, and the equivalent batch of $1/\varepsilon$ workers in the ε-version of the model. Thus, in terms of labor units, the basic data remain the same across the different ε-versions. The important distinction between the different versions is that the employment of a labor unit in an ε-version requires negotiations with $1/\varepsilon$ independent workers.

Let $w^{\varepsilon}(n;N)$, $\Pi^{\varepsilon}(n;N)$ and $W^{\varepsilon}(n;N)$ denote the solution to (5.1-3) with the basic data of the ε-version. Next, define $w(n;N)=\lim_{\varepsilon\to 0}w^{\varepsilon}(n/\varepsilon;N/\varepsilon)/\varepsilon$, $W(n;N)=\lim_{\varepsilon\to 0}W^{\varepsilon}(n/\varepsilon;N/\varepsilon)/\varepsilon$ and $\Pi(n;N)=\lim_{\varepsilon\to 0}\Pi^{\varepsilon}(n/\varepsilon;N/\varepsilon)$. Here, n and N are measured in labor units. The actual numbers n/ε and N/ε increase indefinitely as ε approaches 0, but the equivalent numbers of labor units remain constant. Notice that $W(n;N)$ is obtained as the limit of the ratio $W^{\varepsilon}(n/\varepsilon;N/\varepsilon)/\varepsilon$, while $\Pi(n;N)$ is the limit of $\Pi^{\varepsilon}(n/\varepsilon;N/\varepsilon)$ itself. This reflects the fact that, due to the negligibility of the individual worker, W is now a density, so it is not anymore of the same order as $\Pi(n;N)$ which remains of finite size[6]. Taking the limits[7] of the ε-versions of (5.4-5) yields

$$
(5.6)\qquad w(n;N) = \begin{cases} \dfrac{1}{n^2}\displaystyle\int_o^n xf'(x)\,dx+\frac{1}{2}rW_U & n<N \\[2ex] \dfrac{Nr}{2(Nr+\alpha)}\left[\dfrac{2}{N^2}\displaystyle\int_o^N xf'(x)\,dx\right]+\dfrac{Nr+2\alpha}{2(Nr+\alpha)}rW_U & n=N \end{cases}
$$

Similarly,

$$
(5.7)\qquad \Pi(N;N) = \frac{1}{r}\left\{\frac{\alpha}{Nr+\alpha}\pi(N)+\frac{rN}{Nr+\alpha}\left[\frac{1}{N}\int_o^N \pi(x)\,dx\right]\right\}
$$

It is interesting to observe that the continuous limits of w(n;N) and Π(N;N) are essentially of the same form and on the same order as their discrete counterparts in which n and N are actual numbers of workers rather than labor units. In particular, the downward jump of the wage at a steady state level $N<N^0$ that was recognized before, turns here into a discontinuity,

$w(N^-;N) \equiv \lim_{n\to N} w(n;N) = \int_0^N xf'(x)dx/n^2+rW_U/2 >$

$[2Nr\int_0^N xf'(x)dx/n^2+(Nr+\alpha)rW_U]/2(Nr+\alpha) = w(N;N)$.

6. Discussion

Throughout the discussion we will occasionally refer to the benchmark case of the <u>wage setting</u> firm, which sets the wages of its workers unilaterally. The wage setting firm always pays its workers exactly their reservation wage, i.e., $w^S = (1-\delta)W_U$. Its discounted profit at any stationary employment level n is $\pi(n)/(1-\delta)$ and its optimal steady state level is therefore the level N^S=Argmax$\pi(n)$ pointed out before.

Multiplicity of equilibria

In a sufficiently rich dynamic model as this one, there is no escape from multiplicity of equilibria. We can of course achieve "uniqueness" by imposing the stationarity refinement, which will single out the N^0 equilibrium (see Proposition 6). But this will deprive us of the broader range of insights that are generated by some of the other equilibria. The N^0 equilibrium corresponds to a scenario in which the firm is expected to use over-employment to extract the surplus of the workers. The profit maximizing equilibrium corresponds to a scenario in which it is expected that the firm will not try to extract the workers' surplus by excessive employment. Both of these seem as valid considerations that should be understood and might appear in different relations of this type.

Although the present model is different in important ways from a repeated game model, it is useful to point out the analogies between these models. The N^0 equilibrium is reminiscent of the repeated game SPE in which a unique one-shot equilibrium is played repeatedly. In both models, these equilibria have very robust existence which is reflected in their being the only stationary equilibria and the only equilibria when the periods are long. Despite the inefficiency of these equilibria, it is difficult to dismiss them: they capture natural considerations, and in both models one can envision reasonable circumstances that will make them likely to prevail. The maximum profit equilibria of the present model are reminiscent of the collusive equilibria of the repeated game. They share with them the (near) efficiency and their fragility. In both models, these equilibria are sustained in the face of tempting short run deviations, by the threat of triggering a transition to a less profitable equilibrium.

This analogy perhaps clarifies that the maximum profit equilibrium, on the one hand, and the stationary N^0 equilibrium, on the other hand, capture some of the main considerations of interest. At the same time, this analogy reinforces the sense in which there is no merit in picking one of these equilibria and dismissing the other. In oligopoly theory, both of these equilibria provide useful predictions and this should be the case here as well.

The steady state level effect

In quasi-stationary N equilibria with $N<N^0$ (which include the maximum profit equilibrium for initial $m\leq N^*$), the wage $w(n;N)$ changes sharply at the steady state level N. Inspection of formula (3.4) or (5.6) reveals it immediately in that $w(n;N)$ depends on α at $n=N$, but is independent of α at[8] $n<N$. In the continuous labor case of Section 5 this sharp change is actually translated to a discontinuity at N: $\lim_{n\to N}w(n;N)=w(N^-;N)>w(N;N)$. Notice, however, that there is no discontinuity in the stock values $\Pi(n;N)$ and $W(n;N)$.

There are thus two related features that call for explanation: the sharp change (or discontinuity) of $w(n;N)$ at $n=N$ and the complete independence of $w(n;N)$ of α at $n<N$. The "discontinuity" is a robust phenomenon in this class of equilibria. It is explained by the absence of binding contracts, which is a central feature of the model, and it survives variations on the model, such as the two extensions outlined in Section 7 below, and other similar extensions like publicly observable shocks to the production technology. In contrast, the independence of $w(n;N)$ of α at $n<N$ seems to be a consequence of the constant arrival rate, rather than a consequence of a qualitative feature of the model. It may not survive variations that would make the arrival rate dependent on n. But even in such cases, the "discontinuity" will continue to be present.

Consider the "discontinuity". At a steady state level N, the expected utility that a worker bargains for is $W(N;N)$. The same situation is expected to prevail in perpetuity, so this expected utility is paid through a constant wage,

$$(6.1) \qquad w(N;N) = (1-\delta)W(N;N).$$

At $n<N$, The worker bargains for $W(n;N)$. This sum cannot be paid through the constant average wage $(1-\delta)W(n;N)$, since upon the next arrival the worker's power will be sufficient to secure only utility of $W(n+1;N)<W(n;N)$ and her wage will be renegotiated downwards. Therefore, $w(n;N)$ has to exceed $(1-\delta)W(n;N)$ by a magnitude that compensates for the anticipated drop in the worker's bargaining power. Indeed, from (3.1-3) we have for $n<N$

$$(6.2) \qquad w(n;N) = (1-\delta)W(n;N) - \delta\alpha\Delta W(n+1;N)$$

Thus, the term $-\delta\alpha\Delta W(n+1;N)>0$, which is absent from (6.1), reflects the non-binding nature of the wage agreements. For the steady state level effect to be significant, it has to be that this term does not become exceedingly small when employment gets close to N. This can be examined more sharply in the context of the continuous labor version. The counter-part of (6.2) in the continuous version is

$$(6.3) \qquad w(n;N) = rW(n;N) - \alpha W_1(n;N)$$

Since $W(n;N)$ is clearly continuous in n everywhere (including at $n=N$), $w(n;N)$ is discontinuous at $n=N$ if $\lim_{n\to N}W_1(n;N)$ is bounded away from 0. From the equal split condition in its continuous form, $W(n;N)-rW_U = \Pi_1(n;N)$, we have $W_1(n;N) = \Pi_{11}(n;N)$. Thus,

$$(6.4) \qquad w(n;N) = rW(n;N) - \alpha\Pi_{11}(n;N)$$

Letting $\Pi_{11}(N^-;N)=\lim_{n\to N}\Pi_{11}(n;N)$, we can write (6.4) in the limit of $n\to N$ as

$$(6.5) \qquad w(N^-;N) = w(N;N) - \alpha\Pi_{11}(N^-;N)$$

Thus, the extent of the gap at N is associated with the curvature of $\Pi(n;N)$ near n=N. Notice that, a-priori, there is no reason to expect this curvature to be zero. To see that this is indeed the case, let us calculate directly the continuous version of $\Pi(n;N)$,

$$\Pi(n;N) = \int_0^{(N-n)/\alpha}[f(n(s))-n(s)w(n(s);N)]e^{-rs}ds + e^{-rT}[f(N)-Nw(N;N)]/r.$$

Twice differentiating and evaluating in the limit as $n\to N$ yields

$$\Pi_{11}(N^-;N) = \{rN[w(N^-;N)-w(N;N)] - \alpha[f'(N)-w(N^-;N)-Nw_1(N^-;N)]\}/\alpha^2$$

Combining this with (6.5), we get

$$(6.6)\ \Pi_{11}(N^-;N) = - [f'(N) - w(N^-;N) - Nw_1(N^-;N)]\}/[\alpha+rN]$$

Now, for any $N<N^0$, the RHS of (6.6) is bounded away from zero[9] (it can be verified by noting that $f'(N) - w(N^-;N) - Nw_1(N^-;N) = \Psi(N) - rW_U$ which for $N<N^0$ is strictly positive by the definition of N^0). Thus, the "bonus" which keeps the wage above its long run average remains non-negligible up to N.

The other feature we noted above--the independence of $w(n;N)$ from α for n<N--appears to be a consequence of the uniformity of the arrival rate across states. If we allow these rates to differ and let $\alpha(n;N)$ denote the arrival rate at employment level n, then $w(n;N)$ might depend on the α's as well. In this case the basic difference equation for $w(n;N)$ is

$$w(n;N)=\begin{cases}[\Delta f(n)+(n-1)w(n-1;N)+(1-\delta)W_U+\delta[\alpha(n;N)-\alpha(n-1;N)][\Pi(n;N)-\Pi(n-1;N)]/(n+1) & n<N\\ [\Delta f(N)+(N-1)w(N-1;N)+(1-\delta)W_U-\delta\alpha(N-1;N)(\Pi(N;N)-\Pi(N-1;N))]/(N+1) & n=N\end{cases}$$

Thus, when $\alpha(n;N)\neq\alpha(n-1;N)$, $w(n;N)$ will depend on the arrival probabilities, but as the difference between the n<N and the n=N branches suggests, the

steady state level effect will be still present for the reasons explained above. Obviously, when $\alpha(n;N)=\alpha(n-1;N)=\alpha$, as we assumed throughout, the term $\Delta\alpha(n;N)\Delta\Pi(n;N)$ vanishes[10].

<u>The wage mark-up</u>

At the steady states reached by the maximum profit equilibrium, the wage embodies a mark-up over the reservation wage $(1-\delta)W_U$. This mark-up is most pronounced at N^*, but in general it is present at N^s (and in other steady state levels in $[N^*,N^s]$ if such exist) as well. For the N^* equilibrium (and, in fact, for any quasi-stationary N equilibrium with $N<N^0$) the presence of the mark-up can be directly observed in Figure 1 from the fact that the $(1-\delta)\Pi(N;N)$ curve is strictly below the $\pi(N)$ curve.

Since the wage is determined in bargaining, the existence of the mark-up might not seem too surprising at first. But a further thought would reveal that it is somewhat counter-intuitive. The steady state level of the profit maximum equilibrium might seem incompatible with surplus for the workers, since this surplus can be extracted from the workers through increased employment. To see this point, let N be a steady state of the maximum profit equilibrium (i.e., $N=N^*$ or N^s or some point in between), and consider an equilibrium with steady state employment N+1 such that, if employment falls from N+1 to N, the continuation coincides with the N equilibrium. Letting w denote the steady state wage of this equilibrium, we have from (2.3)

$$[f(N+1)-(N+1)w]/(1-\delta) - [f(N)-Nw(N;N)]/(1-\delta) = w/(1-\delta) - W_U$$

Since $w = \Delta f(N+1)+(1-\delta)W_U+Nw(N;N)]/(N+2) > (1-\delta)W_U$, the profit at the steady state of this equilibrium indeed exceeds the profit at N. Notice, however, that this observation does not contradict the status of N as a steady state of the profit maximum equilibrium. When the (N+1)st worker arrives, it would indeed be profitable for the firm to switch to the N+1 equilibrium, and this is deterred by the equilibrium punishment that would be triggered. But ex-ante this switch is unprofitable, since the anticipation of such excessive hiring would harden the employees' bargaining position and hence would result in a lower profit.

In the stationary N^0 equilibrium, the above described consideration of surplus extraction from the employees is left unhindered, and indeed employment is driven up to a steady state level where the wage falls to the reservation wage. For the same reason, the equilibrium wage in SZ model also coincides with the reservation wage. This contrast underlines the fact that the appearance of the wage mark-up as a robust phenomenon (e.g., it survives the variations outlined in Section 7 below) owes to dynamic considerations and thus requires the dynamic framework adopted in this paper.

The significance of the equilibrium wage mark-up beyond this paper is in adding another potential source of wage mark-up to the existing list in the literature that includes efficiency wage and unionized labor. Obviously, this is not an entirely new consideration, since the mark-up is explained here by the wage bargaining and is thus related to the unionized labor explanation. But it nevertheless refers to a different environment and to industries in which the degree of unionization and collective bargaining might be very low.

<u>The speed of arrival of new workers</u>

Recall that the effective speed of arrival is captured by $\alpha/(1-\delta)$ (or α/r in the continuous case) and that both N^s and N^0 are independent of $\alpha/(1-\delta)$. Observe, however, from the proof of Proposition 2 that N^* depends on $\alpha/(1-\delta)$: when $\alpha/(1-\delta)$ is large, N^* is near N^S and $w(N^*;N^*)\approx(1-\delta)W_U$. In other words, when the firm can find new employees fast, the steady state outcome of the maximum profit equilibrium is near the neo-classical outcome both in terms of employment level and in terms of the wage.

In fact, as can be seen from (3.4), it is true for other quasi-stationary equilibria with steady state levels $N<N^0$ that, when $\alpha/(1-\delta)$ is large $w(N;N)$ is near $(1-\delta)W_U$. This observation might seem to contradict the finding of Section 5 regarding the continuous labor limit. There, as ε approaches 0 in going to the continuous labor limit, the arrival rate of individual workers, α/ε, approaches ∞, but as seen from (5.6) the limit $w(N;N)$ is above $r\bar{W}_U$ for $N<N^0$. The source of the difference is that, when ε approaches 0, the arrival rate α/ε and the worker's size change simultaneously. On the one hand, fast arrival shortens the duration of the loss imposed by a quitting worker, hence diminishing the significance of the worker's threat. On the other hand, the magnitude of the temporary increase of the total wage bill caused by a quit becomes more significant in relation to the wage of a single worker. Thus, the worker's threat is not diminished relative to the gains over which the worker bargains and the wage is not driven to rW_U

<u>Over-employment/Under-employment</u>:

The efficient employment level is of course N^s, where the marginal product is approximately equal to the reservation wage (more precisely, $\Delta f(N^s)>(1-\delta)W_U$ and $\Delta f(N^s+1)<(1-\delta)W_U$). We will refer to employment levels above and below N^s as <u>over-employment</u> and <u>under-employment</u> respectively.

The steady states reached by the maximum profit equilibrium fall in the interval $[N^*,N^s]$ and thus exhibit either under-employment or efficient employment. The under-employment at N^* reflects two considerations. The <u>Long run consideration</u> recognizes that N^s is the most profitable from among the steady state employment levels in $[N^*,N^s]$ that the profit maximizing equilibrium reaches. This simply follows from the strict monotonicity of $\bar{\Pi}(n)$ over this range. The short run consideration is associated with the steady state level effect identified above. The higher wages at employment levels short of the steady state level make it profitable to settle to a steady state level sooner rather than later, even if it is somewhat less profitable in the long run. Indeed, the faster the arrival of new employees, the less significant is the short-run consideration and the closer is N^* to N^s. Obviously, for initial levels above N^s, the short-run consideration is absent and the maximal equilibrium settles to N^s.

The N^0 equilibrium, and all the other quasi-stationary equilibria with steady state levels $N>N^s$, exhibit over-employment. This is because, besides increasing output, an additional worker weakens the bargaining power of the other workers and hence lowers the wage. In some sense, in these equilibria the firm keeps a "reserve army" of inefficiently employed workers to keep the wages low. This over-employment phenomenon appears already in the model of SZ. Since in their model N^0 is the only equilibrium employment level, over-employment appears as a necessary feature of the wage negotiating firm. In contrast, the present model points out that efficient or nearly efficient employment is as plausible in this setting. While the under-employment at N^* is robust and interpretable, it is probably less convincing as a prediction. The more meaningful contrast between the two extreme equilibrium behaviors is between over-employment and (near) efficiency.

Recall the standard monopsony firm model with an upward sloping labor supply curve. There, the firm restricts its employment to keep the wages low. This consideration does not seem related to the under-employment in the present mode, but in some sense it is similar in reverse to the over-employment phenomenon. Here, due to the bargaining, excessive employment lowers the wages.

<u>Distortion of other inputs</u>

The employment of other inputs will also be distorted from their efficient levels. Suppose that the firm employs also capital denoted by K. The production function will now be $f(n,K)$, and p_K will denote the rental rate of capital. Thus,

$$\pi(n,K) = f(n,K) - n(1-\delta)W_U - p_K K$$

Incorporation of capital into the model in a complete way, will require some further modelling decisions about the timing of investments, their reversibility, their observability, etc. Although this might be an interesting extension in itself, we will not develop it here. Instead, we will only examine the stock that maximizes the profit of the firm at the steady state employment level reached at equilibrium.

Thus, for a fixed stock of capital K, let $\Pi(n,K;N)$ denote the counterpart of $\Pi(n;N)$ and let N^* and K^* be defined by $N^*=\mathrm{Argmax}_N\Pi(0,K^*;N)$ and $K^*=\mathrm{Argmax}_K\Pi(N^*,K;N^*)$. Thus, given capital stock K^*, employment N^* is the steady state level reached by the profit maximizing equilibria starting at $m\in[0,N^*]$, and given N^* and the quasi-stationary N^* equilibrium that supports it, K^* is the optimum capital stock. The first order condition satisfied by K^* is

$$\frac{\partial\Pi(N^*,K;N^*)}{\partial K} \equiv \frac{1}{1-\delta}\{\frac{\delta\alpha\pi_2(N^*,K)+(1-\delta)\sum_{i=0}^{N^*}\pi_2(i,K)}{(N^*+1)(1-\delta)+\delta\alpha}\}=0$$

which means that the weighted average of the marginal products is equal to the rental rate

$$\frac{\delta\alpha f_2(N^*,K^*)+(1-\delta)\sum_{i=0}^{N^*}f_2(i,K^*)}{(N^*+1)(1-\delta)+\delta\alpha}=p_K$$

Thus, if capital and labor are complements, $\Delta f_2>0$, then K^* represents under-investment in capital in the sense that $f_2(N^*,K^*)>p_k$; if capital and labor are substitutes, $\Delta f_2<0$, then K^* represents over-investment in capital, $f_2(N^*,K^*)<p_k$. Notice that this means that K^* is distorted relative to the optimal stock given the equilibrium N^*, rather than just relative to the first best level K^s given by $(N^s,K^s) = \text{Argmax}_{n,k}\pi(n,k)$. The incentives for investment in capital are distorted since it directly affects the wage: more capital increases or decreases the marginal product of labor and hence the negotiated wage according to whether $\Delta f_2>0$ or $\Delta f_2<0$.

The distortion in the employment of other inputs is a consequence of determination of the wage in bargaining, rather than any of the other features of the model. Indeed this point already appear in the SZ model as well as in Grout (1984).

7. The dynamic evolution of firm's labor force: two extensions

The dynamical interaction between the hiring and recontracting processes is the central element of this model and indeed it generates the main specific insights. While the simplicity with which this element was modeled above is helpful for the exposition, it leaves unanswered some questions regarding the robustness of the results and the role of other important factors. Two extensions are briefly outlined below (a more detailed analysis is presented in Sections 6 and 7 in Wolinsky (1996)). The first enriches the dynamics of the labor force to include the possibility of employee departures as well, so that non-trivial hiring and replacement go on continually. The second extension endogenizes the arrival process by letting it be determined by the firm's recruitment efforts.

A. The incorporation of departure process: The model is extended by adding a process of departure whereby employees may leave the firm for exogenous reasons. Thus, in addition to the α arrival process, in the end of any period, each employee may depart with probability β.

$$\text{(7.1)} \qquad \text{Prob}(m_t = m \mid n_{t-1} = n < M) = \begin{cases} \alpha & \text{if } m = n+1 \\ n\beta & \text{if } m = n-1 \\ 1-\alpha-n\beta & \text{if } m = n \end{cases}$$

That is, in each period the pool of the potential employees may increase or decrease and, conditional on the occurrence of a departure at some period, each of the employees has equal probability β to depart. To assure that (7.1) is well defined, it is assumed that α and β satisfy $\alpha + M\beta < 1$. The expected utility of a worker who departs through the exogenous departure process is W_D which might but need not be equal to W_U. The employee reservation wage is therefore $(1-\delta)W_U + \delta\beta(W_U - W_D)$, which coincides with its previous value if $W_D = W_U$. The notions of history, policy, stationarity and equilibrium are just as they were defined earlier. The quasi-stationary equilibria here differ from their previous counterparts in that, after the maximum employment level is reached, employment continues to fluctuate due to departures and the continued hiring of replacements. Some of the results and insights are as follows.

(i) There exists a unique stationary equilibrium. It is characterized by a target employment level N^{00}. This equilibrium is the counter-part of the N^0 equilibrium discussed above, but $w(N^{00}; N^{00})$ is below the reservation wage $(1-\delta)W_U + \delta\beta(W_U - W_D)$, since workers take into account the higher wages they may get after colleagues' departures in the future.

(ii) There exist quasi-stationary equilibria with target employment levels below N^{00} which yield higher profit to the firm than the stationary N^{00} equilibrium. In these equilibria the wage exhibits the counterpart of the steady state level effect discussed earlier. But since in this version employment fluctuates on the path, this effect appears repeatedly on the path. When departures reduce the employment to below the target level, the firm enters a rebuilding phase with higher wages. When the target level is reached, the wage drops again and so on.

B. Endogenous arrival process: In the end of each period t the firm chooses the probability α_t of arrival in the beginning of t+1. Probability α is induced at the cost $c(\alpha)$, where c is increasing and convex. So arrivals are affected by a costly recruiting effort. The policy now includes the choice of the arrival rate, but otherwise remains the model remains the same. There is a stationary equilibrium with the same steady state level N^0 and a family of quasi-stationary N equilibria with $N<N^0$. One difference from before is that in these equilibria the arrival probability $\alpha(n;N)$ depends on the current employment n. The following observations can be made.

(i) In quasi-stationary N equilibria with $N<N^0$, the wage continues to exhibit the steady state level effect and the associated mark-up over the reservation wage.

(ii) In the continuous labor limit of any quasi-stationary N equilibrium, the equilibrium arrival probabilities, $\alpha(n;N)$, are independent of n, thus yielding as a result the constant arrival rate which was assumed in the main model.

(iii) In quasi-stationary N equilibria with $N<N^0$, the recruiting effort remains substantial even near the equilibrium steady state level. In contrast, the recruiting effort of a wage setting firm in this environment approaches

zero as employment approaches its steady state level.

8. Remarks on the modeling

The bargaining black-box

The model of this paper is not a fully specified non-cooperative game. The employment decisions are analyzed just as they would be in a non-cooperative game model, but the wage bargaining component is left as a black box. The advantage of this approach is in its simplicity: it avoids the complicated modeling of strategic multi-person bargaining processes, thus allowing to focus the attention on more central issues. There is, however, a conceptual difficulty in invoking the equal split condition when the surplus to be divided depends on future decisions of the firm, since if the firm contemplates a multiple step deviation from its employment policy, it will no longer share the same expectations with its workers. For this reason, the equilibrium definition in Section 2 requires only immunity against single step deviations by the firm. In a model with discounting and bounded payoffs, this limitation itself is harmless, since in a fully specified strategic model, it would still be sufficient to consider only single step deviations from it[11]. But in the context of the present model this is a limitation rather than a result.

However, it should be mentioned that the bargaining component of the present model can be replaced by a strategic bargaining procedure that implements the desired equal split outcome. Wolinsky (1996) and Stole and Zwiebel (1996b) outline alternative bargaining procedures that would accomplish this[12]. Of course, once such an implementation is adopted, the single deviation property is again a result rather than a limitation.

Observable Wages

Recall that the public histories on which behavior may depend consist of the record of employment but not wages. The rationale behind this assumption is that the individual wages in this model are not publicly observed. This of course does not completely justify this modeling decision, since behavior can be conditioned on private information as well. But it at least points out that there is good reason in the model to treat wage history differently. A similar approach was taken by much of the literature on repeated games with imperfect monitoring.

Let us consider briefly the case in which the entire history of wages is observable and can be conditioned upon. In this case the pie over which the parties bargain each period might become highly non-convex, since the wage information can be used to trigger sharp transitions from one phase of an equilibrium to another. Under such circumstances it is not obvious what the equal split rule represents and it makes sense to think on the bargaining in a strategic form such as mentioned above. Without going through a formal setting and analysis of this model, it is straightforward to see that it would have an equilibrium in which the behavior coincides with that of the wage setting firm: the wage is always $(1-\delta)W_U$ and hiring proceeds in the shortest path to N^s. Such equilibrium is supported by a trigger policy that would switch to the minimum profit equilibrium of the model analyzed in this paper (which like all the equilibria of that model continues to be an equilibrium here), if any of the past wages was lower than $(1-\delta)W_U$.

Seniority

The seniority of incumbents which is built into the employment policy is meant to capture a plausible feature of this environment without modeling in detail the training costs or other elements which give natural advantage to the incumbents. It should be noted however that this assumption does not play a major qualitative role in the analysis. Suppose, for example, that the incumbents and the new arrival are treated equally, and consider the quasi-stationary N equilibria. The difference now would be that, at the steady state, workers would face a risk of being unemployed. This would modify the expression for workers' utility but not much else. In fact, a closely related consideration is already being made in the extension outlined in Section 7.A, where workers face an exogenous risk of departure.

The order of moves and the non-binding nature of contracts

Recall that in any period, employment is determined first and then the wages are bargained. This is a central assumption intended to capture the "at will" nature of the employment relations modeled here. This assumption means that the parties can quickly take advantage of changing circumstances and renegotiate.

One can think about an alternative model in which the firm gathers n+1 candidates and then auctions off n one-period employment contracts among them. If these contacts are binding for the one period and arrival is sufficiently fast, say $\alpha=1$, so that when the contracts expire the firm will again have n+1 candidates, then the equilibrium outcome would coincide with that of the wage setting firm, i.e., the wage and employment would be $(1-\delta)W_U$ and N^s respectively.

This contrast, brings out the important features of the environment we are discussing. It has to be that, either employment contracts are not binding (at least along some dimensions), or the normal duration of the contracts is short relative to the normal speed with which new workers can be recruited.

9. Concluding Remarks

There are of course important aspects that have been ignored here and should be considered in further research. First, the heterogeneity of workers both with respect to their skills and with respect to the opportunities to replace them are probably important, particularly in employment relations involving highly skilled labor, which often fit the general type of situation modelled here. Second, the internal organization of the firm might both affect and be affected by employment relations of this type. Jackson and Wolinsky (1995) consider this point briefly in the context of a simple network model of a firm, but this topic requires much more attention.

Appendix 1

Proof of Proposition 1: If N is feasible, the feasible solution to system (3.1-3) also solves the modified system in which the RHS of (3.3) is replaced by $\Pi(n;N) - \Pi(n-1;N)$. This is a linear system of 3N+1 independent equations with the 3N+1 unknowns $w(n;N)$, $W(n;N)$, $\Pi(n;N)$, $n=1...N$ and $\Pi(0;N)$. It therefore has a unique solution. The wage arising in this solution is given by (3.4). This can be verified as follows. For $n<N$, use the modified (3.3) to substitute for W on both sides of (3.2) and rearrange to get

$$(1-\delta+\delta\alpha)\Pi(n;N) - \delta\alpha\Pi(n+1;N) = (1-\delta+\delta\alpha)\Pi(n-1;N) - \delta\alpha\Pi(n;N) + w(n;N) - (1-\delta)W_U$$

Use (3.1) for substitution on both sides to get

$$f(n) - nw(n;N) = f(n-1) - (n-1)w(n-1;N) + w(n;N) - (1-\delta)W_U$$

The solution to this system of difference equations is the $n<N$ branch of (3.4). To get $w(N;N)$, substitute from (3.1) and (3.2) into (3.3) to obtain

$$w(N;N)/(1-\delta) - W_U = [f(N) - Nw(N;N)]/(1-\delta) -$$
$$\{[f(N-1) - (N-1)w(N-1;N) + \delta\alpha[f(N) - Nw(N;N)]/(1-\delta)]\}/(1-\delta+\delta\alpha).$$

Then, substitute for $w(N-1;N)$ from above to get the $n=N$ branch of (3.4). Thus, if N is feasible, the feasible solution is be given by (3.4).

Now, consider $N\le N^0$ and the corresponding solution based on (3.4). Since $\Psi(n)$ is a decreasing function and $\Psi(N^0)=(1-\delta)W_U$, for any $n\le N$, $\Psi(n)\ge(1-\delta)W_U$. This and (3.4) imply that, for any $n\le N$, $w(n;N)\ge(1-\delta)W_U$ and hence $W(n;N)\ge W_U$ and $\Pi(n;N)\ge\Pi(n-1;N)$. Therefore, this is a feasible solution for (3.1-3). It follows that any $N\le N^0$ is feasible.

Suppose to the contrary that $N>N^0$ is feasible. From above, $w(N;N)$ is given by (3.4) and the feasible solution solves the modified system. But, for $N>N^0$, $w(N;N)<(1-\delta)W_U$ implying $W(N;N)<W_U$. Hence, by the modified (3.3), $\Pi(N;N) < \Pi(N-1;N)$, contradicting the feasibility. Thus, N is feasible iff $N\le N^0$. QED

__Proof of Proposition 2__: (i) A straightforward rearrangement yields (see footnote 3) $\pi(n)-\frac{1}{n+1}\sum_{i=0}^{n}\pi(i) = [\Psi(n)-(1-\delta)W_U]n/2$. Since $\Psi(N^0)=(1-\delta)W_U$, the $\frac{1}{n+1}\sum_{i=0}^{n}\pi(i)$ and the π curves intersect at N^0. Since $\Psi(n)$ is strictly decreasing in n, these curves intersect only once. Clearly, $\frac{1}{n+1}\sum_{i=0}^{n}\pi(i)$ is maximized at N^0, since it is increasing when it lies below π and is decreasing when the opposite is true. The $(1-\delta)\Pi(n,n)$ curve is just a convex combination of the other two, hence it also intersects them at N^0.

(ii) From (3.5) and after some rearrangement we get:

$\text{Sign}\{\Pi(n+1;n+1) - \Pi(n;n)\} =$

$$\text{Sign}\{\delta\alpha[(n+2)(1-\delta)+\delta\alpha]\Delta\pi(n+1) + (1-\delta)^2[(n+1)\pi(n+1)-\sum_{i=0}^{n}\pi(i)]\}$$

By the definition of N^S, $\Delta\pi(n+1)$ is positive for $n<N^S$ and negative for $n\geq N^S$. By the definition of N^0, the second term is positive for $n<N^0$ and negative for $n\geq N^0$. Therefore, $\text{Argmax}\Pi(n;n)\subseteq[N^S,N^0)$. Also, for $n>N^S$, the expression on the RHS is strictly decreasing, as can be seen by taking the first difference of that expression, $[(n+2)(1-\delta)+\delta\alpha][\delta\alpha\Delta^2\pi(n+2)+(1-\delta)\Delta\pi(n+2)]$, which is negative since $\Delta^2\pi<0$ for all n and $\Delta\pi(n)<0$ for $n>N^S$. Therefore, $\text{Argmax}\Pi(n,n)=N^{**}$ or $\{N^{**},N^{**}+1\}\subset[N^S,N^0)$.

Next observe that $\text{Sign}\{\Pi(0;n+1)-\Pi(0;n)\} = \text{Sign}\{\Pi(n;n+1) - \Pi(n;n)\}$. After some manipulation of (3.1-5) we get:

$\text{Sign}\{\Pi(n;n+1) - \Pi(n;n)\} =$

$$\text{Sign}\{\delta\alpha[(n+1)(1-\delta)+\delta\alpha]\Delta\pi(n+1) - (1-\delta)[\pi(n)-\frac{1}{n+1}\sum_{i=0}^{n}\pi(i)]\}\ \}$$

which is negative for $n\geq N^s$. This and the definition of N^* imply that $N^*\leq N^s$ (and in fact this is true for any other maximizer of $\Pi(0;n)$). QED

__Proof of Proposition 3__: The uniqueness will follow from Proposition 6. Here we just show the existence of an equilibrium (ν,w) such that $\nu(h)=\min[\ell(h),N^0]$.

Let $w(h) = w(\ell(h);N^0)$, where for $n \leq N^0$, $w(n;N^0)$ given by (3.4) and, for $n > N^0$, $w(n;N^0) = (1-\delta)W_U$.

First, observe that, after any history, w satisfies condition (2.3). For h such that $\ell(h) \leq N^0$, this is so since system (2.1-3) is described by (3.1-3) and w is the feasible solution. For h such that $\ell(h) > N^0$, $w(h) = (1-\delta)W_U$ implies that (2.1-2) are given by

$$\Pi(\ell(h);N^0) = f(n) - n(1-\delta)W_U + \delta\Pi(N^0;N^0) \quad \text{and} \quad W(\ell(h)) = W_U.$$

Therefore,

$$(\#) \quad \Pi(\ell(h);N^0) - \Pi(N^0;N^0) = f(\ell(h)) - f(N^0) - [\ell(h) - N^0](1-\delta)W_U < 0.$$

Since $\mathrm{Max}[\Pi(\ell(h);N^0) - \Pi(N^0;N^0), 0] = 0 = W(\ell(h)) - W_U$, condition (2.3) holds as well.

The feasibility of N^0 implies that, for any h such that $\ell(h) \leq N^0$ and any n, $\Pi(\ell(h);N^0) > \Pi(n;N^0)$ and hence $\nu(h) = \min[\ell(h), N^0] = \ell(h)$ is optimal. Therefore, equilibrium conditions (2.4) and (2.5) are satisfied for such h. For h such that $\ell(h) > N^0$, it follows from (#) that $\nu(h) = \min[\ell(h), N^0] = N^0$ is optimal. QED

<u>Proof Proposition 4</u>: Let $N \in [\hat{N}, N^0]$ and recall that $n_{max}(h)$ denotes the maximal employment level over h. Let $N(h) = N$ if $n_{max}(h) \leq N$ and $N(h) = N^0$ if $n_{max}(h_t) > N$. Define (ν, w) by $\nu(h) = \min[\ell(h), N(h)]$ and $w(h) = w(\ell(h);N(h))$, where for $n > N^0$, $w(n;N^0) = (1-\delta)W_U$. I.e., as long as the employment never surpassed N, the firm is supposed to follow the stationary N policy; if the firm ever deviated and hired more than N, the stationary N^0 equilibrium will be triggered.

Clearly, after histories such that $N(h) = N^0$ the continuation is an equilibrium. When $N(h) = N$ and $\ell(h) \leq N$, system (3.1-3) holds and since $N \leq N^0$ is feasible ν is optimal. Finally, if $N(h) = N$ and $\ell(h) > N$, then it follows from proposition 2 and $N \in [\hat{N}, N^0]$ that $\Pi(N;N) \geq \Pi(N^0;N^0)$ so that $\nu(h) = N$ is optimal. QED

Appendix 2: equilibrium characterization results

Part A: preliminary results

The discussions of the maximum and minimum equilibria share some common arguments presented in this part. The reader is referred to Section 4.A. to recall the description of histories and other notation, like $\Pi^{\nu,w}(h)$ and $W^{\nu,w}(h)$. Also recall from Section 4.D. that $S_m(\nu,w)$ is the expected <u>total net payoff</u> of the firm and its employees, evaluated at the beginning of a period, <u>before</u> the uncertainty about the new arrival was resolved, given that the preceding period had m employees and that the continuation is according to the equilibrium (ν,w) played from its start,

$$S_m(\nu,w)=(1-\alpha)\{\Pi^{\nu,w}(m)+\nu(m)[W^{\nu,w}(m)-W_U]\}+\alpha\{\Pi^{\nu,w}(m+1)+\min[m,\nu(m+1)][W^{\nu,w}(m+1)]-W_U]\}.$$

<u>Proposition A</u>: (i) If (ν,w) is an equilibrium such that $\nu(m)=m$, then

(A.1) $\quad \Pi^{\nu,w}(m) = \{\pi(m) + m\Pi^{\nu,w}(m,m,m-1) + \delta S_m[(\nu,w)|_{(m,m)}]\}/(m+1)$

(ii) If there exists an equilibrium (ν,w) such that

(A.2) $\quad \{\pi(m) + m\Pi^{\nu,w}(m-1) + \delta S_m(\nu,w)\}/(m+1) \geq \Pi^{\nu,w}(m-1)$

then there exists an equilibrium (ν',w') such that $\nu'(m)=m$,

$(\nu',w')|_{(m,m,m-1)}=(\nu,w)|_{m-1}$, $(\nu',w')|_{(m,m),m}=(\nu,w)|_m$, $(\nu',w')|_{(m,m),m+1}=(\nu,w)|_{m+1}$ and $\Pi^{\nu,w}(m)$ is equal to the RHS of (A.2).

<u>Proof</u>: Let (ν,w) be an equilibrium such that $\nu(m)=m$. Condition (2.3) implies

$$f(m) - w(m,m)m + \delta\{(1-\alpha)\Pi^{\nu,w}[(m,m),m] + \alpha\Pi^{\nu,w}[(m,m),m+1]\} - \Pi^{\nu,w}[(m,m,m-1)] =$$
$$w(m,m) + \delta(1-\alpha)\{\nu[(m,m),m]/m]W^{\nu,w}[(m,m),m] + (1-\nu[(m,m),m]/m)W_U\} +$$
$$\delta\alpha\{\min[\nu[(m,m),m]/m,1]W^{\nu,w}[(m,m),m+1]+(1-\min[\nu[(m,m),m]/m,1])W_U\} - W_U$$

Solve it for $w(m,m)$, substitute the result into

$$\Pi^{\nu,w} = f(m) - w(m,m)m + \delta\{(1-\alpha)\Pi^{\nu,w}[(m,m),m] + \alpha\Pi^{\nu,w}[(m,m),m+1]\}$$

and then substitute S_m from above to obtain (A.1).

(ii) To complete the description of (ν',w'), let $(\nu',w')|_{(m,n)}=(\nu,w)|_{(m-1,n)}$ and $(\nu',w')|_{(m,m,k,n)}=(\nu,w)|_{(m-1,n)}$, for all n<m; let (ν',w') coincide with some arbitrary equilibrium after all other histories (not mentioned here or in the statement of the proof); and let

$$w'(m,m) = \{f(m) + \delta[(1-\alpha)\Pi^{\nu,w}(m)+ \alpha\Pi^{\nu,w}(m+1)] - \Pi^{\nu,w}(m-1)$$
$$- \delta(1-\alpha)[(\nu''(m)/m)W^{\nu,w}(m) + (1-\nu''(m)/m)W_U] - \delta\alpha[\min(\nu''(m+1)/m,1)W^{\nu,w}(m+1)$$
$$+ (1-\min(\nu''(m+1)/m,1))W_U] + W_U\}/(m+1)$$

Let us verify now that (ν',w') is indeed an equilibrium. By the choice of $w'(m,m)$, equilibrium condition (2.3) holds after the history (m,m). Since, by construction, the continuations after all histories other than m are equilibria, we only have to verify that the employment decision at m is optimal. Following the steps of part (i) of this proposition, we have that $\Pi^{\nu',w'}(m)$ is given by the LHS of (A.2). Hence, $\Pi^{\nu',w'}(m)\geq\Pi^{\nu,w}(m-1)\geq\Pi(m,n;\nu',w')$, for all n<m, where the first inequality follows from the condition of this proposition while the second follows from the definition of (ν',w') and the fact that (ν,w) is an equilibrium. Thus, $\nu'(m)=m$ is optimal. □□

We now turn to proving the results reported in Section 4.

<u>Proof of Proposition 5</u>: Let $(\nu,w)\in\mathscr{E}$. First, observe that w(h) is bounded from above and below. For any $h\in H^+$, $\Pi(h;\nu,w) \leq \pi(N^s)/(1-\delta)$. Hence, $w(h) \leq \pi(N^s)/(1-\delta)$. Therefore, $W(h)\geq W_U$ implies that $w(h)\geq-\pi(N^s)/(1-\delta)^2$.

By Tychonoff Theorem[13], the set $Y = \{(\nu,w): |w(h)| \leq \pi(N^s)/(1-\delta)^2\}$ is compact in the product topology.

Now, E is closed, since if a sequence $\{(\nu_i,w_i)\}$ is such that $(\nu_i,w_i)\in\mathscr{E}$, for all i, and $\nu_i(h)\to\nu(h)$, $w_i(h')\to w(h')$ for all $h\in H^-$, $h'\in H^+$, then (ν,w) also satisfies (2.3-5) and hence is an equilibrium. Since $E\subseteq Y$, it is a closed subset of a compact set and hence is compact.

Finally, the continuity of $\Pi^{\nu,w}(m)$ and $W^{\nu,w}(m)$ as functions of (ν,w) in the product topology implies that the set of equilibrium payoffs, $\{(\Pi^{\nu,w}(m),W^{\nu,w}(m)): (\nu,w)\in\mathscr{E}\}$ is compact. QED.

Part B: Maximum profit equilibria

The reader should start with Part A of this appendix.

Proof of Theorem I: Parts (i) and (ii). Clearly $\bar{\Pi}(m)\geq\bar{\Pi}(m-1)$. Let $\bar{N}$ be the smallest m such that $\bar{\nu}(m+1)\leq m$.

Proposition B.1: (i) For $m\leq\bar{N}$, $\bar{\Pi}(m)$ is strictly increasing and

$$(B.1)\quad \bar{\Pi}(m) = \mathrm{Max}_{j\leq m,k\leq m+1}\{\pi(m) + m\bar{\Pi}(m-1) + \delta(1-\alpha)[\bar{\Pi}(j) + j[\bar{W}(j)-W_U]] + \delta\alpha[\bar{\Pi}(k) + \min(k,m)[\bar{W}(k)-W_U]]\}/(m+1)$$

$$\text{s.t.}\quad \bar{\Pi}(j)\geq\underline{\Pi}(m) \text{ and } \bar{\Pi}(k)\geq\underline{\Pi}(m+1).$$

Proof: The proof unfolds through three claims.

Claim B.1.1: If $\bar{\nu}(m)=m$, $\bar{\Pi}[(m,m,m-1)] = \bar{\Pi}(m-1)$

Proof: Clearly, from the maximality of $\bar{\Pi}$, $\bar{\Pi}[(m,m,m-1)] \leq \bar{\Pi}(m-1)$. It therefore follows from Proposition A that

$$(B.2)\quad \bar{\Pi}(m) \leq \{\pi(m) + m\bar{\Pi}(m-1) + \delta S_m[(\nu,w)|_{(m,m)}]\}/(m+1)$$

Since $\bar{\Pi}(m)\geq\bar{\Pi}(m-1)$, RHS(B.2)$\geq\bar{\Pi}(m-1)$. Therefore, it follows from Proposition A that there exists an equilibrium starting with m whose payoff is equal to RHS(B.2). By the maximality of $\bar{\Pi}$, $\bar{\Pi}(m)$=RHS(B.2) and hence $\bar{\Pi}[(m,m,m-1)] = \bar{\Pi}(m-1)$. □

We thus conclude that (B.2) holds with equality. It then follows that

$$(B.3)\quad \bar{\Pi}(m) \leq \{\pi(m) + m\bar{\Pi}(m-1) + \delta\mathrm{Max}_{(\nu,w)\in\mathscr{E}}S_m(\nu,w)\}/(m+1)$$

Claim B.1.2: If $\bar{\nu}(m)=m$, (B.3) holds with equality.

Proof: (B.3) and $\bar{\Pi}(m-1)\leq\bar{\Pi}(m)$ immediately imply that RHS(B.3)$\geq\bar{\Pi}(m-1)$. Proposition A then implies that there exists an equilibrium whose profit starting at m is equal to RHS(B.3). The maximality of $\bar{\Pi}$ then implies that (B.3) holds with equality. □

The next step is to show that, if (ν,w) is an equilibrium that maximizes $S_m(\bar{\nu},\bar{w})$, then in its second period after $\nu(m)$ and $\nu(m+1)$, (ν,w) coincides with $(\bar{\nu},\bar{w})$.

<u>Claim B.1.3</u>: For $m\leq\bar{N}$,

$$\mathrm{Max}_{(\nu,w)\in\mathscr{E}}S_m(\nu,w) = \mathrm{Max}\{\bar{\Pi}(i) + i[\bar{W}(i)-W_U] \mid i\leq m \text{ and } \bar{\Pi}(i)\geq\underline{\Pi}(m)\} +$$
$$\mathrm{Max}\{\bar{\Pi}(i) + \min[i,m][\bar{W}(i)-W_U] \mid i\leq m+1 \text{ and } \bar{\Pi}(i)\geq\underline{\Pi}(m+1)\}$$

<u>Proof</u>: <u>Step 1</u>: For $i\leq\bar{N}+1$, $q\in[0,i]$ and $(\nu,w)\in\mathscr{E}$ such that $\nu(i)=i$,

$$\bar{\Pi}(i) + q\bar{W}(i) \geq \mathrm{Max}_{(\nu,w)\in\mathscr{E}}[\Pi(i,i;\nu,w) + qW(i,i;\nu,w)].$$

To see that, let $(\nu',w') \in \mathrm{Argmax}\{\Pi(i,i;\nu,w)+qW(i,i;\nu,w)\text{: } (\nu,w)\in\mathscr{E} \text{ and } \nu(i)=i\}$. Observe that

$$\{\pi(i)+i\bar{\Pi}(i-1)+\delta S_i[(\nu',w')|_{(i,i)}]\}/(i+1)=\{i\bar{\Pi}(i-1)+\Pi^{\nu',w'}(i)+i[W^{\nu',w'}(i)-W_U]\}/(i+1)$$
$$\geq \{i\bar{\Pi}(i-1) + \bar{\Pi}(i)]\}/(i+1) \geq \bar{\Pi}(i-1),$$

where the equality follows from rearrangement of $\pi(i)+\delta S_i[(\nu',w')|_{(i,i)}]$, and the first inequality follows from the choice of (ν',w') which implies $\Pi^{\nu',w'}(i)+q[W^{\nu',w'}(i)-W_U]\geq\bar{\Pi}(i)$ and hence $\Pi^{\nu',w'}(i)+i[W^{\nu',w'}(i)-W_U]\geq\bar{\Pi}(i)$.

Then, by Proposition A, there exists an equilibrium (ν'',w'') such that $\nu''(i)=i$, $(\nu'',w'')|_{(i,i,i-1)}=(\bar{\Pi},\bar{w})|_{i-1}$, $(\nu'',w'')|_{(i,i),i}=(\nu',w')|_i$ and $(\nu'',w'')|_{(i,i),i+1}=(\nu',w')|_{i+1}$. Since on the equilibrium path the only difference between (ν'',w'') and (ν',w') is that $w''(i,i)\leq w'(i,i)$ and since $q\leq i$, it follows that $\Pi^{\nu'',w''}(i) + qW^{\nu'',w''}(i) \geq \Pi^{\nu',w'}(i) + qW^{\nu',w'}(i)$. Equilibrium condition (2.3) implies $W^{\nu'',w''}(i) = \Pi^{\nu'',w''}(i) - \bar{\Pi}(i-1) + W_U$. It then follows that $\Pi^{\nu'',w''}(i) + qW^{\nu'',w''}(i) = (q+1)\Pi^{\nu'',w''}(i) - q\bar{\Pi}(i-1) + qW_U \leq (q+1)\bar{\Pi}(i) - q\bar{\Pi}(i-1) + qW_U = \bar{\Pi}(i) + q\bar{W}(i)$.

Therefore, $(\bar{\nu},\bar{w})$ maximizes $\Pi + qW$ over all equilibria starting with i.

Step 2: Let $j = \mathrm{ArgMax}_{i\leq m}[\bar{\Pi}(i) + i[\bar{W}(i)-W_U]]$ s.t. $\bar{\Pi}(i)\geq\underline{\Pi}(m)$, and let $k = \mathrm{Max}_{i\leq m+1}[\bar{\Pi}(i) + \min[i,m][\bar{W}(i)-W_U]]$ s.t. $\bar{\Pi}(i)\geq\underline{\Pi}(m+1)$. For $m\leq\bar{N}$, there exists an equilibrium (ν,w) such that $\nu(m)=j$, $\nu(m+1)=k$ and thereafter it continues according to $(\bar{\nu},\bar{w})|_j$ and $(\bar{\nu},\bar{w})|_k$ respectively.

To see this, let (ν,w) have the features just mentioned. In addition, after any history h starting with m,i such that $i\neq j$ or with m+1,i such that $i\neq k$, let $(\nu,w)|_h$ coincide with $(\underline{\nu},\underline{w})$. Finally after all histories not covered above let it coincide with some arbitrary equilibrium. Since all continuations are chosen to be equilibria and since $\bar{\Pi}(j)\geq\underline{\Pi}(m)$ and $\bar{\Pi}(k)\geq\underline{\Pi}(m+1)$, (ν,w) is clearly an equilibrium.. Hence, there exists such an equilibrium.

Thus, Step 1 implies that the equality appearing in the statement of Claim B.1.3 holds as inequality, with the left hand side being smaller or equal than the right hand side. Step 2 then implies that this is actually an equality, as required by the claim. □

Claims B.1.1-3 imply Proposition B.1.

Proposition B.2: (i) $\bar{N} \geq N^s$.

(ii) For $m>\bar{N}$, $\bar{\nu}(m)=\bar{N}$.

Proof: (i) Suppose to the contrary that there exists $m<N^s$ such that $\bar{\nu}(m+1)\leq m$. It follows from Proposition B.1 that all employment levels on the path of $(\bar{\nu},\bar{w})|_m$ are smaller or equal to m. This and $m+1\leq N^s$ imply that $\pi(m+1)>(1-\delta)\bar{\Pi}(m)$. Since also $S_{m+1}(\bar{\nu},\bar{w})\geq\bar{\Pi}(m+1)\geq\bar{\Pi}(m)$, we have

$$[\pi(m+1) + (m+1)\bar{\Pi}(m) + \delta S_{m+1}(\bar{\nu},\bar{w})]/(m+2) > \bar{\Pi}(m)$$

From Proposition A-(ii), there exists an equilibrium (ν,w) such that $\bar{\nu}(m+1)=m+1$ and $\Pi^{\nu,w}(m+1)>\bar{\Pi}(m)$, contrary to the supposition that $\bar{\nu}(m+1)\leq\bar{\Pi}(m)$. Therefore, $\bar{N} \geq N^s$.

(ii) Suppose to the contrary that there exists some $j>\bar{N}$ such that $\bar{\nu}(j)>\bar{N}$. It follows that there exists $j>\bar{N}$ such that $\bar{\nu}(j)=j$. Let m be the minimal such j. Since, by the definition of $\bar{N}$, $m>\bar{N}+1$, $\bar{\nu}(m-1)<m-1$. This implies $[\pi(m-1)+(m-1)\bar{\Pi}(m-2)+\delta \mathrm{Max}_{\nu,w}S_{m-1}(\nu,w)]/m<\bar{\Pi}(m-2)$, which in turn implies $\pi(m-1)+\delta \mathrm{Max}_{\nu,w}S_{m-1}(\nu,w)<\bar{\Pi}(m-2)\leq\bar{\Pi}(m-1)$, and hence $\pi(m-1)<(1-\delta)\bar{\Pi}(m-1)$. Now, since $\bar{\nu}(m)=m$, we have from Claim B.1.2, $\bar{\Pi}(m)=[\pi(m)+m\bar{\Pi}(m-1)+\delta \mathrm{Max}_{\nu,w}S_m(\nu,w)]/(m+1)$. It must be that $\bar{\nu}[(m,m),m]<m$, since otherwise $\mathrm{Max}_{\nu,w}S_m(\nu,w)\leq\pi(m)/(1-\delta)$ and hence $\bar{\Pi}(m)\leq[\pi(m)+m\bar{\Pi}(m-1)+\delta\pi(m)/(1-\delta)]/(m+1)<$ $\bar{\Pi}(m-1)$, contradiction. Now, if $\bar{\nu}[(m,m),m+1]\leq m$, then $\mathrm{Max}_{\nu,w}S_m(\nu,w) \leq$ $\mathrm{Max}_{\nu,w}S_{m-1}(\nu,w)$; if $\bar{\nu}[(m,m),m+1]=m+1$, then $\mathrm{Max}_{\nu,w}S_m(\nu,w) \leq$ $(1-\alpha)\mathrm{Max}_{\nu,w}S_{m-1}(\nu,w)+\alpha\pi(m)/(1-\delta) \leq \mathrm{Max}_{\nu,w}S_{m-1}(\nu,w)$. Thus, $\pi(m)+\delta \mathrm{Max}_{\nu,w}S_m(\nu,w) \leq$ $\pi(m-1)+\delta \mathrm{Max}_{\nu,w}S_{m-1}(\nu,w) < \bar{\Pi}(m-1)$. Therefore, $\bar{\Pi}(m)=[\pi(m)+m\bar{\Pi}(m-1)+\delta \mathrm{Max}_{\nu,w}S_m(\nu,w)]/(m+1) < \bar{\Pi}(m-1)$, contrary to the assumption that $\bar{\nu}(m)=m$. Therefore, $\bar{\nu}(m)=\bar{N}$, for all $m>\bar{N}$. ◻◻

Propositions B.1-2 together establish Theorem I-(i-a) and immediately imply I-(i-b).

The proof of Theorem I-(ii) is split into two propositions. Throughout the following analysis, it is assumed that the rate of arrival is sufficiently fast to so that the quasi-stationary equilibrium at N^* exists and that $\Pi(N^*;N^*)>\Pi(N^0;N^0)$. This implies that $\Pi(N^*;N^*)>\underline{\Pi}(m)$ for all m.

<u>Proposition B.3</u>: For $m\leq N^*$, $(\bar{\nu},\bar{w})|_m$ coincides with the quasi-stationary N^* equilibrium. That is, $\bar{\Pi}(m)=\Pi(m;N^*)$, $\bar{W}(m)=W(m;N^*)$, $\bar{w}(m,m)=w(m;N^*)$ and on the path $\bar{\nu}(h)=\min[\ell(h),N^*]$.

<u>Proof</u>: Let N be the smallest $m\leq\bar{N}$ such that $\bar{\Pi}(m) + m[\bar{W}(m)-W_U] > \bar{\Pi}(m+1) + m[\bar{W}(m+1)-W_U]$. If no such m exists, let $N = \bar{N}$. Clearly $N\geq1$, since $\bar{\Pi}(0)<\bar{\Pi}(1)$. The proof will first show that, for any $m\leq N$, $(\bar{\nu},\bar{w})|_m$ coincides with the quasi-stationary N equilibrium, and then show that $N=N^*$.

(i) From the choice of N, for each m<N,

$$\bar{\Pi}(m) + m[\bar{W}(m)-W_U] \le \bar{\Pi}(m+1) + m[\bar{W}(m+1)-W_U]$$

Therefore, for $j\le m$ and $k\le m+1$,

$$\{\pi(m) + m\bar{\Pi}(m-1) + \delta(1-\alpha)[\bar{\Pi}(j)+j[\bar{W}(j)-W_U]] + \delta\alpha[\bar{\Pi}(k)+\min(k,m)[\bar{W}(k)-W_U]]\}/(m+1)$$

$$\le \{\pi(m) + m\bar{\Pi}(m-1) + \delta(1-\alpha)[\bar{\Pi}(m)+m[\bar{W}(m)-W_U]] + \delta\alpha[\bar{\Pi}(m+1)+m[\bar{W}(m+1)-W_U]]\}/(m+1)$$

This together with Theorem I-(i) establishes the proposition for m<N.

(ii) Since $\bar{\Pi}(N) + N[\bar{W}(N)-W_U] > \bar{\Pi}(N+1) + N[\bar{W}(N+1)-W_U]$, essentially the same argument establishes that j=k=N maximize

$$\{\pi(N) + \bar{\Pi}(N-1) + \delta(1-\alpha)[\bar{\Pi}(j)+j[\bar{W}(j)-W_U]] + \delta\alpha[\bar{\Pi}(k)+\min(k,N)[\bar{W}(k)-W_U]]\}/(N+1).$$

(iii) Parts (i) and (ii) of this proof together with Theorem I-(i) imply that, for initial $m\le N$, the following hold. After any history h in which N was not exceeded, $\bar{\nu}(h)=\min[\ell(h),N]$. The values $\bar{\Pi}(m)$, $\bar{W}(m)$ and $\bar{w}(m,m)$ satisfy system (3.1-3). Since this system has a unique feasible solution, the $(\bar{\nu},\bar{w})$ equilibrium (starting from $m\le N$) coincides with the quasi-stationary N equilibrium.

(iv) Obviously, the maximality of $\bar{\Pi}$ implies that $N\in\mathrm{Argmax}_n\Pi(0,n)$. The strict monotonicity of $\bar{\Pi}$ then implies that N must be the minimal element of $\mathrm{Argmax}_n\Pi(0,n)$. Thus, $N=N^*$.

This establishes Proposition B.3 which in turn establishes Theorem I-(ii-a). ◻◻

<u>Proposition B.4</u>: (i) For $m\in(N^*,N^s]$, $\bar{\nu}[(m,m),m]=m$, $\bar{\nu}[(m,m),m+1]\in\{m,m+1\}$, $\bar{\nu}[(N^s,N^s),N^s+1]=N^s$ (i.e., employment reaches a steady state level in $[m,N^s]$ via the shortest path).

(ii) For $m\in(N^s,\bar{N}]$, $\bar{\nu}(m) = m$ and $\bar{\nu}[(m,m),m] = \bar{\nu}[(m,m),m+1] = N^s$ (i.e., m workers are hired only for one period after which only N^s are retained).

$$\bar{\Pi}(m) = \frac{\sum_{i=N^s+1}^{m} \pi(i) + (m-N^s)\frac{\delta}{1-\delta}\pi(N^s) + (N^s+1)\bar{\Pi}(N^s)}{m+1}$$

(iii) For $m > \bar{N}$, $\bar{\nu}(m) = \bar{N}$ and the continuation is as in (ii).

<u>Proof</u>:

<u>**Claim B.4.1**</u>: $\bar{\Pi}(m) + m[\bar{W}(m)-W_U]$ is monotonically increasing in m for $m \le N^s$. Furthermore, $m=N^s$ maximizes it over $[0,\bar{N}]$.

<u>Proof</u>: From Proposition B.3, $\bar{\Pi}(m) + m[\bar{W}(m)-W_U]$ is increasing over $[0,N^*]$, Suppose that, for some $m\in(N^*,N^s]$

$$\bar{\Pi}(m-1) + (m-1)[\bar{W}(m-1)-W_U] > \bar{\Pi}(m) + m[\bar{W}(m)-W_U]$$

It follows from Theorem I-(i) that $\bar{\nu}[(m-1,m-1),m-1] = \bar{\nu}[(m-1,m-1),m] = m-1$, hence $\bar{\Pi}(m-1) + (m-1)[\bar{W}(m-1)-W_U] = \pi(m-1)/(1-\delta)$. But, $\bar{\Pi}(m) + m[\bar{W}(m)-W_U] \ge \pi(m)/(1-\delta) > \pi(m-1)/(1-\delta) = \bar{\Pi}(m-1) + (m-1)[\bar{W}(m-1)-W_U]$, where the first inequality owes to the existence of the quasi-stationary m equilibrium for which $\Pi + m[W-W_U] = \pi(m)/(1-\delta)$, and the second inequality owes to the monotonicity of $\pi(n)$ over $[0,N^s]$. Now, the resulting inequality contradicts the supposition. Therefore $\bar{\Pi}(m) + m[\bar{W}(m)-W_U]$ is monotonically increasing for $m \le N^s$.

Observe now that, if there is $j>N^s$ such that $\bar{\Pi}(j)+j[\bar{W}(j)-W_U] > \bar{\Pi}(N^s)+N^s[\bar{W}(N^s)-W_U]$, then there is also a $j>N^s$ such that, for all $i<j$, $\bar{\Pi}(j)+j[\bar{W}(j)-W_U] \ge \bar{\Pi}(i)+i[\bar{W}(i)-W_U]$. Let m be the maximal $j\ge N^s$ with this property (i.e., for all $i<j$, this inequality holds). The maximality implies that $\bar{\Pi}(m)+m[\bar{W}(m)-W_U] > \bar{\Pi}(m+1)+(m+1)[\bar{W}(m+1)-W_U]$. We can now use again the argument used above. It follows from Theorem I-(i) that $\bar{\nu}[(m,m),m]=\bar{\nu}[(m,m),m+1]=m$, hence $\bar{\Pi}(m)+m[\bar{W}(m)-W_U]=\pi(m)/(1-\delta)$. But, $\bar{\Pi}(m)+m[\bar{W}(m)-W_U] = \pi(m)/(1-\delta) \le \pi(N^s)/(1-\delta) \le \bar{\Pi}(N^s)+N^s[\bar{W}(N^s)-W_U]$, where the first inequality owes to N^s=Argmax$\pi(n)$, while the second owes to the existence of a quasi-stationary N^s equilibrium in which $\Pi+N^s[W-W_U]=\pi(N^s)/(1-\delta)$. Now, if $m>N^s$, the first inequality is strict, contrary to the property that defines m. Therefore, $m=N^s$. □

Claim B.4.1 together with Theorem I-(i) imply Proposition B.4-(i),(ii). For $m\in(N^*,N^s]$, $\bar{\nu}(m)=m$ (since $N^s\leq\bar{N}$) and $\bar{\nu}[(m,m),m+1] \geq \bar{\nu}[(m,m),m] = m$ (since $\bar{\Pi}(m)+m[\bar{W}(m)-W_U]\geq\bar{\Pi}(m-1)+(m-1)[\bar{W}(m-1)-W_U]$). For $m\in(N^s,\bar{N}]$, $\bar{\nu}(m) = m$ and $\bar{\nu}[(m,m),m] = \bar{\nu}[(m,m),m+1] = N^s$ (since $m=N^s$ maximizes $\bar{\Pi}(m)+m[\bar{W}(m)-W_U]$), and employment remains at N^s. Since $\bar{\Pi}(N^s)+N^s[\bar{W}(N^s)-W_U] = \pi(N^s)/(1-\delta)$,

$$\bar{\Pi}(m) = \{\pi(m) + \pi(N^s)\delta/(1-\delta) + m\bar{\Pi}(m-1)\}/(m+1) \tag{B.5}$$

The solution to this simple difference equation is

$$\bar{\Pi}(m) = \frac{\sum_{i=N^s+1}^{m} \pi(i) + (m-N^s)\frac{\delta}{1-\delta}\pi(N^s) + (N^s+1)\bar{\Pi}(N^s)}{m+1}$$

Now, $\bar{N}$ is the lowest value of $m\geq N^s$ such that $\bar{\Pi}(m+1)$ as given by the above formula is smaller than $\bar{\Pi}(m)$.

$$\bar{\Pi}(m+1)-\bar{\Pi}(m) = \frac{(m+1)\pi(m+1) - \sum_{i=1}^{m}\pi(i) + (N^s+1)\,[\frac{\delta}{1-\delta}\pi(N^s) + \sum_{i=1}^{N^s}\pi(i) - \bar{\Pi}(N^s)}{(m+1)(m+2)}$$

Obviously, the difference $\bar{\Pi}(m+1)-\bar{\Pi}(m)$ is monotonically decreasing in m and becomes negative at some finite value of m.

Proposition B.4-(iii) follows immediately from Theorem I-(i). □□

This completes the proof of Theorem I-(ii). Finally, Theorem I-(iii) is established by the following.

<u>Proposition B.5</u>: If $\alpha/(1-\delta)$ is sufficiently small, then: for $m\leq N^0-1$, $(\bar{\nu},\bar{w})$ coincides with the quasi-stationary N^0-1 equilibrium; For $m>N^0-1$, it coincides with it after at most one period.

<u>Proof</u>: The main idea of this proof is to show that, for sufficiently small

$\alpha/(1-\delta)$, $\bar{\Pi}(m-1)<\underline{\Pi}(m)$ for all $m<N^0$. This then implies that on the path of these equilibria employment is increasing at least up to N^0-1.

<u>Claim B.5.1</u>: For any $\varepsilon>0$, there is $\eta>0$ such that, if $\alpha/(1-\delta)<\eta$, then for any equilibrium (ν,w), all $m<N^0$ and any history $h\in H^-$ such that $\ell(h)=m$, we have $\nu(h)=m$ and $|\Pi^{\nu,w}(m) - \sum_{i=0}^{m}\pi(i)/(1-\delta)(m+1)|<\varepsilon$.

<u>Proof</u>: Observe that this property holds for $m=0$, and suppose that it holds up to $m-1<N^0$. Specifically, choose η_1 so that this property holds for $\varepsilon <$ $\mathrm{Min}_{m<N^0}[\pi(m)-\sum_{i=0}^{m-1}\pi(i)/m]/3$ (which is positive). In the proof of Theorem II we show[14] $\underline{\Pi}(m)\geq[\pi(m)+m\underline{\Pi}(m-1)+\delta\mathrm{Min}_{\nu,w}S_m(\nu,w)]/(m+1)$ (see inequality (C.2) in that proof). This and $S_m(\nu,w)\geq\underline{\Pi}(m)$, imply $\underline{\Pi}(m)\geq[\pi(m)+m\underline{\Pi}(m-1)]/(m+1-\delta)$. From the inductive assumption, for $\alpha/(1-\delta)<\eta_1$,

$$\underline{\Pi}(m)>[\pi(m)+\sum_{i=0}^{m-1}\pi(i)/(1-\delta)]/(m+1-\delta)-\varepsilon=[\pi(m)-\sum_{i=0}^{m-1}\pi(i)/m]/(m+1-\delta)+\sum_{i=0}^{m-1}\pi(i)/(1-\delta)m-\varepsilon >$$
$$3\varepsilon + \bar{\Pi}(m-1) - \varepsilon - \varepsilon = \bar{\Pi}(m-1) + \varepsilon$$

This implies that, for any equilibrium (ν,w) and any history $h\in H^-$ such that $\ell(h)=m$, $\nu(h)=m$. Hence, there exists η_2 such that, for $\alpha/(1-\delta)<\eta_2$, $|S_m(\nu,w)-\pi(m)/(1-\delta)|<\varepsilon$. This together with Proposition A and the inductive assumption imply that, for any equilibrium (ν,w) and $\alpha/(1-\delta)<\eta=\min\{\eta_1,\eta_2\}$

$$\Pi^{\nu,w}(m) < [\pi(m)+\sum_{i=0}^{m-1}\pi(i)/(1-\delta)+m\varepsilon+\delta\pi(m)+\delta\varepsilon]/(m+1) < \sum_{i=0}^{m}\pi(i)/(1-\delta)(m+1)+\varepsilon$$
$$\Pi^{\nu,w}(m) > [\pi(m)+\sum_{i=0}^{m-1}\pi(i)/(1-\delta)-m\varepsilon+\delta\pi(m)-\delta\varepsilon]/(m+1) > \sum_{i=0}^{m}\pi(i)/(1-\delta)(m+1)-\varepsilon$$

so that the required property holds for m. □

Next observe that the quasi-stationary N^0-1 equilibrium exists for all values of $\alpha/(1-\delta)$. This follows from $\Pi(N^0-1;N^0-1)\geq\Pi(N^0-1;N^0)=\Pi(N^0;N^0)$, where the latter equality is a property of N^0. This implies that $\bar{\Pi}(N^0-1)\geq\underline{\Pi}(m)$, for all $m\geq N^0-1$. From the Claim B.5.1 above we have that, if $\alpha/(1-\delta)$ is sufficiently small, then for any equilibrium (ν,w) and any $m\geq N^0-1$,

$$S_m(\nu,w)\leq\pi(N^0-1)/(1-\delta)=S_{N^0}(\text{quasi-stationary } N^0-1 \text{ equilibrium}).$$

It then follows from Theorem I-(i) that, for $m \le N^0-1$, $(\bar{\nu},\bar{w})$ satisfies system (3.1-3) with $N=N^0-1$, so that it coincides with the quasi-stationary N^0-1 equilibrium.

Clearly, the above argument also implies that, for any initial $m>N^0-1$, the continuation of $(\bar{\nu},\bar{w})$ after at most period will coincide with the quasi-stationary N^0-1 equilibrium. That is, there exists $N' \ge N^0-1$ such that $\bar{\nu}(m)=\min[m,N']$ and $\bar{\nu}[(\bar{\nu}(m),\bar{\nu}(m)),\bar{\nu}(m)] = \bar{\nu}[(\bar{\nu}(m),\bar{\nu}(m)),\bar{\nu}(m)+1] = N^0-1$. □□

This completes the proof of Theorem I. QED

<u>**Part C: Minimum profit equilibria**</u>

<u>**Proof of Theorem II**</u>: Let $\underline{N}$ be the first $m \ge N^s$ such that $(1-\delta)\underline{\Pi}(m+1)>\pi(m+1)$. Notice that, since there is $m<M$ for which $\pi(m)<0$, such $\underline{N}<M$ exists. Since, for all m, $\underline{\Pi}(m) \le \Pi(m;N^0)$ and N^0 is the first $m \ge N^s$ for which the above relationship holds for the stationary N^0 equilibrium, it follows that $\underline{N} \ge N^0$

<u>Proposition C.1</u>: For $m<\underline{N}$, $\underline{\Pi}(m)$ is strictly increasing in m and

$$\underline{\Pi}(m) = \{\pi(m) + m\underline{\Pi}(m-1) + \delta \mathrm{Min}_{(\nu,w)\in\mathcal{S}} S_m(\nu,w)\}/(m+1)$$

<u>Proof</u>:

<u>Claim C.1.1</u>: If $\underline{\nu}(m)=m$, then $\underline{\Pi}[(m,m,m-1)] = \underline{\Pi}(m-1)$

<u>Proof</u>: Clearly, from the minimality of $\underline{\Pi}$, $\underline{\Pi}[(m,m,m-1)] \ge \underline{\Pi}(m-1)$. It therefore follows from Proposition A that

$$\text{(C.1)} \qquad \underline{\Pi}(m) \ge \{\pi(m) + m\underline{\Pi}(m-1) + \delta S_m[(\underline{\nu},\underline{w})|_{(m,m)}]\}/(m+1)$$

Observe that

$$\begin{aligned}\mathrm{RHS(C.1)}-\underline{\Pi}(m-1) &= \pi(m)+\delta S_m[(\underline{\nu},\underline{w})|_{(m,m)}]-\underline{\Pi}(m-1) \ge \pi(m)+\delta S_m[(\underline{\nu},\underline{w})|_{(m,m)}]-\underline{\Pi}(m,m,m-1)\\ &= \underline{\Pi}(m)-\underline{\Pi}(m,m,m-1) \ge 0\end{aligned}$$

Therefore, it follows from Proposition A that there exists an equilibrium whose payoff is equal to RHS(C.1). By the minimality of $\underline{\Pi}$, $\underline{\Pi}(m)$=RHS(C.1) and hence $\underline{\Pi}[(m,m,m-1)] = \underline{\Pi}(m-1)$. □

We thus conclude that (C.1) holds with equality. It then follows that

(C.2) $\underline{\Pi}(m) \geq \{\pi(m) + m\underline{\Pi}(m-1) + \delta \mathrm{Min}_{(\nu,w)\in\mathcal{E}} S_m(\nu,w)\}/(m+1)$

__Claim C.1.2__: If $\underline{\Pi}(m)>\underline{\Pi}(m-1)$, then (C.2) holds with equality.

__Proof__: If RHS(C.2)$\geq \underline{\Pi}(m-1)$, then Proposition A and the minimality of $\underline{\Pi}$ imply that (C.2) holds with equality. To complete the proof, we will establish that RHS(C.2)$\geq\underline{\Pi}(m-1)$.

Suppose to the contrary that RHS(C.2)$<\underline{\Pi}(m-1)$. Let $(\nu'',w'') \in \mathrm{ArgMin}_{(\nu,w)\in\mathcal{E}} S_m(\nu,w)$. Construct (ν',w') to coincide with $(\underline{\nu},\underline{w})$ except as follows: $\nu'(m)=m-1$ and $(\nu',w')|_{(m,m-1)}$ coincides with $(\underline{\nu},\underline{w})|_{m-1}$; for $i=m,m+1$, $(\nu',w')|_{(m,m),i}$ coincides with $(\nu'',w'')|_i$;

$$w'(m,m) = W_U - \delta(1-\alpha)[(\nu''(m)/m)W^{\nu'',w''}(m) + (1-\nu''(m)/m)W_U]$$
$$- \delta\alpha[\min(\nu''(m+1)/m,1)W^{\nu'',w''}(m+1) + (1-\min(\nu''(m+1)/m,1))W_U]$$

In words: starting from m, the path of (ν',w') involves hiring m-1 and continuing thereafter according to $(\underline{\nu},\underline{w})$; if the firm deviates and hires m in the first period, the continuation is according to (ν'',w'') if it actually employs them, and it is according to $(\underline{\nu},\underline{w})$ if it disagrees with some of them

Let us verify that, under the supposition RHS(C.2)$<\underline{\Pi}(m-1)$, (ν',w') is an equilibrium. Since all the continuations are chosen to be equilibria, we only have to check the equilibrium conditions at m. The details of (ν',w') and the definition of S_m yield

$$\Pi(m,m;\nu',w') = f(m)-mw'(m,m)+\delta[(1-\alpha)\Pi^{\nu'',w''}(m)+\alpha\Pi^{\nu'',w''}(m+1)] = \pi(m)+\delta S_m(\nu',w')$$

Therefore, by the initial supposition, $\Pi(m,m;\nu',w') < \underline{\Pi}(m-1)) = \Pi[(m,m,m-1);\nu',w')$. Condition (2.3) is satisfied after the history (m,m), since $w'(m,m)$ was chosen to satisfy $W(m,m;\underline{\nu},\underline{w})=W_U$. Also, the decision $\underline{\nu}(m)=m-1$ is indeed optimal. It follows that $\Pi^{\nu',w'}(m) = \underline{\Pi}(m-1)$, which contradicts either the minimality of $\underline{\Pi}(m)$ or the assumption $\underline{\Pi}(m)>\underline{\Pi}(m-1)$.

Therefore, the supposition is false, so RHS(C.2)$\geq \underline{\Pi}(m-1)$ and (C.2) holds with equality. □

__Claim C.1.3__: Suppose that m satisfies $\underline{\Pi}(m)>\underline{\Pi}(m-1)$ and $\pi(m)\leq(1-\delta)\underline{\Pi}(m)$. Then

$$\text{(C.3)} \qquad \mathrm{Min}_{(\nu,w)\in\mathscr{E}} S_m(\nu,w) \leq \mathrm{Min}_{(\nu,w)\in\mathscr{E}}\{\Pi^{\nu,w}(m+1) + \min[\nu(m+1),m][W^{\nu,w}(m+1)-W_U]\}$$

and if $\pi(m)<(1-\delta)\underline{\Pi}(m)$, then (C.3) holds with strict inequality.

__Proof__: Let $(\nu',w')\in \mathrm{ArgMin}_{(\nu,w)\in\mathscr{E}} S_m(\nu,w)$ and note that it therefore minimizes the RHS of (C.3) as well. Consider an equilibrium (ν'',w'') such that $\nu''(m)=m$, $(\nu'',w'')|_{(m,m,m-1)} = (\underline{\nu},\underline{w})|_{m-1}$, $(\nu'',w'')|_{(m,m),i} = (\nu',w')|_i$, for i=m,m+1, and $(\nu'',w'')|_{m+1} = (\nu',w')|_{m+1}$. Notice that, after m, (ν'',w'') coincides with $(\underline{\nu},\underline{w})$, since (ν',w') is a continuation of $(\underline{\nu},\underline{w})$; after m+1, it coincides with (ν',w'). Therefore, it immediately follows that such an equilibrium exists. Observe that

$$\Pi^{\nu'',w''}(m) + m[W^{\nu'',w''}(m)-W_U] = \pi(m) + \delta S_m(\nu',w') \leq S_m(\nu',w')(1-\delta) + \delta S_m(\nu',w') = S_m(\nu',w').$$

where the inequality follows from $\pi(m) \leq (1-\delta)\underline{\Pi}(m) \leq (1-\delta)S_m(\nu',w')$.

Therefore,

$$S_m(\nu'',w'') \leq (1-\alpha)S_m(\nu',w') + \alpha[\Pi^{\nu',w'}(m+1) + \min[\nu'(m+1),m][W^{\nu',w'}(m+1)-W_U]]$$

Now, (C.3) must hold, for otherwise the last inequality would lead to a contradiction $S_m(\nu'',w'') < S_m(\nu',w')$. Clearly, if $\pi(m)<(1-\delta)\underline{\Pi}(m)$, the above argument implies that (C.3) holds strictly. □

__Claim C.1.4__: (i) For all $m<\underline{N}$, $\underline{\Pi}(m)$ is strictly increasing.

(ii) For all $m>\underline{N}$, $\underline{\Pi}(m) = \underline{\Pi}(\underline{N})$.

__Proof__: (i) The proof consists of two steps.

__Step 1__: If $\pi(m)>(1-\delta)\underline{\Pi}(m-1)$, then $\underline{\Pi}(m)>\underline{\Pi}(m-1)$.

To see this, observe that the following inequality holds

$\{\pi(m) + m\underline{\Pi}(m-1) + \delta \mathrm{Min}_{(\nu,w)\in\mathscr{E}} S_m(\nu,w)\}/(m+1) > \underline{\Pi}(m-1)$,

since, for any equilibrium (ν,w), $S_m(\nu,w) \geq \underline{\Pi}(m) \geq \underline{\Pi}(m-1)$. Proposition A-(ii) then implies that there exists an equilibrium (ν',w') such that $\nu'(m)=m$ and $\Pi^{\nu',w'}(m)$ is equal to the LHS. Proposition A-(i) immediately implies that $\Pi^{\nu',w'}(m) = \mathrm{Min}\{\Pi^{\nu,w}(m) \mid (\nu,w)\in\mathscr{E} \text{ and } \nu(m)=m\}$.

Therefore, $\underline{\Pi}(m) = \Pi^{\nu',w'}(m) > \underline{\Pi}(m-1)$ □

<u>Step 2</u>: If $m<N^s$, then $\underline{\Pi}(m)<\underline{\Pi}(m+1)$.

Clearly, $\underline{\Pi}(1)>\underline{\Pi}(0)$. Let $m<N^s$ be such that $\underline{\Pi}(m)>\underline{\Pi}(m-1)$. If $\pi(m) > (1-\delta)\underline{\Pi}(m)$, then $m<N^s$ implies $\pi(m+1) > (1-\delta)\underline{\Pi}(m)$ and, by Step 1, $\underline{\Pi}(m)<\underline{\Pi}(m+1)$. Consider m such that $\pi(m)\leq(1-\delta)\underline{\Pi}(m)$. Suppose to the contrary that $\underline{\Pi}(m+1)=\underline{\Pi}(m)$. This implies that $\underline{W}(m+1)=W_U$ and hence

$\mathrm{Min}_{(\nu,w)\in\mathscr{E}}\{\Pi^{\nu,w}(m+1) + \min[\nu(m+1),m][W^{\nu,w}(m+1)-W_U]\} = \underline{\Pi}(m)$.

This together with Claim C.1.3 imply

$$\mathrm{Min}_{(\nu,w)\in\mathscr{E}} S_m(\nu,w) \leq \underline{\Pi}(m) \leq \mathrm{Min}_{(\nu,w)\in\mathscr{E}} S_{m+1}(\nu,w).$$

This together with (C.2), Claim C.1.2 and the monotonicity of $\pi(m)$ for $m<N^s$ yield

$$\begin{aligned}\underline{\Pi}(m+1) &\geq \{\pi(m+1) + (m+1)\underline{\Pi}(m) + \delta \mathrm{Min}_{(\nu,w)\in\mathscr{E}} S_{m+1}(\nu,w)\}/(m+2)\\ &= \{(m+1)[\pi(m+1) + m\underline{\Pi}(m) + \delta \mathrm{Min}_{(\nu,w)\in\mathscr{E}} S_{m+1}(\nu,w)]/(m+1) + \underline{\Pi}(m)\}/(m+2)\\ &> \{\pi(m) + m\underline{\Pi}(m-1) + \delta \mathrm{Min}_{(\nu,w)\in\mathscr{E}} S_m(\nu,w)\}/(m+1) = \underline{\Pi}(m),\end{aligned}$$

where the strict inequality follows from a direct comparison between $\underline{\Pi}(m)$ and $\{\pi(m+1) + m\underline{\Pi}(m) + \delta \mathrm{Min}_{(\nu,w)\in\mathscr{E}} S_{m+1}(\nu,w)\}/(m+1)$. Both are weighted averages using the same weights, but in the latter these weights multiply larger terms. This contradicts the supposition that $\underline{\Pi}(m)=\underline{\Pi}(m+1)$. Therefore, $\underline{\Pi}(m)<\underline{\Pi}(m+1)$.

By the definition of $\underline{N}$ and the monotonicity of π, for all $m\in[N^s,\underline{N})$, $\pi(m)>(1-\delta)\underline{\Pi}(m)$. Thus, Steps 1 and 2 together cover the entire range $[0,\underline{N})$ required in part (i).

(ii) Suppose to the contrary that there is $m>\underline{N}$ such that $\underline{\Pi}(m)>\underline{\Pi}(m-1)$. Since by the definition of $\underline{N}$, $\pi(m)<(1-\delta)\underline{\Pi}(m)$, it follows from Claim C.1.3 that (C.3) holds strictly. This implies in turn that $\underline{\Pi}(m+1)>\underline{\Pi}(m)$, for otherwise any equilibrium continuation after m is also an equilibrium continuation after m+1 implying that (C.3) holds with equality, contrary to the above conclusion. This reasoning then goes on to imply $\underline{\Pi}(M)>\underline{\Pi}(M-1)$. But the argument of Claim C.1.3 adapted for this case leads to the contradiction $\mathrm{Min}_{(\nu,w)\in\mathscr{E}}S_m(\nu,w)<\mathrm{Min}_{(\nu,w)\in\mathscr{E}}S_m(\nu,w)$. Therefore, for any $m>\underline{N}$, $\underline{\Pi}(m)=\underline{\Pi}(m-1)$ and hence $\underline{\Pi}(m)=\underline{\Pi}(\underline{N})$. □

Claims C.1.2 and C.1.4 together establish Proposition C.1. □□

__Proposition C.2__: For $m\leq\underline{N}$,

$$\underline{\Pi}(m) = f(m) - m\underline{w}(m,m) + \delta[(1-\alpha)\underline{\Pi}(m) + \alpha\underline{\Pi}(m+1)]$$

__Proof__: __Step 1__: For $i\leq\underline{N}+1$, $q\in[0,i]$ and $(\nu,w)\in\mathscr{E}$ such that $\nu(i)=i$,

$$\underline{\Pi}(i) + q\underline{W}(i) \leq \mathrm{Min}_{(\nu,w)\in\mathscr{E}}[\Pi(i,i;\nu,w) + qW(i,i;\nu,w)].$$

To see that, let $(\nu',w') \in \mathrm{Argmin}\{\Pi(i,i;\nu,w)+qW(i,i;\nu,w):\ (\nu,w)\in\mathscr{E} \text{ and } \nu(i)=i\}$. From Proposition A-(i) and since $\underline{\Pi}(i-1)\leq\Pi^{\nu',w'}(i,i,i-1)$,

$$\{\pi(i) + i\underline{\Pi}(i-1) + \delta S_i[(\nu',w')|_{(i,i)}]\}/(i+1) \geq \underline{\Pi}(i-1).$$

Therefore, by Proposition A-(ii), there exists an equilibrium (ν'',w'') such that $\nu''(i)=i$, $(\nu'',w'')|_{(i,i,i-1)}=(\underline{\nu},\underline{w})|_{i-1}$, $(\nu'',w'')|_{(i,i),i}=(\nu',w')|_i$ and $(\nu'',w'')|_{(i,i),i+1}=(\nu',w')|_{i+1}$. Since on the equilibrium path the only difference between (ν'',w'') and (ν',w') is that $w''(i,i)\geq w'(i,i)$ and since $q\leq i$, it follows that $\Pi^{\nu'',w''}(i) + qW^{\nu'',w''}(i) \leq \Pi^{\nu',w'}(i) + qW^{\nu',w'}(i)$. Equilibrium condition (2.3) implies $W^{\nu'',w''}(i) = \Pi^{\nu'',w''}(i) - \underline{\Pi}(i-1) + W_U$. Hence,

$\Pi^{\nu'',w''}(i) + qW^{\nu'',w''}(i) = (q+1)\Pi^{\nu'',w''}(i) - q\underline{\Pi}(i-1) + qW_U \geq (q+1)\underline{\Pi} - q\underline{\Pi}(i-1) + qW_U = \underline{\Pi}(i) + q\underline{\Pi}(i)$.

Therefore, $(\underline{\nu},\underline{w})$ minimizes $\Pi + qW$ over all equilibria starting with i.

Step 2: There exists $(\nu',w') \in \mathrm{Argmin}_{(\nu,w)\in\mathscr{E}} S_m(\nu,w)$ such that $\Pi^{\nu',w'}(m)=\underline{\Pi}(m)$ and $\Pi^{\nu',w'}(m+1)=\underline{\Pi}(m+1)$.

Consider an equilibrium (ν,w) in the above Argmin. Suppose that $\nu'(m)=j$ and $\nu'(m+1)=k$. Recall from Step 1 that $\underline{\Pi}(j)+j[\underline{W}(j)-W_U]\le\Pi^{\nu,w}(j)+j[W^{\nu,w}(j)-W_U]$ and $\underline{\Pi}(k)+\min[k,m][\underline{W}(k)-W_U]\le\Pi^{\nu,w}(k)+\min[k,m][W^{\nu,w}(k)-W_U]$. Let $x,y\in[0,1]$ be such that $x\underline{\Pi}(j)+(1-x)\Pi^{\nu,w}(j)=\underline{\Pi}(m)$ and $y\underline{\Pi}(k)+(1-y)\Pi^{\nu,w}(k)=\underline{\Pi}(m+1)$. By the convexity of the equilibrium payoff sets[15], there exists an equilibrium (ν',w') such that $\Pi^{\nu',w'}(j)=\underline{\Pi}(m)$ and $\Pi^{\nu',w'}(k)=\underline{\Pi}(m+1)$. Furthermore, $S_m(\nu',w')\le S_m(\nu,w)$ and hence $(\nu',w') \in \mathrm{Argmin}_{(\nu,w)\in\mathscr{E}} S_m(\nu,w.)$

Proposition C.2 now follows from Proposition A. ☐

Propositions C.1-2 together establish Theorem II-(i)

Proposition C.3: If the arrival rate is fast enough, then

$$\underline{\Pi}(m) = \{\pi(m) + m\underline{\Pi}(m-1) + \delta(1-\alpha)\underline{\Pi}(m) + \delta\alpha\underline{\Pi}(m+1)\}/(m+1)$$

Proof: If the "arrival rate" is fast enough, there exist equilibria (ν,w) such that $\Pi^{\nu,w}(0)>\underline{\Pi}$. By the assumed convexity[14] of the equilibrium payoff sets, there exist equilibria (ν',w') and (ν'',w'') such that $\Pi^{\nu',w'}(0) = \underline{\Pi}(m)$ and $\Pi^{\nu'',w''}(0) = \underline{\Pi}(m+1)$. This implies that there exists an equilibrium (ν,w) such that $\nu(m)=\nu(m+1)=0$, $(\nu,w)|_{(m,0)}=(\nu',w')|_0$ and $(\nu,w)|_{(m,0)}=(\nu'',w'')|_0$. Now, $\Pi^{\nu,w}(m)+\nu(m)W^{\nu,w}(m)=\underline{\Pi}(m)$ and $\Pi^{\nu,w}(m+1)+\nu(m+1)W^{\nu,w}(m+1)=\underline{\Pi}(m+1)$. Obviously, this together with the observation

$$\mathrm{Min}_{(\nu,w)\in\mathscr{E}} S_m(\nu,w) \ge \delta[(1-\alpha)\underline{\Pi}(m) + \alpha\underline{\Pi}(m+1)]$$

imply that this inequality holds in fact as an equality. Substituting this in (C.1) when it holds as equality we get the result. ☐

This concludes the proof of Theorem II-(i),(ii). The proof of II-(iii) follows.

Proposition C.4: If $\alpha/(1-\delta)$ is sufficiently small, $(\underline{\nu},\underline{w})$ coincides with the stationary N^0 equilibrium.

<u>Proof</u>: We first show that, if $\alpha/(1-\delta)$ is sufficiently small, then $\underline{N}=N^0$. Let $\varepsilon=[\pi(N^0) - \pi(N^0+1)]/2 > 0$. From Claim B.5.1 (proof of Theorem I), there is sufficiently small $\alpha/(1-\delta)$ such that $\underline{\Pi}(N^0-1)\geq\sum_{i=0}^{N^0-1}\pi(i)/(1-\delta)N^0-\varepsilon$. From the definition of N^0, $\sum_{i=0}^{N^0-1}\pi(i)/N^0 = \pi(N^0)$. Therefore,

$(1-\delta)\underline{\Pi}(N^0+1) \geq (1-\delta)\underline{\Pi}(N^0-1) \geq \sum_{i=0}^{N^0-1}\pi(i)/N^0-(1-\delta)\varepsilon = \pi(N^0)-(1-\delta)\varepsilon > \pi(N^0+1)$.

Thus, by the definition of $\underline{N}$ (see beginning of the proof of Theorem II), $\underline{N}=N^0$.

Now, this together with Claim B.5.1 and with Proposition C.2 imply that, for $m\leq N^0$, $\underline{w},\underline{W}$ and $\underline{\Pi}$ satisfy system (3.1-3) for $N=N^0$. Therefore, $(\underline{\nu},\underline{w})$ coincides with the N^0 stationary equilibrium. □□

This completes the proof of Theorem II. QED

<u>Remark: The convexity of V(m)</u>

The convexity of equilibrium payoff sets V(m) is required to establish the equality $\underline{\Pi}(m) = f(m) - m\underline{w}(m,m) + \delta[(1-\alpha)\underline{\Pi}(m) + \alpha\underline{\Pi}(m+1)]$ in Theorem II-(i), and to establish the existence of an equilibrium (ν,w) such that $\Pi^{\nu,w}(0) = \underline{\Pi}(m)$ in Theorem II-(ii). Both of these results are relatively minor for this paper. The former observation is used later in Theorem II-(iii) to establish the exact coincidence of the minimum profit equilibrium and the stationary N^0 equilibrium when arrival is slow. However, without that observation, we can still argue that these two equilibria are arbitrarily close. Thus, convexity of V(m) is not crucial for this paper, but for completeness let us discuss it.

V(m) can be convexified by assuming that everybody observes the realization of a public randomizing device at the beginning of each round. With the signals, the component added to the history in period t is of the form $(m_{t,1},\theta_{t,1},n_{t,1},\ldots,m_{t,k_t},\theta_{t,k_t},n_{t,k_t})$, where the θ's are independent random draws from the uniform distribution on [0,1]. Let H^{θ} denote the set of histories ending with a signal, so that $H=H^{-}\bigcup H^{\theta}\bigcup H^{+}$, where H^{-} and H^{+} are as before. The employment decision is made now after a history in H^{θ}. The notation will be kept close to the previous analysis as follows. For $h\in H^{-}$, think of $\nu(h)$ as a function from [0,1] to $\{0,\ldots M;C\}$ meaning $\nu(h)(\theta)$ is the choice prescribed by ν after history $h,\theta\in H^{\theta}$. Similarly, given (ν,w) and a history $h\in H^{-}$, let $\Pi^{\nu,w}(h)$ and $W^{\nu,w}(h)$ be functions defined on [0,1] which give the expected profit and employee utility respectively, for each realization of the public signal after h, i.e., $\Pi^{\nu,w}(h)(\theta)=\Pi[h,\theta,\nu(h)(\theta);\nu,w]$ and $W^{\nu,w}(h)(\theta)=W[h,\theta,\nu(h)(\theta);\nu,w]$. Let $E[\Pi^{\nu,w}(h)]$, $E[W^{\nu,w}(h)]$ and $E[\theta(h)W^{\nu,w}(h)]$ denote the expectations of these functions with respect to θ, e.g., $E[\Pi^{\nu,w}(h)]=E_{\theta}\{\Pi[h,\theta,\nu(h)(\theta);\nu,w]\}$. Obviously, the set of expected payoffs, given an initial pool m, $\{(E[\Pi^{\nu,w}(m)],E[W^{\nu,w}(m)]):\ (\nu,w)\in\mathscr{E}\}$ is now convex.

It is a routine matter to verify that <u>all</u> the arguments of Parts A, B and C of this appendix go through to the public signals case after the following simple modifications. Replace everywhere the expressions $\Pi^{\nu,w}(h)$, $W^{\nu,w}(h)$, $\nu(h)W^{\nu,w}(h)$ and $\min[m,\nu(h)]W^{\nu,w}(h)$ by their expectations w.r.t. the last signal: $E[\Pi^{\nu,w}(h)]$, $E[\nu(h)W^{\nu,w}(h)]$ and $E\{\min[m,\nu(h)]W^{\nu,w}(h)\}$. Wherever $\nu(h)$ is treated as a number, think of it as a constant function, e.g., read $\nu(h)=n$ as saying $\nu(h)(\theta)=n$ for all θ.

<u>Part D: Stationary equilibria</u>

<u>Proof of Proposition 6</u>: Let (ν,w) be a stationary equilibrium. Let $\hat{\Pi}(\ell(h))$, $\hat{W}(\ell(h))$ and $\hat{w}(\ell(h))$ denote $\Pi(h;\nu,w)$, $W(h;\nu,w)$ and $w(h)$ respectively. Consider the equilibrium path starting with m=0. Stationarity implies that this path must eventually reach some stationary level or stationary cycle. That is, there are numbers k and K, $k\leq K$, such that $\nu(m)=m$, for $m\leq K$, and $\nu(K+1)=k$.

Suppose $k=K$. This implies that, for $m \leq K+1$, the path of (ν, w) is that of a stationary K equilibrium and that $K \leq N^0$. In other words, $\hat{\Pi}(K)=\Pi(K;K)$, $\hat{W}(K)=W(K;K)$ and $\hat{w}(K)=w(K;K)$. Being an equilibrium implies that $\hat{\Pi}(K+1) \leq \hat{\Pi}(K)$. Thus, from condition (2.3), $\hat{W}(K+1)=W_U$. Therefore,

$$\hat{w}(K+1)+\delta[(1-\alpha)K/(K+1)]\hat{W}(K)+\delta[1-(1-\alpha)K/(K+1)]W_U \leq W_U$$

which upon rearrangement and substitution yields

$$(\#) \qquad \hat{w}(K+1) \leq (1-\delta)W_U-\delta(1-\alpha)[w(K;K)-(1-\delta)W_U]K/(K+1)(1-\delta).$$

Hence,

$$\begin{aligned}(\#\#) \quad & \hat{\Pi}(K+1)-\hat{\Pi}(K) = \hat{\Pi}(K+1)-\Pi(K;K) \geq f(K+1) - f(K) - (K+1)\hat{w}(K+1) + Kw(K;K) \geq \\ & \Delta f(K+1) - (1-\delta)W_U + [w(K;K)-(1-\delta)W_U]K(1-\delta\alpha)/(1-\delta) = \\ & \Delta f(K+1) - (1-\delta)W_U + [\Psi(K)-(1-\delta)W_U](1-\delta\alpha)K(K+1)/2[(K+1)(1-\delta)+\delta\alpha] = \\ & \{[\Psi(K)-(1-\delta)W_U](1-\delta\alpha)(K+1)(K+2) - \delta[2(K+1)-\alpha(2K+1)][\Delta f(K+1)-(1-\delta)W_U]\} \\ & /2[(K+1)(1-\delta)+\delta\alpha]\end{aligned}$$

The first inequality makes use of the fact that, according to the policy, the continuation profit is $\Pi(K)$ after K and at least that after K+1; the second inequality follows from (#); the equality that follows is obtained by substitution from (3.4); the fourth is gotten from rearrangement using the definition of Ψ. The last inequality is obvious for $K \in [N^S, N^0)$, since then $\Delta f(K+1)-(1-\delta)W_U<0$ while $[w(K;K)-(1-\delta)W_U]>0$ by the definition of N^0; for $K<N^S$, this is seen from the third expression in the chain since then both $\Delta f(K+1)-(1-\delta)W_U>0$ and $[w(K;K)-(1-\delta)W_U]>0$.

Thus, $\hat{\Pi}(K+1)-\hat{\Pi}(K)>0$ for all $K<N^0$, implying that $K \geq N^0$. Hence, $K=N^0$.

Suppose $k<K$. Since $\hat{\Pi}(k) \leq \hat{\Pi}(k+1) \leq \ldots \leq \hat{\Pi}(K) \leq \hat{\Pi}(k)$, we have $\Pi(i) = \Pi(j)$, for all $i,j \in [k,K]$. This implies $\hat{W}(k+1)=W_U$ and hence

$$\hat{\Pi}(k) = f(k) - k\hat{w}(k) + \delta\hat{\Pi}(k)$$

$$\hat{W}(k) = \hat{w}(k) + \delta(1-\alpha)\hat{W}(k) + \delta\alpha W_U$$

Since for $m<k$, the path of (ν,w) is that of a stationary k equilibrium, it follows from the above equations that $\hat{w}(k)=w(k;k+1)$. We now can repeat the same argument used in the first part to show and that $\hat{\Pi}(k+1)-\hat{\Pi}(k)>0$ for all $k<N^0$, implying that $k=N^0$ (the argument is in fact easier since $w(k;k+1)>w(k;k)$). Therefore, $\hat{W}(i)=W_U$ for all $i\in[k,K]$ and hence $\hat{w}(i)=(1-\delta)W_U$ for all $i\in[k,K]$. But this contradicts $\Pi(i) = \Pi(j)$, for all $i,j\in[k,K]$. Therefore, on the path starting at $m=0$, there may not be a cycle $[k,K]$ with $k<K$.

Thus, the path starting from $m=0$ coincides with the path of stationary N^0 equilibrium. Finally, it remains to establish that there cannot be another cycle $[k,K]$, $N^0<k\leq K$, which would be reached by a path starting from some $m>N^0$. If there were such a cycle, then for $i\in[k,K]$, $f(i)-i\hat{w}(i) \geq f(N^0)-N^0(1-\delta)W_U$. This implies $\hat{w}(i)<(1-\delta)W_U$, for all $i\in[k,K]$ in contradiction to $\hat{W}(i)\geq W_U$. QED

Footnotes

1. Another class of intermediate situations, between the neo-classical firm and the fully unionized one, is addressed by the small literature on unionized firms that face two separate unions, e.g., Davidson (1988), Horn and Wolinsky (1988a,b) and Ben Porat (1989).

2. SZ discuss this issue in more detail and provide references to the legal literature on such "at will" employment contracts.

3. These extensions are discussed in more detail in an earlier working paper version Wolinsky (1996).

4. The upper bound M saves us some extra work on establishing the boundedness of the problem, but it is almost obvious that in no equilibrium will the employment get near M and that this assumption is not crucial.

5. Rearrangement yields the alternative formula

$$\Psi(n) = \frac{2}{n}\left[f(n) - \frac{1}{n+1}\sum_{i=1}^{n} f(i)\right]$$

6. Indeed, observe that the RHS of equation (3.3), which is the one equation that brings together $W(n;N)$ and $\Pi(n;N)$, approaches 0 as ε goes to zero. The limit version of (3.3) is $W(n;N) - W_U = \partial\Pi(n;N)/\partial n$.

7. There are two limit operations here: first with respect to the length of the period and then with respect to the size of a worker. Notice, however, that the same continuous labor limit can be reached by going directly from the discrete version to the limit with respect to the worker's size. To keep the equivalence between successive scenarios, as we go to this limit, the speed of arrival must increase at roughly the same rate at which the worker's size shrinks. This means that the effective length of a period shrinks to zero in this process anyway. So there is no real issue about the order of the limits. If we break this connection and let the worker's size shrink, but keep the speed of arrival constant, it would be like the exercise of varying the speed of arrival that we discuss later.

8. An alternative way to state the same observation is that, for any feasible n and N such that $n<N$, $w(n;n)<w(n;N)$. That is, at a given n, the wage is lower if this n happens to be the target level than it would be otherwise.

9. It is clear why the curvature of the profit function is proportional to the rate of change of the instantaneous profit $f(n)-nw(n;N)$ near N. Since $\Pi_1(n;N)$ captures the increase in profit due to trading instantaneous profit on the path to N for earlier attainment of N, and this in turn changes at the rate at which the instantaneous profit does.

10. It is interesting to note, however, that in the continuous limit of a variation on the model mentioned in Section 7 below in which arrival rates are selected optimally by the firm at each instant, the independence of $w(n;N)$ of the arrival rates emerges as a result.

11. This follows from well known results in dynamic programming which have already been applied repeatedly in the repeated games literature.

12. The reader who will look at these references will also surely appreciate the substantial complexity that was avoided by the modeling approach adopted above.

13. See, for example, Ash (1972).

14. That part of the proof of Theorem II does not require the convexity of V(m), so no extra assumptions are added here.

15. Recall that for now we are simply assuming the convexity of the equilibrium payoff sets. But in a remark following the proof of Theorem II we will explain how the convexity can be achieved through public randomization.

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Figure 1