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PROOF THAT PRICES WHICH ARE PRESENT-DISCOUNTED CERTAINTY EQUIVALENTS FLUCTUATE RANDOMLY*

by

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1. CONSISTENT BELIEFS

Consider a time sequence $\ldots, \tilde{\Phi}_t, \tilde{\Phi}_{t+1}, \ldots$ of random vectors. As in Samuelson 1965 [4], the price of spot # 2 wheat in Chicago at time t might be some component, say, $\tilde{X}_t$ of the vector $\tilde{\Phi}_t$; another component, say, $\tilde{V}_t$ of the vector $\tilde{\Phi}_t$ might denote, as in Samuelson 1973 [5], the price of common stock of General Motors at time t; and so on. At any time t, the values $\ldots, \tilde{\Phi}_{t-1} = \varphi_{t-1}, \tilde{\Phi}_t = \varphi_t$ are already history and, so, are fixed forever. But the same cannot be said about the values of $\tilde{\Phi}_{t+1}, \tilde{\Phi}_{t+2}, \ldots$. These values are still in the future, and we may suppose that at time t they cannot be known with certainty.

Now, fix attention on $\tilde{X}_T$, where T is some arbitrarily chosen date; and suppose that at any time $t \leq T$ an individual assigns to $\tilde{X}_T$ a probability distribution

$$\text{Prob}\{\tilde{X}_T \leq x_T \mid \varphi_t, \varphi_{t-1}, \ldots\} = P_t(x_T \mid \varphi_t, \varphi_{t-1}, \ldots) . \tag{1}$$

The probability distribution that the individual assigns to $\tilde{X}_T$ at time $t+1$ would depend, in general, upon the value that $\tilde{\Phi}_{t+1}$ takes. Suppose that at time t the individula assigns to $\tilde{\Phi}_{t+1}$ the probability distribution

$$\text{Prob}\{\tilde{\Phi}_{t+1} \le \varphi_{t+1} \mid \varphi_t, \varphi_{t-1}, \ldots\} = P_t(\varphi_{t+1} \mid \varphi_t, \varphi_{t-1}, \ldots) ; \qquad (2)$$

and to $\tilde{X}_T$ conditional upon the value that $\tilde{\Phi}_{t+1}$ takes, he assigns the (conditional) probability distribution

$$\text{Prob}\{\tilde{X}_T \le x_T \mid \tilde{\Phi}_{t+1} = \varphi_{t+1}, \varphi_t, \ldots\} = P_t(x_T \mid \varphi_{t+1}, \varphi_t, \ldots) . \qquad (3)$$

ASSUMPTION 0: The individual's probability beliefs are consistent in the sense that they accord with the fundamental logic of probability calculus.

Then (1), (2) and (3) above necessarily obey the relation

$$P_t(x_T \mid \varphi_t, \varphi_{t-1}, \ldots) = \int_{-\infty}^{\infty} P_t(x_T \mid \varphi_{t+1}, \varphi_t, \varphi_{t-1}, \ldots) \, P_t(d\varphi_{t+1} \mid \varphi_t, \varphi_{t-1}, \ldots) . \qquad (4)$$

Herein, $\int_{-\infty}^{\infty} f(x)\, g(dx)$ denotes a Stieltjes integral; and when x is a vector, it denotes a multiple Stieltjes integral.

2. ECONOMIC BEHAVIOR: A REVIEW

Suppose, now, that the individual owns a futures contract in # 2 wheat for delivery in Chicago at time T; and, for this contract, let $\tilde{Y}_t$ be his minimum asking price at any time $t \le T$. Then at any time

$t < T$, the asking prices $\ldots, \tilde{Y}_{t-1} = y_{t-1}, \tilde{Y}_t = y_t$ are already history and, hence, fixed. But the asking prices $\tilde{Y}_{t+1}, \tilde{Y}_{t+2}, \ldots, \tilde{Y}_T$ are still in the future, and we may suppose that they cannot at time t be stated with certainty. We may, however, presume that when the due date T for the futures contract arrives arbitrage will ensure that

$$\tilde{Y}_T = x_T \text{ iff } \tilde{X}_T = x_T \text{ commissions aside.} \tag{5}$$

For all times $t < T$, the relation between y_t and $\tilde{X}_T$ will depend upon what we posit about how the individual sets his asking price y_t.

For example, it might be posited that at any time $t < T$ the individual sets y_t equal to the <u>now-expected level</u> of the terminal spot price $\tilde{X}_T$. That is.

$$\begin{aligned} y_t &= \int_{-\infty}^{\infty} x_T \, P_*(dx_T \mid \varphi_t, \varphi_{t-1}, \ldots) \\ &= E_t(\tilde{X}_T), \end{aligned} \tag{6}$$

where E_t denotes the "<u>expectation operator</u>" with respect to the probability distribution $P_*(x_T \mid \varphi_t, \varphi_{t-1}, \ldots)$ which the individual assigns to $\tilde{X}_T$ conditional upon $\tilde{\Phi}_t = \varphi_t, \tilde{\Phi}_{t-1} = \varphi_{t-1}, \ldots$. This implicitly assumes that the individual has a linear utility for cash flow or income; further, it ignores the availability (to the individual) of risk-free investments yielding a positive interest.

Accordingly, it might more generally be posited that at any time $t < T$ the individual sets y_t equal to the <u>present-discounted</u> expected value of $\tilde{X}_T$, the discount rate $(r_{t+1}, r_{t+2}, \ldots, r_T)$ being equal to the risk-free interest rate $(\rho_{t+1}, \rho_{t+2}, \ldots, \rho_T)$ suitably adjusted for the individual's risk attitude toward holding out the futures contract for the next period ((t to t+1), (t+1 to t+2), ..., (T-1 to T)). That is,

$$y_t = \lambda_{t+1}^{-1} \cdot \lambda_{t+2}^{-1} \cdot \ldots \cdot \lambda_T^{-1} \cdot E_t(\tilde{X}_T)$$

$$= {}_t\pi_T \circ E_t(\tilde{X}_T) , \qquad (7)$$

where $\lambda_i = 1 + r_i$, with $\lambda_i > \rho_i$ [resp. $\lambda_i < \rho_i$] $(i = t+1, t+2, \ldots, T)$ according as the individual is risk averse [resp. risk loving]; and ${}_t\pi_T$ denotes the "discounting operator" from time T to t at the individual's discount rate schedule.

A little reflection shows that both (6) and (7) above are subsumed under the more general behavioral assumption that at any time $t \leq T$ the individual sets y_t equal to the present-discounted certainty equivalent of the terminal spot price $\tilde{X}_T$, with the discount rate $(r_{t+1}, r_{t+2}, \ldots, r_T)$ being equal to the risk-free interest rate $(\rho_{t+1}, \rho_{t+2}, \ldots, \rho_T)$ for the period ((t to t+1), (t+1 to t+2), ..., (T-1 to T)).

The question then is what can we say about the sequence $\ldots, y_{t-1}, y_t, \tilde{Y}_{t+1}, \ldots, \tilde{Y}_T$ $(t < T)$?

Samuelson 1965 [4] -- with a slight reinterpretation -- provides an answer to this question when y_t is related to $\tilde{X}_T$ $(t < T)$ by (6) and (7), respectively. The more general case, when y_t is equal to the present-discounted certainty equivalent of $\tilde{X}_T$, I shall now investigate.

3. CERTAINTY EQUIVALENT: CONSISTENT PREFERENCES WITH TIME DISCOUNTING

Make the following sufficient assumptions to guarantee the existence of cardinal utilities u_t for cash flow or income at time t $(t \in \mathbb{Z})$, where $\mathbb{Z}$ is the set of natural numbers.

ASSUMPTION 1: For the purpose of the individual's preferences, a risky alternative is completely characterized by the probability distributions for cash flow or income at time t $(t \in \mathbb{Z})$.

ASSUMPTION 2: The individual has, over all risky alternatives, preferences which are consistent in the sense that he cannot, so to speak, make book against himself and end up winning -- or losing -- money! In other words, posit the Axiom of Complete Ordering of all risky alternatives, and the Axiom of "Strong Independence" (see, for example, Samuelson 1952 [3]).

This much assumption implies the existence of cardinal utility functions u_t for sure cash flow or income at time t $(t \in \mathbb{Z})$ and, hence, also the expected utility maximization rule for choice among risky alternatives. The following further assumption should be acceptable to all but the mystical few.

ASSUMPTION 3: All the utility functions u_t $(t \in \mathbb{Z})$ are strictly increasing monotonic in their argument, cash flow or income.

This now allows definition of the individual's "<u>certainty equivalent operator</u>" C_t :

$$C_t(\tilde{X}_\tau) = \zeta_\tau \quad \text{iff} \quad u_\tau(\zeta_\tau) = \int_{-\infty}^{\infty} u_\tau(x_\tau)\ P_*(dx_\tau \mid \varphi_t, \varphi_{t-1}, \ldots), \tag{8}$$

where $\tilde{X}_\tau$ $(\tau \in \mathbb{Z})$ is any random cash flow at time τ and $P_*(x_\tau \mid \varphi_t, \varphi_{t-1}, \ldots)$ is the probability distribution which the individual assigns to $\tilde{X}_\tau$ conditional upon $\tilde{\Phi}_t = \varphi_t,\ \tilde{\Phi}_{t-1} = \varphi_{t-1}, \ldots$ $(t \le \tau)$.

Thus, the behavioral axiom that at any time $t \le T$ the individual sets y_t equal to the present-discounted certainty equivalent of the terminal spot price $\tilde{X}_T$ may be formally stated as

$$\begin{aligned} y_t &= \lambda_{t+1}^{-1} \cdot \lambda_{t+2}^{-1} \cdot \ \ldots \ \cdot \lambda_T^{-1} \cdot C_t(\tilde{X}_T) \\ &= {}_t\pi_T \circ C_t(\tilde{X}_T), \end{aligned} \tag{9}$$

where, in this case, $\lambda_i = 1 + \rho_i$ $(i = t+1, t+2, \ldots, T)$.

Lastly, make the following

<u>ASSUMPTION</u> <u>4</u>: The individual's preferences among <u>sure</u> cash flows at different times accord with the usual present-discounted value calculus.

The following Fundamental Consistency Theorem may now be recaptured without proof from Prakash 1974 [1] and [2].

<u>THEOREM</u> (<u>Prakash</u> <u>1974</u>): <u>Grant Assumptions 1 through 4 above</u>. <u>Then, the family</u> $\{u_t \mid t \in \mathbb{Z}\}$ <u>of the individual's cardinal</u>

<u>utility functions is such that, for any</u> $\tau \in \mathbb{Z}$, <u>and any</u> $t \leq \tau$,

$$ {}_t\pi_\tau \circ C_t \equiv C_t \circ {}_t\pi_\tau . \tag{10}$$

Not too roughly, this says that, if an individual has consistent preferences, then it must be that his present-discounted certainty equivalent of any random cash flow $\tilde{X}_\tau$ is the same as his present certainty equivalent of the random cash flow obtained by discounting $\tilde{X}_\tau$ to the present. Using (8) above, (10) translates into

$$ u_t({}_t\pi_\tau(\zeta_\tau)) = \int_{-\infty}^{\infty} u_t({}_t\pi_\tau(x_\tau))\ P_t(dx_\tau \mid \varphi_t, \varphi_{t-1}, \ldots) , \tag{11}$$

where $\zeta_\tau = C_t(\tilde{X}_\tau)$.

4. PRESENT-DISCOUNTED CERTAINTY EQUIVALENTS FLUCTUATE RANDOMLY

Toward enunciating the main theorems, let $\{\tilde{\Phi}_t\}$ be a "time" sequence of random vectors $\tilde{\Phi}_t$ of which some component $\tilde{X}_t$ denotes random cash flow at time t . Fix a date T arbitrarily. For any $t \leq T$, let the probability laws (1) through (4) hold. Further, grant Assumptions 1 through 4, and let u_t be a cardinal utility function for random cash flows at time t $(t \leq T)$.

<u>THEOREM</u>: <u>For</u> $t < T$, <u>the sequence</u> $y_t, \tilde{Y}_{t+1}, \ldots, \tilde{Y}_T$ <u>defined by</u> (9) <u>has the property</u>

$$ \begin{aligned} C_t(\tilde{Y}_{t+1} \mid \varphi_t) &= \lambda_{t+1} \cdot y_t , \\ C_t(\tilde{Y}_{t+k} \mid \varphi_t) &= \lambda_{t+1} \cdot \lambda_{t+2} \cdot \ \ldots \ \cdot \lambda_{t+k} \cdot y_t , \end{aligned} \tag{12}$$

where λ_i *is the discount rate for the period* $(i - 1)$ *to* i $(i = t+1, t+2, \ldots, t+k)$ *and* $(t + k) \leq T$.

Proof: By definition (9), $(y_{t+1} | \varphi_{t+1}) = {}_{t+1}\pi_T \circ C_{t+1}(\tilde{X}_T | \varphi_{t+1})$. Then, using (3) and (11),

$$u_{t+1}(y_{t+1} | \varphi_{t+1}) = \int_{-\infty}^{\infty} u_{t+1}({}_{t+1}\pi_T(x_T)) \; P_t(dx_T | \varphi_{t+1}, \varphi_t, \varphi_{t-1}, \ldots) .$$

Now denote $C_t(\tilde{Y}_{t+1} | \varphi_t) = \zeta_{t+1}$ for short. Then, by (2) and (8),

$$\begin{aligned} u_{t+1}(\zeta_{t+1}) &= \int_{-\infty}^{\infty} u_{t+1}(y_{t+1} | \varphi_{t+1}) \; P_t(d\varphi_{t+1} | \varphi_t, \varphi_{t-1}, \ldots) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_{t+1}({}_{t+1}\pi_T(x_T)) \; P_t(dx_T | \varphi_{t+1}, \varphi_t, \varphi_{t-1}, \ldots) \\ &\qquad\qquad P_t(d\varphi_{t+1} | \varphi_t, \varphi_{t-1}, \ldots) . \\ &= \int_{-\infty}^{\infty} u_{t+1}({}_{t+1}\pi_T(x_T)) \; P_t(dx_T | \varphi_t, \varphi_{t-1}, \ldots) , \text{ using (4).} \end{aligned}$$

Denote $C_t(\tilde{X}_T) = \zeta_T$ for short. Then, using (11), the right side of the above equality is identified to be equal to $u_{t+1}({}_{t+1}\pi_T(\zeta_T))$. Hence, $u_{t+1}(\zeta_{t+1}) = u_{t+1}({}_{t+1}\pi_T(\zeta_T))$, so that $\zeta_{t+1} = {}_{t+1}\pi_T(\zeta_T)$; and ${}_t\pi_{t+1}(\zeta_{t+1}) = {}_t\pi_{t+1} \circ {}_{t+1}\pi_T(\zeta_T) = {}_t\pi_T(\zeta_T) = y_t$ by definition. Recalling that ${}_t\pi_{t+1} = \lambda_{t+1}^{-1}$, we may rearrange the terms to yield the result $C_t(\tilde{Y}_{t+1} | \varphi_t) = \lambda_{t+1} \cdot y_t$. The second part of (12) now follows by repeating the above argument k times. ◊

COROLLARY (*Samuelson's Theorem 2, 1965*): *For* $t < T$, *the sequence* $y_t, \tilde{Y}_{t+1}, \ldots, \tilde{Y}_T$ *defined by (7) has the property*

$$\begin{aligned} E_t(\tilde{Y}_{t+1} | \varphi_t) &= \lambda_{t+1} \cdot y_t ; \\ E_t(Y_{t+k} | \varphi_t) &= \lambda_{t+1} \cdot \lambda_{t+2} \cdot \ldots \cdot \lambda_{t+k} \cdot y_t \end{aligned} \qquad (13)$$