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THE GENERIC EXISTENCE OF A CORE FOR q-RULES

by

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Abstract. A q-rule is where, for n-voters, a winning coalition consists of q or more voters. An important question is to determine when, generically, core points exist; that is, when the core exists in other than highly contrived settings. As known, the answer depends upon the dimension of issue space. McKelvey and Schofield found bounds on these dimensions, but Banks found a subtle, but critical error in their proofs. The sharp dimensional values along with results about the structure of the core are derived.

A "q-rule" is where a winning coalition consists of at least q of the n voters. So, if $[x]$ denotes the "greatest integer function," the "majority rule" is where $q = [\frac{n}{2}] + 1$. A standard assumption, which asserts that if C is a winning coalition then the remaining voters (the complementary set C^c) cannot form another winning coalition, restricts the q values to range between the majority and unanimity ($q = n$) rules.

A central issue in the application of these rules to economics and spatial voting is to understand when they can be stable. To illustrate with an historical example, several times when the selection of a pope for the Catholic Church required only a simple majority of the eligible Cardinals (so $q = [\frac{n}{2}] + 1$), the precarious instability of the system was manifested by the church erupting into dissension and conflict complete with a pope and anti-pope vying for power. Stability became the issue, so in 1179 the Third Lateran Council changed the selection procedure to the $q = [\frac{2n}{3}] + 1$ rule that remains in use (Saari, [S1, p 15-16]).

The *core* is one accepted way to examine "stability." Recall, $\mathbf{x}$ is a core point with a q-rule if, for all other proposals $\mathbf{y}$, it is impossible to find q voters who prefer $\mathbf{y}$ to $\mathbf{x}$; the core is the set of all core points. Again using the pope selection example, the flavor of this definition is captured by R. Kieckhefer's (a noted expert on the history of religion) explanation for the two-thirds procedure. As he argues, for a candidate ($\mathbf{y}$) to replace a pope ($\mathbf{x}$), the pope would have to bungle affairs sufficiently badly on enough issues to alienate at least half of his original supporters. Eventually this happened ([S1, p. 16]).

When there are new concerns, we have to expect, as the church example suggests, that the stability of a q-rule might diminish or even vanish. Each issue defines a direction in "issue space," so the introduction of new topics is measured by an

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increase in the dimension, k, of issue space. Should the new concerns be sufficiently attractive to enough agents, we should expect stability to be jeopardized. This is the general situation.

To see what can happen, let $\mathbf{x}_j$ be the ideal point for the jth agent where her utility function is $u_j(\mathbf{y}) = -\|\mathbf{y}-\mathbf{x}_j\|$. In words, with Euclidean distance preferences, the closer a point is to her ideal point, the more she likes it. Now, for n odd, $q = [\frac{n}{2}] + 1$, and $k = 1$, the only core point is the ideal point for the median voter. To prove this standard conclusion, place the ideal points along a line and observe that the median point divides all remaining ideal points into two sets; half are on each side. Thus the median voter's bliss point (the global maximum for her utility function) is the core point. Moreover, this description is robust; the assertion and argument hold even if the voters' ideal points are slightly altered.

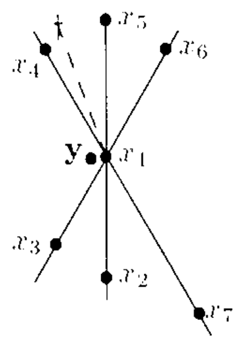


Figure 1. Plott's construction.

Even for $k \geq 2$, where n is odd and $q = [\frac{n}{2}] + 1$, core points that are bliss points exist. A trivial example is unanimity where all ideal points have the same position. This situation is highly unlikely, so, to justify the core, robust examples are needed. One approach is to use the Plott [P] construction of pairing voters' ideal points. Start by placing the first agent's ideal point, $\mathbf{x}_1$, somewhere in R^k and then pass $[\frac{n}{2}]$ lines through $\mathbf{x}_1$. Each line is divided by $\mathbf{x}_1$ into two sides; place a voter's ideal point on each segment. (This is depicted in Fig. 1 for $k = 2$.) The proof that $\mathbf{x}_1$ is a core point is a minor modification of the "median voter proof." Moreover, notice that if $\mathbf{x}$ is a core point for a q-rule, it is a core point for a q_1-rule where $q_1 > q$. (If q voters cannot be found to vote against $\mathbf{x}$, then it is impossible to find even more that are willing to do so.) Consequently, Plott's construction establishes the existence of a core point for all q-rules for odd values of n and $k \leq \frac{n-1}{2}$.

While this pairing of ideal points makes $\mathbf{x}_1$ a core point for the majority rule, the example is not robust. To see this, in Fig. 1 slightly vary voter five's preferences as denoted by the dagger on the dashed line. With this new configuration, $\mathbf{y}$ is preferred by a majority (voters two through five) to $\mathbf{x}_1$. To prove this assertion, draw the convex hull defined by the new ideal points of voters two through five. Draw a line perpendicular to an edge of this hull. It follows from elementary trigonometry that when this coalition compares two points on this line, each voter prefers the point closer to the edge. As $\mathbf{y}$ and $\mathbf{x}_1$ are on such a line, the conclusion follows.

Actually, the core is empty for this alignment. To prove this, draw the convex hulls defined for all four voter coalitions and observe that no point is common to all hulls. So, for any proposal $\mathbf{x}$, there is a $\mathbf{y}$ preferred by a decisive coalition. In other words, while this construction establishes the existence of a core point for all q-rules for odd values of n and $k \leq \frac{n-1}{2}$, this conclusion may fail to survive even the slightest changes in these preferences. As this fickle behavior is due to larger k values, we need to find the dimensions of issue space which do, or do not, permit existence assertions to be robust. In mathematical terms, for a given n and q-rule, we seek those k values whereby, generically, the core is empty. Similarly, we seek those k values whereby core points exist for an open set of preferences (i.e., it exists when preferences are slightly perturbed in any manner.) I solve this problem for all n, q rules.

The problem of determining the k values where the core is generically empty has received considerable research attention. Using a singularity theory argument (that I found to be insightful with its careful embedding of various classes of singularities within other classes), Schofield [Sc] and McKelvey and Schofield [MS1] published assertions stating that the core is generically empty for certain k values. While the embedding approach simplifies their analysis, the issue space dimension they compute is for an object that differs from the core. Therefore, these conclusions may describe only a subset of the relevant k values. Nevertheless, the assertions were correctly and widely treated as a major advance.

Recently, after a particularly careful analysis of these papers, Banks [B] found a subtle error in their argument, which, unfortunately, invalidated the conclusions. By correcting their argument, Banks found that the correct estimates may differ from what was previously believed. (Banks also used the embedding approach.) Thus, with Banks' paper, the core problem briefly was reopened; in particular, the issue became to determine the appropriate k values. In this current paper, we close this aspect of the problem by finding correct, sharp values.

It is interesting to note that the values obtained here differ significantly from the originally speculated values and that they are sharp rather than approximate bounds. The mathematical source of my improvements is that, instead of using the embedding argument, I characterize the behavior of the "core-singularities" (Prop. 1 is key to my approach) so that singularity theory can be applied to this specific geometry.

As for creating examples, there probably exist several papers in addition to McKelvey and Schofield [MS2]. However, I am unaware of any previous paper proposing general assertions about robust existence. Therefore, the assertions given below, and the approach developed in Sect. 5, appear to be the first general results of this kind where we know that the assertions are best possible.

2. Dimensions for the existence of the core

Assume that each agent has C^∞ smooth, strictly convex preferences. Namely, for each $\mathbf{x} \in R^k$, the set

$$M(\mathbf{x}) = \{\mathbf{y} \in R^k \mid u(\mathbf{y}) \geq u(\mathbf{x})\} \tag{2.1}$$

is strictly convex. In the following theorem, "generically" means a residual set (a set that can be expressed as a countable intersection of open-dense sets) of utility functions where the topology on the space of these functions is the Whitney

C^∞ topology (see (Golubitsky and Guillemin [GG] and Saari and Simon [SS]). (By using the [SS] arguments, when issues are restricted to a compact subset of R^k, "generic" can be extended to "open-dense.") For a first reading, when "generic" refers to nonexistence, interpret it as meaning "everything except improbable, carefully concocted examples where the conclusion changes with an arbitrarily slight change in the preferences." When "generic" describes existence, it means that examples exist where the conclusion remains true even after an example is slightly modified. Namely, they hold for a "C^2 open set" in the Whitney topology. (The utility functions can be slightly perturbed along with its first and second derivatives and the conclusions remain.) Also, while the "smoothness" conditions can be relaxed, rather than developing the attendant technicalities I prefer to concentrate on the structure of the core conditions.

To simplify notation, a core point that is a bliss point for some agent is called a "bliss-core point." Other core points are called "nonbliss-core points." While these results are directed toward strictly convex preferences, with minor changes, they hold for all smooth preferences. An interpretation and description follows the statement of the theorem.

Theorem 1. *a. Generically, for a q-rule where* $\frac{n}{2} < q < n$, *bliss-core points exist if and only if*

$$k \leq 2q - n. \tag{2.1}$$

b. For any k and n, there exists a q-rule where core points exist generically. Indeed, if $n = q$ (that is, for an unanimous decision), then, for any dimension $k \geq 1$, there exist open sets of preferences giving rise to bliss and nonbliss core points.

c. Let the "excess size of issue space dimension" be $\beta = k - [2q - n]$. Generically, there exists nonbliss core points for a q-rule if and only if q satisfies

$$\frac{1}{2\beta + 4} + n\left(1 - \frac{1}{2\beta + 4}\right) \leq q. \tag{2.3}$$

In terms of the dimension of policy space, generically, a given q-rule with n voters has a nonbliss-core point if and only if $\beta \leq \frac{4q-3n-1}{2(n-q)}$; that is, iff

$$k \leq 2q - n + \frac{4q - 3n - 1}{2(n - q)}. \tag{2.4}$$

d. Consider the ratio $\alpha = \frac{q}{n}$ where a candidate must receive at least α of the vote to be selected. For a given α rule, $\frac{1}{2} < \alpha \leq 1$, and a dimension for issue space, k, there exists a positive integer n_0 so that for all $n \geq n_0$, generically, the core is nonempty.

To see what this theorem means, start with the simple majority rule $q = [\frac{n}{2}] + 1$ where n is even. As $q = \frac{n}{2} + 1$, we have that $2q - n = 2$. According to Eq. 2.2, this means that robust, simple majority examples with a bliss point exist only up to a two-dimensional issue space. For nonbliss core points, the value of the right-hand side of Eq. 2.4 is $2 - \frac{n-3}{n-1} < 2$, so, for even values of n, nonbliss simple majority core points exist (generically) only for a one-dimensional issue space.

Compare these outcomes with what happens when n has an odd value so $q = \frac{n+1}{2}$, or $2q - n = 1$. As this gives the value for Eq. 2.2, we have that with an odd number of voters, robust bliss-core points exist only for a single-dimensional issue space. For nonbliss core points, the dimensional bound is given by Eq. 2.4, or by $1 - \frac{n-1}{n-1} = 0$. Therefore, for odd values of n, generically, simple majority nonbliss core points never exist.

To better understand this behavior, notice from Eqs. 2.2, 2.4 that a measure of stability is the $2q - n$ value; the larger the value, the more we can expect from core points. (This is further developed in Theorem 2.) This measure has its smallest value when n is odd and $q = [\frac{n}{2}] + 1$, so we should expect, and it is true, that with odd numbers of voters, simple majority core points have a precarious existence.

It seems reasonable to expect someone who voted for an accepted proposal will defect only when presented an offer that cannot be refused. If each defecting voter needs a particular issue, then the bound on the dimension of issue space allowing stability should roughly agree with the number of voters that need to defect to change the outcome. To relate this intuition to the $2q - n$ measure, observe that if k is the maximal admissible dimension of issue space, Eq. 2.2 becomes

$$(2.5) \qquad q - \frac{n}{2} = \frac{k}{2}$$

which indicates that the limiting dimension of issue space ensuring that a core exists is given by the difference between q and the simple majority rule.

Applying this argument to the pope-selection problem, we find that stability of an outcome can be expected for $\frac{k}{2} = \frac{2n}{3} - \frac{n}{2}$ or $k \leq \frac{n}{3}$. In other words, as long as there are not enough issues — one issue per defecting Cardinal — to alienate a third of the voting Cardinals, then the original choice should remain stable. As the defecting Cardinals are those who voted for the sitting pope, half of the pope's original supporters defect. Observe how this mathematical argument parallels Kieckhefer's insightful explanation reported in the introductory paragraphs.

Extending this argument to α rules (as defined in the theorem), we find that the issue space dimension always roughly correlates with the number of voters that need to be alienated, or allured, away from $\mathbf{x}$ to support another proposal. First, observe for any α rule that the same analysis allows stability as long as $k \leq (2\alpha - 1)n$. Compare this dimensional limitation with the number of voters, x, a losing coalition of $(1 - \alpha)n$ voters must lure away from a winning coalition of αn voters to form a new winning coalition; i.e., $(1 - \alpha)n + x = \alpha n$, or $x = (2\alpha - 1)n$. The assertion follows from the equality of these numbers. To illustrate with $\alpha = \frac{3}{4}$, the number of defecting voters must constitute half of all voters.

The $\frac{3}{4}$ rule arises in a different context. As computed below, if a q-rule requires less than a $\frac{3}{4}$ vote, the nonbliss core points generically exist only up to dimension $k \leq 2q - n - 1$. Thus, the "excess dimension" assertion is operative only for q-rules starting with the $\frac{3}{4}$ rule. Notice that the "excess dimension of issue space," β, is defined in terms of the measure $2q - n$. This choice emphasizes (see the proof) that as soon as $\beta > 0$, the coalition action always must rely upon specialized configurations that are restricted to lower dimensional subspaces of issue space. For nonbliss points, this is true for $\beta \geq 0$.

These last comments have important implications for Euclidean preferences. To explain, by specifying a voter's ideal point $\mathbf{x}_j$, we know that the gradient evaluated at $\mathbf{x}$ points in the $\mathbf{x}_j - \mathbf{x}$ direction and we know all second derivative terms. Consequently, Euclidean preferences belong to the excluded set of "non-generic behavior," so Theorem 1 is not applicable. But, because of their wide use, a parallel theory is needed for them. As the "ideal points" are the only relevant parameters, "robustness" has to mean that their positions are not highly restricted. So, when using Euclidean preferences, interpret "generic" as referring to conclusions which include an open set of locations for each voter's ideal point. To illustrate, conclusions about median voter where $k = 1$ are robust; the example of Fig. 1 is not. In particular, as the discussion of the previous paragraph claims that when $\beta \geq 0$, most gradient vectors must be in a lower dimensional subspace, this forces the ideal points to be restricted to a lower dimensional setting. In turn, this violates the robustness of the conclusions.

Corollary 1. *For Euclidean preferences, bliss-core points exist generically iff* $k \leq 2q - n$. *Nonbliss core points exist generically iff* $k \leq 2q - n - 1$.

Additional consequences of Theorem 1 follow.

Examples: a. Part b follows immediately from c. To see why, we only need to determine whether $q = n$ (which is an admissible choice) satisfies Eq. 2.3. But, this only involves determining whether the equivalent inequality

$$\frac{1}{2\beta + 4} \leq \frac{n}{2\beta + 4}$$

is satisfied. This is trivially true for $n \geq 1$. (Even easier, use Eq. 2.4 and note that $n = q$ forces the denominator on the right hand side to vanish.)

b. By converting the $n(1 - \frac{1}{2\beta+4})$ term from Eq. 2.3 to the mixed number $a + \frac{r}{2\beta+4}$, it follows that $q = a + 1$. Therefore, dropping the equality and the first term on the left side of Eq. 2.3, we obtain the equivalent

$$\frac{2\beta + 3}{2\beta + 4} < \frac{q}{n}. \tag{2.6}$$

To illustrate Eq. 2.6, the q-rules where issue space exceeds the $2q - n$ dimension by $\beta = 100$, are those satisfying

$$q > n\frac{203}{204}.$$

This suggests using n values that are integer multiples of 204, say, $n = 4(204) = 816$. The minimal q value is $1 + 4(203) = 813$, so the 813-rule generically admits core points where $k \leq 2q - n + \beta = 1626 - 816 + 100 = 910$.

c. If q is the smallest value solving these inequalities for $n = \gamma(2\beta + 4)$, then $q = n - \gamma + 1$, so the restriction on the dimension of issue space is

$$k \leq n + (\beta - 2\gamma + 2) = (\beta + 1)(2\gamma + 1) + 1. \tag{2.7}$$

This inequality provides a sense of the k growth rates relative to the q-rule.

d. With Eq. 2.6, we discover the β-bifurcation roles played by different rules. For instance, with $\beta = 0$, we have from Eq. 2.6 that the $\frac{3}{4}$ rule is the bifurcation point. (More precisely, "one more than a three-fourths vote"). With $\beta = 1$, the bifurcation occurs at the $\frac{5}{6}$ rule. Continuing, we see that the bifurcations arise at the (one more than) $\frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \dots, \frac{\text{odd integer}}{1+\text{odd integer}}, \dots$ of the votes. From this, it follows that for any β, there are supporting n and q values where $q < n$. But by specifying n and q, Eq. 2.4 bounds k. To illustrate with $q = 30$ and $n = 35$, it follows from Eq. 2.2 that bliss core points exist, generically, up to dimension

$$k \leq 60 - 35 = 25$$

and nonbliss core points exist, generically, up to dimension

$$k \leq (60 - 35) + \frac{120 - 106}{10} = 25 + \frac{14}{10},$$

or $k = 26$. This example where the nonbliss points exist longer than the bliss points suggests examining when $(4q - 3n - 1)/2(n - q)$ is negative (to force the bliss point to persist longer than the nonbliss core points). Again, with simple arithmetic, we recover the earlier assertion that this is when the decision rule is bounded above by the $\frac{3}{4}$ rule.

A related issue is to determine whether, generically, the nonbliss points could vanish at a dimension two or three before the bliss points. If possible, it would require q and n values where

$$\frac{4q - 3n - 1}{2(n - q)} < -1,$$

or, by solving, where $2q < n + 1$. As such a phenomenon requires using less than a majority vote, it is not relevant.

e. To prove part d of the theorem, notice that Eq. 2.4 can be represented as

$$k \leq n(2\alpha - 1) + \frac{4\alpha - 3 - \frac{1}{n}}{2(1 - \alpha)}, \quad \alpha = \frac{q}{n}.$$

For fixed k and α, we can ignore the $\frac{1}{n}$ term, so what is left on the right-hand side defines a linear equation in n. The conclusion now follows. This assertion means that even rules close to majority rule (i.e., $\alpha \approx \frac{1}{2}$) can be supported in, say, a 100 dimensional issue space with enough voters. To illustrate with $\alpha = 0.5001$, a core point can be supported in a hundred dimensional issue space with around a half million voters. So, for a city about the size of Minneapolis, as long as the number of issues doesn't exceed a hundred, stability could exist. $\square$

3. Comments on the structure of the core

Theorem 1 tells us when the core exits, but it does not address the structure of this set. For instance can the core be the union of disjoint sets? (It can when convexity is dropped, but not with strict convexity.) Of the many important issues, the ones I describe provide added support for the theme that the stability of a core improves with larger values of q. These results describe how the core changes when the values of q, n and k vary.

For intuition about what kinds of conclusions to expect, start with a R^1 example of eleven voters, where the jth voter's ideal point is located at the integer j, $j = 1, \dots, 11$. For the simple majority rule ($q = 6$), $x_6 = 6$ is the only core point. With $q = 7$, the core is the interval $[5, 7]$ where the endpoints are bliss core points. Indeed, for any q value, the core is $[11 - q, q]$. This suggests that the statement where a q core point also is a $q_1 > q$ core point probably extends to require the q_1 core to strictly contain the q core. This is the general case.

Another measure of stability is the dimension of the core. For instance, in the eleven voter example with $q = 8$, the fact the core includes interval $(3, 8)$ introduces a strong sense of stability – near any core point is another one. Similarly, in the seven voter example of Fig. 2, the core is the shaded region, so, near any core point in the interior is another core point. This underscores the importance of the assertions in Theorem 2 specifying the dimension of the core.

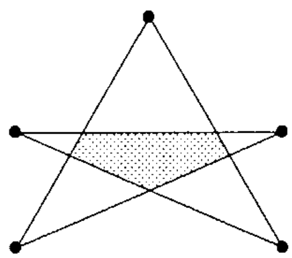


Figure 2. Core for $n = 5$, $q = 4$, $k = 2$

In Fig. 2, the core is any point in the shaded region and its boundary. (So, there are no bliss-core points.) For instance, a boundary line connects the bliss points of two agents, and a vertex is where two bliss-connecting lines intersect. By using singularity theory, it follows that this picture describes the general situation. Namely, singularities form a stratified structure where a restricted setting is in the closure of the previous setting. Thus, in general the boundary of a core consists of core points of restricted types. The boundaries may not be straight lines, but they are points where preferences line up as indicated.

Implicit in these comments is that when the core exists for dimensions $k-1$ and k, then, in some manner, the core for dimension k is a subset of the core for dimension $k-1$. This requires comparing cores by deleting, rather than adding, issues. So, for a k dimensional issue space with a k_1-dimension subspace, $k > k_1 > 0$, let P_{k,k_1} be the natural projection mapping. (For instance, if $k = 3$ and $k_1 = 2$ represents the x-y plane, then $P_{3,2}((x, y, z)) = (x, y)$ where the z component is dropped.) The k_1-dimensional preferences are assumed to be the k-dimensional preferences restricted to the lower dimensional plane. If this lower dimensional plane is obtained by dropping $k - k_1$ issues, then it is a coordinate plane of the k-dimensional space. Otherwise, the plane represents where certain issues are combined into a single issue. (With minor technical changes, this holds for a k_1 dimensional smooth manifold.) The following tells us that a core point persists when issues are restricted.

Theorem 2. *a. Suppose n and k are such that non-bliss core points exist gener-*

ically for a q-rule: $q < n$. The core for the $q+1$ rule always contains the q-rule core. It is generically unlikely that the two cores are the same.

b. Generically, bliss-core points are isolated points. Generically for those q, n, and k values that satisfy Eq. 2.3 for $\beta = -1$, the set of nonbliss core points has a nonempty interior.

c. Suppose $\beta \geq 0$ is needed to satisfy Eq. 2.3 for specified k, n, q values. Generically, the core is a union of submanifolds with dimension $k - (\beta + 1)(2q - n + 1)$.

d. If $\mathbf{x}$ is a core point for a q-rule in a k dimensional space and if there is a k_1 dimensional plane passing through $\mathbf{x}$, then $P_{k,k_1}(\mathbf{x})$ is a core point for the k_1 dimensional issue space.

We now encounter an interesting conflict. This discussion demonstrates that larger q values provide a wider selection of core points with added stability. The message seems to be that larger q-values are better. So, why don't we insist on always using unanimity as the deciding rule? The reason, of course, is obvious: very little can be accomplished with an unanimous rule because it is so difficult to design anything that makes everyone happy.

This instinct is supported by the technical statements. With unanimity and Euclidean preferences, the core consists of the convex hull defined by the voter's ideal points. (To see this, let $\mathbf{y}$ be outside of this hull and $\mathbf{x}$ the nearest point on this hull. As the line $\mathbf{y} - \mathbf{x}$ is orthogonal to this edge, it follows from trigonometry that all voters prefer $\mathbf{x}$ to $\mathbf{y}$. With minor modifications, the same argument holds when $\mathbf{x}$ is a vertex. But, if $\mathbf{y}$ is in the hull, then, as any other $\mathbf{x}$ is farther from some voter's ideal point than $\mathbf{y}$, that voter will veto the move to $\mathbf{x}$.) Generally, as asserted by Thm. 2a, the unanimity core strictly contains the core for any other q-rule. Thus, it is easier for the status quo to be in the unanimity core than in the core for any other q-rule. But, once the status quo is in the core, change is impossible. So, rather than being a desirable feature, maybe we should avoid core stability because it retards progress.

Restating this concern, why should we care about core-stability? To examine this issue, start with any $\mathbf{x}_1$ and notice that if the core is empty, there must exist a $\mathbf{x}_2$ that is supported by a winning coalition over $\mathbf{x}_1$. Similarly, there is a $\mathbf{x}_3$ that is supported by a decisive coalition over $\mathbf{x}_2$. The argument continues - forever. So, without core stability, we incur the "chaos" behavior of spatial voting carefully described by Kramer [K], McKelvey [M] and others. (Richards [R] provides a new explanation.) In fact, as McKelvey showed, there are situations where with sincere voting, we can start with any initial proposal, pass through any other specified proposal, and then return to the original one. Without stability, we cannot trust that the outcome truly reflects the views of the voters.

What we need is a compromise between enjoying the stability of the core while preventing gridlock which arises whenever the core contains the status quo. With q-rules, this means we need to choose a q value allowing a robust core that is not "too large." If we know the number of issues typically involved in a decision process, this k value determines minimal q-rules with core stability but without gridlock. But, perhaps the real problem is the tacit assumption that we should be restricted to binary comparisons and q-rules. Elsewhere this question is explored in more detail.

4. A CHARACTERIZATION OF CORE POINTS

To motivate the basic technical tool of this paper, start with Euclidean preferences. As shown in the Sect. 3, the core for the unanimity rule is the convex hull defined by the voters' ideal points. A similar argument shows that with a q-rule and a specified coalition of q-voters, "their core" consists of the convex hull of their q ideal points. So, the core for a q-rule is the intersection of the convex hulls defined by all possible q-voter coalitions. If this intersection is empty, the core is empty.

This geometry explains why the core vanishes with larger k values. For instance, with $n = 7$ and $q = 4$, if $k = 1$, all points must lie on a straight line, so the core exists. However, even for $k = 2$, we cannot expect three or more ideal points to be on any line. Consequently, the extra freedom provided to position ideal points makes it difficult for all convex hulls to have a common intersection point. One remedy is to require the hulls to have more vertices; this is equivalent to increasing the q value.

This intuition extends to describe what happens with strictly convex smooth preferences. ("Smoothness" ensures that the curved utility surfaces can be approximated by the planes defined by the derivative conditions; strict convexity requires all preferred points to be on one side of this plane.) Thus, basic to our arguments is Prop. 1 which characterizes core points for q-rules in terms the derivative properties the utility functions must have in order for this intersection to be nonempty. In this description, $Co_{\mathbf{x}}(\{\mathbf{v}_j\}_{j\in C})$ denotes the convex hull of the (vertices of the) vectors $\{\mathbf{x} + \mathbf{v}_j\}_{j\in C}$ and $\overline{Co}_{\mathbf{x}}(\{\mathbf{v}_j\}_{j\in C})$ is this convex hull minus the vertices. For this proposition, "smooth" can be relaxed to C^2 smoothness.

Proposition 1. *Assume the voters have smooth, strictly convex preferences. A necessary and sufficient condition for $\mathbf{x}$ to be a core point is if for any set of q agents, C, either*

$$\mathbf{x} \in \overline{Co}_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j\in C}) \tag{4.1}$$

or $\mathbf{x}$ is both a vertex of $Co_{\mathbf{x}}(\{\mathbf{v}_j\}_{j\in C})$ and a bliss point for some voter in C.

This assertion makes sense; if q or more voters prefer alternatives that are in the same general direction from $\mathbf{x}$, then they can block the selection of $\mathbf{x}$. The directional derivative is determined by a scalar product, $\frac{\partial u(\mathbf{x})}{\partial \mathbf{v}} = (\nabla u(\mathbf{x}), \mathbf{v})$, so the sense of "the same general direction" is captured by passing a plane through $\mathbf{x}$ where, for all agents in this coalition, the $\nabla u_j(\mathbf{x})$ vectors are strictly on the same side of the plane. Indeed, choosing $\mathbf{v}$ as the normal for this plane where $\mathbf{v}$ is on the same side as the gradients, it follows from the positive value of $(\nabla u_j(\mathbf{x}), \mathbf{v})$ that all utilities are improved by moving in this direction. Conditions that ensure $\mathbf{x}$ is a core point are those that prevent this scenario from occuring. This is the content of the proposition; the proof is a formal expression of this intuition.

The proof uses an important relationship between local and global comparisons of alternatives. Call $\mathbf{x}$ an "infinitesimal core point" if it is a core point when the admissible choices are restricted to a sufficiently small open neighborhood of $\mathbf{x}$. While a core point always must be an infinitesimal core point, it is easy to construct examples where an infinitesimal core point is not a core point. (The idea is similar to constructing examples where a local maximum is not a global maximum: just use utility functions where the level sets have many "wiggles.") On the other hand, just as appropriate convexity assumptions force a local maximum of a function to be a global maximum, the following lemma asserts that our strict convexity assumption on utility functions forces infinitesimal core points to be core points.

Lemma 1. *If all agents have smooth, strictly convex preferences, then an infinitesimal core point is a core point.*

Proof of Lemma 1. Assume that $\mathbf{x}$ is an infinitesimal core point, but not a core point because a decisive coalition C prefers $\mathbf{y}$. This means that $\mathbf{y} \in \cap_{i \in C} M_i(\mathbf{x})$. But, by being the intersection of convex sets, $\cap_{i \in C} M_i(\mathbf{x})$ is convex. Consequently, any point $\mathbf{y}_t = t\mathbf{x} + (1-t)\mathbf{y}$ on the straight line connecting $\mathbf{y}$ and $\mathbf{x}$ is in this set. Thus, for any t sufficiently close to unity (so $\mathbf{y}_t$ is sufficiently close to $\mathbf{x}$), this same coalition would prefer $\mathbf{y}_t$ to $\mathbf{x}$. This means that $\mathbf{x}$ is not an infinitesimal core point. The contradiction completes the proof. □

Proof of the proposition. To prove that the stated condition is necessary, it suffices to show that if Eq. 4.1 does not hold, then $\mathbf{x}$ is not an infinitesimal core point. If Eq. 4.1 fails to hold, there exists a coalition of q voters, C, where $\mathbf{x}$ is not in its convex hull. This geometry permits the construction of plane passing though $\mathbf{x}$ with the convex hull strictly on one side. If $\mathbf{v}$ is the normal vector for this plane that points toward the side with the convex hull, then $\frac{\partial u_j(\mathbf{x})}{\partial \mathbf{v}}$ is positive for all voters in the decisive coalition C. Consequently, $\mathbf{x}$ cannot be a core point.

If $\mathbf{x}$ is a vertex of the hull for some q-voter coalition C, then, by construction, $\nabla u_j(\mathbf{x}) = \mathbf{0}$ for some voter in this coalition. Since $\mathbf{x}$ is a vertex, a plane can be passed through $\mathbf{x}$ which does not meet any other point of the convex hull. Now, if $\mathbf{x}$ is not the bliss point for agent j, then, by the assumption of strict convexity, $M_j(\mathbf{x})$ contains an open set. (This open condition is not necessary; all we need is that $M_j(\mathbf{x})$ contains a point other than $\mathbf{x}$.) By a rotation of the preferences of this agent, if necessary, the set $M_j(\mathbf{x})$ can be made to intersect the convex hull. (That is, let Ω be a rotation matrix. Construct a new set of preferences that are defined by $u_j^*(\mathbf{x}) = u_j(\Omega(\mathbf{x}))$. According to the chain rule, $\nabla u_j^*(\mathbf{x}) = \mathbf{0}$ so the conditions of the proposition continue to apply.) This means there are alternatives that are preferable to $\mathbf{x}$ for all members of the decisive coalition C. Consequently, if $\mathbf{x}$ is not a bliss point, it is possible to find preferences satisfying the conditions of the proposition where $\mathbf{x}$ is not a core point. This completes the proof of necessity.

The proof of sufficiency involves three cases: the first one has $\mathbf{x}$ as an interior point of $Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C})$ for all possible decisive coalitions, and the other two are the two possible ways $\mathbf{x}$ can be a boundary point of this hull for some decisive coalition. For the interior point situation, any plane passing though $\mathbf{x}$ must have vertices of $Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C})$ strictly on each side of the plane — these vertices are defined by gradient vectors of the agents from this coalition. Now, if $\mathbf{x}$ is not a core point, there exists an alternative $\mathbf{y}$ that is preferred by this coalition to $\mathbf{x}$. Let $\mathbf{v} = \mathbf{y} - \mathbf{x}$ be a normal vector for a plane passing though $\mathbf{x}$. For those gradient vectors on the side of the plane opposite of $\mathbf{v}$, the directional derivative $\frac{\partial u}{\partial \mathbf{v}} = (\nabla u, \mathbf{v})$ is negative. As these agents have a lower utility for any such change, they will not vote for such a move. Thus, $\mathbf{x}$ is an infinitesimal core point.

The analysis is essentially the same when $\mathbf{x}$ is a boundary point. If $\mathbf{x}$ is a vertex of $Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j\in C})$ for a q-voter coalition C, then, by assumption in the Proposition, $\mathbf{x}$ must be a "bliss point" for some agent in this coalition. Clearly, whenever $\mathbf{x}$ is a bliss point for some voter, this voter does not wish to change. Consequently, this coalition cannot change the outcome.

For the remaining case, if $\mathbf{x}$ is on the boundary of $\overline{Co}_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j\in C})$ but not a vertex (but, it could be a bliss point), then $\mathbf{x}$ could belong to several boundary components. (For instance, if $\mathbf{x}$ is on the intersection of two boundary faces, it is on both faces and on the defined edge.) Choose the lowest dimensional boundary component. Because $\mathbf{x}$ is not a vertex, it *must* be in the interior of the convex hull defined by the gradient vectors in this component. The gradient vectors (and agents) on this component are the ones we analyze.

Note that a slight modification of the above argument (where $\mathbf{x}$ is an interior point) proves that a change $\mathbf{v}^*$ in this boundary component, or a change with a nonzero component in this component, is unacceptable to some voter. This is because, as $\mathbf{x}$ is in the interior of the hull defined by vectors on the boundary component, for some voter with a gradient in this component, the directional derivative $\frac{\partial u_k}{\partial \mathbf{v}^*}$ is negative. Consequently, the only potential admissible changes for this coalition must be orthogonal to the boundary component, i.e., the changes must be in a direction $\mathbf{v}$ that is orthogonal to all of the gradient vectors on this component. Now, by strict convexity, for any of these voters with gradient in the plane, the only point of the set $M_k(\mathbf{x})$ that is in the tangent plane passing though $\mathbf{x}$ with normal $\nabla u_k(\mathbf{x})$ is $\mathbf{x}$. In other words, after point $\mathbf{x}$, the level set $u_k(\mathbf{x})$ is strictly on the $\nabla u_k(\mathbf{x})$ side of the tangent plane. So, for each voter with a gradient vector in the boundary plane, any change in the $\mathbf{v}$ direction constitutes a change of lower utility. This means that $\mathbf{x}$ is an infinitesimal core point. □

Proposition 1, which is critical for what follows, should be viewed as providing an higher order extension of an application of Smale's [Sm] characterization of Pareto points to core points. The connection with Pareto points is that a core point is a Pareto point for every decisive coalition. The higher order derivative conditions are snuck in with the strict convexity arguments. As it will become clear, these strict convexity properties are critical for our analysis (when $\beta \geq 0$) because it involves analyzing what happens when $\mathbf{x}$ is on the boundary of this hull. (Indeed, this insight is what permits me to avoid the embedding argument of Banks, McKelvey, and Schofield.) For this to occur, we need strict convexity. It is important to stress that this assumption does not restrict our conclusions because level sets with flat spots belong to the nongeneric setting.

Proof of Thm. 2d. The projection of a convex set to a lower dimension subspace is a convex set. As the projection of $\nabla u_j(\mathbf{x})$ is the gradient of u_j restricted to the lower dimensional submanifold, the proof follows directly from Prop. 1. □

5. Constructing examples

In this section I show how to construct examples with bliss-core points where $k \leq 2q - n$ and nonbliss core points where $k \leq 2q - n - 1$. Examples for the "excess dimension setting" are discussed in Sect. 6.

Observe that, with the exception of a bliss-core point, Prop. 1 emphasizes the directions, rather than the lengths, of the gradient vectors $\{\nabla u_j(\mathbf{x})\}_{j\in C}$ evaluated at a core point. Thus, examples can be constructed by first finding appropriate gradient directions and then choosing the lengths in any desired non-zero manner. As "directions" are identified with unit vectors, they can be treated as points on a unit sphere, S^{k-1}, with center $\mathbf{x}$. When constructing bliss-core point examples, the bliss point is positioned at $\mathbf{x}$.

Already, additional insights are available. For instance, while Theorem 1 ensures that open sets of preferences support a bliss core point for the simple majority rule, $k = 2$, and $n = 1000$, it is clear from Prop. 1 and this gradient direction description that, at best, the gradient directions are approximately $2\pi/999$ radians apart. This does not provide much room to vary. Therefore, while Theorem 1 correctly asserts that these directions can be slightly changed, it also is clear they cannot be altered very much without forcing the core to vanish. More generally, it is not difficult to extend the sense of this argument to show that for fixed maximal value of k, when n and the corresponding q value become arbitrarily large, then the permissible amount of variation in preferences becomes arbitrarily small. Namely, while the core exists for an open set of preferences, the size of this open set approaches zero; while the core is generically possible, it approaches becoming unlikely.

Before finding gradient directions, I show how to use them to construct agent's preferences. An easy way is to use Euclidean preferences that are defined by specifying an agent's ideal point. As a gradient direction defines a ray emanating from $\mathbf{x}$, place on a ray an agent's ideal point. According to the theorem, even if these preferences are slightly varied, so level sets no longer are spheres, the conclusion holds. An easy way to construct examples without Euclidean preferences is to use Taylor series. To illustrate with $\mathbf{x} = (1, 2)$ and the abbreviated Taylor expansion

$$u(x_1, x_2) = u(1, 2) + \nabla u(1, 2)\cdot(x_1 - 1, x_2 - 2) + \frac{1}{2}D^2 u_{(1,2)}((x_1 - 1, x_2 - 2), (x_1 - 1, x_2 - 2))$$

construct examples by choosing $\nabla u(1, 2)$ and $D^2_{(1,2)}u$ consistent with the strict convexity assumption. For instance, for the direction $\frac{1}{5}(-4, 3)$, we could choose $\nabla u = (-4, 3)$ and $D^2 u$ to be the diagonal matrix with entries $-1, -1$ to define the utility function $u(x_1, x_2) = -4(x_1 - 1) + 3(x_2 - 2) - (x_1 - 1)^2 - (x_2 - 2)^2$. A similar construction holds for any dimension.

Gradient directions. To identify all decisive coalitions that prefer another alternative (in k-dimensional issue space) to $\mathbf{x}$, pass a $(k-1)$-dimensional "dividing plane" through $\mathbf{x}$. If q or more points are on one side of this dividing plane, then these q-voters define a decisive coalition that denies $\mathbf{x}$ the honor of being an infinitesimal or core point. The basic tool for the proof of the theorem and the construction of examples, then, is to analyze all dividing planes that pass through $\mathbf{x}$, and count the number of points on either side. This is the first condition.

[1] If none of the $\nabla u_j(\mathbf{x})$ vectors are on the $(k-1)$-dimensional dividing plane passing through $\mathbf{x}$, then, according to Prop. 1, there can be no more than $q - 1$ such vectors on either side.

For the next condition, observe that any $k-1$ of these vectors that are linearly independent define a dividing plane. But, an arbitrarily small change in the orientation of this dividing plane can force all $k-1$ of the vectors onto one side or the other (of its new position). To satisfy [1], only a limited number of points can be on either side of its original orientation.

[2] A dividing plane that contains $k-1$ linearly independent gradient vectors can have no more than

$$s^* = (q-1) - (k-1) = q-k \tag{5.1}$$

gradient vectors on either side.

These two conditions are used repeatedly to construct examples and prove the theorems. Indeed, because the only way the number of points on either side can change is when a dividing plane is rotated through a point, it follows from Prop. 1 that a sufficient condition for an example to have $\mathbf{x}$ as a core point in a k-dimensional space is if it satisfies [1] for one dividing plane without gradient vectors and [2] for all $(k-1)$-dimensional dividing planes where a dividing plane has, other than a bliss point, just $k-1$ of these gradient vectors.

An underlying theme in this construction is that if $\mathbf{x}$ is a core point for a q-rule, then it is a core point for a q_1-rule where $q_1 > q$. So, it suffices to create examples supporting the minimal q value. To do so, I design examples that come as close as possible to satisfying the majority rule: namely, the goal is to allow no more than half of the remaining points to be on each side of any dividing plane.

From conditions [1] and [2], to create "majority rule" examples, the points must be spread as far apart as possible. (This makes intuitive sense because a clustering of points indicates a commonality of interest that forms the basis of a decisive coalition.) Because a maximum separation of points is associated with their symmetric distribution, a natural approach is to symmetrically position the n points on S^{k-1}. But, while it is trivial to symmetrically locate n points on a circle (place them $2\pi/n$ radians apart), it is not clear how to do this for higher dimensional spheres. For instance, for S^3 (in R^4) and $n=5$, a symmetric configuration is a five-gon with ten equal edges, and for $n = 2^4$ it is a (four-dimensional) cube. But, what is the symmetric configuration on S^3 for $n = 30$, or even $n = 10$? Actually, the complexity of this problem plagues other mathematical concerns such as, for instance, the study of the Newtonian N-body problem, or the vortex formation of cyclones. (Here the symmetry issue is identified with "central configurations," [S2, X]) To avoid these complexities, a simpler method is devised to capture this geometry. Specific cases are described before the general argument is given.

One-dimensional issue space. Although we know the answer for $k=1$, I show how to use [1] and [2] to create examples with $\mathbf{x}$ as a core point. Start by positioning m points according to the "alternating rule" where the first point is placed to the right of $\mathbf{x}$, the second to the left, the third to the right, etc. until all points are positioned and no two points occupy the same position.

If m is an even integer, there are $\frac{m}{2}$ points on each side of $\mathbf{x}$. Thus, according to [1], this choice supports $q \geq [\frac{m}{2}] + 1$ rules. For $k = 1$, a "dividing plane" is the point $\mathbf{x}$, so [2] is not applicable. Now, choosing $n = m$ as the number of voters, the example supports $\mathbf{x}$ as a nonbliss core point for all $q \geq [\frac{n}{2}] + 1$ rules with an even number of voters. (The example is robust as the points can be slightly altered without changing the argument. In fact, the nonbliss core points consist of the interval between the first points on either side of $\mathbf{x}$.) If $n = m + 1$, so $\mathbf{x}$ is a bliss-core point for an odd number of voters where one voter's gradient is assumed to be zero, this construction supports the $q \geq [\frac{n-1}{2}] + 1 = [\frac{n}{2}] + 1$ rules. (This is because n is an odd integer, so $[\frac{n}{2}] = [\frac{n-1}{2}]$.)

If m has an odd value, there are at most $[\frac{m}{2}] + 1$ points on one side of $\mathbf{x}$, so this construction supports all $q \geq ([\frac{m}{2}] + 1) + 1 = [\frac{m}{2}] + 2$ rules. If $n = m$, so $\mathbf{x}$ is a nonbliss core point, this example supports all $q \geq [\frac{n}{2}] + 2$ rules. When $n - 1 = m$, so $\mathbf{x}$ is to be a bliss core point for an even number of voters, the example supports all $q \geq [\frac{n-1}{2}] + 2 = [\frac{n}{2}] + 1$ rules. This conclusion uses the fact that when n is even, $[\frac{n-1}{2}] = [\frac{n}{2}] - 1$.

To complete the construction by finding preferences, assign each voter a utility preference by using one of the earlier described methods. Observe that these examples support all q, n rules allowed by Theorem 1 for $k = 1$. (The computation is trivial; particularly by starting with Eq. 2.5.)

Two dimensions. For all k, the goal is to modify the "alternate side" construction used for $k = 1$ to make it applicable. For $k = 2$, this requires positioning m points on S^1, the circle with center $\mathbf{x}$, in an alternating fashion. Whatever the positions of these m points, there is a line through $\mathbf{x}$ that misses all of them. Let this line be the x_2-axis and its orthogonal complement the x_1-axis. (See Fig. 3.) To simplify the argument, identify each point on the circle with a point on either the "left line" (the line $x_1 = -1$) or the "right line" ($x_1 = 1$). This is done by drawing a ray from $\mathbf{x}$ through the point on the circle and determining where and on which line this ray intersects. Conversely, a point on either the right or left lines uniquely identifies a point on the circle.

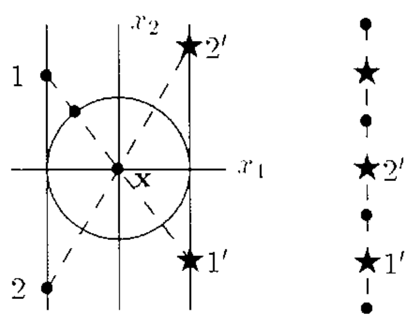


Figure 3. The two-dimensional case.

To further simplify the construction, define for each point on the left its *dual point* on the right line; it is where the line through this point and $\mathbf{x}$ intersects the $x_1 = 1$ line. Denote each such point by star. (See Fig. 3.) Conversely, for a specified dual point, the original point on the left side can be uniquely reconstructed. Therefore, all points and dual points can be placed on the right line. In doing so, observe that a point and its dual lie on opposite sides of a dividing plane. For instance, in the figure, 2 and 2′ are on opposite sides of the 1 1′ line. Also notice that if a point is on a dividing plane, then so is its dual.

For a two-dimensional example, the gradients must span a two-dimensional space. So, to satisfy [1], this requires $m \geq 3$ points. Arbitrarily place half of them ($\frac{m}{2}$ if m is even, $\frac{m+1}{2}$ if m is odd) as regular points (dots) on the right line. If m is odd, this placement defines the same number ($\frac{m-1}{2}$) of bounded intervals as points that have yet to be assigned. Place each remaining point in a unique bounded interval and denote it with a star; this is to be the dual of the actual point. (This construction is indicated on the vertical line on the right-hand side of Fig. 3.) When m is even, there are $\frac{m}{2}$ remaining points and $\frac{m}{2}-1$ bounded intervals. So, place the dual of all but one of the remaining points in a bounded intervals and denote this position with a star (to indicate it is the dual of the actual point). Place the last star (dual point) in an unbounded interval; assume it is the top one. Observe by symmetry that this construction could start by first positioning the dual points. This is a consequences of the "alternating selection" process of assigning dual and regular points.

To analyze these positions, start with m odd and the $x_1 = 0$ dividing plane. One side has $[\frac{m}{2}] + 1$ points so, according to [1], at this stage the construction supports $q - 1 \geq [\frac{m}{2}] + 1$ or all $q \geq [\frac{m}{2}] + 2$ rules. If a dividing plane passes through the top dot on the right line (see Fig. 3), then, by construction, precisely $[\frac{m}{2}]$ dots and $[\frac{m}{2}]$ stars are below this line. Consequently, there are precisely $[\frac{m}{2}]$ right points below and $[\frac{m}{2}]$ left points above the dividing plane. According to [2] and Eq. 5.1 where $k = 2$ and $s^* = [\frac{m}{2}]$, so far this example supports all $q \geq [\frac{m}{2}] + 2$ rules.

Next, pass a dividing plane through the jth point down the line. With the alternating assignment of dual and regular points, if j is even, then the jth point is a dual point with $\frac{j}{2} - 1$ dual points and $[\frac{m}{2}] + 1 - \frac{j}{2}$ regular points below it. Consequently, there are $[\frac{m}{2}]$ points both above and below this dividing plane. As the same conclusion holds for odd values of j, this construction provides examples for $k = 2$ core points for any $q \geq [\frac{m}{2}] + 2$ when m is odd.

To summarize, when m is odd, $k = 2$, this construction supports $q \geq [\frac{m}{2}] + 2$ rules. If $n = m$ (so n is odd and $\mathbf{x}$ is a nonbliss core point), this construction supports all $q \geq [\frac{n}{2}] + 2$ rules. If n is even where $\mathbf{x}$ is to be a bliss core point (so $n = m + 1$), this construction supports all $q \geq [\frac{m}{2}] + 2 = [\frac{n-1}{2}] + 2 = ([\frac{n}{2}] - 1) + 2 = [\frac{n}{2}] + 1$ rules.

It remains to analyze even values of m. By using the dividing plane $x_1 = 0$ (where there are $\frac{m}{2}$ points on each side) we have from [1] the $q \geq [\frac{m}{2}] + 1$ constraint. When the dividing plane passes through a point, then there are $[\frac{m}{2}]$ points on one side and $[\frac{m}{2}] - 1$ points on the other. According to Eq. 5.1 and [2], the associated bounds are $q \geq s^* = [\frac{m}{2}] + 2$. To translate these bounds into terms of n, notice that when n is even and the core is not a bliss point (so $n = m$), this construction supports all $q \geq [\frac{n}{2}] + 2$ rules. When n is odd and the core is a bliss point (so $n = m + 1$) then, because $[\frac{m}{2}] = [\frac{n-1}{2}] = [\frac{n}{2}]$, this construction supports all $q \geq [\frac{m}{2}] + 2 = [\frac{n}{2}] + 2$ rules.

As above, for all k the four cases are when m is odd or even, and when $\mathbf{x}$ is or is not a bliss point (corresponding, respectively, to $n = m + 1$, $n = m$). A factor always changing the outcome is that $[\frac{n-1}{2}] = [\frac{n}{2}] - 1$ when n is even; this always arises with an even number of voters and a bliss-core point. Finally, a simple computation (starting with Eq. 2.5) shows this construction provides $k = 2$

supporting examples for all of the n, q rules admitted by Theorem 1.

Three dimensions. The construction of $k = 3$ examples proceeds in the same manner. Start with $\mathbf{x}$ at the origin of an axis system where no points are on the $x_1 = 0$ plane. Skip the step of placing points on the sphere S^2 by immediately placing them on left and right planes defined, respectively, by $x_1 = -1$, $x_1 = 1$. A true $k = 3$ example requires enough points so that their convex hull has a positive three-dimensional volume; i.e., $m \geq 4$. The alternating assignment approach is to place half of the points on each plane, and then represent them, in appropriate positions, as points and dual points on the right plane.

Without a natural ordering on R^2, it is not obvious how to position points in an alternating manner. To resolve the problem, use the unit circle in the right plane with center $(1, 0, 0)$. As indicated in Fig. 4, start with $m \geq 4$ points and place $\frac{m}{2}$ of them as regular points in an arbitrary fashion on this circle. As their convex hull has the same number of edges as vertices, the alternating assignment procedure now is easy: place each dual point in an arc of the circle defined by an edge. If m is even, there are as many arcs as dual points, so each arc has a dual point. If m is odd, there is one more arc than dual points, so (as in Fig. 4) one arc has no dual point. Observe that even after slightly perturbing these points (even off the circle), this same construction holds where dual and regular points alternate as determined by the connecting cords. Also notice that if $m = 4$, when the four points are converted into regular points, their convex hull is a tetrahedron. Finally, observe that the alternating assignment symmetry (indicated in the figure with the dashed and solid lines and imposed by the alternating assignment process) allows this construction to start with either the dual (the starred) or the regular points.

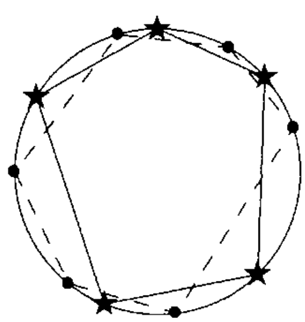


Figure 4. Points and dual points.

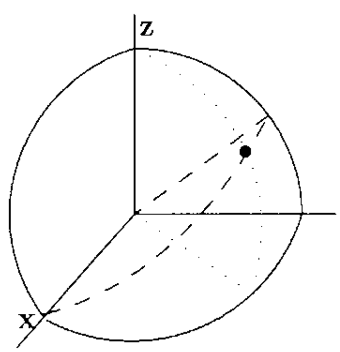
Start with $m \geq 4$ even and pass a dividing plane through two points of the same kind, say, dual points; so $[\frac{m}{2}] - 2$ other dual points remain. If j of them are on one side of this dividing plane, then the "alternating point" construction requires $j + 1$ regular points to be on the same side. After the dual points are converted to regular points on the left side, we have $[\frac{m}{2}] - 1$ points on each side of this dividing plane. Similarly, if the points defining the dividing plane are of opposite types, then if j dual points are on one side of this plane, there also are j regular points. In any case, there are $[\frac{m}{2}] - 1$ points on either side of the dividing plane.

According to Eq. 5.1, this construction supports $q \geq ([\frac{m}{2}] - 1) + 3 = \frac{m}{2} + 2$. Therefore, if $n = m$ (so $\mathbf{x}$ is not a bliss point) it supports $q \geq [\frac{n}{2}] + 2$ rules in a three-dimensional issue space. If n is odd, so $\mathbf{x}$ is to be a bliss point, and because $[\frac{n}{2}] = [\frac{n-1}{2}]$, the construction supports all $q \geq [\frac{n}{2}] + 2$ rules. Again, by starting with

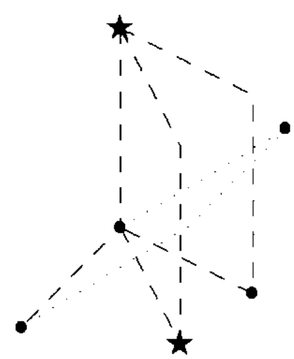
Eq. 2.5, it is easy to see that this construction supports all values of q and n which, according to the theorem, admit a three-dimensional issue space without $\beta \geq 0$.

The only change in the argument for odd values of $m \geq 5$ is to consider the effect of the sector without a dual point. By choosing the dividing plane to pass through these two regular points, all $[\frac{m}{2}]$ dual points and the rest of the regular points are on one side. Consequently, one side of this dividing plane has $[\frac{m}{2}]$ points (and the other side has $[\frac{m}{2}] - 1$). Using the alternating point construction, it follows that this is true for any dividing plane defined by two points. Therefore, $q \geq [\frac{m}{2}] + 3$. So, if $m = n$ (that is, when **x** is not a bliss point), this example supports all $q \geq [\frac{n}{2}] + 3$ rules. Where **x** is to be a bliss-core point, n is even, and $m = n - 1$, the relationship $[\frac{m}{2}] = [\frac{n-1}{2}] = [\frac{n}{2}] - 1$ shows that this construction supports all $q \geq [\frac{n}{2}] + 2$ rules. Again, with a simple computation, it follows that these examples support all rules described in Theorem 1 that do not involve "excess dimension" arguments.

Higher dimensions. New problems arise with $k \geq 4$ because the simplification of choosing the points and dual points in the unit sphere in the right plane fails to reduce the problem to a one-dimensional setting where "alternate positions" is well-defined. Instead, the sphere S^{k-2} has dimension $k - 2 \geq 2$. Therefore, the new approach is to iteratively choose "alternate" locations with respect to each of the $k - 2$ different directions.



a.



b.

Figure 5. Rotations to define directions.

The basic idea comes from modifying spherical coordinates as indicated in Fig. 5a. Start with the plane defined by the origin (the "base point") and the points **x** and **z** on, respectively, the positive x and z axes. A way to describe the position of a point on the sphere is to specify how to rotate this plane about the two axes, $\mathbf{x} - \mathbf{0}$ and $\mathbf{z} - \mathbf{0}$, to meet the point. First rotate this plane about the z-axis until it hits the specified point; this angular distance defines the angular difference from the x axis which, for this example, corresponds to the longitude of the point. To complete the specification of the point, rotate the plane about the x-axis to discover the angular change of this point from the z-axis. The intersection of the two rotated planes defines the directional line defining the point so the corresponding two angles completely specify the point. (This is not spherical coordinates because a point in the original plane cannot be identified in this manner.) In a higher dimensional sphere, the same argument is used except that a b-dimensional plane has b points for each axis of rotation. Notice, the axes of rotation need not be orthogonal, only linearly independent.

A similar construction is used to define gradient directions except that the base point is a point on the sphere. To capture the alternating assignment process, points and dual points are assigned, in an alternating fashion, different orientations of the plane when rotated about different axes. This is indicated in Fig. 5b.

To create k-dimensional examples, the hull defined by the points must have a positive k-dimensional volume, so $m \geq k+1$ points are positioned on the unit sphere in the right hyperplane. Start with an even value for m and an odd value for k where the same number of points and dual points are on the unit sphere. Place $k-1$ linearly independent points on this sphere where half are regular and half are dual and designate one regular point, denoted by $\mathbf{x}_1$, as the base point. Label the rest of the points as $\mathbf{x}_j$, $j = 2, \dots, k-1$, where the odd and even subscripts denote, respectively, a regular and a dual point.

The plane defined by $\{\mathbf{x}_j\}_{j=1}^{k-1}$ divides S^{k-2} into two parts; choose one of these halves to place the remaining points. The placement involves $k-2$ steps. First, the points $\{\mathbf{x}_j\}_{j=1}^{k-2}$ define an axis of rotation for a $(k-1)$-dimensional plane where the rotation is uniquely determined by specifying another point on the sphere. For instance, the removed point $\mathbf{x}_{k-1}$ defines the starting orientation. Rotate this plane away from $\mathbf{x}_{k-1}$ into the designated half of S^{k-2} by specifying $m-(k-1)$ different orientations strictly within the designated half of S^{k-2}. Designate each position as a plane for either a dual or a regular point according to the alternating rule. Namely, because $\mathbf{x}_{k-1}$ is a dual point, the first orientation of the plane belongs to a regular point, denoted by $\mathbf{x}_k$, the next to a dual point, denoted by $\mathbf{x}_{k+1}$, etc. Because there are an even number of remaining terms, the last orientation belongs to a dual point denoted by $\mathbf{x}_m$. So far, only one $\mathbf{x}_j$ component has been specified; $k-2$ more remain.

The second stage of the argument requires dropping $\mathbf{x}_{k-2}$ from the set $\{\mathbf{x}_j\}_{j=1}^{k-1}$ to define another axis of rotation for a different $k-1$ dimensional plane. Starting at x_{k-2}, rotate this plane into the designated half of S^{k-2} and select $m-(k-1)$ orientations. As $\mathbf{x}_{k-2}$ is a regular point, the first orientation is designated for a dual point. Therefore, choose the plane from step one assigned to dual point $\mathbf{x}_m$. The intersection of these two planes determine the first two "coordinates" for this dual point. Because both planes are $k-1$ dimensional and because they meet transversely (e.g., the vectors $\mathbf{x}_{k-1} - \mathbf{x}_1$ and $\mathbf{x}_{k-2} - \mathbf{x}_1$ are linearly independent, they are, respectively, not in the first and second planes, but, respectively, in the second and first planes) their intersection defines a $k-2$ dimensional "ray." (This construction is illustrated in Fig. 5b for regular points.)

As the next orientation is reserved for a regular point, choose the plane from the first stage assigned to $\mathbf{x}_{m-1}$; the intersection of these two planes define the first two coordinates for this regular point. The argument continues in the same alternating fashion while preserving the order established at stage one. As the numbers of regular and dual points to be determined agree and as each plane from the first step must intersect each plane from the second in a $k-2$ dimensional plane, the necessary intersections arguments are immediate. It is important that the ordering for both steps are related.

The induction argument is obvious. Assume this construction holds up to the $(s-1)$th stage of the construction, $s \leq k-1$. Thus, each point to be assigned is specified up to the particular $k-(s-1)$ dimensional ray. Use the axis defined by dropping the point $\mathbf{x}_{k-s}$, and rotate this plane starting at $\mathbf{x}_{k-s}$. Again, choose $m-(k-1)$ orientations for this plane that are strictly within the designated half of S^{k-2}. Each orientation is designated as either a dual or a regular point according to the alternating assignment approach where the assignment starts either with the $k-(s-1)$ dimensional ray containing $\mathbf{x}_m$ (if the first point is to be a dual point) or the ray for $\mathbf{x}_k$ (if the first point is to be a regular point). Because the vectors $\mathbf{x}_{k-1}-\mathbf{x}_1,\dots,\mathbf{x}_{k-s}-\mathbf{x}_1$ are linearly independent where the ith vector is not in the ith plane but in all others, this intersection defines a $(k-s)$- dimensional ray. Clearly, with the flexibility of choosing the position for each plane, we can assume that a plane defined by any $k-1$ points contains only these points.

It remains to show that any dividing plane defined by $k-1$ points (and $\mathbf{x}$) has half of the remaining points on each side. To do this, assign each regular and dual point, respectively, the value $v(\mathbf{x}_j)=+1$ and -1. It is easy to see that if the sums of point values on each side of a dividing plane agree, then there are an equal number of points on each side of this plane after the dual points are converted into regular points. With the original plane, the sums for both sides (as well as for the points on the plane) equal zero. (The sum for one side is trivially zero as it has no points.)

Choose $k-1$ points $\{\mathbf{x}_\alpha\}_{\alpha\in D}$ where D is the set of $k-1$ subscripts. When the first plane is rotated, it passes through a first and a last point from this set; denote them by $\mathbf{x}_{\alpha_1}$, $\mathbf{x}_{\alpha_2}$; $\alpha_1<\alpha_2$. (If the indices in D are greater than $k-2$, then, from the construction, all of them must be between α_1 and α_2) The orientations of the plane defined by these extreme values defines a sector, S_1. We must show that $a_3=\sum_{j=\alpha_1+1}^{\alpha_2-1}v(\mathbf{x}_j)$, the sum of point values inside this sector, agrees with the sum of those from outside. Positioning this rotating plane on $\mathbf{x}_{k-1}$ means that the sum of points on either side is zero. Rotate it until it reaches the regular point $\mathbf{x}_k$, so this regular point replaces the dual point $\mathbf{x}_{k-1}$ on the rotating plane. Thus, the sum of values of points on the $\mathbf{x}_{k-1}$ side is -1, the sum is $+1$ for points on the plane, so the sum on the other side of the plane also equals -1. When this plane reaches the second point, this dual point replaces the previous regular point so the sum of points on each side returns to zero. Clearly, because of the alternating form of this construction, when this plane is on $\mathbf{x}_{\alpha_1}$ or $\mathbf{x}_{\alpha_2}$, the sum of points on each side of this orientation of the plane is the same.

If $a_1=\sum_{j=1}^{k-2}v(\mathbf{x}_j)$, $a_2=\sum_{j=k-1}^{\alpha_1-1}v(\mathbf{x}_j)$, $a_4=\sum_{j>\alpha_2}v(\mathbf{x}_j)$, the above arguments for the two endpoints $\mathbf{x}_{\alpha_1}$ and $\mathbf{x}_{\alpha_2}$ and the fact that the sum of all values is zero correspond to the equations

$$
\begin{aligned}
a_2 &= a_3+v(\mathbf{x}_{\alpha_2})+a_4\\
a_4 &= a_2+v(\mathbf{x}_{\alpha_1})+a_3\\
a_1+a_2+v(\mathbf{x}_{\alpha_1})&+a_3+v(\mathbf{x}_{\alpha_2})+a_4=0
\end{aligned}
\tag{5.2}
$$

The desired conclusion $a_3=a_1+a_2+a_4$ follows by eliminating the $v(\mathbf{x}_{\alpha_j})$ terms from Eq. 5.2. That is, the sum of values of points in S_1 agrees with the sum for points outside of this sector. This value can be -1, 1, or 0 depending on whether the endpoints are both regular, both dual, or one of each kind.

The argument continues by using the second plane. Find the extreme points $\mathbf{x}_{\alpha_3}, \mathbf{x}_{\alpha_4}$ for orientations of this plane. (By construction, one or both of these points could agree with the extreme points from the first stage. If not, then $\alpha_1 < \alpha_3, \alpha_4 < \alpha_2$.) The rotation of the second plane defined by these two points define a sector, denote the intersection of this sector with S_1 by S_2. Because of the manner in which points are defined, all of the points between (in the sense of the change of the orientation of the second plane) $\mathbf{x}_{\alpha_3}$ and $\mathbf{x}_{\alpha_4}$ are in S_1. Therefore, the same counting argument applies showing that the sum of values of points in S_2 agrees with the sum of values of points outside of this sector. (This statement need not be true if, when choosing the second coordinate for each point, we selected *any* regular point from step one. Without the imposed ordering, the points in S_2 can lose the alternating structure critical for our construction.)

The same argument now is applied to each plane. Because of the ordering imposed on the choice of the points, the conclusion follows. All that remains to show that the final sector contains all points on one side of a dividing plane and only these points. But, this follows from the ordering imposed in the construction. Notice that the sum of the values of the points in the final sector can vary between $\pm(k-1)/2$ where the precise value is determined by the the number of regular and dual points on the dividing plane. (Notice, this argument is just an iterated version of the one used with Figs. 3 and 4.)

To summarize, for even m and odd k the construction defines gradient directions where, for any dividing plane defined by $k-1$ points, there are $s^* = (m-k+1)/2$ points on either side. From [2], this example supports all $q \geq k + \frac{m-k+1}{2} = \frac{k+1}{2} + \frac{m}{2} = [\frac{m}{2}] + \frac{k+1}{2}$ rules. When $m = n$ (so $\mathbf{x}$ is to be a nonbliss core point for an even number of voters), this example supports all $q \geq [\frac{n}{2}] + \frac{k+1}{2}$ rules. When $m + 1 = n$ (so $\mathbf{x}$ is to be a bliss core point for an odd number of voters), this example supports all $q \geq [\frac{n}{2}] + \frac{k+1}{2}$ rules.

For the remaining cases, the distribution of points and the supporting arguments are essentially as described above. When m and k are both even, use $\frac{k}{2}$ regular and $\frac{k}{2} - 1$ dual points to define the original plane. Here, in the distribution of points in the designated half of S^{k-2}, the alternating assignment requires one more dual than regular point as both $\mathbf{x}_k$ and $\mathbf{x}_m$ represent dual points. (For $k = 4$, this construction is partially indicated in Fig. 5b.) As the sum of values of points on the original dividing point is unity, the sum of points on one side is -1 and on the other (where there are no points) is zero. In fact, in this situation, that goal is to show that the magnitude of the difference between the sum of the values of points on either side of a dividing plane defined by $k-1$ points differs by no more than unity: this indicates that no more than $[\frac{m-k+1}{2}] + 1$ points (the rounding upwards of the fraction $\frac{m-k+1}{2}$) points are on either side. This proof is essentially the same as above with Eq. 5.2 once the modification is made that when a plane rotates, the difference between summed values on each side differs from unity. (In carrying out the computations, I found it is easier to consider separately the three cases where the two $v(\mathbf{x}_{\alpha_j})$ values are positive, negative, or differ in sign.) Thus, after dual points are translated into regular points, this example has at most $\frac{m}{2} - \frac{k}{2} + 1$ points on either side. As $s^* = [\frac{m}{2}] - \frac{k}{2} + 1$, we have from [2] that this example supports all $q \geq [\frac{m}{2}] + \frac{k}{2} + 1$ rules. When $n = m$, so $\mathbf{x}$ is to be a nonbliss point with an even number of voters, this supports all $q \geq [\frac{n}{2}] + \frac{k}{2} + 1$ rules. Where $\mathbf{x}$ is to be a bliss core point with an odd number of voters ($n - 1 = m$), this example supports the $q \geq [\frac{n}{2}] + \frac{k}{2} + 1$ rules.

If both m and k are odd, start with $\frac{k-1}{2}$ regular and dual points on the original plane. Of the $m - k + 1$ points left to distribute, let $[\frac{m-k+1}{2}] + 1$ be regular points and the rest ($[\frac{m-k+1}{2}]$) dual points where the assignment is in the usual alternating order. (Similar to the previous case, two regular points are next in the rotation, but never three. As indicated in Fig. 4, we encountered this situation already for $k = 3$.) Again, an obvious modification of the proof represented in Eq. 5.2 shows that the difference in sums of values of points on either side differs in magnitude by at most unity. Thus, after the same conversion of dual to regular point construction, a dividing plane can have no more than $[\frac{m-k+1}{2}] + 1$ points on either side. Thus, $q \geq [\frac{m}{2}] + \frac{k+1}{2} + 1$. When $m = n$ ($\mathbf{x}$ is a nonbliss core point for an odd number of voters) this example supports all $q \geq [\frac{n}{2}] + \frac{k+1}{2} + 1$ rules. When $n = m + 1$ ($\mathbf{x}$ is to be a bliss core point for an even number of voters), this example supports all $q \geq [\frac{n-1}{2}] + \frac{k+1}{2} + 1 = [\frac{n}{2}] + \frac{k+1}{2}$ rules.

The final setting is where m is odd and k is even. Here, the starting plane has $[\frac{k-1}{2}] + 1$ regular and $[\frac{k-1}{2}]$ dual points. The remaining even number of points that are to be distributed, in the usual alternating fashion, are evenly divided into dual and regular points. This construction allows $\frac{m-k+1}{2}$ points one each side of any dividing plane. Thus, according to [2], it supports $q \geq [\frac{m}{2}] + \frac{k+1}{2} + 1$ rules. With nonbliss settings for $\mathbf{x}$ and an odd number of voters, $n = m$, the example supports all $q \geq [\frac{n}{2}] + \frac{k+1}{2} + 1$ rules. For the nonbliss settings of $n - 1 = m$, it supports all $q \geq [\frac{n-1}{2}] + \frac{k+1}{2} + 1 = [\frac{n}{2}] + \frac{k+1}{2}$ rules. Again, in all cases, it follows by simple computations that these examples support all rules allowed by Theorem 1 that are not supplemented by "excess dimensions".

6. Upper bounds

In Sect. 5, examples are constructed to support the assertions of Thm. 1 where $\beta = -1$. It remains to construct $\beta \geq 0$ examples, to show that the bounds of Thm. 1 are tight (i.e., that one cannot do better), and to prove Thm. 2a. All are done with singularity theory.

The reader unfamiliar with this important tool can view singularity theory (e.g., [GG] or [SS]) as a sophisticated implicit function theorem. Let $f : R^n \to R^m$, be a smooth mapping and Σ be a smooth b dimensional manifold in R^m (so, the codimension of Σ is $m - b$). According to the implicit function theorem, if f satisfies appropriate conditions, then, at least locally, $f^{-1}(\Sigma)$ is a codimension $m - b$ (or dimension $n - (m - b)$) submanifold of R^n. The needed *transversality condition* requires for $\mathbf{x}$ where $f(\mathbf{x}) \in \Sigma$ that the span of the tangent spaces $D_{\mathbf{x}}f(R^n)$ and $T_{f(\mathbf{x})}\Sigma$ is R^m. (The usual inverse function theorem where Σ is a point only requires the $D_{\mathbf{x}}f(R^n)$ to span R^m to have rank m as the tangent space of a point is the zero vector.)

To use this tool to analyze first and second derivative conditions imposed upon a function f, use the "jet" map. This is the mapping $j^2f(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}), Df_{\mathbf{x}}, D^2_{\mathbf{x}}f)$. The domain of this mapping is R^n, and the range is $J^2 = R^n \times R^m \times L(R^n, R^m) \times B(R^n, R^m)$ where $L(R^n, R^m)$ are the linear maps from R^n to R^m and $B(R^n, R^m) \approx$

$L(R^n, L(R^n, R^m))$ are the bilinear symmetric maps. If the derivative conditions define a manifold Σ in J^2 for $f = (u_1, \dots, u_n)$ where $j^2 f$ meets Σ transversely, then the implicit function theorem ensures that the set of points satisfying these conditions is (locally) smooth submanifold with codimension equal to the codimension of Σ. The difficulty is to verify the transversality condition.

This problem is resolved by extending the notion of a transverse intersection to allow $j^2 f$ to miss Σ. Here, the Thom Transversality Theorem ensures that for a generic set of functions, the intersection is transverse. (When issue space is restricted to a compact subset, this assertion holds for an open-dense set of functions. [SS]) In words, the general situation is that the preferences either fail or satisfy the core conditions robustly. Thanks to this important assertion, the proof of the theorem reduces to (1) representing the core derivative conditions of Prop. 1 in terms of a manifold Σ in J^2, (2) finding the codimension of Σ, and (3) showing that we are not discussing the empty set.

Task (3) requires showing that there exists a f where $j^2 f \in \Sigma$. This is important because there are many examples where $j^2 f$ never meets Σ; they correspond to settings where the alternatives force the voters' preferences to cluster in ways that do not satisfy Prop. 1. In other words, rather than asserting that the core always exists, Theorem 1 only ensures that there exist robust examples in the indicated issue spaces. It is important to note that we do not need to verify that $j^2 f$ meets Σ transversely. The assertion that there exist functions that do so, hence robust examples exist (not necessarily this particular f) is a gift from Thom. The idea is that if $j^2 f \in \Sigma$ is a boundary point, then, as Thom's result has the conclusion holding for an open set about f, near-by functions satisfy the transversality condition. To illustrate with Plott's construction, choose the $[\frac{n}{2}]$ lines so their span is a $[\frac{n}{2}]$-dimensional space. Even if this example is not robust, it demonstrates that some f satisfies the core condition, so, from singularity theory, we know there is an open set of preferences with this property for the specified dimensions of issue space where n is odd and the q rules are bounded above by the $\frac{3}{4}$ rule. (From Eq. 2.5, the equation is $\frac{n-1}{2} = 2q - n$ or $q = \frac{3n-1}{4}$.)

Proof of Thm. 1. Suppose n and q are specified. To construct a manifold Σ in $L(R^k, R^n)$ (the space of linear maps from R^k to R^n) corresponding to the core conditions specified in Prop. 1, use the identification

$$L(R^k, R^n) \approx \{\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n) \mid \mathbf{A}_j \in R^k\}.$$

Namely, think of $\mathbf{A}_j$ as a dummy variable where its range of R^k represents all possible choices for $\nabla u_k(\mathbf{x})$. An advantage of using this identification, rather than discussing properties of matrices, is that it is a simple parametric representation of a manifold in $L(R^k, R^n)$ in terms of the $\mathbf{A}_j$ vectors. Since $\mathbf{A}_j$ is meant to capture conditions on $\nabla u_j(\mathbf{x})$, these Σ manifold definition follows directly from the core conditions on the gradients. (This representation is another way my method differs significantly from the approach used in [B, Sc, MS1].)

Bliss-core points. To start with bliss-core points, assume that the core point is the bliss point for the first agent. This means we must examine all points $\mathbf{x}$ where $\nabla u_1(\mathbf{x}) = \mathbf{0}$. All such situations are captured by the manifold $\Sigma_1 = \{\mathbf{A} \mid \mathbf{A}_1 = \mathbf{0}\}$.

Because there are no restrictions on any other J^2 coordinates, if there is any $\mathbf{x} \in R^k$ where $\nabla u_1(\mathbf{x}) = \mathbf{0}$, then $j^2(u_1, \ldots, u_n) \in \Sigma_1$. The codimension of Σ_1 is k (reflecting that all k components of $\mathbf{A}_1$ are completely specified), so, generically, the set of $\mathbf{x}$'s satisfying this condition is $k - k$, or zero-dimensional; it is a set of isolated points.

Imposing any other condition on Σ_1 which increases the codimension, so it is larger than k, corresponds to where the set of points satisfying the conditions is generically empty. For instance, if the core point is the bliss point for two or more agents where one is the jth, $j \neq 1$, then the manifold capturing this situation is $\Sigma_2 = \{\mathbf{A} \mid \mathbf{A}_1 = \mathbf{A}_j = \mathbf{0}\}$ with codimension $2k$. Thus, the set of points satisfying this condition generically form a union of $k - 2k$ dimensional manifolds. As this dimension is negative, this behavior is generically impossible. Generically, a point is a bliss point for at most one agent.

A condition motivated by Plott's construction is to suppose that at the bliss-core point the gradients of two other agents lie along the same line. If these agents are, say, 2 and 3, then the manifold in $L(R^n, R^m)$ capturing these conditions is $\Sigma_3 = \{\mathbf{A} \mid \mathbf{A}_1 = \mathbf{0}, \mathbf{A}_2 = \lambda \mathbf{A}_3, \lambda \in R\}$ with codimension $2k - 1$. (k codimensions come by specifying $\mathbf{A}_1 = \mathbf{0}$ and $k - 1$ from the fact that the direction of $\mathbf{A}_2$ is specified.) As the codimension of Σ_3 is larger than the dimension of issue space when $k \geq 2$, such behavior is generically impossible outside of a one-dimensional issue space.

To generalize Σ_2, Σ_3, consider where $\nabla u_1(\mathbf{x}) = \mathbf{0}$ and there are $1 \leq b \leq k$ agents, with index set D, so that $\{\nabla u_j(\mathbf{x})\}_{j \in D}$ is in a $b - 1$ dimensional subspace. To see that this is generically impossible, consider the manifold $\Sigma_{1,D} = \{\mathbf{A} \mid \mathbf{A}_1 = \mathbf{0}$, there exists scalars λ_j, not all zero, so that $\sum_{j \in D} \lambda_j \mathbf{A}_j = \mathbf{0}\}$. As the sum ensures that one $\mathbf{A}_j$ is determined by the others, another value is added to the codimension; as $\Sigma_{1,D}$ has codimension $k + 1$, this defines a generically empty situation.

To use $\Sigma_{1,D}$ to prove Theorem 1, recall from [2] and Prop. 1 that a plane defined by $k - 1$ linearly independent vectors can have no more than $(q-1) - (k-1) = q - k$ gradient vectors on either side. If no other gradient vectors are on this plane, then by counting the maximum number of vectors that can be on both sides, the number on the dividing plane and the bliss point, we have $2(q - k) + k - 1 + 1 \geq n$ or $k \leq 2q - n$. That is, if $k > 2q - n$, there is another gradient vector on this plane. Such a situation requires these core points to be in $\Sigma_{1,D}$ for some D with k indices.

All possible situations are given by the $\binom{n-1}{k}$ distinct sets of k indices from $\{2, 3, \ldots, n\}$. If D denotes such a set, $j^2 f$ must be in some $\Sigma_{1,D}$. Thus, the set of points satisfying such situations is generically empty. This condition includes the bliss-core points for $k > 2q - n$. As all bliss core points are obtained by changing the choice of D and the identity of the voter whose bliss point is the core point, this is a finite condition. Namely, it corresponds to a finite union of submanifolds, all with the same codimension. Each submanifold is obtained from the first by a permutation of the indices. Thus, the assertion follows. □

Nonbliss core points. The argument showing that Eqs. 2.3, 2.4 determine upper bounds for the generic existence of nonbliss core points resembles the bliss-core point setting but the counting argument is more difficult. To help the reader, the ideas are introduced with a special case.

The $q = 4, n = 5$ rule. The first choice allowing $\beta \geq 0$ is the $q = 4, n = 5$ rule:

Theorem 1 asserts that issue space can have dimension up to $k = 3$. To show this is an upper bound, let $\mathbf{x}$ be a nonbliss ideal point and consider the plane defined by two gradient vectors $\nabla u_j(\mathbf{x})$. Using the arguments leading to [2], no more than one gradient vector can be on either side of this dividing plane. Counting one for each side, and the two that define the plane, we have accounted for four of the five gradient vectors. Thus the dividing plane also must contain the last gradient vector. Moreover, $\mathbf{x}$ is in the convex hull defined by these three gradient vectors. If not, then there is a line passing through $\mathbf{x}$ where all three gradient vectors are on the same side. If the dividing plane is rotated about this line, all three gradient vectors will end up on the same side as one of the remaining gradient vectors: this violates the assumption that $\mathbf{x}$ is a core point.

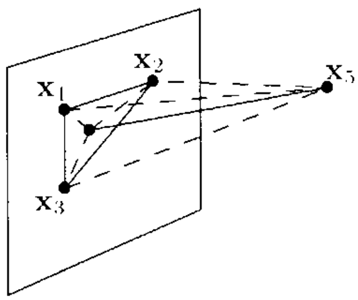


Figure 6. New planes

This condition, where a plane defined by two gradient vectors has a third and their convex hull contains $\mathbf{x}$, imposes strict conditions on all five points! To see why, suppose $\{\mathbf{x}_j\}_{j=1}^3$ are on the plane, $\mathbf{x}$ is in the interior of this hull, and $\mathbf{x}_5$ is off of the plane. (See Fig. 6.) The pairs of points $\{\mathbf{x}_1, \mathbf{x}_5\}$, $\{\mathbf{x}_2, \mathbf{x}_5\}$, and $\{\mathbf{x}_3, \mathbf{x}_5\}$ (along with $\mathbf{x}$) define three more planes which intersect along the line connecting $\mathbf{x}$ and $\mathbf{x}_5$. With this condition, another point must be on each plane. This extra point, $\mathbf{x}_4$, cannot be $\mathbf{x}$ as $\mathbf{x}$ is not a bliss point. If $\mathbf{x}_4$ is not on the $\mathbf{x}$–$\mathbf{x}_5$ line, then $\mathbf{x}_4$ is on one of the three planes, so we need two more points. With $n = 5$, this is impossible. Consequently, $\mathbf{x}_4$ is on the $\mathbf{x}$–$\mathbf{x}_5$ line. Whatever the orientation of the $\mathbf{x}$–$\mathbf{x}_5$ line, there is a plane though $\mathbf{x}$ with two of $\{\mathbf{x}_j\}_{j=1}^3$ and $\mathbf{x}_5$ on the same side. If $\mathbf{x}_4$ were between $\mathbf{x}_5$ and $\mathbf{x}$, four points would be on the same side of a dividing plane: this violates [1]. Thus, $\mathbf{x}_4$ is on the other side of the plane.

In J^2, these conditions are captured by $\Sigma = \{\mathbf{A} \mid \{\mathbf{A}_j\}_{j=1}^3$ are linearly dependent; there exists $\lambda < 0$ so that $\mathbf{A}_4 = \lambda \mathbf{A}_5\}$. To find the codimension, notice that $k - 2$ dimensions are imposed by the dependency condition on the first three vectors, and $k - 1$ dimensions are obtained from the second. Thus, as the codimension is $2k - 3$, such examples are generically possible only when $2k - 3 \leq k$ or $k \leq 3$. If $\mathbf{x}$ is on the boundary of the hull, same argument holds except that an extra codimension is added to reflect that two vectors are along the same line. This situation, then, holds generically only for $k \leq 2$. As the construction requires $k \geq 3$, it is generically empty.

The final possibility is if all five vectors are on the plane. Here, $\Sigma = \{\mathbf{A} | \mathbf{A}_j, j = 3, 4, 5$, is a linear combination of $\mathbf{A}_1$ and $\mathbf{A}_2\}$. There are no restrictions on the choice of $\mathbf{A}_1, \mathbf{A}_2$, but a codimension $k - 2$ imposed on the choice of all remaining vectors. Thus, the total codimension is $3k - 6$. Such behavior is generically possible only if $3k - 6 \leq k$, or if $k \leq 3$.

To obtain all possible cases, use all possible permutations of the indices. As this number is finite, the conclusion holds. To create examples, use the gradient directions defined by the construction. Notice, for Euclidean preferences, both cases restrict the positioning of ideal points so, here, the conclusion is not generic.

General case. The proof for the general case extends the above construction. Let $\mathbf{x}$ be a nonbliss core point, and choose any $k-1$ linearly independent gradient vectors. If no other gradient vectors are on this plane, then, according to Prop. 1 and [2], no more than $q-k$ gradient vectors can be on either side of this plane. Using this maximum as the number on each side and counting all gradient vectors on the plane, we have that $2(q-k)+(k-1) \geq n$, or $2q-n-1 \geq k$. Thus, if $k > 2q-n-1$, another gradient vector must be on this dividing plane. To underscore this assertion, denote the dimension of issue space as $k = \beta + (2q-n)$ where the admissible values for the "excess dimension" are $\beta = -1, 0, \ldots$. First we consider what happens should all points be on this hyperplane, then we analyze what happens when some points are off of it.

If all gradients are in the same $2q-n-1$ dimensional space, then this space is defined by a basis; assume it is given by the first $2q-n-1$ vectors. As all other vectors are in this linear space, the J^2 representation is $\Sigma = \{\mathbf{A} \mid \mathbf{A}_i \in Span(\{\mathbf{A}_j\}_{j=1}^{2q-n-1})\ \forall i \geq 2q-n$, the plane defined by any $k-1$ vectors from $(\{\mathbf{A}_j\}_{j=1}^{2q-n-1})$ has no more than $q-k$ vectors on either side$\}$. The dependency condition imposes a $\beta+1$ dimensional constraint on the choice of $\mathbf{A}_j$, $j \geq 2q-n-1$. (The dividing plane condition is an open one that, from Sect. 5 can occur, so it does not contribute to the codimension.) Thus, the codimension of Σ is $(1+\beta)(2n-2q+1)$. Consequently, such a condition exists generically only if the inequality

$$(\beta+1)(2n-2q+1) \leq k = 2q-n+\beta$$

is satisfied. Collecting terms and solving for q we obtain Eq. 2.3; solving for k leads to Eq. 2.4. Considering all possible permutations of indices completes the proof.

(This construction does not explicitly consider where no set of $2q-n-1$ vectors are linearly independent. But, this condition increases the codimension for each $\mathbf{A}_j$ so the total codimension of the new version of Σ is larger. Therefore, such situations become generically unlikely with smaller k values.)

Vectors out of the plane. It remains to consider where not all vectors are in the same $2q-n-1$ dimensional subspace. This analysis uses the next statement.

Lemma 2. *Suppose $\mathbf{x}$ is a nonbliss core point and a set of $2q-n-1$ linearly independent gradient vectors define a plane $\mathcal{L}$ passing through $\mathbf{x}$. The convex hull of the gradient vectors on $\mathcal{L}$ contains $\mathbf{x}$ but not as a vertex. If $\mathbf{x}$ is a boundary point of this convex hull, and if there are $2q-n-1+c$ vectors in $\mathcal{L}$, then at least $c+1$ of the vectors lie on the same boundary surface of this hull and the hull they define has $\mathbf{x}$ as an interior point.*

Proof. Replace gradients with points $\mathbf{x}_j$ where directions are $\mathbf{x}_j - \mathbf{x}$. As n is finite, we can extend $\mathcal{L}$ to a codimension one plane $\mathcal{L}_1$ which contains no point off of $\mathcal{L}$. If $\mathbf{x}$ is not in the convex hull, then, because it is separated from this hull, a hyperplane $\mathcal{H}$ passes through $\mathbf{x}$ where all points from $\mathcal{L}$ are strictly on one side. With $2q - n - 1 + \epsilon$ points on $\mathcal{L}$, and, hence, on $\mathcal{L}_1$, $\epsilon \geq 1$. A slight change in the orientation of $\mathcal{L}_1$, using $\mathcal{H} \cap \mathcal{L}_1$ as as an axis of rotation, can force all points on $\mathcal{L}_1$ to one side of the new dividing plane. As $\mathbf{x}$ is a core point, no more than $(q-1) - (2q - n - 1 + \epsilon) = n - q - \epsilon$ points are on either side of $\mathcal{L}_1$, so, at most, $2(n - q - \epsilon)$ points can be off of $\mathcal{L}_1$. The number of points not on $\mathcal{L}$ is $n - (2q - n - 1 + \epsilon) = 2n - 2q + 1 - \epsilon$, so $2(n - q - \epsilon) \geq 2n - 2q - \epsilon + 1$ or $-1 \geq \epsilon$. This contradiction proves the assertion.

If $\mathbf{x}$ were a vertex, it would be a bliss point which violates the assumption. If $\mathbf{x}$ is a boundary point of the hull, it may be on one or several boundary components of the hull; choose the one, B, with minimal dimension γ. Because $\mathbf{p}$ is not a vertex, the number of points, r, on B satisfies $r \geq \gamma + 1$ where (because B has minimal dimension) $\mathbf{x}$ is in the interior of the hull defined by these points. To find a bound on r in terms of ϵ, notice that (n is finite) there is a plane $\mathcal{P}$ passing through $\mathbf{x}$ and intersecting $B \subset \mathcal{L}_1$ so that at least $\frac{r}{2}$ of the B points are on the same side as the $2q - n - 1 + \epsilon - r$ remaining points of $\mathcal{L}_1$. By using the dividing plane argument, where the plane comes from rotating $\mathcal{L}_1$ about the $\mathcal{P} \cap \mathcal{L}_1$ axis, and the fact that one side of $\mathcal{L}_1$ has at least half, $n - q - \frac{\epsilon-1}{2}$, of the remaining points, it must be that $(n - q - \frac{\epsilon-1}{2}) + (2q - n - 1 + \epsilon - r) + (\frac{r}{2}) \leq q - 1$ or $\epsilon + 1 \leq r$. □

While there exist configurations allowing core points where all points are not on the $2q - n - 1$ dimensional space, we show how the rapid growth of the binomial coefficient forces the codimension to grow so fast that this is generically possible only in very special cases. Assume that $\mathbf{x}_n$ (representing $\nabla u_n(\mathbf{x})$) is not in $\mathcal{L}$ and that $\mathbf{x}$ is in the interior of the hull of the $2q - n$ points $\{\mathbf{x}_j\}_{j=1}^{2q-n}$ on $\mathcal{L}$. Because $\{\mathbf{x}_j\}_{j=1}^{2q-n}$ defines a convex hull with interior, any of $\mathbf{x}_j$ is removed, the remaining directions (i.e., $\mathbf{x}_i - \mathbf{x}$) are linearly independent. Moreover, because $\mathbf{x}_n \notin \mathcal{L}$, when $\mathbf{x}_n$ replaces any two points from $\{\mathbf{x}_j\}_{j=1}^{2q-n}$, the new set defines a $2q-n-1$ dimensional plane. As there are $\binom{2q-n}{2}$ ways to choose pairs from $\{\mathbf{x}_j\}_{j=1}^{2q-n}$, there are $\binom{2q-n}{2}$ planes. From the linear independence statement, each plane does not include the pair dropped to define the basis and these planes meet only along the $\mathbf{x}$–$\mathbf{x}_n$ line.

This dimension requires each plane to have one more point. One possibility is to place a point, $\mathbf{x}_{n-1}$, on the $\mathbf{x}$-$\mathbf{x}_n$ line to simultaneously satisfy this condition for all planes. (Again, $\mathbf{x}$ must be between $\mathbf{x}_{n-1}$ and $\mathbf{x}_n$.) The J^2 manifold containing this condition is $\Sigma = \{\mathbf{A} \mid \text{vectors } \mathbf{A}_j, j = 2q - n, \ldots, n-2 \text{ are in the space spanned by } \{\mathbf{A}_j\}_{j=1}^{2q-n-1}; \text{ there exists a scalar } \lambda < 0 \text{ so that } \lambda\mathbf{A}_n = \mathbf{A}_{n-1}\}$. Each $\mathbf{A}_j, j = 2q - n, \ldots, n-2$ vector adds $(1 + \beta)$ to the codimension and only the length of $\mathbf{A}_{n-1}$ can be chosen, so the codimension of Σ is $(1 + \beta)(2n - 2q - 1) + (k-1)$. To be generically possible, this value must be bounded above by k which means that $(1+\beta)(2(n-q)-1) \leq 1$. In turn, this inequality is satisfied only for $n = q+1$, $\beta = 0$, or $q = n$. Notice, a larger β value is obtained when all points remain on the plane.

When the extra point for each plane is not chosen to satisfy all planes simultaneously, the codimension increases more rapidly. To minimize the number of extra points is to choose points that satisfy as many planes as possible. As $\gamma = 2q-n \geq 3$, this occurs by choosing a point that is in the intersection of $\gamma - 2$ planes; in $\mathcal{L}$, this intersection is the $\mathbf{x}_j$–$\mathbf{x}$ line for some j. As there are $\binom{\gamma}{2}$ planes, we need at least $\delta = [\binom{\gamma}{2}/(\gamma - 2)] \geq 3$ points chosen in this way. First assume that all these points are in $\mathcal{L}$. The J^2 manifold including this situation is $\Sigma = \{\mathbf{A} \mid$ any $\gamma - 1$ vectors from $\Gamma = \{\mathbf{A}_j\}_{j=1}^{\gamma}$ are linearly independent, each vectors $\{\mathbf{A}_{\gamma+j}\}_{j=1}^{\delta}$ is a scalar multiple of a vector from $\Gamma\}$. The codimension of Σ is bounded above by $(1+\beta)+\delta(k-1)$. As $\delta \geq 3$, $k \geq \gamma \geq 3$, this value always is larger than k, so this setting never is generically possible.

More generally, if points are chosen to be on $\gamma - s$ planes, where $k = \gamma + \beta \geq \gamma \geq s+1$, then, as there are $\binom{\gamma}{s+1}$ ways to choose these planes, there must be at least $\delta_s = \lceil \binom{\gamma}{s+1}/(\gamma-(s+1)) \rceil$ points. Thus, the J^2 set containing this condition is $\Sigma_s = \{\mathbf{A} \mid$ any $\gamma - 1$ vectors from $\Gamma_s = \{\mathbf{A}_j\}_{j=1}^{\gamma}$ are linearly independent, each vectors $\{\mathbf{A}_{\gamma+j}\}_{j=1}^{\delta_2}$ can be expressed as a linear combination of s vectors from $\Gamma_s\}$. The codimension of Σ_s is bounded above by $(1+\beta)+\delta_s(k-s)$. Therefore, to be generically possible, this number must be bounded above by k, which, by using $k = \gamma+\beta$, requires satisfying $1 + (\delta_s - 1)\gamma + \delta_s\beta \leq \delta_s s$. As $\delta_s > 2, \gamma > s$, this never is satisfied. (I leave it as a simple exercise to show that if points are chosen to be on different number of planes, then the same conclusion holds. This is because the more planes a point is on, the higher its codimension. The other extreme, of choosing more points with lower codimension requires so many more points that the codimension grows faster.)

The remaining possibility is to choose extra points off of $\mathcal{L}$. This modifies Σ by allowing the $\mathbf{A}_j$ points to include $\mathbf{A}_n$ its representation. Thus the new codimension is $(1+\beta)+\delta_s(k-s-1)$. To be generically possible, this number must be less than k, or $1+(\delta_s-1)\gamma+\delta_s\beta \leq \delta_s(s+1)$. Again, because $\delta_s \geq 3$ and $\gamma \geq s+1$, this never can be satisfied.

Finally, if $\mathbf{x}$ is on the boundary of the convex hull of the points on $\mathcal{L}$, then the same argument applies, but the codimension escalates more rapidly because, as Lemma 2 asserts, most of the points are on a lower dimensional subset of $\mathcal{L}$ and all added points also are on this set. Thus, the same computations show that this setting is not generically possible.

Constructing examples. The construction of examples is simple. Choose any point $\mathbf{x}$. Using the construction of Sect. 5, find gradient directions for points where these gradient directions define a $2q-n-1$ dimensional space and where they satisfy the alternating rules for the specified n, q values. Now, use these directions to define gradients. The only difference in choosing second derivative terms is that they must include all variables.

To illustrate with $n = 4, q = 5$ and $k = 3$, use Fig. 2. Translate and rotate the figure in R^3 so that a specified $\mathbf{x}$ from the shaded region is at the desired location of a nonbliss core point, and the orientation of the plane is consistent with the desired plane of gradient vectors. Define the gradient directions by the directions from $\mathbf{x}$ to the vertices of the star. This construction may suggest that a two-dimensional core results; this would be in conflict with Theorem 2c which asserts it should be zero-dimensional isolated points. The explanation is that in R^2, the chosen gradient directions have only one degree of freedom when the base point is varied. Consequently, the same general star figure is defined by the gradients at neighboring points. With the added degrees of freedom from R^3, when the base point is varied, the gradient need not lie in the same plane. Once this happens, [1] and [2] are violated, so a core does not exist. Therefore, this construction does, in fact, define the core to be a collection of isolated, zero-dimensional points.

Dimensions. The dimensional statements follow directly from the codimension statements earlier in this section. This is because, generically, the sets are the union of isolated smooth submanifolds with the indicated codimension of appropriate Σ. Another approach is to note that the construction developed in Sect. 5 allows for freedom in the choice of the gradient directions (an open set about each direction), so the construction is robust. The same comment applies to when the construction of Sect. 5 applies to the $\beta \geq 0$ settings.

I stated that, generally, the core has a stratified structure. As this statement is a direct consequences of singularity theory, it is not formally asserted nor proved. Yet, related ideas are in the proof of Thm. 2a. First, however, I need that the core is closed. This is a consequence of continuity of the gradients and the fact that Prop. 1 defines a closed condition. The only difference in the J^2 representation is that some of the $\mathbf{A}_j$ vectors are on the same plane. This adds to the codimension which, in turn, reflects that the boundary is a lower dimensional object.

Theorem 2a. Only the proof of Theorem 2a remains. To show that if $\mathbf{x}$ is a q core point then it is a $q+1$ core point, notice that the convex hull defined by $q+1$ gradient vectors includes the convex hull defined by a subset of them. The conclusion follows from Prop. 1.

Next, consider n, k, q values where, generically, the core for the q rule must be in a lower dimensional submanifold. If the generic situation for the $q+1$ rule allows the core to have a nonempty interior (e.g., $q = n-1$ or where $q+1$ admits $\beta = -1$), then, generically, the two cores cannot agree as they have different dimensions.

Suppose the core for the $q+1$ rule is, generically, in a lower dimensional set. The dimension for the q rule is $d_q = k - (\beta_1 + 1)(2q - n + 1)$ where $\beta_1 = k - (2q - n - 1$. The excess dimension for the $q+1$ rule is $\beta = k - (2(q+1) - n - 1) = \beta_1 - 2$. The above argument handles $\beta_1 < 2$, so let $\beta_1 \geq 2$. The core dimension for the $q+1$ rule is $d_{q+1} = k - (\beta + 1)(2q - n + 3) = k - (\beta_1 - 1)((2q - n + 1) + 2)$. To see that $d_{q+1} > d_q$, so the conclusion will follow because the dimensions disagree, compute $d_{q+1} - d_q = 2((2q - n + 1) - \beta_1) + 2$. As $q \leq n - 2$, the value of β_1 is bounded above by $q - \frac{3}{4}n$, (see the expression for β found between Eqs. 2.3, 2.4) so $d_{q+1} - d_q > 0$.

The remaining case is where the core for the q rule has a nonempty interior. Assume that the core for the q and $q+1$ rules are the same. Generically, the core is a closed set where if $\mathbf{x}$ is a boundary point, then $\mathbf{x}$ satisfies Eq. 4.1 by being on the boundary of the convex hull for some coalition. Suppose not; suppose $\mathbf{x}$ is in the interior of each $Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C}$ for each decisive coalition C and that $\mathbf{v}$ is such that $\mathbf{x} + t\mathbf{v}$ is not a core point for any $t > 0$. By continuity and the fact that $\mathbf{x}$ is an interior point, there is an open neighborhood of $\mathbf{x}$ so that any point in this neighborhood also is in the convex hull of the gradients defined at that point. Any such point is a core point. This contradiction proves the assertion.

So, assume both cores agree and they have a nonempty interior. We consider points on the $k-1$-dimensional boundary. This corresponds to a q-rule core point $\mathbf{x}$ and a coalition C, say $C = \{1, 2, \ldots, q\}$, where $\mathbf{x}$ is on the boundary of $Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C}$. This boundary is $k-1$ dimensional and it contains k of the gradient vectors. (If it contained more, then this would add to the codimension violating the fact that $\mathbf{x}$ is on a $k-1$ dimensional component.) To represent this as a J^2 condition. let $\Sigma_q = \{\mathbf{A} | k$ vectors from $\{\mathbf{A}_j\}_{j=1}^q$ are linearly dependent $\}$. As this has codimension one. the set of points satisfying this condition is, generically. a collection of $k-1$ dimensional manifolds. Indeed. it includes the boundary of the core. (To make it explicitly the boundary of the core. add the convexity conditions of Prop. 1.)

Because the cores agree. $\mathbf{x}$ is on the $k-1$ dimensional boundary of the $q+1$ rule core. In particular. this means that there is some agent i so that when C is augmented to C_i by adding agent i. $i > q$. then. $\mathbf{x}$ is a boundary point of $Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C_i}$. From C_i. $q+1$ different coalitions of q voters can be constructed; q of them are formed by replacing an agent from C with i. When $\nabla u_i(\mathbf{x})$ replaces one of the gradient vectors from C. then either there are $k-1$ gradient vectors on this plane. or $\nabla u_i(\mathbf{x})$ is on this plane. The first case means that a plane can be passed through $\mathbf{x}$ using this new q voter coalition where $\mathbf{x}$ is not in its convex hull. As this means $\mathbf{x}$ is not a core point. it must be that $\nabla u_i(\mathbf{x})$ is in this plane. In J^2. this is captured by the manifold $\Sigma_{q,i} = \{\mathbf{A} \in \Sigma_q | \mathbf{A}_i$ is in the span of the k linearly dependent vectors $\}$. This adds another codimension. so. generically. $\mathbf{x}$ belongs to a $k-2$ dimensional manifold. The dimension contradiction proves the theorem. Indeed, with slight extra care. we have that. in general. the q-core is in the interior of the $q+1$ core.

Removing strict convexity. If the convexity assumption is removed. the portions of Prop. 1 where $\mathbf{x}$ is on the boundary of the convex hull need not hold. These conditions are needed primarily for the $\beta \geq 0$ analysis. Second. an infinitesimal core point need not be a core point. So. in the more general setting. all infinitesimal core points that are not core points must be removed. The main change in the conclusion is that the core may be the union of several submanifolds of the indicated dimensions.

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