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Dynamic Voluntary Contribution to a Public Project

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ABSTRACT: We consider the dynamic private provision of funds to a project that generates a flow of public benefits. Examples include fund drives for public television or university buildings. The games we study have complete information about payoffs, allow each player to contribute each period, and let each player observe only the aggregate of the other players' past contributions. The symmetric Nash equilibrium outcomes are characterized and shown to be also perfect Bayesian equilibrium outcomes. If the number of periods in which contributions are accepted is large enough, and the players are patient or the period length is short enough, equilibria exist in which the project is eventually or asymptotically completed. Some equilibria with these features are Markov perfect. In some, the time to completion shrinks to zero with the period length — free riding vanishes in the limit. These results are in contrast to those of other models in which allowing repetitive contributions worsens the free riding problem.

KEYWORDS: dynamic games, public goods, private provision, voluntary contribution, free riding

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1. Introduction

We study a dynamic game of voluntary contribution to a public project. The game consists of a number of rounds in which individuals may contribute to the capital stock of a public project that generates a flow of public benefits. The game is a stylized model, for example, of a campaign to raise funds for public television or university buildings. A key feature is that individuals may contribute whenever, however much, and as often as they wish. The project may generate partial benefits even before it is completed, as is typical of ongoing projects for building a road network, park system, or library.

We show that if the public is kept abreast of the progress made towards achieving a contribution goal, then equilibria which reach the goal exist if the number of periods in which contributions are allowed is large, and discounting is low or the period length small. The only necessary inefficiency caused by free riding is a delay in the completion of the project. Even this inefficiency may be inconsequential, as the delay vanishes as the period length shrinks to zero. Thus, the results indicate that dynamic considerations can alleviate the free riding inefficiencies exhibited by static voluntary contribution games.[1]

This positive result must of course be tempered by the observation that our game has multiple equilibria, some of which are not efficient. On the other hand, our positive result is surprising in view of an intuition for why allowing players to contribute repetitively might increase the incentive to free ride: it effectively creates more players (future ones) upon which current players can free ride. Fershtman and Nitzan (1991) obtain a negative result based on this logic; our results differ because we consider a discrete rather than a continuous time game, and a broader class of equilibria. Admati and Perry (1991) also reach a negative conclusion; our results differ mostly because we allow any player to contribute in any period, rather than requiring them to contribute in alternate periods. These and other related papers are discussed later in Section 7.

We consider for simplicity a public benefit function that is linear in the aggregate cumulative contribution until it reaches a "provision point" that completes the project.

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[1] Static games are studied, e.g., by Andreoni (1988), Bergstrom, et al. (1986), Bernheim (1986), Cornes and Sandler (1996), Palfrey and Rosenthal (1984), and Varian (1994).

Benefits may jump upwards at the provision point. A familiar polar case is a binary project which generates no benefits until it is completed, like a bridge that cannot be used until the last girder is in place. A more general example is the construction of a road network. Marginal benefits prior to completion are positive because each new road can be used immediately, but benefits rise discontinuously at completion because the linking road increases benefits disproportionately. Another source of a benefit jump occurs if the provision point is the goal set for a fund-raising campaign, and a third party has committed to contributing a "challenge bonus" if the goal is met. If the benefit jump is positive, equilibria are obtained that complete the project in finite time; otherwise, equilibria are obtained that complete the project asymptotically in the sense that the aggregate cumulative contribution converges to the provision point.

A complication arises because each player observes only the *aggregate* of the past contributions of the others. In the literature such imperfect observability is typically circumvented by restricting attention to Markov perfect strategies, which in this paper are strategies that require a player's contribution to depend only on the aggregate cumulative contribution to date. But this restriction impedes the ability to punish free riders. We accordingly focus first on the broader class of perfect Bayesian equilibria, so that a player's contributions can depend on his inferences about past individual contributions.

This imperfect observability aggravates a perfection problem caused when benefits jump at completion. In this case, once contributions have risen enough so that only a small contribution is needed to reach the provision point, every continuation equilibrium must complete the project. Deviators then cannot be punished by "grim" strategies that call for all players to never again contribute. However, if the deviator can be identified, the most punishing continuation equilibrium still requires the deviator to complete the project on his own. We show that unilateral deviators can effectively be identified, and so always be punished by the others withholding all future contributions. Consequently, every Nash equilibrium outcome is also a perfect Bayesian equilibrium outcome.[2]

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[2] We give the formal argument only for symmetric outcomes, but it will be clear that it holds also for asymmetric outcomes. See Remark 2 in Section 4.

The paper proceeds with a description of the dynamic environment in Section 2. In Section 3 the game is defined and its symmetric Nash equilibrium outcomes are characterized. They are shown to be perfect Bayesian equilibrium outcomes in Section 4. In Section 5 are presented existence conditions and descriptions of equilibria which complete, or asymptotically complete, the project. This section also contains "Coase conjecture" results showing that as the period length shrinks, the completion delay disappears. In Section 6 are considered Markov perfect equilibria that are characterized by a sequence of increasing "goals"; in each period contributions raise the aggregate cumulative contribution to the smallest goal so far unachieved, until the project is completed. Symmetric equilibria of this form exist only under more parameter restrictions than are required for completion by perfect Bayesian equilibria. However, this is not true of *asymmetric* Markov perfect equilibria. Related literature is discussed in Section 7, and conclusions appear in Section 8. Proofs for Section 4 (6) are in Appendix A (B).

2. Environment

The set of players is $N \equiv \{1,\ldots,n\}$, with $n \geq 2$. Player i contributes $z_i(t) \geq 0$ of a private good to a public project in periods $t = 0,1,\ldots$. Contributions become the non-depreciating capital the project uses to create a public good or service consumed each period, and they are nonrefundable. If the aggregate cumulative contribution reaches the *provision point*, $X^* > 0$, the project is *complete* and benefits are constant in the remaining periods.

Let $z(t) \equiv (z_1(t),\ldots,z_n(t))$, $Z(t) \equiv \sum_{j\in N} z_j(t)$, and $Z_i(t) \equiv Z(t) - z_i(t)$. At the end of period t, the individual and aggregate cumulative contributions are, respectively,[3]

$$x_i(t) \equiv \sum_{\tau \leq t} z_i(\tau) \text{ and } X(t) \equiv \sum_{j\in N} x_j(t).$$

For the sake of brevity, we sometimes refer to $X(t)$ simply as the *cumulation*.

Each player's discount factor is $\delta = e^{-r\ell} \in (0,1]$, where $r \geq 0$ is the discount rate and $\ell \geq 0$ is the period length, i.e., the minimum time required between contributions.

[3] Set $x_i(-1) \equiv 0$.

The players have the same quasilinear preferences. A player's valuation of the public good is given by a *benefit function* f, where $f(X)$ is the player's discounted present value of the benefit flow from the public goods that would be generated if the capital of the project were to equal X in all periods. Thus, in period t the public good generated by $X(t)$ yields a benefit of $(1-\delta)f(X(t))$. Given a contribution sequence $\{z\} = \{z(t)\}_{t=0}^{\infty}$, player i's payoff is

$$U_i(\{z\}) \equiv \sum_{t=0}^{\infty} \delta^t \left[(1-\delta)f(X(t)) - z_i(t)\right]. \tag{2.1}$$

It will be useful to rewrite this as

$$U_i(\{z\}) = (1-\delta) \sum_{t=0}^{\infty} \delta^t \left[f(X(t)) - x_i(t)\right]. \tag{2.2}$$

One special case is the static situation in which contributions can be made only in period 0. Then the expressions above reduce to $U_i = (1-\delta)\big(f(X(0)) - x_i(0)\big)$. Modulo the constant $1-\delta$, this is the payoff function of the familiar static game in which each person's strategy is a single contribution. Another special case is that of no discounting. Taking $\delta \to 1$ yields $U_i = \lim_{t\to\infty}\left[f(X(t)) - x_i(t)\right]$, which is appropriate for modeling short fund drives at the end of which all costs and benefits are borne. The $\delta < 1$ case is more appropriate for projects that take time to complete and generate benefits along the way.

Remark 1. Expressions (2.1) and (2.2) yield different interpretations. According to a *flow interpretation*, each player receives a private good income each period, and can bank at the interest rate $1/\delta - 1$. In period t player i consumes $c_i(t)$ and contributes $z_i(t)$ of his income, and consumes a public benefit of $(1-\delta)f(X(t))$. His budget constraint is $\sum_{t=0}^{\infty}\delta^t(c_i(t) + z_i(t)) = w$, where w is the present value of his income. Substituting this into the expression $U_i = \sum_{t=0}^{\infty}\delta^t\big((1-\delta)f(X(t)) + c_i(t)\big)$, and dropping w, yields (2.1).

On the other hand, expression (2.2) corresponds to a *stock interpretation*, according to which each player starts with a stock w of an indestructible good. In period t player i chooses how much to contribute from the amount he then has of this stock. At the end of the period his remaining stock, $w - x_i(t)$, generates a private benefit on a one-to-one basis, and the aggregate cumulative amount contributed to the project, $X(t)$, generates a public benefit of $f(X(t))$. The player's utility in period t is $f(X(t)) + w - x_i(t)$. Normalizing by $1-\delta$, and dropping w, the discounted sum of these utilities is (2.2).

Though they generate the same payoff function, the two interpretations imply different constraints. According to the stock interpretation, a player's contribution should satisfy $z_i(t) \le w - x_i(t-1)$. According to the flow interpretation, $\Sigma_{t=0}^{\infty} \delta^t z_i(t) \le w$ is required (assuming consumption is nonnegative). These are different constraints unless $\delta = 1$. However, if $w > X^*$ then it is feasible for each player to fund the project alone, and these constraints do not bind and can be ignored. We do ignore them.

We assume a specific form for the benefit function:

$$f(X) \equiv \begin{cases} \lambda X & \text{for } X < X^* \\ V & \text{for } X \ge X^*. \end{cases} \tag{2.3}$$

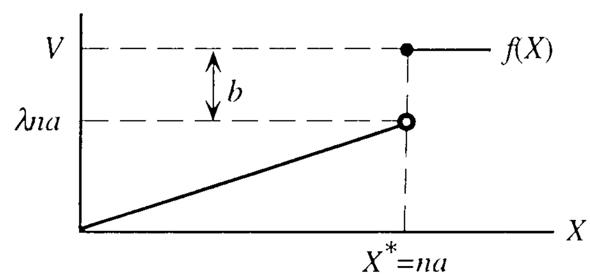

Figure 1

The personal marginal benefit of a contribution prior to completion is λ, and the benefit from a completed project is V. The possible *benefit jump* at completion is

$$b \equiv V - \lambda X^*. \tag{2.4}$$

We assume $\lambda \ge 0$ and $b \ge 0$, which together are the same as

$$0 \le \lambda \le \frac{V}{X^*}. \tag{2.5}$$

The per-capita contribution required for completion is $a \equiv X^*/n$. We assume

$$a < V < X^*. \tag{2.6}$$

The first inequality insures that the project is worthwhile; the second insures that no player is willing to fund it alone. Assumptions (2.5) and (2.6) together imply $\lambda < 1$: the personal marginal benefit of a non-completing contribution is less than its marginal cost.

One polar case of (2.5) is $\lambda = 0$, a *binary* project yielding a positive benefit of $b = V$ if and only if it is completed. The other polar case is $\lambda = V/X^*$, so that $b = 0$; in this case benefits rise linearly and continuously to their maximum.

A positive benefit jump is a form of strong increasing returns at completion.[4] It can be due to a technological convexity, such as the linking road that completes a network, or it may be due to design. For example, consider a fund drive producing a public good equal to λ times the dollars contributed. Suppose some party has publicly committed to contributing a "challenge bonus" of b/λ dollars if contributions reach the goal $X^* - b/\lambda$. The public good level then jumps by $\lambda(b/\lambda) = b$ when this goal is met. Given the upcoming result that a positive jump can allow the project to be completed in finite time, a campaign organizer may well want to persuade a donor to commit to such a bonus.

Payoffs can now be written in a way useful for dynamic programming. Let the *completion date* T be the smallest t for which $X(t) \geq X^*$, and set $T = \infty$ if the project is never completed. Refer to the contribution sequence as *wasteless* if $X(t) \leq X^*$ for all t. For a wasteless contribution sequence, (2.1) and (2.2) are equivalent to the following:[5]

$$\begin{aligned} U_i(\{z\}) &= \sum_{t=0}^{T} \delta^t \left[\lambda Z(t) - z_i(t) \right] + \sum_{t=T}^{\infty} \delta^t \left[(1-\delta) b \right] \\ &= \sum_{t=0}^{T} \delta^t \left[\lambda Z(t) - z_i(t) \right] + \delta^T b. \end{aligned} \tag{2.7}$$

The first line of (2.7) views the payoff as a discounted sum of benefits and costs that are each borne in just one period; it is as though $\lambda Z(t) - z_i(t)$ is received in each period prior to completion, and $(1-\delta)b$ is received in the completion and each subsequent period.[6]

Given the maintained assumptions, Pareto efficiency requires the project to be completed wastelessly, and to be completed without delay if $\delta < 1$.

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[4] The results for $b > 0$ would be similar if the benefit function were continuous everywhere, but sufficiently convex on a small interval $[X^* - \Delta, X^*]$.

[5] For $T = \infty$ and $\delta = 1$, let $\delta^T = 0$.

[6] If $b = 0$, (2.7) seems to be time separable, like a repeated game payoff. The difference is that here the completion date T is endogenously determined, and a player's set of undominated actions, $[0, X^* - X(t-1)]$, depends on the contribution history.

3. Equilibrium Contributions

The *dynamic contribution game* is defined by the above payoff function, a (*contributing*) *horizon* $\overline{T} \leq \infty$, and the following extensive form:

(i) contributions can be made only in periods $t = 0, 1, \ldots, \overline{T}$;

(ii) the players contribute simultaneously each period; and

(iii) at the start of period t, player i observes only his own and aggregate past contributions: $\{z_i(\tau), Z(\tau)\}_{\tau=0}^{t-1}$.

We consider finite as well as infinite contributing horizons or order to consider whether it is beneficial for the players to (somehow) credibly commit to a deadline. (It is not.)

In this section we characterize the symmetric Nash equilibrium outcomes, starting with the benchmark *static game* in which $\overline{T} = 0$.

The Static Game

Player i's strategy in the static game is a single contribution. We write it now as z_i, dropping the time argument. A strategy profile z results in an aggregate contribution Z, and a payoff of $f(Z) - z_i$ for player i.

The incentive to free ride in the static game increases with λ. To see this, observe that the best reply of player i to a contribution $Z_i < X^*$ by the others is either to contribute nothing, or to complete the project by contributing $z_i = X^* - Z_i$. (Intermediate contributions are inferior because $\lambda < 1$.) The marginal benefit of completing the project, $f(X^*) - f(Z_i) = V - \lambda Z_i$, decreases with λ. The marginal cost of completing the project is $z_i = X^* - Z_i$. Thus, the player completes the project if $\lambda < 1 - (X^* - V)/Z_i$, and does not contribute if this inequality is reversed.

Summarizing, the reaction function is

$$z_i(Z_i) = \begin{cases} 0 & \text{if } X^* - Z_i > c(0) \\ X^* - Z_i & \text{if } X^* - Z_i < c(0), \end{cases} \tag{3.1}$$

where

$$c(0) \equiv \frac{V - \lambda X^*}{1 - \lambda} = \frac{b}{1 - \lambda}. \tag{3.2}$$

(The "0" in $c(0)$ is explained later.) A player wants to complete the project if and only if the required contribution does not exceed $c(0)$. We accordingly refer to $c(0)$ as the *maximal static contribution.*

In the polar case $\lambda = V/X^*$, each player's unique dominant strategy is $z_i = 0$, since $c(0) = b = 0$. In the polar case $\lambda = 0$, profile $(a, \ldots, a)$ is an equilibrium, since $c(0) = b = V$ and $V > a$. Theorem 0 summarizes (its proof is left to the reader).

Theorem 0. *One equilibrium of the static game is always* $(0, \ldots, 0)$. *Any* $z \in \Re^n_+$ *is an equilibrium if and only if* $Z = X^*$ *and each* $z_i \le c(0)$. *Efficient equilibria exist if and only if* $c(0) \ge a$, *and this is the case if and only if* $(a, \ldots, a)$ *is an equilibrium.*

If λ is large, free riding prevents the existence of efficient equilibria. To see this, write the condition for an efficient equilibrium not to exist, $c(0) < a$, as

$$\lambda(n-1)a > V - a. \tag{3.3}$$

The left side of (3.3) is a player's payoff if he contributes zero when the others each contribute a. The right side is the player's payoff if he joins them by contributing a. So if (3.3) holds, a player who believes the others are contributing their share of the project's cost should still not contribute.

The Dynamic Game

Turning now to the case $\bar{T} \ge 0$, we restrict attention to pure-strategy symmetric equilibrium outcomes. In such an outcome, each player's contribution sequence is the same, say $g = \{g(t)\}_{t=0}^{\infty}$. The project is completed once the sum of these contributions reaches a. Let $T(g)$ be the completion period, as defined by

$$\sum_{t=0}^{T(g)-1} g(t) < a \le \sum_{t=0}^{T(g)} g(t). \tag{3.4}$$

If the project is not completed, set $T(g) = \infty$.

A necessary condition for g to be a symmetric equilibrium outcome is that no contributions occur after $\bar{T}$:

$$g(t) = 0 \text{ for all } t > \bar{T}. \tag{3.5}$$

Another requirement is that if the project is completed, it is completed wastelessly, since otherwise a player could gain by reducing his contribution in period $T(g)$ or later. So another necessary condition is

$$T(g) < \infty \quad \Rightarrow \quad \sum_{t=0}^{T} g(t) = a \text{ for all } T \geq T(g). \tag{3.6}$$

Refer to a sequence $g = \{g(t)\}_{t=0}^{\infty}$ of individual contributions as a *candidate outcome* if it is nonnegative and satisfies (3.5) and (3.6).

A candidate outcome g is a Nash equilibrium outcome if and only if no player wishes to unilaterally deviate from it when doing so is met by a maximal feasible punishment. A strategy profile in which all the other players never contribute imposes the maximal conceivable punishment on a unilateral deviator. This punishment is also feasible, since it is imposed by the *grim-g strategy profile*, defined to be the profile in which g is played every period unless an event of the form $Z(\tau) \neq ng(\tau)$ is witnessed, in which case no player ever contributes again. The grim-g profile is feasible because it is based only on observations of aggregate contributions. Thus, g is a Nash equilibrium outcome if and only if the grim-g profile is a Nash equilibrium.

This observation is used below in Theorem 1 to characterize equilibrium outcomes. We shall see that the set of completing equilibrium outcomes enlarges as the contributing horizon and the discount factor increase. We pause now to give an intuition for this. Recall that $(a, \ldots, a)$ is not an equilibrium of the static game if

$$\lambda(n-1)a > V - a. \tag{3.3}$$

In fact, for any $\overline{T} \geq 0$, $(a, \ldots, a)$ is not a first-period equilibrium contribution vector. To see why, assume each player $j \neq i$ contributes a in the first period. Then the left side of (3.3) is a lower bound on player i's payoff from deviating in the first period by not contributing, since it would be his payoff if the deviation caused every player never to contribute again. The right side is the player's payoff from not deviating. So (3.3) implies that the player will deviate in the first period from any profile specifying $z(0) = (a, \ldots, a)$.

Suppose now that g is a candidate outcome which completes the project, with $g(0)$ positive but so small that inequality (3.3) is reversed when $g(0)$ replaces a:

$$\lambda(n-1)g(0) < V - a. \tag{3.7}$$

Under the grim-g profile, the left side of (3.7) is a player's payoff if he contributes zero in the first period. This deviation payoff is lower than the deviation payoff shown on the left of (3.3) because the others are now contributing less: a player's incentive to free ride in a period is diminished by reducing the contributions of the others in that period. Contributions in one period can be reduced without preventing completion, so long as later contributions can be increased. The resulting completion delay is of little matter to a patient player — his payoff is still nearly $V-a$. So (3.7) shows that if $\delta \approx 1$, a player who believes g will be played after period zero will not deviate in the first period. In the no-discounting case, (3.7) puts an upper bound of $(V-a)/(\lambda(n-1))$ on the set of symmetric equilibrium first-period contributions. A similar argument puts an upper bound on each $g(t)$, and the satisfaction of these bounds is a necessary and sufficient condition for a candidate outcome to be an equilibrium.[7]

Since it applies to the general case $\delta \le 1$, the proof of Theorem 1 takes a slightly different tack. Two sets of constraints on g are required for it to be an equilibrium outcome. The first constraints are needed, by an argument like that above, to deter downward deviations (free riding). It bounds the contributions from above, and we refer to them as the *under-contributing constraints*:

$$g(t) \le \delta^{T(g)-t}c(0) + \left(\frac{n\lambda - 1}{1-\lambda}\right)\sum_{\tau=t+1}^{T(g)} \delta^{\tau-t} g(\tau) \text{ for } t \le T(g). \tag{3.8}$$

The second set of constraints bound from below each remaining amount required to complete the project, $X^* - \sum_{\tau \le t} g(\tau)$. These constraints are needed to deter impatient players from contributing too much in order to complete the project prematurely. We refer to them as the *over-contributing constraints*:

$$X^* - n\sum_{\tau=0}^{t} g(\tau) \ge (1-\delta^{T(g)-t})c(0) - \left(\frac{n\lambda - 1}{1-\lambda}\right)\sum_{\tau=t+1}^{T(g)} \delta^{\tau-t} g(\tau) \text{ for } t \le T(g)-1. \tag{3.9}$$

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[7] The extension of (3.7) to period t is $(n\lambda - 1)\sum_{\tau<t} g(\tau) + (n-1)\lambda g(t) < V - a$.

Theorem 1. *A candidate outcome g is a symmetric Nash equilibrium outcome if and only if it satisfies the under- and over-contributing constraints,* (3.8) *and* (3.9).

Proof. Let g be a candidate outcome and $T = T(g)$. The grim-g profile gives the payoff

$$U^{eq} \equiv (n\lambda - 1)\sum_{\tau=0}^{T}\delta^{\tau}g(\tau) + \delta^{T}b \tag{3.10}$$

to each player. Three kinds of deviation must be considered. The first is for a player to contribute in some period a wrong amount that does not complete the project, and then to never contribute again. If this deviation is $z_i \neq g(t)$ at $t \leq T$, it yields payoff

$$U^{d1}(z_i,t) \equiv (n\lambda - 1)\sum_{\tau=0}^{t-1}\delta^{\tau}g(\tau) + \delta^{t}\left[\lambda(n-1)g(t) - (1-\lambda)z_i\right]. \tag{3.11}$$

As $\lambda < 1$ and $z_i \geq 0$, we see that $U^{d1}(z_i,t) \leq U^{d1}(0,t)$. A player is thus deterred from these deviations if and only if $U^{eq} \geq U^{d1}(0,t)$ for all $t \leq T$.

The second kind of deviation a player could make, at a date $t \leq T-1$, is to over-contribute exactly enough to immediately complete the project, i.e., to contribute

$$\bar{z}(t) \equiv g(t) + X^{*} - n\sum_{\tau=0}^{t}g(\tau).$$

Since no further contributions will be made, the player's payoff from this deviation is

$$U^{d2}(t) \equiv U^{d1}(0,t) + \delta^{t}\left[b - (1-\lambda)\bar{z}(t)\right]. \tag{3.12}$$

A player is deterred from these deviations if and only if $U^{eq} \geq U^{d2}(t)$ for all $t \leq T-1$.

The final kind of deviation to consider is for a player to contribute a non-completing amount $z_i \neq g(t)$ in some period $t \leq T-1$, and also to contribute later. It is easy to show that any such deviation is dominated by a deviation of one of the previous two kinds. If $b \leq (1-\lambda)\bar{z}(t)$, the deviation is dominated by contributing zero in period t and all subsequent periods. If $b > (1-\lambda)\bar{z}(t)$, the deviation is dominated by completing the project immediately by contributing $\bar{z}(t)$ in period t. Thus, $U^{eq} \geq U^{d1}(0,t)$ for $t \leq T$, and $U^{eq} \geq U^{d2}(t)$ for $t \leq T-1$, are jointly necessary and sufficient for g to be an equilibrium outcome. Rearranging these inequalities yields (3.8) and (3.9), respectively. ■

If the under-contributing constraints hold and the per-capita amount left to be contributed is not less than the maximal static contribution $c(0)$, the over-contributing constraints must be satisfied. This observation is the proof of the following corollary.

Corollary 1. *Let g be a candidate outcome satisfying (3.8), and let $T = T(g)$. If (i) $c(0) = 0$, (ii) $T = 0$, or (iii) $T < \infty$ and $g(T-1) + ng(T) \geq c(0)$, then g is a Nash equilibrium outcome.*

Proof. If any of the conditions (i) - (iii) hold, then

$$g(t) + n \sum_{\tau=t+1}^{T} g(\tau) \geq c(0) \text{ for } t \leq T-1. \tag{3.13}$$

Theorem 1 therefore implies the result once we show that (3.8) and (3.13) imply (3.9). Given (3.8), the right side of (3.9) is no greater than $c(0) - g(t)$. The left side of (3.9) is no less than $n\sum_{\tau=t+1}^{T} g(\tau)$. Thus, (3.8) and (3.13) indeed imply (3.9). ■

4. Perfect Bayesian Equilibrium

We now show that all symmetric Nash equilibrium outcomes are perfect Bayesian equilibrium outcomes. The perfection issue arises because the grim strategies used to prove Theorem 1 are not sequentially rational if the benefit jump b, and hence the maximal static contribution $c(0)$, is positive. For, suppose the cumulation X is so large that less than $c(0)$ is needed to complete the project. Then a player would complete it alone if the others were expected never again to contribute. The grim strategies are thus not sequentially rational if at this X they require each player to never contribute. Furthermore, they cannot be made sequentially rational here by, say, altering them so that each player contributes $(X^* - X)/n$ to complete the project. Given this alteration, if in some period no player is ever supposed to contribute and the cumulation is slightly less than $X^* - c(0)$, a player would prefer to contribute the small amount required to put the cumulation over $X^* - c(0)$ and hence induce completion in the next period.

However, the maximal conceivable punishment of a unilateral deviator consists of the other players never contributing again — the deviator himself should play a best reply. If the identity of the deviator is known, sequentially rational strategies imposing the maximal punishment are feasible. Just alter a grim-g equilibrium so that in any period in which the players are not supposed to contribute, if a unilateral deviation raises the cumulation enough to put it for the first time into the interval

$$\hat{C} \equiv [X^* - c(0), X^*),$$

the strategies call for the deviator to complete the project alone in the next period. This makes the deviation unprofitable, as it causes the deviator to complete the project alone by contributing more than $c(0)$ (over two periods). If $n = 2$, so that the identity of a unilateral deviator is common knowledge, this argument shows that every Nash equilibrium outcome g is a subgame perfect equilibrium outcome.

If $n > 2$, we must consider a player's beliefs about who deviated. We use the concept of a *perfect Bayesian equilibrium* (PBE): a belief-strategy combination that satisfies Bayes' rule whenever possible, and sequential rationality always.[8] An argument in Appendix A shows that given any Nash equilibrium outcome, strategies and beliefs can be found that support it as a PBE outcome. This yields the result of this section.

Theorem 2. *Every symmetric Nash equilibrium outcome is a PBE outcome.*

In the remainder of this section we introduce material used to prove this result in Appendix A, and give some intuition for it. First, we find it convenient to view a strategy as a "machine" that is in one of several possible "states" each period.[9] The strategy of player i specifies a contribution each period as a function of his machine's state and public information, and the next state of his machine as a function of its current state and the $(z_i(t), Z(t))$ he observes. A machine's current state is private information to its owner if $n > 2$.

The set of possible machine states is $\{q^0, q^{00}, q^{01}, q^{10}\}$. If the player's machine is in state $q_i(t)$ in period t, he contributes $z_i(t) = s(q_i(t), X(t-1), t)$, where s is the *contribution*

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[8] Fudenberg and Tirole (1991) define perfect Bayesian equilibrium concepts.

[9] See, e.g., Osborne and Rubinstein (1994), Ch. 8

rule. In terms of the Nash equilibrium outcome g that the PBE is to generate, the contribution rule is defined, for $X < X^*$ and $t \leq \overline{T}$, by,

$$s(q_i, X, t) \equiv \begin{cases} g(t) & \text{if } q_i = q^0 \\ 0 & \text{if } q_i = q^{00}, q^{01} \\ na - X & \text{if } q_i = q^{10}. \end{cases} \tag{4.1}$$

For $X \geq X^*$ or $t > \overline{T}$, set $s(q_i, X, t) \equiv 0$.

The structure of the equilibrium is the following. If no player deviates, all machines stay at q^0, the *normal state*. They all move to q^{00}, the *grim state*, if a deviation occurs and afterwards the cumulation is less than $X^* - c(0)$. If a deviation occurs and afterwards $X \in \hat{C}$, every machine moves to q^{01} or q^{10}, and at least one moves to q^{10}. If it is a unilateral deviation that causes the cumulation to enter $\hat{C}$, the machines of the non-deviators move to q^{01}, the *innocent state*, and the deviator's machine moves to q^{10}, the *guilty state*. A player whose machine is in the guilty state immediately contributes enough to complete the project alone, and players whose machines are in the innocent state do not contribute.

These properties are obtained by appropriately specifying for each player a *transition function* F_i that determines the next state of the player's machine as a function of his private and public information at the end of the current period:

$$q_i(t+1) = F_i(q_i(t), z_i(t), Z(t), X(t), t).$$

The pair (s, F_i) is a strategy for player i.

Beliefs for each player about past contributions are represented as a probability distribution that depends on his machine's state and the aggregate contribution of the others. In period t, if $q_i(t) = q_i$ and $Z_i(t) = Z_i$, player i considers the contribution vector $z_{-i}(t)$ to be a random variable $\tilde{z}_{-i} \in \Re_+^{n-1}$ having a probability measure $P_i(\cdot \mid q_i, Z_i, t)$. The desired PBE is then the assessment $\left((s, F_i), P_i\right)_{i \in N}$.

The transition and belief functions are crafted so that any player whose machine is in the guilty state believes the machines of all the other players are in the innocent state. This is why it is a best reply for such a player to contribute immediately enough to complete the project, as he believes the others will not contribute and he is thus willing to complete it alone (as $X \in \hat{C}$). Conversely, a player whose machine is in the innocent state believes that at least one of the machines of the other players is in the guilty state. This is why such a player is willing not to contribute, as he believes another player (or more) will complete the project without his help.

Remark 2. The proof of Theorem 2 is complex because the identity of a unilateral deviator from a symmetric equilibrium path is not generally common knowledge. If the outcome is asymmetric, detecting the identity of a unilateral deviator is easier. For example, if only player 1 is supposed to contribute at some date, and the observed aggregate contribution is less than his equilibrium contribution, all players will know that he deviated. Thus, though we have not given a formal argument, Theorem 2 clearly holds for asymmetric as well as symmetric outcomes: every Nash equilibrium outcome is a PBE outcome.

5. Completing the Project

We now show that if the discount factor and contributing horizon are large, equilibria exist that essentially complete the project. If $b > 0$, the project can be actually completed: $X(T) = X^*$ for some $T < \infty$. If $b = 0$, it can be asymptotically completed: $X(t) \to X^*$ as $t \to \infty$. In either case the necessary completion delay and the efficiency loss vanish as the period length shrinks to zero.

We put the results in terms of the private marginal benefit of a non-completing contribution, $\lambda \in [0, V/X^*]$. By Theorem 0, the static game has equilibria in which contributions are made if and only if,

$$\lambda \leq \frac{V - a}{(n-1)a}.$$

In this section we show that for any $\lambda \in [0, V/X^*)$ (and so for $b > 0$), equilibria exist that complete the project for all large δ and $\overline{T}$. The closer λ is to V/X^* (the closer b is to zero), the more periods it takes to complete, and hence the larger $\overline{T}$ must be. For $\lambda = V/X^*$ ($b = 0$), the project can only be completed asymptotically. However, even in this case project can be completed almost instantly as the period length shrinks to zero.

Completing Equilibria

We first construct a completing equilibrium in the case $\lambda < V/X^*$ $(b > 0)$. The idea is straightforward. As the maximal static contribution $c(0)$ is positive, the project can be completed immediately if less than $nc(0)$ is required to do so. To keep the players contributing until this is true, the threat of halting all contributions deters free riding. However, this deters a player from free riding in the early periods only if the contributions of the others are small, as we discussed in Section 3, and the player does not discount the future too much. The larger is λ, the less the others can contribute if one of them is not to free ride, and hence the more periods are required.

Let T be the completion date, as determined below. At $t = T$, inequality (3.8) is $g(T) \le c(0)$. If this binds, then (3.8) at $t = T-1$ gives an upper bound for $g(T-1)$, which we denote as $c(1)$. Proceeding inductively, making the under-contributing constraints (3.8) bind this way defines a sequence $\{c(k)\}_{k=0}^{\infty}$, where

$$c(k) \equiv \delta^k c(0) + \left(\frac{n\lambda - 1}{1-\lambda}\right)\sum_{\kappa=0}^{k-1} \delta^{k-\kappa} c(\kappa) \text{ for } k > 0. \tag{5.1}$$

Quantity $c(k)$ is the maximal contribution each player can make, in a symmetric equilibrium, at period $t = T - k$ if in each period $\tau > t$, each player contributes $c(T-\tau)$. Using the definition of $c(0)$ in (3.2), the solution of (5.1) is

$$c(k) \equiv \left(\frac{(n-1)\lambda\delta}{1-\lambda}\right)^k \left(\frac{b}{1-\lambda}\right) \text{ for } k \ge 0. \tag{5.2}$$

The completion date T must satisfy

$$\sum_{k=0}^{T-1} c(k) < a \le \sum_{k=0}^{T} c(k). \tag{5.3}$$

It is easily verified that when $b > 0$, a finite T, which we refer to as T^*, is defined by (5.3) if and only if $\delta > \delta_n^*$, where

$$\delta_n^* \equiv \frac{(1-\lambda)a - b}{(n-1)\lambda a} = 1 - \frac{V-a}{(n-1)\lambda a.} \tag{5.4}$$

Because $V > a$ (by (2.6)), $\delta_n^* < 1$.

Proposition 1. *Suppose* $b > 0$, $\delta \in (\delta_n^*, 1]$, *and* $\bar{T} \geq T^*$. *Then* (5.5) *below defines an equilibrium outcome, and it completes the project at the finite date* T^*:

$$g^*(t) \equiv \begin{cases} a - \sum_{k<T^*} c(k) & \text{for } t = 0 \\ c(T^* - t) & \text{for } 0 < t \leq T^* \\ 0 & \text{for } t > T^*. \end{cases} \tag{5.5}$$

Proof. By construction, g^* is a candidate outcome that satisfies (3.8) and completes the project at the finite date T^*. Because $T^* = 0$ or $g^*(T^*) = c(0)$, either (ii) or (iii) in Corollary 1 holds, and the proposition is proved. ■

In the equilibrium of Proposition 1, The number of periods to completion, $T^* + 1$, decreases in δ, since (5.2) implies that each $c(\kappa)$ increases in δ. Thus, the project is completed nearly instantly if the period length is nearly zero:

Corollary 2. *Under the conditions of Proposition* 1, *with* $\delta = e^{-r\ell}$, *if the equilibrium outcome* g^* *is played, then* (i) *the project is finished nearly instantaneously if* ℓ *is nearly zero, and* (ii) *the efficiency loss is nearly zero if either* r *or* ℓ *is nearly zero.*

Asymptotically Completing Equilibria

We turn now to the case $\lambda = V/X^*$ ($b = 0$), showing that then the project can be asymptotically completed if $\bar{T} = \infty$ and δ is large. This result is a special case of Proposition 2 below, which treats the broader case $\lambda \geq 1/n$.[10] The proposition concludes that if $\bar{T} = \infty$ and δ is large, then for any $\alpha \in [0, a - \frac{1}{n}c(0)]$, an equilibrium exists in which each player's cumulative contribution converges to α. The condition $\lambda \geq 1/n$ is needed so that a player's utility, $(\lambda n - 1)x_i(t)$, increases as $x_i(t)$ increases with time. Otherwise, since the completion benefit b is never received, a player would be better off never contributing. If $\lambda = V/X^*$, the limiting per-capita cumulative contribution can be chosen to be $\alpha = a$ (since $c(0) = 0$), which yields the asymptotic completion result.

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[10] If $\lambda = V/X^*$, then $\lambda \geq 1/n$, since $V > a = \frac{1}{n}X^*$.

The proof is again constructive. Each player's contribution sequence is the following geometric sequence defined in terms of a parameter α:

$$g(t,\alpha) \equiv \left(\frac{\underline{\delta}}{\delta}\right)^t \left(\frac{\delta - \underline{\delta}}{\delta}\right)\alpha \text{ for } t \geq 0, \tag{5.6}$$

where

$$\underline{\delta} \equiv \frac{1-\lambda}{(n-1)\lambda}. \tag{5.7}$$

If $\delta > \underline{\delta}$, the partial sums of $g(\cdot,\alpha)$ converge to α. Note that $\underline{\delta} < 1$ if and only if $\lambda > 1/n$.

This $g(\cdot,\alpha)$ is obtained by converting the inequalities in (3.8) to equalities, and solving the resulting system subject to $\sum_{t=0}^{\infty} g(t) = \alpha$. This insures that the under-contributing constraints are satisfied with equality.[11] As the proof below indicates, the over-contributing constraints are also satisfied if the cumulation does not grow too large. If it ever exceeds $X^* - c(0)$, no player can be prevented from over-contributing to complete the project. This is why $n\alpha \leq X^* - c(0)$ is required.

Proposition 2. *Suppose* $\overline{T} = \infty$ *and* $\lambda \geq 1/n$. *Then* (5.6) *defines an equilibrium outcome if* $\delta \in (\underline{\delta}, 1]$ *and* $\alpha \in [0, a - c(0)/n]$.

Proof. Observe that $\Sigma_{\tau \geq t} \delta^{\tau} g(\tau,\alpha) = \delta^t g(t,\alpha)/(1-\underline{\delta})$. Hence, for $t \geq 0$,

$$g(t,\alpha) = \left(\frac{1-\underline{\delta}}{\underline{\delta}}\right) \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} g(\tau,\alpha) = \left(\frac{n\lambda - 1}{1-\lambda}\right) \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} g(\tau,\alpha). \tag{5.8}$$

Because the project is not completed when each player's contribution sequence is $g(\cdot,\alpha)$ (as $\alpha \leq a$), $T(g(\cdot,\alpha)) = \infty$. So (5.8) implies that (3.8) holds with equality. Condition (3.9) is also satisfied by $g(\cdot,\alpha)$, since

$$X^* - n\sum_{\tau=0}^{t} g(\tau,\alpha) \geq n(a-\alpha) \geq c(0) \geq c(0) - \left(\frac{n\lambda - 1}{1-\lambda}\right) \sum_{\tau=t+1}^{T} \delta^{\tau-t} g(\tau,\alpha),$$

using $\alpha \leq a - c(0)/n$ and $\lambda \geq 1/n$. Now the result follows from Theorem 1. ■

[11] We actually do not need to make the under-contributing constraints bind. We could instead multiply the right side of (3.8) by a fraction $\beta \in (0,1)$ before solving (3.8) as a system of equalities. This procedure would yield a similar equilibrium in which, conditional on any history, each player's strategy is a strict best reply.

If $\lambda = V/X^*$, we can choose $\alpha = a$ in Proposition 2, since $c(0) = 0$. This proves the existence of equilibria in which players contribute according to the geometric sequence $g(\cdot, a)$, thereby completing the project asymptotically. Even though the project is never completed, the efficiency loss due to delay vanishes as $\delta \to 1$, as we now show.

Corollary 3. *Assume* $b = 0$, $\overline{T} = \infty$, *and* $\delta = e^{-r\ell} > \underline{\delta}$. *Then* $g(\cdot, a)$ *is an equilibrium outcome that asymptotically completes the project. In this equilibrium,* (i) *after any fixed, positive amount of time has passed, the contribution needed to complete the project is nearly zero if* ℓ *is nearly zero;* (i) *the efficiency loss is nearly zero if* r *or* ℓ *is nearly zero.*

Proof. Given Proposition 2, we need prove only (i) and (ii). Let $Y > 0$. Time Y is reached in no less than $\iota(Y/\ell)$ periods, where $\iota(Y/\ell)$ is the integer part of Y/ℓ. A player's cumulative contribution at time Y is thus no less than

$$\sum_{\tau=0}^{\iota(Y/\ell)} g(\tau, a) = \left(1 - \left(\underline{\delta} e^{r\ell}\right)^{\iota(Y/\ell)+1}\right) a.$$

This converges to a as $\ell \to 0$, proving (i). To prove (ii), note that a player's payoff,

$$U(g(\cdot, a)) = (n\lambda - 1) \sum_{\tau=0}^{\infty} \delta^{\tau} g(\tau, a) = \frac{(n\lambda - 1)(\delta - \underline{\delta})a}{(1 - \underline{\delta})\delta}.$$

converges to its upper bound, i.e., $U(g(\cdot, a)) \to (n\lambda - 1)a = V - a$, as $\delta \to 1$. ■

Non-Completing Equilibria

We end this section by considering non-completing equilibria. Proposition 3 lists several results; some are roughly converses to Propositions 1 and 2.

Proposition 3. (i) *Equilibria exist in which no player contributes.* (ii) *If* $\overline{T} < \infty$ *and* $\lambda = V/X^*$ ($b = 0$), *no contributions are made in any equilibrium.* (iii) *If* $\overline{T} = \infty$ *and* $\lambda = V/X^*$ ($b = 0$), *the project is not completed in any equilibrium.* (iv) *If* $\overline{T} < \infty$, *or* (v) *if* $\lambda < 1/n$, *then in no equilibrium do players contribute without completion occurring.*

Proof. To prove (i) we apply Theorem 1 by showing that $g(\cdot) \equiv 0$ satisfies (3.8) and (3.9). Trivially, (3.8) is satisfied. To show (3.9), note that the definition of $c(0)$ in (3.2), and the parameter restrictions in (2.5) and (2.6), together imply $X^* > c(0)$. Thus, since (3.9) reduces to $X^* \geq c(0)$ for $g(\cdot) \equiv 0$, (3.9) is satisfied.

To prove (ii) – (iv), assume one is false. Then an equilibrium exists such that for some $T < \infty$, the cumulative contribution at the end of period T is $X(T) > 0$, some player i contributes $z_i(T) > 0$ in period T, and no contributions are made thereafter. The worst that can happen to player i if he deviates to zero in period T is also that no contributions are made thereafter. Since $b = 0$ (cases (ii) and (iii)) or $X(T) < X^*$ (case (iv)), this deviation yields a continuation payoff no less than $\lambda(X(T) - z_i(T)) - (x_i(T) - z_i(T))$. If the player does not deviate, his continuation payoff is $\lambda X(T) - x_i(T)$. As $\lambda < 1$, the player strictly prefers to deviate, contradicting the equilibrium assumption.

To prove (v), consider an equilibrium which gives rise to a sequence $\{z(t)\}_{t=0}^{\infty}$ of contribution vectors. Assume some contributions are positive, but the project is not completed. Then, setting $T = \infty$ in (2.7) and summing over i yields

$$\sum_{i=1}^{n} U_i(\{z(t)\}) = (n\lambda - 1)\sum_{t=0}^{\infty} \delta^t Z(t) < 0,$$

where the inequality follows from $\lambda < 1/n$ and $Z(t) > 0$ for some t. This contradicts the fact that each player's equilibrium payoff must be nonnegative, since a player can obtain a nonnegative payoff by never contributing. ■

6. Markov Perfect Equilibria

The cumulation $X(t-1)$ is a natural state variable observed by all players in the game with an infinite contributing horizon. A popular equilibrium refinement is thus *Markov perfect equilibrium* (MPE), which can be defined here as a PBE in which strategies are Markovian, so that each player's contribution in any period t depends only on $X(t-1)$. A player's beliefs about the past individual contributions of the others are then irrelevant: if the others use Markovian strategies, a player's payoff depends on their contributions only via the observed aggregate. Markov perfect equilibria are arguably plausible and/or tractable (Fudenberg and Tirole, 1991; Maskin and Tirole, 1994). Other work on dynamic voluntary contribution has focused on them (Fershtman and Nitzan, 1991; Wirl, 1996).

In this section we restrict attention to the game with an infinite contributing horizon, and study completing and asymptotically completing Markov perfect equilibria. The equilibria are characterized by a sequence of *contribution goals*: equilibrium play in any period raises the cumulation to the smallest goal so far unachieved. Punishments are highly forgiving: a player who free rides in one period delays the achievement of the current and subsequent goals by just one period.

Symmetric Markov Perfect Equilibria

We first consider symmetric equilibria: starting from anywhere, each player makes the same contribution. We start with the case $b > 0$. The goals then form a decreasing sequence $\{X_k\}_{k=0}^{\infty}$, starting with $X_0 = X^*$. If $X \in [X_{k+1}, X_k)$, the current goal is X_k and each player contributes an equal share of the amount required to reach it. A player's Markov strategy $z_i(t) = \psi(X(t-1))$ thus takes the form

$$\psi(X) \equiv \begin{cases} (\frac{1}{n})(X_k - X) & \text{for } X \in [X_{k+1}, X_k) \\ 0 & \text{for } X \geq X^*. \end{cases} \tag{6.1}$$

The goals are defined by

$$X_{k+1} \equiv X^* - Kb \sum_{\kappa=0}^{k} (\delta C)^{\kappa} \text{ for } k \geq 0, \tag{6.2}$$

where

$$K \equiv \frac{(1-\delta)n}{1-(1-\delta)\lambda - \delta(\frac{1}{n})} \geq 0 \tag{6.3}$$

and

$$C \equiv 1 + (\lambda - \tfrac{1}{n})K = \frac{(1-\delta)(n-1)\lambda + \delta(\frac{n-1}{n})}{1-(1-\delta)\lambda - \delta(\frac{1}{n})} > 0. \tag{6.4}$$

This equilibrium is derived by backwards induction: given X_k, we define X_{k+1} to be the smallest $X < X_k$ such that a player is willing to contribute $\frac{1}{n}(X_k - X)$ rather than zero, assuming the others collectively contribute $\frac{n-1}{n}(X_k - X)$. A value function is derived jointly with the goals, using (2.7). For example, if a contribution sequence $\{z\}$ results in $X(t-1) = X < X^*$, and it is an equilibrium contribution sequence in periods $\tau \geq t$, the value $V(X)$ is the continuation payoff satisfying

$$U_i(\{z\}) = \sum_{\tau=0}^{t-1}\delta^{\tau}\left[\lambda Z(\tau) - z_i(\tau)\right] + \delta^{t}V(X). \tag{6.5}$$

The first step of the induction is to let $X_0 = X^*$, and to define $V(X)$ for $X \geq X^*$. In this region, no contributions occur in equilibrium. Viewing the completion benefit b as though it were paid out in increments of $(1-\delta)b$ each period upon completion (see (2.7)), we obtain $V(X) = \sum_{\tau \geq t}\delta^{\tau - t}(1-\delta)b = b$.

Now suppose that for some $k \geq 0$, a decreasing sequence $\{X_\kappa\}_{\kappa=0}^{k}$ has been defined. The value function is defined on $[X_k, X^*)$ by

$$V(X) \equiv (\lambda - \tfrac{1}{n})(X_\kappa - X) + \beta(X_\kappa)(1-\delta)b + \delta V(X_\kappa) \text{ for } X \in [X_{\kappa+1}, X_\kappa), \tag{6.6}$$

where

$$\beta(Y) \equiv \begin{cases} 0 & \text{if } Y < X^* \\ 1 & \text{if } Y \geq X^*. \end{cases} \tag{6.7}$$

Expression (6.6) is the equilibrium continuation payoff when X_κ is the current goal: it can be viewed as the current benefit from the current aggregate contribution, $\lambda(X_\kappa - X) + \beta(X_\kappa)(1-\delta)b$, less the player's own contribution, $\frac{1}{n}(X_\kappa - X)$, plus the discounted future value, $\delta V(X_\kappa)$.

We now define X_{k+1} and extend V to $[X_{k+1}, X_k)$. The latter is immediate: when $X < X_k$ and the goal is X_k, a player's equilibrium continuation payoff is as shown in (6.6), with $\kappa = k$. So (6.6) also defines V on $[X_{k+1}, X_k)$. Now, when the goal is X_k, but a player makes a one-shot deviation by contributing zero, the cumulation is raised only to $X + (\frac{n-1}{n})(X_k - X) < X_k$. The deviator's continuation payoff is

$$V^d(X) = \lambda(\tfrac{n-1}{n})(X_k - X) + \delta V[X + (\tfrac{n-1}{n})(X_k - X)]. \tag{6.8}$$

For ψ to be a MPE, $V(X) \geq V^d(X)$ is required. We let X_{k+1} be the smallest X for which this inequality holds, which is the X that makes it an equality. This procedure results in the following difference equations, which are easily solved to yield (6.2):

$$V(X_{k+1}) = C\left[\delta V(X_k) + (1-\delta)\beta(X_k)b\right], \tag{6.9}$$

$$X_{k+1} \equiv X_k - K\left[\delta V(X_k) + (1-\delta)\beta(X_k)b\right]. \tag{6.10}$$

Note that strategy ψ is defined for all $X \geq 0$ if and only if $X_\infty < 0$, where $X_\infty \equiv \lim_{k \to \infty} X_k$. In this case the project is completed in finite time when the players use ψ. Lemma B1 in Appendix B shows that $X_\infty < 0$ is sufficient as well as necessary for ψ to be a symmetric MPE.

When is $X_\infty < 0$? One requirement is $b > 0$: the project cannot be completed in finite time if $b = 0$. (Note that $X_k = X^*$ for all $k \geq 0$ if $b = 0$.) Also required is $\delta < 1$, since $\delta = 1$ implies that $X_k = X^*$ for all $k \geq 0$. Thus, unlike in Propositions 1 and 2, discounting is required. The reason is clear: a player who free rides in a finite number of periods, but plays ψ otherwise, does not change the ultimate aggregate cumulation (it is still X^*), but he does lower his own ultimate contribution. If $\delta = 1$, the player's payoff depends only on the limits of $X(t)$ and $x_i(t)$, and so this free riding is beneficial.[12]

Further conditions for $X_\infty < 0$ are given in Lemma B2 in Appendix B, which immediately implies the following theorem.

Theorem 3. *Assume $b > 0$ and $\overline{T} = \infty$. Then:*

(i) For $c(0) > a$: $\delta_0 \in (0,1]$ exists such that ψ is a MPE if and only if $\delta \in (0, \delta_0)$. Furthermore, $\delta_0 = 1$ if and only if $V > (2 - \frac{1}{n})a$.

(ii) For $0 < c(0) \leq a$: If $V \leq (2 - \frac{1}{n})a$, then ψ is not a MPE for any $\delta \in (0,1)$. If $V > (2 - \frac{1}{n})a$, then $\delta_0 \in [0,1)$ exists such that ψ is a MPE if and only if $\delta \in (\delta_0, 1)$.

Theorem 3 (i) refers to those cases in which the static game has equilibria that complete the project, such as when $\lambda = 0$. In these cases ψ is an equilibrium of the dynamic game if and only if the discount factor is low. (The sufficiency of a small δ is not surprising, as the dynamic game is very much like the static game if players discount the future heavily.)

Theorem 3 (ii) refers to the case in which the static game does not have completing equilibria. Now ψ is an equilibrium only if the discount factor is high enough, and only if the per-capita value of the project, $V = \lambda na + b$, exceeds $(2 - \frac{1}{n})a$. In fact, $V > (2 - \frac{1}{n})a$

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[12] Any equilibrium of the game with $\delta = 1$ in which the project is completed even after free riding occurs must induce the free rider to make up his contribution shortfall.

is needed in both cases (i) and (ii) in order for ψ to be an equilibrium for all discount factors near one. This is in contrast to Propositions 1 and 2, which establish the existence of completing equilibria, for all large discount factors, given any V greater than a.

Assuming $V > (2-\frac{1}{n})a$, so that ψ is a MPE for high discount factors, we can ask whether the Coase conjecture holds. It does not — the following corollary is proved in Appendix B.

Corollary 4. *Assume* $V > (2-\frac{1}{n})a$, $b > 0$, *and* $\overline{T} = \infty$. *Then as* $\ell \to 0$, *the time the MPE defined by* (6.1) – (6.4) *takes to complete the project converges to*

$$L^* \equiv \frac{a\left(\frac{n-1}{n}\right)\left[\ln\left(V-(2-\frac{1}{n})a\right)-\ln(b)\right]}{r\left[V-(2-\frac{1}{n})a-b\right]} > 0. \tag{6.11}$$

If $b = 0$, a result similar to Theorem 3 holds. Specifically, if and only if $V > (2-\frac{1}{n})a$ and δ is sufficiently large, a similar construction yields a sequence of goals $\{X_k\}_{k=0}^{\infty}$, which now increase and converge to X^*, and a symmetric MPE strategy ψ that requires a player to contribute $(\frac{1}{n})(X_{k+1}-X)$ in any period in which $X \in [X_k, X_{k+1})$. Because of space limitations, we spare the reader the derivation.[13] This equilibrium completes the project asymptotically. However, as in Corollary 4, even an asymptotic version of the Coase conjecture is false: for any $\theta \in (0,1)$, the time required for these equilibria to raise the aggregate cumulation to θX^* converges to a positive number L_θ^* as $\delta \to 1$. Again we leave the details to the reader.[14]

Asymmetric Markov Perfect Equilibria

In contrast to the PBE outcomes discussed in Section 5, the completing Markov perfect equilibria in the previous subsection exist for small period lengths only if $V > (2-\frac{1}{n})a$, and the time they take to complete the project is bounded away from zero

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[13] The procedure is to first reverse the subscripts k+1 and k in the difference equations (6.9) and (6.10), and then solve them in terms of $V(0)$. Then $V(0)$ is set so that $X_k \to X^*$. This yields $X_k \equiv (1-(\delta C)^{-k})X^*$.

[14] The formula is $L_\theta^* \equiv r^{-1}\left[a\left(\frac{n-1}{n}\right)\ln(1-\theta)\right]\Big/\left[(2-\frac{1}{n})a-V\right]$.

as the period length shrinks. It appears that symmetry is a serious impediment to the efficiency of Markov perfect equilibria.[15]

Markov perfection is not the culprit *per se*. For all $V > a$, asymmetric Markov perfect equilibria do exist that complete (asymptotically, if $b = 0$) the project if δ is sufficiently large, and the time they take to complete (approximately complete, if $b = 0$) vanishes as the period length shrinks to zero. We prove this here for the case $b = 0$. (The arguments are similar but messier when $b > 0$.)

The equilibrium is characterized by a sequence of goals, $\{\hat{X}_k\}_{k=0}^{\infty}$, which start at $\hat{X}_0 = 0$, increase in k, and converge to X^*. Only one player contributes each period. If the current X is in the interval $[\hat{X}_{k-1}, \hat{X}_k)$, then the current goal is $\hat{X}_k$, and only player k (mod n) is responsible for raising X to $\hat{X}_k$. The Markov strategy of player i is thus

$$\sigma_i(X) \equiv \begin{cases} \hat{X}_k - X & \text{for } X \in [\hat{X}_{k-1}, \hat{X}_k) \text{ and } i =_n k \\ 0 & \text{otherwise,} \end{cases} \tag{6.12}$$

where $i =_n k$ is shorthand for $i = k \,(\text{mod}\, n)$.[16] The goals are specified in terms of the unique positive root of the polynomial

$$J(p) \equiv p^{n-1} + p^{n-2} + \cdots + p - \tfrac{1-\lambda}{\lambda}.$$

Denote this root as γ, and observe that $\gamma \in (0,1)$.[17] The goals are defined for $k \geq 0$ by

$$\hat{X}_k \equiv \left(1 - (\tfrac{\gamma}{\delta})^k\right) X^*. \tag{6.13}$$

These goals are positive provided $\delta > \gamma$, and this is sufficient for σ to be a MPE that asymptotically completes the project:

Theorem 4. *Assume* $b = 0$ *and* $\overline{T} = \infty$. *Then* (6.12) *and* (6.13) *define a MPE provided* $\delta \in (\gamma, 1]$. *This equilibrium asymptotically completes the project.*

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[15] We suspect that (i) no symmetric MPE completes (or, if $b = 0$, asymptotically completes the project) if $V < (2 - \frac{1}{n})a$ and δ is large, and (ii) no symmetric MPE nearly completes the project nearly instantaneously when the period length is arbitrarily short.

[16] That is, $i =_n k$ provided that k is equal to i plus a multiple of n.

[17] This is because $J(0) < 0 < J(1)$, which is true because $b = 0$ implies $\lambda = V/na \in (\frac{1}{n}, 1)$.

We prove Theorem 4 in Appendix B, but here we derive the equilibrium. Let $Z_k \equiv \hat{X}_k - \hat{X}_{k-1}$. For $k \geq 1$, denote by H_k the equilibrium continuation payoff of player $i =_n k$ starting from $X = \hat{X}_k$, which in equilibrium is the period after this player raises the cumulation to $\hat{X}_k$. In the subsequent $n-1$ periods, each of the other players $j \neq i$ in turn contribute $Z_{k+1}, Z_{k+2}, \ldots, Z_{k+n-1}$, and in the n^{th} period, player i contributes Z_{k+n}. His continuation payoff starting in the $(n+1)^{\text{th}}$ subsequent period is H_{k+n}. Thus,

$$H_k = \lambda \sum_{\kappa=k+1}^{k+n-1} \delta^{\kappa-k-1} Z_\kappa + (\lambda - 1)\delta^{n-1} Z_{k+n} + \delta^n H_{k+n}. \tag{6.14}$$

We let $V_i(X)$ be the equilibrium value function of player i. Thus, $V_i(\hat{X}_k) = H_k$ if $i =_n k$, $V_i(X) = 0$ for $X \geq X^*$, and for $X \in [\hat{X}_{k-1}, \hat{X}_k)$,

$$V_i(X) \equiv \begin{cases} (\lambda - 1)(\hat{X}_k - X) + \delta H_k & \text{if } i =_n k \\ \lambda(\hat{X}_k - X) + \delta V_i(\hat{X}_k) & \text{if } i \neq_n k. \end{cases} \tag{6.15}$$

Since a player has the option of contributing zero forever to obtain a zero continuation payoff, $V_i(\hat{X}_k) \geq 0$ for all $k \geq 0$. Now suppose $k \geq 1$ and $i =_n k+1$. If $X = \hat{X}_{k-1}$, player i is supposed to let another player contribute Z_k in the current period before he contributes Z_{k+1} in the next period. His equilibrium continuation payoff from this strategy must be no less than what he would get from "jumping the gun" by contributing Z_{k+1} in the current period, at the same time as Z_k is contributed. Thus,

$$V_i(\hat{X}_{k-1}) \geq \lambda Z_k + (\lambda - 1) Z_{k+1} + \delta H_{k+1}.$$

Applying (6.15) to each side, we obtain $\lambda Z_k + \delta V_i(\hat{X}_k) \geq \lambda Z_k + V_i(\hat{X}_k)$. This implies $V_i(\hat{X}_k) \leq 0$. We conclude that

$$V_i(\hat{X}_k) = 0 \text{ for all } k \geq 1 \text{ and } i =_n k+1. \tag{6.16}$$

In equilibrium, a player who contributes in period $t > 0$ contributes so much that his continuation payoff (before he contributes) is zero — otherwise he would have wanted to make the contribution earlier. If (6.16) holds also for $k = 0$, then the same is true of the contributor (player 1) at $t = 0$, and his equilibrium payoff is zero. This is the equilibrium we construct.

So, using (6.15) and (6.16) for $k \geq 0$, we have

$$H_k = \delta^{-1}(1-\lambda)Z_k \text{ for all } k \geq 1. \tag{6.17}$$

We solve the system consisting of (6.14), (6.17), and

$$\sum_{k=1}^{\infty} Z_k = X^*. \tag{6.18}$$

Using (6.17) to remove H_k and H_{k+n} from (6.14), and letting $\rho_k \equiv \delta^k Z_k$, yields a linear homogeneous difference equation:

$$-\left(\tfrac{1-\lambda}{\lambda}\right)\rho_k + \sum_{\kappa=k+1}^{k+n-1} \rho_\kappa = 0,$$

Since $J(\gamma) = 0$, this difference equation has a solution of the form $\rho_k = C\gamma^k$, or rather, $Z_k = C(\frac{\gamma}{\delta})^k$. Then (6.18) fixes the constant at $C = X^*(\delta - \gamma)/\gamma$, and requires $\delta > \gamma$. This yields $Z_k = (\frac{\gamma}{\delta})^{k-1}(1-\frac{\gamma}{\delta})X^*$, and (6.13) comes from $\hat{X}_k = \sum_{\kappa=1}^{k} Z_\kappa$.

The asymmetric MPE of Theorem 4 exists for all large discount factors, even when the symmetric MPE of Theorem 3 does not, i.e., even when $V < (2 - \frac{1}{n})a$. The reason is that in the symmetric MPE, a free rider gains from the contributions made by the other players in the period in which the free-rider withholds his. This gain to free riding is small only if the equilibrium contributions in each period are small. If δ is large, the gain to free riding must be made small in this way, since the only cost to free riding is the completion delay it causes. The delay cost is lower the lower is the net gain from completion, which is $V - a$ if all players contribute a. Thus, if V is sufficiently close to a, the delay cost is so small that if the contributions each period are small enough to deter free riding, then the contribution rate is too small to reach X^*.

The incentive to free ride is lower in the asymmetric MPE of Theorem 4. A player who contributes too little in a period in which he is supposed to contribute, and who then plays according to the equilibrium, makes up the shortfall himself next period. The only gain to the free rider is to shift a contribution into the discounted future, as opposed to lowering his total contribution. Thus, now the gain as well as the delay cost of free riding shrink as δ increases. The result is that the contributions each period do not become arbitrarily small as $\delta \to 1$. In fact, $\hat{X}_k \equiv \left(1 - (\frac{\gamma}{\delta})^k\right)X^*$, the equilibrium cumulation at the end of period $k+1$, increases with δ, and the strategy is an MPE even if $\delta = 1$.

It is now clear that the asymptotic version of the Coase conjecture holds for the asymmetric equilibrium of Theorem 4.

Corollary 5. *Assume* $b = 0$ *and* $\overline{T} = \infty$, *and let* $\theta \in (0,1)$. *Then the equilibrium defined by* (6.12) *and* (6.13) *raises the aggregate cumulation to* θX^* *in a number of periods that converges, as* $\ell \to 0$, *to within one of* $\ln(1-\theta)/\ln(\gamma)$. A fortiori, *the time taken to achieve* θX^* *converges to zero as* $\ell \to 0$.

Proof. The equilibrium raises the cumulation to θX^* in a number of periods equal to the first k for which $\hat{X}_k \geq \theta X^*$. By (6.13), this is within one of the τ defined by $1-(\frac{\gamma}{\delta})^\tau = \theta$. Hence, since $\delta \to 1$ as $\ell \to 0$,

$$\tau = \frac{\ln(1-\theta)}{\ln(\gamma)-\ln(\delta)} \to \frac{\ln(1-\theta)}{\ln(\gamma)} \text{ as } \ell \to 0. \blacksquare$$

Remark 3. As the number of players increases, each player contributes less frequently in the equilibrium of Theorem 4. The limiting strategies are an equilibrium of the game in which the set of players is $N = \{1,2,\ldots\}$. In this equilibrium each player contributes once only, doing so in order to induce the higher-indexed players to contribute later. The strategies have the same form as (6.12), except that $i =_n k$ is replaced by $i = k$. When $n = \infty$, the root of $J(p)$ is $\gamma = 1-\lambda$. Substituting this into $Z_k = (\frac{\gamma}{\delta})^{k-1}(1-\frac{\gamma}{\delta})X^*$ yields

$$x_{k-1} \equiv \left(\frac{1-\lambda}{\delta}\right)^{k-1}\left(\frac{\delta+\lambda-1}{\delta}\right)X^* \qquad (6.19)$$

as the contribution of player k when it is his turn to contribute. Provided $\delta > 1-\lambda$, this defines an equilibrium of the $n = \infty$ (and $b = 0$ and $\overline{T} = \infty$) game which asymptotically completes the project. The analog of Corollary 5 holds, so that efficiency is obtained in the limit as the period length vanishes.

This equilibrium roughly approximates what occurs in real fund drives that last a short time, such as a week-long public radio or television drive. Contributors in such drives tend to contribute only once during the drive. Potential contributors tend to be kept abreast of the progress made towards the drive's goal, which suggests that the level of previous contributions is an important determinant of when an individual contributes.

Whether later contributions tend to be smaller than earlier ones, as they are in the equilibrium, is unknown to us. In any case, assuming an infinite number of potential contributors seems to appropriately capture how each participant in a fund drive in a sizable city views the number of potential contributors.

7. Related Literature

Voluntary contribution in a dynamic, complete information setting has been studied in three related works.[18] Like us, Fershtman and Nitzan (1991) and Admati and Perry (1991) study private provision, i.e., descriptive models of project funding that do not utilize a third party (center, government) with coercive or commitment capabilities. Unlike us, they reach the negative conclusion that allowing contributions to be made over time aggravates free riding. Bagnoli and Lipman (1989) reach a more positive conclusion, but their focus is on the normative issue of designing a mechanism to be run by an entity with some coercive/commitment capabilities.

Fershtman and Nitzan (1991) consider a continuous-time differential game in which contributions at each date become the capital a project uses to generate a flow of public benefits. The open-loop equilibrium is analogous to the equilibrium of a static contribution game, and it accordingly yields low contributions. Surprisingly, contributions are no higher, and they are often lower, in the symmetric linear closed-loop equilibrium. In this equilibrium a player's contribution decreases with the level of the cumulation. This creates a negative feedback which increases free riding, since a decrease in one player's current contribution is partially offset by an increase in all players' future contributions. The dynamics allow current players to free ride not only on each other, but

[18] Less related are the following. A repeated game with a voluntary contribution stage game is studied in McMillan (1979); repeated game folk theorems apply. Dynamic contribution games in incomplete information settings with discrete public good and contribution levels are studied in Bliss and Nalebuff (1984), Gradstein (1992), and Vega-Redondo (1995). Delay is caused in these games by the incentive to wait for low-cost types to contribute first, as in wars of attrition.

also on their future incarnations. The authors conclude that free riding inefficiencies are aggravated in a dynamic context.

The model of Fershtman and Nitzan (1991) differs in several ways from ours: e.g., contributions decay over time rather than last forever, and payoffs are quadratic rather than linear up to a provision point. They also examine only symmetric and linear closed-loop equilibria.[19] However, we believe the key difference is that theirs is a continuous-time differential game, rather than a discrete-time dynamic game. Our game has, for a range of parameter values, symmetric Markov perfect equilibria in which contributions are made, even in the limit as the period length shrinks to zero (see the discussion following Corollary 4). The differential game of Fershtman and Nitzan (1991), with similar parameter values, has a unique closed-loop equilibrium, and in it no contributions are made (cf. fn. 19). This is not a contradiction, as the strategies of our MPE do not converge to well-behaved strategies as the discrete time is made continuous.

The second related paper, Admati and Perry (1991), is like ours in that it too concerns a dynamic, discrete-time model of contribution to a public project. They restrict attention to the case of an infinite contributing horizon and a binary technology. Their game differs from ours in two ways: (*i*) the (two) players can contribute only in alternate periods rather than in any periods they wish; and (*ii*) a player's cost function for contributing, denoted $W(z_i)$, is strictly convex in the contribution z_i, rather than being the identity function.[20] This game has an (essentially) unique subgame perfect equilibrium, and it is generally inefficient. If W is approximated by the identity function

[19] Wirl (1996) shows that contributions are larger in some *non-linear* closed-loop equilibria. However, if the quadratic payoffs are made linear like ours (with no jump), the game has a unique closed-loop equilibrium (if each player's wealth is bounded — the equilibrium is bang-bang). No contributions are made in this equilibrium, under our assumption that $V < X^*$ (i.e., $\lambda = V/X^* < 1$).

[20] If completion occurs in period $T \geq 0$ and player i contributes $z_i(0),\ldots,z_i(T)$, his payoff is $U_i = \delta^T V - \sum_{t=0}^{T} \delta^t W(z_i(t))$, where W is increasing, strictly convex, and satisfies $W(0) = 0$. This is the same as our payoff function (2.7) when $\lambda = 0$ (and so $b = V$), if W is replaced by the identity function.

at small contributions,[21] then under our assumption that no player is willing to fund the project on his own ($V < X^*$), no contributions are made. This is quite different from our result that if the technology is binary, then our game has a PBE that completes the project in one period.[22]

Admati and Perry's negative result depends on both distinguishing features of their game, (*i*) and (*ii*). Assume the project is worth completing ($V > a$). Then, if (*i*) is changed so that each player can contribute in any period, equilibria exist that complete the project.[23] Completing equilibria also exist if instead (*ii*) is changed so that W is the identity function.[24] Of course, Admati and Perry's results show that these latter equilibria are not robust to the addition of even a slight amount of strict convexity to W. Thus, the difference in results should be attributed essentially to the difference (*i*) in assumptions.[25]

The third related paper, Bagnoli and Lipman (1989), considers a mechanism design problem: a game form is constructed that fully implements, via a refinement of subgame perfect equilibrium, the core of a public goods economy in which the public good is

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[21] I.e., if $|W'(0)-1| < \varepsilon$ for a small $\varepsilon > 0$.

[22] Proposition 1 implies this: since $\lambda = 0$, $c(0) = V > a$, and so $T^* = 0$ for any $\delta \geq 0$.

[23] It is easy to check that the equilibria we study, such as those of Proposition 1, are approximated by equilibria of perturbations of our game in which the identity cost-of-contributing function is replaced by a nearby but strictly convex W.

[24] Let $W(z_i) = z_i$. The project should be completed, but neither player is willing to do it alone, if $\frac{1}{2}X^* < V < X^*$. Assuming this, the project cannot be completed in one period, since the players must alternate in their contributions. But the following is an equilibrium that completes the project in two periods. Let $R = R(t)$ be the remaining amount needed to complete the project at the start of period t (so R is Admati and Perry's X). If $R < R_1 \equiv (1-\delta)V$, completing the project by contributing R is the dominant strategy of the player whose turn it is. If it is player 1's turn, his strategy is to contribute $z_1 = R$ if $R < R_1$, $z_1 = 0$ if $R_1 \leq R \leq V$, and $z_1 = R - V$ if $V < R$. If it is player 2's turn, he contributes $z_2 = R$ if $R \leq V$, and $z_2 = 0$ if $R > V$. It is easy to show that for any $\delta \geq (X^* - V)/V$, these are MPE strategies. According to them, player 1 contributes $X^* - V$ in the first period, and player 2 completes the project by contributing V in the second period. The equilibrium payoffs are $U_1 = (1+\delta)V - X^*$ and $U_2 = 0$.

[25] Nonetheless, it is interesting to recall, from Theorem 4, that our game can have simple equilibria in which the players do take turns contributing, on the equilibrium path.

available only in discrete levels. The game form resembles a dynamic voluntary contribution game, but with a central authority required (committed) to halt the process in any period in which contributions are too small, and to refund in any period contributions that fall short of the amount needed to increase the public good to the next level. Hence, their result shows that adding a third party with these relatively small commitment capabilities can overcome free riding. In contrast, our purpose has been to determine what happens when a central authority is entirely absent.

The discreteness of the public good in Bagnoli and Lipman (1989) serves to alleviate free riding: a player has no incentive to under-contribute if he believes the others are contributing so much that he needs to contribute only a little in order to make the public good jump to its next level. This is similar to the role played by our benefit jump at completion, which allows the cumulation to reach a size sufficiently large that the project can be completed in one period. Our asymptotic completion result for the case $b=0$, Proposition 2, is foreshadowed by an unpublished result in the Appendix of Bagnoli and Lipman (1987), proved there for a strictly concave (and hence non-discrete) public good production function and two players.

8. Conclusions

We have shown that in a simple, complete information model of voluntary contribution to a public project, dynamics can allow the alleviation of free-riding inefficiency. This is despite the fact that creating future contributors upon which to free ride creates incentives for current individuals to lower their contributions, as previous authors have observed. In the equilibria we examine, this increased incentive to free ride is countered by the ability of the future potential contributors to punish past free riders by lowering their own contributions.

Public institutions often hold fund-raising campaigns over periods of weeks or months, and often the public is periodically exhorted to contribute by informing it of the amount that is currently required to meet the campaign goal. This is roughly consistent with the nature of the equilibria we have studied. Another set of consistent evidence arises in experimental work. In particular,[26] Dorsey (1992) conducted experiments in which players could contribute to a public good in each of several periods, with the current level of the aggregate cumulation posted each period. One of the findings is that allowing contributions to be made over time increases their ultimate level, especially if they are nonrefundable and the production function has a "provision point."

The games we have studied are probably the most obvious simple models of dynamic contribution to a public project by a sizable number of individuals. The assumptions that each player can see the aggregate but not the individual contributions, and that each player can contribute in any period, seem realistic. Less realistic, at least in some settings, is the assumption that information is complete. However, allowing preferences to be private information in our dynamic model will make it very complicated, as then a player's current contribution will serve the auxiliary role of signaling his private information. We choose to make no contribution now to this extension, leaving it instead to future contributors.

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[26] The general experimental literature on public goods is surveyed by Ledyard (1995).

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Appendix A: Proof of Theorem 2

Given the Nash equilibrium outcome g, the contribution rule s is defined by (4.1). We now define the transition function F_i of player i. His machine starts at $q_i(0) = q^0$. It stays at this state if no publicly detectable deviation from g occurs:

$$F_i(q_i, z_i, Z, X, t) \equiv q^0 \text{ if } q_i = q^0 \text{ and } Z = ng(t). \tag{a1}$$

It changes from the normal to the grim state if a deviation occurs and subsequently the cumulation is still less than $X^* - c(0)$:

$$F_i(q^0, z_i, Z, X, t) \equiv q^{00} \text{ if } Z \neq ng(t) \text{ and } X < X^* - c(0). \tag{a2}$$

It remains in the grim state if the cumulation is less than $X^* - c(0)$:

$$F_i(q^{00}, z_i, Z, X, t) \equiv q^{00} \text{ if } X < X^* - c(0). \tag{a3}$$

Transitions to q^{01} or q^{10} occur if an aggregate contribution reveals a deviation and results in $X \in \hat{C}$. These transitions depend on whether $i = 1$ or $i > 1$. In the latter case, player i's machine moves to the guilty state if the aggregate contribution of the others is what it should be if none of them deviated (and so player i must have deviated). But if their aggregate contribution proves that one or more of them did deviate, player i's machine moves to the innocent state. Thus, for $Z \neq ng(t)$, $X \in \hat{C}$, and $i > 1$,

$$F_i(q^0, z_i, Z, X, t) \equiv \begin{cases} q^{01} & \text{if } Z_i \neq (n-1)g(t) \\ q^{10} & \text{if } Z_i = (n-1)g(t), \end{cases} \tag{a4}$$

and for $Z \neq 0$, $X \in \hat{C}$, and $i > 1$,

$$F_i(q^{00}, z_i, Z, X, t) \equiv \begin{cases} q^{01} & \text{if } Z_i \neq 0 \\ q^{10} & \text{if } Z_i = 0. \end{cases} \tag{a5}$$

In contrast, player 1's machine state depends directly on his own previous contribution, not that of the other players. If his machine starts at q^{00}, it moves to the guilty state if he deviated and to the innocent state if he did not: for $Z \neq 0$ and $X \in \hat{C}$,

$$F_1(q^{00}, z_i, Z, X, t) \equiv \begin{cases} q^{01} & \text{if } z_1 = 0 \\ q^{10} & \text{if } z_1 \neq 0. \end{cases} \tag{a6}$$

If player 1's machine starts at q^0, it moves to the innocent state if he did not deviate and possibly only one of the others did; otherwise, with one knife-edged exception,[1] it moves to the guilty state: for $Z \neq ng(t)$ and $X \in \hat{C}$,

$$F_1(q^0, z_i, Z, X, t) \equiv \begin{cases} q^{01} & \text{if } z_1 = g(t) \text{ and } Z_1 \geq (n-2)g(t) \\ q^{01} & \text{if } z_1 = (n-1)g(t) \text{ and } Z_1 = 0 \\ q^{10} & \text{otherwise.} \end{cases} \tag{a7}$$

The guility and innocent states are sinks:

$$F_i(q_i, z_i, Z, X, t) \equiv q_i \text{ if } q_i \in \{q^{01}, q^{10}\}. \tag{a8}$$

For completeness, we set $F_i(q_i, z_i, Z, X, t) \equiv q_i$ for $X \geq X^*$.

Note that in any period t, precisely one of the following holds: (a) $q_i(t) = q^0$ for all $i \in N$, (b) $q_i(t) = q^{00}$ for all $i \in N$, or (c) $q_i(t) \in \{q^{10}, q^{01}\}$ for all $i \in N$. Hence, given that the transition functions are known, the following statements are true:

(F1) *If* $q_i(t) = q^0$ ($q_i(t) \neq q^0$), *player i knows* $q_j(t) = q^0$ ($q_j(t) \neq q^0$) *for all* $j \neq i$.

(F2) *If* $q_i(t) = q^{00}$ ($q_i(t) \neq q^{00}$), *player i knows* $q_j(t) = q^{00}$ ($q_j(t) \neq q^{00}$) *for all* $j \neq i$.

We now specify the probability function $P_i(\cdot \mid q_i, Z_i, t)$ representing player i's beliefs about the contribution vector $z_{-i}(t)$ of the other players in period t given that his machine is in state $q_i(t) = q_i$, and he observed that the aggregate contribution of the others was $Z_i(t) = Z_i$. The first requirement is needed for structural consistency:

$$P_i\Big(\sum_{j \neq i} \tilde{z}_j = Z_i \mid q_i, Z_i, t\Big) = 1. \tag{a9}$$

We next specify that if a player's machine is at q^0, and the player observes the others contribute $Z_i = (n-1)g(t)$, then he believes each of them contributed $g(t)$:

$$P_i\left(\tilde{z}_j = g(t) \mid q^0, (n-1)g(t), t\right) = 1 \text{ for all } j \neq i. \tag{a10}$$

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[1] The middle line of (a7) is the exception. It has a role, and is distinct from the first line, only if all players $j \neq 1$ deviate to $z_j = 0$, and player 1 deviates to $z_1 = (n-1)g(t)$. (This needs $n > 2$.) In this case players $j \neq 1$ see $Z_j = (n-1)g(t)$, and so each one thinks he is the only deviator. By (a4), their machines move to q^{10}. Player 1, realizing this, realizes that in the next period the others will contribute enough to finish the project, and so he will not contribute, in accordance with his machine moving to q^{01}.

This is necessary for the beliefs to be Bayesian consistent with the strategies: a player whose machine is in state q^0 in period t knows that the other players' machines are also in state q^0, and hence that each of them is supposed to contribute $g(t)$. If their aggregate contribution does not prove any of them deviated, player i's beliefs must put all probability on each of them having contributed $g(t)$.[2] Condition (a10) thus insures that beliefs are correct on the equilibrium path.

If $q_i \in \{q^0, q^{00}\}$, and Z_i proves to player i that at least one of the other players deviated, his beliefs depend on his identity. If $i > 1$, he believes player 1 deviated:

$$P_i\left(\tilde{z}_1 = g(t) \mid q^0, Z_i, t\right) = 0 \text{ for } Z_i \neq (n-1)g(t) \text{ and } i > 1, \tag{a11}$$

$$P_i\left(\tilde{z}_1 = 0 \mid q^{00}, Z_i, t\right) = 0 \text{ for } Z_i \neq 0 \text{ and } i > 1. \tag{a12}$$

If $i = 1$, and possibly just one of the others deviated, he believes only player 2 deviated:

$$P_1\left(\tilde{z}_j = g(t) \mid q^0, Z_1, t\right) = 1 \text{ for } Z_1 \geq (n-2)g(t) \text{ and } j > 2, \text{ and} \tag{a13}$$

$$P_1\left(\tilde{z}_j = 0 \mid q^{00}, Z_1, t\right) = 1 \text{ for } j > 2. \tag{a14}$$

The case not covered by (a13) and (a14) is that in which player 1 learns that two or more of the other players deviated, which happens if $q_i(t) = q^0$ and $Z_1 < (n-2)g(t)$. In this case we require, so long as Z_1 is positive, that the beliefs of player 1 be atomless:

$$P_1\left(\tilde{z}_j = \alpha \mid q^0, Z_1, t\right) = 0 \text{ for all } \alpha \geq 0,\ j > 1, \text{ and } 0 < Z_1 < (n-2)g(t). \tag{a15}$$

Conditions (a9) and (a10) insure structural consistency and, after histories ending with each machine being in either the normal or the grim state, Bayesian consistency. Conditions (a11) - (a15) concern beliefs after deviations, for which Bayesian consistency is not an issue. We have imposed no restrictions on the belief functions after histories ending with $q_i(t) \in \{q^{10}, q^{01}\}$ for all $i \in N$, since a player's beliefs about the contributions of the others after such a history are irrelevant (as we shall see). We henceforth let $(P_1, \ldots, P_n)$ be any belief profile satisfying (a9) - (a15).

---

[2] Similarly, (a9) implies Bayesian consistency at the end of a period t in which each machine is at q^{00}. For then player i expects the others to contribute zero, and (a9) requires him to put all probability on this event if indeed $Z_i(t) = 0$.

If the others use $((s,F_j))_{j\neq i}$, player i's optimal action after any history depends only on his beliefs about the current states of their machines. His beliefs if $q_i(t)\in\{q^0,q^{00}\}$ are given by (F1) and (F2) above. The following lemma establishes them otherwise.

Lemma A1. *For all $i\in N$, if player i knows that each player $j\neq i$ uses strategy (s,F_j), and player i has beliefs P_i satisfying* (a9) – (a15), *then for all $t\leq \overline{T}$,*

(F3) *If $q_i(t)=q^{10}$, player i believes $q_j(t)=q^{01}$ for all $j\neq i$, and*

(F4) *If $q_i(t)=q^{01}$, player i believes $q_j(t)=q^{10}$ for some $j\neq i$.*

Proof of (F3). Since q^{01} is a sink, if player i ever believes the machines of the others are at q^{01}, he will subsequently always believe they are at q^{01}. We can thus let t be the first date at which player i's machine is at q^{10}. Thus $q_i(t-1)\in\{q^0,q^{00}\}$. Let $Z=Z(t-1)$, $Z_i=Z_i(t-1)$, and $q_i=q_i(t-1)$. Also let $s=s(q_i,X(t-2),t-1)$ (which is either 0 or $g(t-1)$). By (F1) and (F2), player i knows the states of the other machines are all the same as his in period $t-1$. Hence, he knows that both he and each other player was supposed (in equilibrium) to have contributed s in period $t-1$. By (a4) – (a7), a deviation occurred that caused $Z\neq ns$, and it put $X(t-1)\in\hat{C}$.

Suppose $i>1$. Then by (a4) or (a5), player i deviated in period $t-1$, and his evidence regarding the others, $Z_i=(n-1)s$, is consistent with none of them having deviated. So by (a10) or (a9), player i believes no other player deviated: $P_i(z_j=s\mid q_i,Z_i,t-1)=1$ for all $j\neq i$. He thus believes that each player $j\neq i$ saw an aggregate contribution by the players other than j of $\tilde{Z}_j=z_i+(n-2)s\neq(n-1)s$. This implies, by (a4) or (a5) if $j\neq 1$, and by the first row of (a7) (note that $\tilde{Z}_j=z_i+(n-2)s\geq(n-2)s$) or by (a6) if $j=1$, that player i believes $q_j(t)=q^{01}$.

Now suppose $i=1$ and $q_i=q^{00}$. Then $z_1\neq 0$ by (a6), and so player 1 knows $Z_j\neq 0$ for each $j\neq 1$. Player 1 thus knows, by (a5), that $q_j(t)=q^{01}$.

Now suppose $i=1$ and $q_i=q^0$. Three cases must be considered, depending on Z_1. For the sake of brevity, we continue to use s for $g(t-1)$.

Case 1: $Z_1 = 0$. In this case player 1 knows $z_j = 0$ for all $j \neq 1$, and hence that $Z_j = z_1$ for all $j \neq 1$. Because $Z_1 = 0$, the middle line of (a7) implies $z_1 \neq (n-1)s$. Hence $Z_j \neq (n-1)s$, and so by (a4), player 1 knows $q_j(t) = q^{01}$ for all $j \neq 1$.

Case 2: $0 < Z_1 < (n-2)s$. By (a15), for each $j \neq 1$,

$$P_1\left[\tilde{Z}_j = (n-1)s \mid q_1, Z_1, t-1\right] = P_1\left[\tilde{z}_j = Z-(n-1)s \mid q_1, Z_1, t-1\right] = 0.$$

So by (a4), player 1 believes with probability one that $q_j(t) = q^{01}$ for all $j \neq 1$.

Case 3: $Z_1 \geq (n-2)s$. Now $z_1 \neq s$, by $q_1(t) = q^{10}$ and (a7). By (a13), player 1 believes player $j > 2$ contributed $\tilde{z}_j = s$, and so believes $\tilde{Z}_2 = z_1 + (n-2)s \neq (n-1)s$ is what player 2 sees. By (a4), player 1 thus believes $q_2(t) = q^{01}$. He also believes $j > 2$ sees $\tilde{Z}_j = Z - \tilde{z}_j = Z - s \neq (n-1)s$, and so by (a4) believes $q_j(t) = q^{01}$. ■

Proof of (F4). As in the proof of (F3), we can assume t is the first date at which player i's machine is at q^{01}, and so $q_i(t-1) \in \{q^0, q^{00}\}$. We again let $Z = Z(t-1)$, $Z_i = Z_i(t-1)$, $q_i = q_i(t-1)$, and $s = s(q_i, X(t-2), t-1)$. Again player i knows the states of the other machines are all the same as his in period $t-1$, and hence that each was supposed to have contributed s. By (a4) – (a7), a deviation occurred so that $Z \neq ns$ and $X(t-1) \in \hat{C}$.

Case 1: $i > 1$ and $q_i = q^{00}$. Now (a5) implies $Z_i \neq 0$, and so (a12) implies that player i is sure that $\tilde{z}_1 \neq 0$. Hence, by (a6), player i believes $q_1(t) = q^{10}$.

Case 2: $i > 1$ and $q_i = q^0$. (Again we use s instead of $g(t-1)$.) By (a4), $Z_i \neq (n-1)s$. So by (a11), player i believes with probability one that $\tilde{z}_1 \neq s$. Player i also believes with probability one that $\tilde{z}_1 \neq (n-1)s$ or $\tilde{Z}_1 \neq 0$, for otherwise $Z_i = \tilde{z}_1 + \tilde{Z}_1 - z_i = \tilde{z}_1 = (n-1)s$, a contradiction. So by (a7), player i believes $q_1(t) = q^{10}$.

Case 3: $i = 1$ and $q_i = q^{00}$. By (a6), $z_1 = 0$. By (a14), player 1 believes with probability one that $\tilde{z}_j = 0$ for $j > 2$. Hence, player 1 believes $\tilde{Z}_2 = z_1 + \sum_{j>2} \tilde{z}_j = 0$, and so by (a5), believes $q_2(t) = q^{10}$.

Case 4: $i = 1$ and $q_i = q^0$. By (a7), there are two subcases to consider. The first is $z_1 = s$ and $Z_1 \geq (n-2)s$, in which case (a13) implies that player 1 believes $\tilde{z}_j = s$ for all $j > 2$. Then player 1 believes $\tilde{Z}_2 = (n-1)s$, and so $q_2(t) = q^{10}$ by (a4). The second subcase is $z_1 = (n-1)s$ and $Z_1 = 0$. In this case player 1 knows $z_j = 0$, and hence that $Z_j = z_1 = (n-1)s$, for all $j \neq 1$. So by (a4), player 1 knows $q_j(t) = q^{10}$ for all $j \neq 1$. ■

We have now defined the assessment $((s,F_i),P_i)_{i\in N}$, and it satisfies Bayes consistency (and more) by construction. We complete the proof of Theorem 2 by showing it is sequentially rational. In particular, we show that no player believes he can gain from a one-shot deviation after any history. We can restrict attention to histories that leave time to contribute and have not finished the project. (After other histories, not contributing is a dominant strategy, and this is in accordance with s.) So, let $(z_i(\tau),Z(\tau))_{\tau<t}$ be a history observed by player i with $t<\overline{T}$ and $X(t-1)<X^*$. If the subsequent contribution sequence is $(\hat{z}(\tau))_{\tau\geq t}$, and it wastelessly completes the project in period $T\leq\infty$, the player's (present value) continuation payoff (see (2.8)) is

$$U_i^c \equiv \sum_{\tau=t}^{T}\delta^{\tau-t}\left[\lambda\hat{Z}(\tau)-\hat{z}_i(\tau)\right]+\delta^{T-t}b.$$

Given a one-shot deviation $z_i(t)$ after this history, the beliefs of player i about $(\hat{z}(\tau))_{\tau\geq t}$ and T depend on his beliefs about the states of the other players' machines in period t, and the publicly known t, $X(t-1)$, and $Z(t)$. By (F1) – (F4), his beliefs about the machines of the others depend only on his machine's state.

Case $q_i(t)=q^0$.

By (F1), player i knows $q_j(t)=q^0$ for all $j\in N$. He thus knows that if he does not deviate, every player's continuation contribution sequence will be $(g(\tau))_{\tau\geq t}$ and the project will be completed at $T=T(g)$. This gives the player the same continuation payoff as in the grim-g equilibrium after $t-1$ periods. If instead player i unilaterally deviates to $z_i(t)\neq g(t)$, each $j\neq i$ will contribute $z_j(t)=g(t)$ and observe $Z_j(t)=z_i(t)+(n-2)g(t)$. Thus, $Z(t)\neq ng(t)$, $Z_j(t)\neq(n-1)g(t)$, and $Z_1(t)\geq(n-2)g(t)$, This implies, by (a2), (a4), and (a7), that either $q_j(t+1)=q^{00}$ for all $j\neq i$, or $q_j(t+1)=q^{01}$ for all $j\neq i$. In either case, (4.1) implies that the deviation brings to a halt all future contributions by players $j\neq i$, just as it would in the grim-g equilibrium. In period t player i knows this will be the consequence of his deviation, since he knows the machines of the others are presently at q^0. So he must believe the deviation is unprofitable, since it is unprofitable in the grim-g equilibrium.

Case $q_i(t)=q^{00}$.

By (F2), player i knows $q_j(t)=q^{00}$ for all $j\in N$. So he knows that if he does not deviate, no more contributions will be made and his continuation payoff will be $U_i^c=0$.

A one-shot deviation $z_i(t) > 0$ yielding $X(t) < X^* - c(0)$ causes all machines to stay at q^{00}, and no subsequent contributions will be made. Player i, realizing this, knows his continuation payoff will be $(\lambda - 1)z_i(t) < 0$, and so regards the deviation as unprofitable. If the deviation causes $X(t) \in \hat{C}$, then (a5) and (a6) imply that $q_j(t+1) = q^{01}$ for all $j \neq i$, since every player $j \neq i$ will contribute $z_j(t) = 0$ and see $Z_j(t) = Z(t) = z_i(t) \neq 0$. Player i then knows his continuation payoff will be (recall that $c(0) = b/(1-\lambda)$)

$$\begin{aligned}\hat{U}_i^c &= (\lambda - 1)z_i(t) + \delta\Big[(\lambda - 1)\big(X^* - z_i(t) - X(t-1)\big) + b\Big] \\ &= -(1-\lambda)\Big\{(1-\delta)z_i(t) + \delta\big(X^* - c(0) - X(t-1)\big)\Big\} < 0,\end{aligned}$$

and so this deviation is unprofitable. Finally, a deviation that yields $X(t) \geq X^*$ must be worse than the smaller contribution that yields $X(t) = X^*$, i.e., the wastelessly completing contribution $z_i = X^* - X(t-1)$. Player i knows the latter contribution is unprofitable because he knows it yields a continuation payoff of

$$\begin{aligned}\hat{U}_i^c &= (\lambda - 1)\big(X^* - X(t-1)\big) + b \\ &= -(1-\lambda)\big(X^* - c(0) - X(t-1)\big) < 0.\end{aligned}$$

Case $q_i(t) = q^{10}$.

By (F3), player i believes $q_j(t) = q^{01}$ for all $j \neq i$. Since q^{01} is a sink, he believes no other player will contribute in any period $\tau \geq t$. As $X(t-1) \in \hat{C}$, the amount it takes to complete the project, $X^* - X(t-1)$, does not exceed $c(0)$. So by (3.1), player i's best reply is to complete the project immediately: $s(q^{10}, X(t-1), t) = X^* - X(t-1)$.

Case $q_i(t) = q^{01}$.

By (F4), player i believes $q_j(t) = q^{10}$ for some $j \neq i$. So he believes the project will be completed without his help, and so his best reply is $s(q^{01}, X(t-1), t) = 0$. ■

Appendix B: Proofs for Section 6

Lemma B1. *Expressions* (6.1)–(6.4) *define a MPE strategy of the game with* $\overline{T} = \infty$ *and* $b > 0$ *if and only if* $X_\infty < 0$.

Proof. Expressions (6.1)–(6.4) define a Markov strategy if and only if $X_\infty < 0$. We must show only that if $X_\infty < 0$, then ψ is a MPE, i.e., that starting from any X, a player should use ψ if the others do.

Let $k \geq 0$ and suppose $X \in [X_{k+1}, X_k)$. Define

$$\hat{X}(X) \equiv X_k - (\tfrac{1}{n})(X_k - X). \tag{b1}$$

If a player contributes $z_i \geq 0$ in the current period, and the other players use ψ, the cumulation rises to $Y = \hat{X}(X) + z_i$. We view the player as choosing Y instead of z_i, subject to the constraint $Y \geq \hat{X}(X)$. Choosing $Y > X^*$ is strictly dominated, and so we can restrict attention to $Y \in [\hat{X}(X), X^*]$. If the player chooses Y from this interval, and subsequently joins the others in playing ψ, his continuation payoff is

$$W(Y, X) \equiv \lambda(Y - X) + \beta(Y)(1-\delta)b - (Y - \hat{X}(X)) + \delta V(Y), \tag{b2}$$

where V and β are shown in (6.6) and (6.7). According to ψ, the player should choose $Y = X_k$. Hence, we must show that $Y = X_k$ maximizes $W(\cdot, X)$ on $[\hat{X}(X), X^*]$. This finishes the proof, as it shows that no one-shot unilateral deviation from ψ is profitable.

On the intervals $[X_{\kappa+1}, X_\kappa)$, V and hence $W(\cdot, X)$ are continuous. By (6.6), $V'(Y) = \frac{1}{n} - \lambda$ on these intervals. So for $Y \in [X_{\kappa+1}, X_\kappa)$,

$$W_Y(Y, X) = [(1-\delta)\lambda + \delta \tfrac{1}{n}] - 1 < 0. \tag{b3}$$

This shows that on $[\hat{X}(X), X^*]$, $W(\cdot, X)$ has a maximizer, and each of its maximizers is in the set $\{\hat{X}(X), X_k, X_{k-1}, \ldots, X_1, X_0 = X^*\}$.

Using (6.6) and (6.8), we see that $W(X_k, X) = V(X)$ and $W(\hat{X}(X), X) = V^d(X)$. By construction, $V(X_{k+1}) = V^d(X_{k+1})$. For any $\alpha \in (X_{k+1}, X_k)$,

$$V'(\alpha) - V^{d\prime}(\alpha) = \tfrac{1}{n}[1 - (1-\delta)\lambda - \delta \tfrac{1}{n}] > 0. \tag{b4}$$

Thus $W(\hat{X}(X), X) \leq W(X_k, X)$, and so $\{X_k, \ldots, X_0\}$ contains a maximizer of $W(\cdot, X)$. For $\kappa \in \{0, \ldots, k-1\}$,

$$\begin{aligned} W(X_{K+1},X)-W(X_K,X) &= (1-\lambda)(X_K - X_{K+1}) - (1-\delta)\beta(X_K)b \\ &\quad + \delta[V(X_{K+1}) - V(X_K)] \\ &= (1-\lambda)K\big[\delta V(X_K) + (1-\delta)\beta(X_K)b\big] \\ &\quad + \delta\big[C\big[\delta V(X_K)+(1-\delta)\beta(X_K)b\big] - V(X_K)\big], \end{aligned} \tag{b5}$$

using (b2) for the first equality, and (6.9) and (6.10) for the second. From (6.3) and (6.4), $(1-\lambda)K - 1 + C\delta = (n-1)(1-\delta)$. Using this in (b5) yields

$$\begin{aligned} W(X_{K+1},X)-W(X_K,X) &= (n-1)(1-\delta)\big[\delta V(X_K)+(1-\delta)\beta(X_K)b\big] \\ &> 0, \end{aligned} \tag{b6}$$

where the inequality follows from $b > 0$ and $V(X_K) > 0$ (the latter follows from (6.9)). As $\{X_k,\ldots,X_0\}$ contains a maximizer of $W(\cdot,X)$, (b6) shows that X_k is a maximizer. ■

Lemma B2. *Define, for* $\lambda \neq \frac{1}{n}$,

$$\hat{\delta} \equiv \frac{(1-\lambda)a - b}{(n-1)(\lambda - \frac{1}{n})a}. \tag{b7}$$

Then if $b > 0$,

(i) for $\lambda > \frac{1}{n}$: $X_\infty < 0$ *iff* $\delta > \hat{\delta}$; $\hat{\delta} > 0$ *iff* $c(0) < a$; *and* $\hat{\delta} < 1$ *iff* $V > (2 - \frac{1}{n})a$.

(ii) for $\lambda < \frac{1}{n}$: $X_\infty < 0$ *iff* $\delta < \hat{\delta}$; $\hat{\delta} > 0$ *iff* $c(0) > a$; *and* $\hat{\delta} < 1$ *iff* $V < (2 - \frac{1}{n})a$.

(iii) for $\lambda = \frac{1}{n}$: $X_\infty < 0$ *iff* $c(0) > a$ *iff* $V > (2 - \frac{1}{n})a$.

Proof. From (6.2), $X_\infty = -\infty$ if $\delta C \geq 1$, and otherwise $X_\infty = X^* - A(\delta)$, where

$$A(\delta) \equiv \frac{Kb}{1-\delta C} = \frac{nb}{1-\lambda-(n-1)(\lambda-\frac{1}{n})\delta}. \tag{b8}$$

Hence, $X_\infty < 0$ if and only if $A(\delta) < 0$ or $A(\delta) > X^*$.

If $A(\delta)$ is finite, $\operatorname{sign} A'(\delta) = \operatorname{sign}(\lambda - \frac{1}{n})$. The equation $A(\delta) = X^*$ has a unique solution, the $\hat{\delta}$ given in (b7), if and only if $\lambda \neq \frac{1}{n}$. Note that $A(0) = nc(0)$. Figure A1 illustrates, assuming $c(0) < a$.

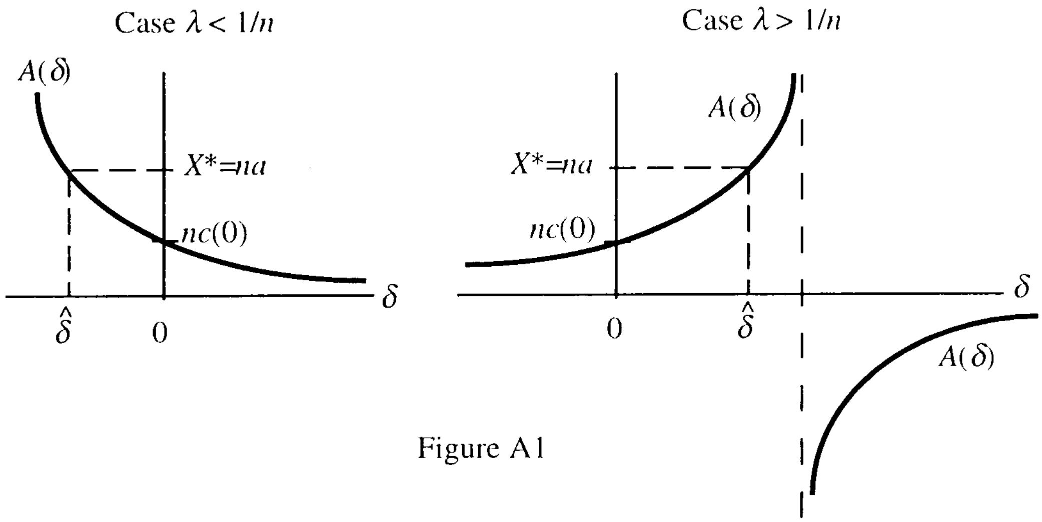

Figure A1

If $A(1)$ is positive, then $A(1) > X^*$ if and only if $b > (2-\frac{1}{n})a - n\lambda a$, and this last inequality is the same as $V > (2-\frac{1}{n})a$. Each part of the lemma now follows. ■

Before proving Corollary 4, we record the following fact:

$$\lim_{\delta\to 1}\left(\frac{\ln(\delta)}{\ln(\delta C)}\right) = \lim_{\delta\to 1}\left(\frac{1/\delta}{1/\delta + \frac{\partial C}{\partial \delta}/C}\right) = \frac{-a\left(\frac{n-1}{n}\right)}{V-(2-\frac{1}{n})a-b}. \tag{b9}$$

The first inequality in (b9) follows from L'Hôpital's rule, justified by the observation that (6.3) and (6.4) imply $\delta C \to 1$ as $\delta \to 1$.

Proof of Corollary 4. By Theorem 3, we need only show that the time to completion if ψ is used converges to L^*. By (6.2), the completion date is the smallest k satisfying

$$A(\delta)\left[1-(\delta C)^{k+1}\right] \geq X^*,$$

where $A(\delta)$ is defined in (b8). The completion date is the first integer greater than or equal to the k for which the above inequality is an equality, which is

$$\tau(\delta) \equiv \frac{\ln\left[1-X^* A(\delta)^{-1}\right]}{\ln(\delta C)} - 1. \tag{b10}$$

(The numerator is well-defined since, by the proof of Lemma B2, either $A(\delta) < 0$ or $A(\delta) > X^*$.) Expressing ℓ in terms of δ yields $\ell(\delta) = -r^{-1}\ln(\delta)$. The time to completion is between $(\tau(\delta)+1)\ell(\delta)$ and $(\tau(\delta)+2)\ell(\delta)$, and hence has the same limit as

$$L(\delta) \equiv (\tau(\delta)+1)\ell(\delta) = \frac{\ln\left[1 - X^* A(\delta)^{-1}\right]\ln(\delta)}{-r\ln(\delta C)}.$$

Using (b8) and (b9), we see that

$$\lim_{\delta\to 1} L(\delta) = \frac{a\left(\frac{n-1}{n}\right)\ln\left[1 - X^* A(1)^{-1}\right]}{r\left[V - (2-\frac{1}{n})a - b\right]}$$

$$= \frac{a\left(\frac{n-1}{n}\right)\left[\ln\left(V - (2-\frac{1}{n})a\right) - \ln(b)\right]}{r\left[V - (2-\frac{1}{n})a - b\right]} = L^*. \blacksquare$$

Proof of Theorem 4 for $\delta \in (\gamma, 1)$. With $\delta < 1$, we need to show only that one-shot deviations are unprofitable. Let the current cumulation be $X \in [\hat{X}_{k-1}, \hat{X}_k)$ for some $k \geq 1$, and consider player $i \in N$.

Case $i =_n k$. In this case player i is supposed to unilaterally bring the cumulation up to $\hat{X}_k$. He can raise it to any level $Y \geq X$. As $Y \geq X^*$ is strictly dominated, we restrict attention to $Y \in [X, X^*)$. Choosing such a Y and then joining the others in playing σ gives the player a continuation payoff of

$$W^i(Y, X) \equiv (\lambda - 1)(Y - X) + \delta V_i(Y). \qquad \text{(b11)}$$

On each interval $[\hat{X}_{\kappa-1}, \hat{X}_\kappa)$, $W^i(\cdot, X)$ is continuous and decreasing.[a] A maximizer of $W^i(\cdot, X)$ on $[X, X^*)$ is thus in $\{X, \hat{X}_k, \hat{X}_{k+1}, \ldots\}$. We show that $Y = \hat{X}_k$ is a maximizer.

Note that $W^i(X, X) = \delta V_i(X)$, and $W^i(\hat{X}_k, X) = V_i(\hat{X}_k)$. Thus, $V_i(\hat{X}_k) \geq 0$ implies $W^i(X, X) \leq W^i(\hat{X}_k, X)$. This shows that a maximizer of $W^i(\cdot, X)$ is in $\{\hat{X}_k, \hat{X}_{k+1}, \ldots\}$.

For $\kappa \geq k$, let $\Delta_\kappa \equiv W(\hat{X}_{\kappa+1}, X) - W(\hat{X}_\kappa, X)$. Hence,

$$\Delta_\kappa = (\lambda - 1)(\hat{X}_{\kappa+1} - \hat{X}_\kappa) + \delta[V_i(\hat{X}_{\kappa+1}) - V_i(\hat{X}_\kappa)]. \qquad \text{(b12)}$$

If $i =_n \kappa + 1$, then $V_i(\hat{X}_\kappa) = 0$ by (6.16). In this case, using (6.15),

$$\Delta_\kappa = (\lambda - 1)(\hat{X}_{\kappa+1} - \hat{X}_\kappa) + \delta V_i(\hat{X}_{\kappa+1}) = V_i(\hat{X}_\kappa).$$

This shows that $\Delta_\kappa = 0$ if $i =_n \kappa + 1$. Now suppose $i \neq_n \kappa + 1$. Then by (6.15),

$$V_i(\hat{X}_\kappa) = \lambda(\hat{X}_{\kappa+1} - \hat{X}_\kappa) + \delta V_i(\hat{X}_{\kappa+1}). \qquad \text{(b13)}$$

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[a] By (6.15), $W_Y^i(\cdot, X) = (1-\delta)(\lambda - 1) < 0$ if $i =_n \kappa$, and $W_Y^i(\cdot, X) = (1-\delta)\lambda - 1 < 0$ if $i \neq_n \kappa$.

Substituting this into (b12) yields

$$\Delta_\kappa = V_i(\hat{X}_\kappa) - \left[\hat{X}_{\kappa+1} - \hat{X}_\kappa + \delta V_i(\hat{X}_\kappa)\right]. \tag{b14}$$

Note that $\hat{X}_{\kappa+1} > \hat{X}_\kappa$ and $\lambda = V/na \in (0,1)$, and so, using (b13), $V_i(\hat{X}_\kappa) > V_i(\hat{X}_{\kappa+1})$. Therefore, (b14) and (b13) and imply

$$\Delta_\kappa < V_i(\hat{X}_\kappa) - \left[\lambda(\hat{X}_{\kappa+1} - \hat{X}_\kappa) + \delta V_i(\hat{X}_{\kappa+1})\right] = 0.$$

We conclude that $\Delta_\kappa \le 0$ for all $\kappa \ge k$, and hence $Y = \hat{X}_k$ maximizes $W^i(\cdot, X)$.

Case $i \neq_n k$. In this case, in the current period, player i is not supposed to contribute and some other player raises the cumulation $\hat{X}_k$. If player i contributes, he raises the cumulation some $Y \ge \hat{X}_k$. As $Y \ge X^*$ is strictly dominated, we restrict attention to $Y \in [\hat{X}_k, X^*)$. Choosing such a Y and then joining the others in playing σ gives the player a continuation payoff of

$$\begin{aligned}\hat{W}^i(Y,X) &\equiv \lambda(\hat{X}_k - X) + (\lambda - 1)(Y - \hat{X}_k) + \delta V_i(Y) \\ &= \lambda(\hat{X}_k - X) + W^i(Y, \hat{X}_k),\end{aligned} \tag{b15}$$

where $W^i(\cdot,\cdot)$ is defined in (b11). We must show that $Y = \hat{X}_k$ maximizes $\hat{W}^i(\cdot, \hat{X}_k)$ on $[\hat{X}_k, X^*)$. We just showed that $Y = \hat{X}_k$ maximizes $W^i(\cdot, X)$ on $[X, X^*)$ for any $X \in [\hat{X}_{k-1}, \hat{X}_k)$. Hence, as $W^i(Y,X)$ is continuous in X, $Y = \hat{X}_k$ maximizes $W^i(\cdot, \hat{X}_k)$ on $[\hat{X}_k, X^*)$. So (b15) implies that $Y = \hat{X}_k$ maximizes $\hat{W}^i(\cdot, \hat{X}_k)$ on $[\hat{X}_k, X^*)$. This finishes the proof for $\delta < 1$.

Proof of Theorem 4 for $\delta = 1$. Even though $\delta = 1$, σ is well-defined and yields $\hat{X}_k = (1-\gamma^k)X^* \to X^*$. Each V_i is also well-defined, with finite nonnegative values.

Let $i \in N$, and let player i's and the aggregate current cumulative contributions be x_i and X. We show that conditional on starting from (x_i, X), σ_i is a best reply to σ_{-i}.

If $X \ge X^*$, σ_i agrees with the conditionally dominant strategy of never contributing, and so it is a conditional best reply to σ_{-i}. We now assume $X < X^*$.

When σ is played starting from (x_i, X), $X(t) \to X^*$. Let x_i^∞ be the corresponding limit of player i's cumulative contribution. Hence, conditional on having reached (x_i, X) and then playing σ, player i's payoff is $U_i = \lambda X^* - x_i^\infty$.

Let $\tilde{\sigma}_i$ be a pure strategy best reply to σ_{-i}, conditional on having reached (x_i, X). Let the sequences of player i's and the aggregate cumulative contributions when $(\tilde{\sigma}_i, \sigma_{-i})$

is played, starting from (x_i, X), be $\{\tilde{x}_i(\tau)\}$ and $\{\tilde{X}(\tau)\}$. Let the limits of these sequences be $\tilde{x}_i^\infty$ and $\tilde{X}^\infty$. If $\tilde{X}^\infty > X^*$ then, given the nature of σ_{-i}, at some date τ player i contributes more than $X^* - \tilde{X}(\tau - 1)$, which is strictly dominated. Hence, $\tilde{X}^\infty \leq X^*$.

Suppose $\tilde{X}^\infty = X^*$. The nature of σ_{-i} then implies that starting from (x_i, X), player i contributes no less when he uses $\tilde{\sigma}_i$ than when he uses σ_i: $\tilde{x}_i^\infty \geq x_i^\infty$. His conditional payoff when he uses $\tilde{\sigma}_i$, $\tilde{U}_i = \lambda X^* - \tilde{x}_i^\infty$, is thus no greater than it is when he uses σ_i, $U_i = \lambda X^* - x_i^\infty$. Hence, σ_i is a conditional best reply to σ_{-i}.

Now suppose $\tilde{X}^\infty < X^*$. Then for some $k \geq 1$, $\tilde{X}^\infty \in [\hat{X}_{k-1}, \hat{X}_k)$. The nature of σ_{-i} implies that $i =_n k$, and that $\tilde{\tau} < \infty$ exists for which $\tilde{X}(\tau) \in [\hat{X}_{k-1}, \hat{X}_k)$ for all $\tau \geq \tilde{\tau}$; only player i ever contributes after date $\tilde{\tau}$. Now, modify $\tilde{\sigma}_i$ to a strategy $\overline{\sigma}_i$ by replacing it with σ_i at all dates $\tau > \tilde{\tau}$. According to $\overline{\sigma}_i$, player i at date $\tilde{\tau} + 1$ raises the cumulation from $\tilde{X}(\tilde{\tau})$ to $\hat{X}_k$, whereupon it is raised successively to $\hat{X}_{k+1}$, $\hat{X}_{k+2}$, ..., and converges to X^*. This gives the player a continuation payoff of $V_i(\tilde{X}(\tilde{\tau}))$, and so his payoff conditional on having reached (x_i, X) and then $(\overline{\sigma}_i, \sigma_{-i})$ being played is

$$\begin{aligned}
\overline{U}_i &= \lambda \tilde{X}(\tilde{\tau}) - \tilde{x}_i(\tilde{\tau}) + V_i(\tilde{X}(\tilde{\tau})) \\
&= \lambda \tilde{X}^\infty - \tilde{x}_i^\infty - \lambda\left(\tilde{X}^\infty - \tilde{X}(\tilde{\tau})\right) + \left(\tilde{x}_i^\infty - \tilde{x}_i(\tilde{\tau})\right) + V_i(\tilde{X}(\tilde{\tau})) \\
&= \tilde{U} + (1 - \lambda)\left(\tilde{X}^\infty - \tilde{X}(\tilde{\tau})\right) + V_i(\tilde{X}(\tilde{\tau})),
\end{aligned}$$

where the last equality uses $\tilde{x}_i^\infty - \tilde{x}_i(\tilde{\tau}) = \tilde{X}^\infty - \tilde{X}(\tilde{\tau})$. Since $V_i(\tilde{X}(\tilde{\tau})) \geq 0$, this shows that $\overline{U}_i \geq \tilde{U}_i$. Hence, $\overline{\sigma}_i$ is also a conditional best reply to σ_{-i}. The argument of the previous paragraph can therefore be applied with $\overline{\sigma}_i$ replacing $\tilde{\sigma}_i$, since the aggregate cumulation does converge to X^* when $(\overline{\sigma}_i, \sigma_{-i})$ is played after (x_i, X) is reached. This finishes the proof that σ_i is a conditional best reply to σ_{-i}. ■