Northwestern University
2001 Sheridan Road 580 Leverone Hall Evanston, IL 60208-2014 USA

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R. Preston McAfee
University of Texas

Daniel Vincent
Northwestern University

www.kellogg.nwu.edu/research/math

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SEQUENTIALLY OPTIMAL AUCTIONS

by

R. Preston McAfee[1]

and

Daniel Vincent[2]

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Abstract: We examine equilibria in sequential auctions where a seller can post a reserve price but, if the auction fails to result in a sale, can commit keeping the object off the market only for an exogenously fixed period of time. We restrict attention to environments where bidders have independent private values and where the support of the bidder types lies strictly above the valuation of the seller. In the case where the seller sells by second price auction in each period, there is a unique perfect Bayesian equilibrium. A form of revenue equivalence is shown. There exists a perfect Bayesian equilibrium of repeated first price auctions with the feature that in every period, the seller's expected revenue from the continuation is the same in either auction mechanism. As the length of time the seller can commit to keeping the object off the market goes to zero, seller expected revenues converge to those of a static auction with no reserve price. As the number of bidders becomes large, the seller expected revenue approaches the revenue from an optimal static auction. We also characterize a parametrized auction game in which the simple equilibrium reserve price policy of the seller mirrors a policy commonly used by many auctioneers.

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1. Department of Economics, University of Texas at Austin, Austin, TX 78712 and Department of Economics, MIT.

2. MEDS, J.L. Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208.

1. Introduction

Regular participants in the now defunct Christies' auctions of fine wines in Chicago often experienced the sensation of deja vu. The same rare bottles of wine seemed to make an appearance in many different auctions. Similar phenomena occurred in government auctions of lumber tracts, oil tracts and distressed sales of real estate, though with somewhat less mystery -- by policy, properties that failed to sell at earlier auctions were put up for bids at the following auction. Either implicitly or by explicit policy, auctioneers were acknowledging the impossibility of resisting the temptation to try to resell an object that failed to meet a reserve price in an earlier auction.

It has long been recognized in the literature on bargaining that solution concepts of dynamic bargaining games must recognize the constraints on agents imposed by sequential rationality. Although in many environments, bargainers would like to impose take-it-or-leave-it offers, they often cannot credibly commit never to attempt to renegotiate in the event of no sale. This inability often prevents a trader from extracting much surplus from the transaction. Solutions to dynamic bargaining games, therefore, frequently impose as an additional constraint some form of sequential rationality. A well-known result from the literature on auctions is that in many environments, a seller does best in the class of Bayesian incentive compatible mechanisms by conducting a standard auction with a reserve price. Given both the formal and intuitive similarities between reserve prices and take-it-or-leave-it offers, it is curious then, that so little attention has been paid to the question of a seller's inability to commit to taking the object off the market if no sale occurs.

In this paper, we wed the literature on one-sided offer sequential bargaining with that of optimal auctions to characterize the dynamic path of reserve prices in auctions in which a seller can commit only to an exogenously fixed period of time. We show that if bidder types are independently and identically distributed such that the value of the lowest possible bidder type exceeds the seller's use value, then in a game consisting of repeated second price auctions with reserve prices, there is a unique perfect Bayesian equilibrium path of reserve prices which decline deterministically over time. We also show that there is an equilibrium in the repeated first price auctions with reserve prices which generates the same expected revenue for the seller as the sequentially optimal repeated second price auction. In both cases, as the length of time which the seller can commit to keeping the object off the market goes to zero, her revenue converges to her expected revenue from an auction with no reserve price. In contrast to the dynamic monopoly case, however, as the time between auctions shrinks to zero, the reserve price remains above the lowest possible bidder valuation. As the number of bidders becomes large, the reserve prices converge to the static optimal reserve price. In a recent study of auction mechanisms by Bulow and Klemperer (1993) it is shown that an auctioneer may opt to seek more bidders and impose no reserve price rather than attempt to impose an optimal reserve price. Our results in Section 3 provide an explanation of why a seller may just as well forgo any attempt to post reserve prices. In Section 4, we illustrate an equilibrium path of an auction game in the case of "no gap" with the characteristic that reserve prices fall in fixed proportion -- a feature of sequential reserve price policies actually followed by some auctioneers.

2.. Equilibrium in Two Sequentially Optimal Auction Games

The seller of a single good for which she has zero use-value attempts to sell it to a market of n potential buyers. Each buyer, b, values the object in monetary units, x_b which is ex ante independently and identically distributed according to the distribution function, F(●) defined over support $[1,v_H]$, $v_H < \infty$. It is assumed that F(●) has a strictly positive density f(●). The seller can commit in any given period to sell the good via a second price auction with a reserve price. However, she cannot commit to never putting the object up for sale again one period later if bids fail to meet the reserve price in the current period. A sequential auction trading game thus emerges consisting of a potentially infinite sequence of second price auctions with reserve prices. In any period $t = 0,1,2,\ldots$, if the seller obtains prices, p_t, her payoff is given by $\delta^t p_t$, if a bidder with valuation, x_b obtains the object and pays p_t in period t, his payoff is $\delta^t(x_b - p_t)$, otherwise he receives zero. Incorporating both the demand for sequential rationality and for sophisticated learning by the seller, the solution concept we focus on is perfect Bayesian equilibrium.[3]

In the following, we introduce the notation, X_1, to denote the random variable which is the highest of the n bidders' valuations and Y_1 to denote the random variable which is the highest of n-1 bidders' valuations. The corresponding distribution and density functions of Y_1 are

$$F_{Y_1}(Y_1)=F^{n-1}(Y_1),\ \ dF_{Y_1}(Y_1)=f_{Y_1}(Y_1)=(n-1)F^{n-2}(Y_1)f(Y_1).$$

Often the phrase "beliefs v_t" will be used as shorthand for the state of a game in which the seller believes that all remaining bidder valuations lie in $[1,v_t)$ in period t. The skimming behaviour this terminology implies is justified by the following lemma.

Lemma 0:*i) In any pBe, if a bidder submits a bid above the posted reserve price, R_t, his unique*

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[3] For a definition of perfect Bayesian equilibrium, see Freixas, Guesnerie and Tirole (1985).

weakly dominant strategy is to bid $\beta(v) = v$.

ii) (Successive skimming) In any pBe following any history h_t with posted reserve price, R_t, for any bidder, if it is a best response to submit a serious bid for a bidder with valuation v, then it is a strict best response for a bidder with valuation v' > v to submit a bid.

Proof: Proofs of Lemmas are provided in the Appendix.

We begin by iteratively defining a sequence of optimization problems. The idea (similar to that of Fudenberg,Levine and Tirole(1986)) is to consider games which artificially must end after at most i periods with the imposition of a reserve price of one. It is then shown that for some seller beliefs, in all equilibria, the game will end in at most i periods and yield outcomes equivalent to the solution of the optimization problem.

Fix

$$\gamma_0 \equiv \gamma_0^* \equiv 1, \;\; \Pi_0(v) \equiv \int_1^v \int_1^{X_1} nY_1 f(X_1) dF_{Y_1} dX_1, \;\; r_0 \equiv 1.$$

Define the sequences,[4]

$$\{\gamma_j\}_{j=0}^{i-1}, \;\; \{\gamma_j^*\}_{j=0}^{i-1}, \;\; \{r_j\}_{j=0}^{i-1}, \;\; \{\Pi_j\}_{j=0}^{i-1}, \;\; \{g_j\}_{j=1}^{i-1},$$

iteratively as follows:

$$r_j(x,w)F_{Y_1}(x) = (1-\delta)xF_{Y_1}(x) + \delta(r_{j-1}(w,\gamma_{j-2}^*(w))F_{Y_1}(w) + \int_w^x Y_1 dF_{Y_1}),$$

[4] The optimization problem is stated in terms of choosing bidder types who submit serious bids in a given period rather than choosing reserve prices. Since it will be shown that for each reserve price there is a unique partition of bidder types who submit serious bids, this behaviour will correspond to equilibrium behaviour.

$$g_j(v_t,x,w) = r_j(x,w) n F_{Y_1}(x)[F(v_t) - F(x)]$$
$$+ \int_x^{v_t} \int_x^{X_1} n Y_1 f(X_1) dF_{Y_1} dX_1 + \delta \Pi_{j-1}(x),$$

$$\Pi_j(v_t) = \max_{x \le v_t, w \in \gamma_{j-1}(x)} g_j(v_t,x,w),$$

$$\gamma_j(v_t) = \ argmax_{x \le v_t} \{ g_j(v_t,x,w) \mid for\ some\ w \in \gamma_{j-1}(x) \},$$

$$\gamma_j^*(v_t) = sup\{\gamma_j(v_t)\}.$$

The sequence of functions, $r_j(v, \gamma^*(v))$ denote the lowest reserve price which would induce a bidder of type v to submit a serious bid. Observe that, assuming this function is increasing, the lowest bidder type to win at the current reserve price trades off winning at the reserve price this period against the probability weighted sum of the next period reserve price and the second highest bid. The functions $g_j(v,x,w)$ characterize the return to a seller when the lowest bidder types who submit bids this period and the next are x and w respectively. The functions Π_j are the maximized values of this function subject to what will be the sequential rationality constraint on subsequent choices of bidder cut-offs. The sequence of correspondences $\gamma_j(v)$ are derived from the same seller optimization problem and determine the seller's sequentially optimal cutoff bidder type when her beliefs are that the remaining types lie in the interval [l,v]. The upper semicontinuous function $\gamma^*_j(v)$ is constructed by choosing the maximum from $\gamma_j(v)$ for every v.

For any i, assume that this sequence is defined up to i-1 and make the following induction hypotheses for all $j < i$:

(H1) Π_j increasing and continuous.

(H2) $\gamma_j(x) < x$ and γ_j is compact-valued, increasing and upper hemi-continuous so γ^*_j is increasing and upper semi-continuous.

(H3) $r_j(x,w)$ is strictly increasing in both of its arguments, continuous in x and upper semi-continuous in w and satisfies $r_j(x,v) < x$ for $v < x$ and, where defined,

$$\frac{d[r_j(x,\gamma^*_{j-1}(x))F_{Y_1}(x)]}{dx} \geq x f_{Y_1}(x). \tag{1}$$

Observe that **(H1-H3)** hold for $i-1 = 1$.

Lemma One: *If (H1-H3) hold for i-1, (H1-H3) also hold for i.*

Define

$$z_1 = sup\{v_t | \gamma^*_1(v_t)=1\}$$
$$z_i = \min\{sup\{v_t|\gamma^*_i(v_t) \leq z_{i-1}\}, v_H\}$$

Lemma Two: *$z_1 > 1$ and there exists an $N < \infty$, such that $z_N = v_H$.*

Observe that for any $v_t < z_i$, $\gamma_i(v_t) = \gamma_{i-1}(v_t)$, $\Pi_i(v_t) = \Pi_{i-1}(v_t)$ and $r_i(x,\gamma^*_{i-1}(x)) = r_{i-1}(x,\gamma^*_{i-2}(x))$ for $x \leq v_t$. Thus, by Lemma Two, we can define some γ, γ^*, Π and r independent of i over $[1,v_H]$. Fix a v_H and define v_t so that $v_1 \in \gamma(v_H)$, $v_t = \gamma^*(v_{t-1})$ and R_t so that $R_t = r(v_{t+1},\gamma^*(v_{t+1}))$. Observe that since $\gamma^*(\bullet)$ is increasing, such a sequence is generically unique in v_H.

Theorem One: *In the sequential second price auction game, in any perfect Bayesian equilibrium, in any period $t > 1$, if the belief is v_t, the seller's best response reserve price is R_t*

$= r(v_{t+1}, \gamma^*(v_{t+1}))$ for $v_{t+1} \in \gamma(v_t)$. All bidders with type $x \geq \gamma^*(v_t)$ submit bids equal to their own value. All other bidders do not submit serious bids. In period $t = 1$, any reserve price $R_1 = r(v_2, \gamma^*(v_2))$ for $v_2 \in \gamma(v_H)$ is an equilibrium reserve price offer. Along the equilibrium path, for $t > 2$, the unique equilibrium reserve price is $R_t = r(\gamma^*(v_t), \gamma^*(\gamma^*(v_t)))$.

Proof of Theorem One: The proof proceeds by defining necessary conditions of bidder and seller strategies iteratively over the support of bidder types via two lemmas. Let $R^\sigma(v_t, h_t)$ the seller's best response reserve price in some pBe, σ, following a history, h_t and with beliefs that bidder types lie in $[1, v_t]$ and let $P^\sigma(v_t, h_t)$ be her expected payoff. Condition C1(a,j) partly characterizes the strategy of the seller.

$$C1(a,j)\!: \forall\sigma, \forall v_t < a, \forall h_t,\;\; R^\sigma(v_t,h_t) = r_j(x, \gamma_j^*(x)),\;\; for\; x \in \gamma_j(v_t).$$

Condition C2(j) characterizes strategies of bidder types below z_j and partially for types above z_j.

$$\begin{gathered} C2(j)\!: \forall\sigma, \forall h_t, \forall v_t < z_j,\;\; if\; R_t > r_j(v_t, \gamma_j^*(v_t)),\;\; No\ Bid \\ \forall v_t,\;\; if\; R_t \leq r_j(v_t, \min\gamma_j(v_t)),\;\; Bid\; B(v_t) = v_t. \end{gathered}$$

Lemma Three[5]: *If C1 and C2 hold for $j = i-1$ and $a = z_{i-1}$, then C1 holds for $j = i$ and $a = z_i$.*

[5] Observe that the optimization problems as stated are "as if" the seller can also choose in period t her most favourable cutoff level (among her optimal responses) in period t+1 if the object fails to sell. This is not true in general since $\gamma(v_{t+1})$ may not be single valued. Lemma Three illustrates that since if there were a possibility that the future belief is unfavourable (that is, v too low) then the upper hemicontinuity and monotonicity of the optimal choice function would have allowed the seller to do better by selecting a slightly higher cutoff level this period. This would yield only first order costs in the probability of a sale but increase the reserve price by an amount bounded above zero. Therefore, for $t > 2$, the equilibrium reserve price is $R_t(v_t) = r(\gamma^*(v_t), \gamma^*(\gamma^*(v_t)))$.

Lemma Four: *If C2 holds for $j = i\text{-}1$ and C1 holds for $j = i$ and $a = z_i$, C2 holds for $j = i$.*

A simple adaptation of Lemmas Three and Four show that **(C1)** and **(C2)** hold for $i = 1$. Therefore, we can now apply Lemmas Three and Four iteratively to specify necessary conditions of equilibrium behaviour over the whole interval, $[1,v_H]$. Sufficiency is not quite shown. The result would follow simply if $\gamma(v_t)$ was known to be single-valued. In general it is not, but the argument given in Gul, Sonnenschein, Wilson (1986) is easily adapted to show sufficiency as well. Suppose the seller posts an out of equilibrium reserve price R', $r(v,\min\gamma(v)) < R' < r(v,\gamma^*(v))$, for some v. Subsequent randomization of the reserve price off the equilibrium path is a characteristic of the equilibrium in order to convexify the correspondence, $\gamma(\bullet)$. All bidder types $v' > v$ submit bids and all bidder types $v'' < v$ do not submit bids. In the next period, the seller randomizes between her two best response choices of v_{t+1}, $\gamma^*(v)$ and $\min \gamma(v)$ by offering either $r(\min \gamma(v),\gamma^*(\min \gamma(v)))$ or $r(\gamma^*(v),\gamma^*(\gamma^*(v)))$ so as to make bidder types v and higher willing to submit bids in the current period. ■

Corollary One: *For any seller belief, v_t, let $\{v_{t+i}\}$, $i = 1,2,\ldots,N$, be the subsequent seller beliefs along the (unique) equilibrium continuation. The seller's expected equilibrium revenue can be expressed as*

$$n\sum_{i=0}^{N} \delta^i \int_{v_{t+i+1}}^{v_{t+i}} \left(v - \frac{F(v_t)-F(v)}{f(v)}\right) F^{n-1}(v) f(v)\, dv \tag{2}$$

Proof of Corollary One: An adaptation of an argument in Myerson and Satterthwaite (1983) shows that, if U(v) is the expected utility of a buyer in any Bayesian equilibrium, then dU(v)/dv = $\delta^{t(v)}F^{n-1}(v)$ almost everywhere, where t(v) is the equilibrium period of trade of a bidder of type

v. By Theorem One, this period is deterministic (up to a selection of v_{t+1}). Integrating by parts yields

$$n\int_1^{v_t} U(v)f(v)dv = \sum_{i=0}^{N} \delta^i \int_{v_{t+1+i}}^{v_{t+i}} F^{n-1}(v)(1-F(v))dv$$

Using the definition of total expected surplus as the sum of seller's expected revenue and total expected buyer surplus and rearranging terms yields (2). ■

Now consider a sequential auction game in which the seller conducts first price auctions in every period. It is a well-known result that in static independent private values auctions with the same reserve price, both auctions yield the same expected seller revenue. Theorem Two shows that a version of this result extends to the sequentially optimal auctions and also illustrates that along the equilibrium, the reserve prices are the same in both mechanisms.

Theorem Two *(Revenue Equivalence): There exists a perfect Bayesian equilibrium of the sequential first price auction such that along the equilibrium path, for every seller belief [1,v_t], the equilibrium reserve price and the seller's expected revenue along the equilibrium is the same as the Sequentially Optimal Second Price Auction.*

Proof of Theorem Two: The proof proceeds by characterizing strategies and showing that they comprise a pBe of the sequential first price auction. Let $r(v,\gamma^*(v))$ and $\gamma(v)$ be as in the proof of Theorem One.

Seller Strategies: For every seller belief, $[1,v_t)$, if $R_{t-1} = r(v_t,\gamma^*(v))$, post a reserve price $R_t = r(\gamma^*(v_t),\gamma^*(\gamma^*(v_t)))$. If $r_* = r(v_t,\min\gamma(v_t)) \leq R_{t-1} < r(v_t,\gamma^*(v_t)) = r^*$ post reserve price $R_t = r_*$ with probability β and $R_t = r^*$ with probability $1 - \beta$ where β satisfies

$$R_{t-1} = \beta * r(\min\gamma(v_t), \gamma^*(\min\gamma(v_t))) + (1-\beta) r(\gamma^*(v_t), \gamma^*(\gamma^*(v_t)))$$

Seller Beliefs: For any beliefs $[1, v_{t-1})$ in period t-1, if $r(v_t, \min\gamma(v)) \leq R_{t-1} \leq r(v_t, \gamma^*(v))$, no bid is submitted, beliefs in period t are $[1, v_t]$. If $1 > R_{t-1}$, and no bids are submitted, beliefs in period t are the same as in period t-1. If $R_t > r(v_{t-1}, \gamma^*(v_{t-1}))$, beliefs are $[1, v_{t-1})$.

Buyer Strategies: In every period, if $v_{t+1} = \rho(R_t)$ where ρ is the inverse of $r(v, \gamma^*(v))$, all $v < v_{t+1}$ do not submit bids. All $v \geq v_{t+1}$ submit bids, $B(v; R_t)$, where

$$B(v;R_t) F_{Y_1}(v) = \int_1^v x dF_{Y_1} + c, \qquad B(v_{t+1}; R_t) = R_t \tag{3}$$

First suppose that this profile of strategies comprise a pBe. In this case, then for any seller belief, $[1, v_t]$, the sequence of bidder types who bid in all subsequent periods is the same as in the second price equilibrium, that is, trade occurs with the same bidder type in the same period as the second price auction. A simple adaptation of the Myerson-Satterthwaite reasoning then implies that the expected seller revenues are the same. The definition of the seller strategies implies that the equilibrium reserve prices are the same. Thus as long as we can show that the strategies form a pBe, the theorem is proved.

Note that the seller beliefs satisfy Bayes' rule given the buyers' strategies.

To show the optimality of buyer strategies, let the seller beliefs be $[1, v_t)$ and the reserve price be R_t and $v_{t+1} = \rho(R_t)$. Let $r_* = r(\min\gamma(v_{t+1}), \gamma^*(\min\gamma(v_{t+1})))$ and $r(\gamma^*(v_{t+1}), \gamma^*(\gamma^*(v_{t+1}))) = r^*$. Finally suppose that buyers follow the proposed strategies in all later periods. Then, by definition of the bidding functions next period, if the reserve price is r^*, bidder type v_{t+1} bids where the constant term is determined by (3). A similar equation holds for a next period reserve

$$B(v_{t+1})F_{Y_1}(v_{t+1}) = \int_1^{v_{t+1}} x dF_{Y_1} + c$$
$$= \int_{\gamma^*(v_{t+1})}^{v_{t+1}} x dF_{Y_1} + F_{Y_1}(\gamma^*(v_{t+1}))r^*$$

price of r_*. Therefore, given the strategies of the other bidders, a bidder of type v_{t+1} who bids when the current reserve is R_t will only bid R_t and receives expected utility

$$(v_{t+1} - R_t)F_{Y_1}(v_{t+1}) \tag{4}$$

If v_{t+1} waits until next period, his expected utility is

$$\begin{aligned}
&\delta(v_{t+1} - \beta B(v_{t+1};r_*) - (1-\beta)B(v_{t+1};r^*))F_{Y_1}(v_{t+1}) \\
&= \delta(v_{t+1} - \beta(\int_{\min\gamma(v_{t+1})}^{v_{t-1}} x dF_{Y_1} + r_* F_{Y_1}(\min\gamma(v_{t+1}))) \\
&\quad - (1-\beta)(\int_{\gamma^*(v_{t+1})}^{v_t+1} x dF_{Y_1} + r^* F_{Y_1}(\gamma^*(v_{t+1}))))
\end{aligned} \tag{5}$$

By definition of v_{t+1}, and β, (4) equals (5) so bidder type v_{t+1} is just indifferent between bidding this period and next. Since this period utility increases faster in bidder type than next period utility, that means that all bidder types above v_{t+1} strictly prefer bidding and those below, strictly prefer not to bid. Finally, standard arguments from first price auctions illustrate that the bid function (3) is a best response for bidders who bid in period t.

Finally, to show the sequential optimality of the seller's strategy, suppose that the seller's strategy is sequentially rational for all beliefs $[1, v]$ with $v \leq z_i$. Then, for any beliefs satisfying this restriction, a further application of Myerson-Satterthwaite illustrates that the expected payoff for the seller from the pBe is the same as $\Pi(v)$. An argument similar to that of Lemma Three for Theorem One, then shows that there is an $\epsilon > 0$, such that for all $v \leq z_i + \epsilon$, a reserve price such that $v_{t+1} \leq z_i$ is optimal and therefore, given proposed bidder behavior, expected equilibrium payoffs are again given by $\Pi(v)$. The same argument then is applied to $v \leq z_1 + 2\epsilon$ and so on. That the proposed seller behavior is optimal for $v \leq z_1$ is straightforward to show. ■

3. Comparative Statics

When a single seller faces a single buyer and has the strategic power to make take-it-or-leave-it offers in every period, Gul\Sonnenschein\Wilson (1986) prove, formally, a conjecture of Coase that as the time costs of waiting until the next period go to zero, the expected profits of the seller converge to the profits she would enjoy against only the buyer with the lowest valuation. That environment, of course, is a special case of the environment analyzed here. And a generalized version of the Coase Conjecture also holds. Theorem Three shows that as the time costs go to zero, the expected seller revenues converge to the expected revenues from an auction with a reserve price set at the lowest valuation. In the case of more than one bidder, this corresponds to the revenues earned in a no-reserve price auction.

Theorem Three: *(Coase Conjecture)*

$$\lim_{\delta \to 1} \Pi(v_H) = \int_1^{v_H} \int_1^{X_1} Y_1 n f(X_1) dF_{Y_1} dX_1$$

That is, as δ approaches one, the expected revenues of the seller is the same as in a game with no reserve price.

Proof of Theorem Three: (Part of this proof follows Tirole (1988)) We show first that for any $\epsilon > 0$ there is an N such that all equilibria of games with $1 > \delta > \epsilon$ end with probability one after N periods, independent of δ. For any δ, Theorem One shows there exists a unique equilibrium in which the decision of bidders whether to participate given a current period reserve price is time independent and the seller's profits depends only on the current state, v_t. For any δ and pBe, let v_t, v_{t+1}, v_{t+2} be the equilibrium cutoff levels of participating bidders in periods t,t+1 and t+2 respectively with $v_{t+2} \geq z_1$ and define $F_{t+i} \equiv F(v_{t+i})$. Note that given the current state, v_t, since bidder strategies are stationary, a seller could always have chosen a reserve price to induce a next period state v_{t+2} instead of v_{t+1} so we must have

$$\begin{gathered} R_t F_{t+1}^{n-1} n[F_t - F_{t+1}] + \int_{v_{t+1}}^{v_t}\int_{v_{t+1}}^{X_1} nY_1 f(X_1) dF_{Y_1} dX_1 + \\ \delta R_{t+1} F_{t+2}^{n-1} n[F_{t+1} - F_{t+2}] + \delta \int_{v_{t+2}}^{v_{t+1}}\int_{v_{t+2}}^{X_1} nY_1 f(X_1) dF_{Y_1} dX_1 + \delta^2 \Pi(v_{t+2}) \\ \geq R_{t+1} F_{t+2}^{n-1} n[F_t - F_{t+2}] + \int_{v_{t+2}}^{v_{t+1}}\int_{v_{t+2}}^{X_1} nY_1 f(X_1) dF_{Y_1} dX_1 + \\ \int_{v_{t+1}}^{v_t}\int_{v_{t+2}}^{v_{t+1}} nY_1 f(X_1) dF_{Y_1} dX_1 + \int_{v_{t+1}}^{v_t}\int_{v_{t+1}}^{X_1} nY_1 f(X_1) dF_{Y_1} dX_1 + \delta \Pi(v_{t+2}) \end{gathered}$$

where the second integral has been broken to facilitate rearranging terms. This can be written as

$$\begin{gathered} [R_t F_{t+1}^{n-1} - R_{t+1} F_{t+2}^{n-1}] n[F_t - F_{t+1}] \\ \geq (1-\delta) R_{t+1} F_{t+2}^{n-1} n[F_{t+1} - F_{t+2}] + (1-\delta) \int_{v_{t+2}}^{v_{t+1}}\int_{v_{t+2}}^{X_1} nY_1 f(X_1) dF_{Y_1} dX_1 + \\ \int_{v_{t+1}}^{v_t}\int_{v_{t+2}}^{v_{t+1}} nY_1 f(X_1) dF_{Y_1} dX_1 + \delta(1-\delta)\Pi(v_{t+2}) \end{gathered}$$

By definition of R and v_{t+i},

$$R_t F_{t+1}^{n-1} - R_{t+1} F_{t+2}^{n-1} = (1-\delta)(v_{t+1} F_{t+1}^{n-1} - R_{t+1} F_{t+2}^{n-1}) + \delta \int_{v_{t+2}}^{v_{t+1}} Y_1 dF_{Y_1}$$

so substituting into the above inequality, rearranging terms (by subtracting the last term in the above equation and dropping the last integral in the right side of the inequality) and dividing by $1-\delta > 0$ yields

$$[v_{t+1}F_{t+1}^{n-1}n[F_t-F_{t+1}]-R_{t+1}F_{t+2}^{n-1}n[F_{t+1}-F_{t+2}]-\int_{v_{t-1}}^{v_t}\int_{v_{t-2}}^{v_{t-1}}nY_1f(X_1)dF_{Y_1}dX_1\geq\delta\Pi(v_{t+2})$$

Rearranging once more gives

$$n[F_t-F_{t+1}][v_{t+1}(F_{t+1}^{n-1}-F_{t+2}^{n-1})-\int_{v_{t+2}}^{v_{t+1}}nY_1dF_{Y_1}]$$
$$+[v_{t+1}-R_{t+1}]F_{t+2}^{n-1}n[F_t-F_{t+2}]\geq\delta\Pi(v_{t+2})$$

The first term is positive so the inequality can be written

$$n[F_t-F_{t+2}][v_{t+1}(F_{t+1}^{n-1}-F_{t+2}^{n-1})-\int_{v_{t+2}}^{v_{t+1}}nY_1dF_{Y_1}+[v_{t+1}-R_{t+1}]F_{t+2}^{n-1}]\geq\delta\Pi(v_{t+2})$$

Since $v_H - 1 \geq v_{t+1} - p_{t+1}$ for any price paid by a bidder in any equilibrium, this implies

$$n[F_t-F_{t+2}][v_H-1]F_{t+2}^{n-1}\geq\delta\Pi(v_{t+2})\geq F_{t+2}^{n}$$

The last inequality comes from the fact that the seller can always post a reserve price of one in any period. By Lemma 2, $F(z_1) > 0$ for any $v_{t+2} > z_1$, (for $v_t < z_3$ the game ends in two periods) so

$$F_t\geq(\frac{\delta}{n(v_H-1)}+1)F_{t+2}$$

or

$$F_{N-M}\geq(\frac{\delta}{n(v_H-1)}+1)^{M/2}F(z_1)$$

for M even. Since $F_0 = 1$, this gives for N such that

the game must end before N periods. This upper bound is decreasing in δ. Combined with Corollary One, this implies that as δ approaches one, the seller's expected revenue approaches

$$N \approx 2 * \frac{\log(\frac{1}{F(z_1)})}{\log(\frac{\delta}{n(v_H - 1)} + 1)} \tag{6}$$

the expected revenue in a game with no reserve price. ■

The next result uses Theorem Three to provide a bound on seller revenues when she cannot commit to keeping the object off the market in the event the reserve price is not met,

Corollary Two: *Let $\hat{P}$ denote the seller's expected revenue in a static optimal auction. Let P_δ denote the expected revenue in the sequential second price auction, and let P denote the expected revenue in an auction with no reserve price. For any δ,*

$$\frac{P_\delta - P}{P} \leq \frac{\hat{P} - P}{P} * (1 - \delta^N)$$

where N is defined in (6).

Proof of Corollary Two: By Myerson (1981), P and $\hat{P}$ satisfy

$$P = n \sum_{i=0}^{N} \int_{v_{t+1}}^{v_{t+i}} (v - \frac{F(v_t) - F(v)}{f(v)}) F^{n-1}(v) f(v) dv \tag{8}$$

and

$$\hat{P} = n \int_{\hat{v}}^{v_t} (v - \frac{F(v_t) - F(v)}{f(v)}) F^{n-1}(v) f(v) dv \tag{9}$$

where $\hat{v}$ is the optimal reserve price for the static auction. By Corollary One and by definition of $\hat{v}$,

$$P_\delta \leq \int_{\hat{v}}^{v_t}(v-\frac{F(v_t)-F(v)}{f(v)})nF^{n-1}(v)f(v)dv+\delta^N\int_1^{\hat{v}}(v-\frac{F(v_t)-F(v)}{f(v)})nF^{n-1}(v)f(v)dv.$$

where N is defined in (6). Rearranging terms yields the result. ■

Corollary Two illustrates that information about the upper and lower end of the support of bidder types, enough information about the distribution at the lower end to give an estimate of z_1 and δ and n are enough to put some bounds on the maximum value of imposing a reserve price when the seller cannot commit.

The next Theorem illustrates that as the number of bidders becomes large, seller revenue approaches that achievable in an auction in which the seller can commit to a static auction with a reserve price. More significantly, for the differentiable solution, it shows that the equilibrium reserve price approaches the optimal reserve price in a static auction.

Theorem Four: *If for all n, there is a number M such that $\partial\gamma^*(v)/\partial v \leq M$, then as n becomes large, the sequentially optimal reserve prices in each period approach the static optimal reserve price.*

Proof of Theorem Four: Fix δ and current seller beliefs, v_t. From Corollary One, the expected revenue of a seller with beliefs v_t, can be expressed solely as a function of her beliefs in subsequent periods, $\{v_{t+i}\}$. Let v_{t+1} be the seller's next period beliefs and $\{v_{t+i}\}$, i = 2,3,...N, be the subsequent beliefs determined by the unique equilibrium continuation.

$$g(v_t,v_{t+1}) = n\sum_{i=0}^{N-1} \delta^i \int_{v_{t+i+1}}^{v_{t+i}} (v - \frac{F(v_t)-F(v)}{f(v)}) F^{n-1}(v) f(v) dv$$

$$= \int_{v_{t+1}}^{v_t} (v - \frac{F(v_t)-F(v)}{f(v)}) dF_{X_1}(v) + \sum_{i=0}^{N-2} \delta^i \int_{v_{t-2+i}}^{v_{t-1+i}} (v - \frac{F(v_{t+1})-F(v)}{f(v)} - \frac{F(v_t)-F(v_{t+1})}{f(v)}) dF_{X_1}(v)$$

Differentiating with respect to v_{t+1},

$$\frac{\partial g(v_t,v_{t+1})}{\partial v_{t+1}} = (1-\delta)(F(v_t)-F(v_{t+1})-v_{t+1}f(v_{t+1})) +$$

$$(1-\delta)\delta\sum_{i=0}^{N-2} \delta^i (F(v_t)-F(v_{t+1})) \frac{F^{n-1}(v_{t+2+i})}{F^{n-1}(v_{t+1})} \Pi_{j=0}^{i} \frac{\partial v_{t+2+j}}{\partial v_{t+1+j}}$$

$$+\delta \frac{\partial v_{t+2}}{\partial v_{t+1}} \frac{\partial g(v_{t+1},v_{t+2})}{\partial v_{t+2}} \frac{1}{nF^{n-1}(v_{t+1})}$$

The last term is zero for all n since v_{t+2} must be chosen optimally when the seller beliefs are v_{t+1} and the second term goes to zero as n becomes large if the condition on the derivative of γ is satisfied. The first term is the same as the necessary condition for the static optimization problem. ■

Remark: Theorem Four illustrates that at least for large n, in every period, reserve prices rise in n and as n becomes large, r(v) approaches v. Notice also, that unlike the static auction result, typically reserve prices in sequentially optimal auctions will vary with changes in the number of bidders.[6]

Theorem Four implies a monotonicity of the reserve prices in the limits as n becomes large. One might expect that as δ approaches one, the equilibrium reserve price falls, however,

[6] This feature is also present in McAfee's (1993) analysis of a dynamic model where sellers compete for bidders by offering auctions with reserve prices.

analytic comparative statics in δ do not appear to be available. Theorem Five yields some information on the behavior of the reserve prices and cutoff bidder types for informationally "small" games (which end within two periods).

Theorem Five: *Let $v_H \in [1,z_2)$. Then in the unique pBe of the sequential auction game,*

i) The first period reserve price R_1 falls as δ increases and rises as n increases.

ii) The second period reserve price R_2 is independent of δ and n.

iii) There is a number v satisfying $0 = F(v_H) - F(v) - vf(v)$ such that the probability that trade occurs in the first period is given by $1- F^n(v)$. In particular the probability trade occurs in the first period is independent of δ and depends on n only as $1- F^n(v)$ depends on n.

Proof of Theorem Five: By Theorem One, trade occurs with probability one by the second period, for $v_H < z_1$ the initial reserve price is one and trade occurs immediately. For $v_H \in [z_1,z_2)$, the optimal cutoff level of the seller is a $v_2 < w_1$ and bidders in this range submit serious bids only if R_1 is less than $r_1(v,1)$ given by

$$r_1(v,1)F_{Y_1}(v)=(1-\delta)vF_{Y_1}(v)+\delta\int_1^v Y_1 dF_{Y_1}$$

The expected utility for a seller who chooses cutoff level x is

$$\begin{aligned}g_1(v_t,x,1)&=r_1(x,1)nF_{Y_1}(x)[F(v_t)-F(x)]\\&+\int_x^{v_t}\int_x^{X_1}nY_1f(X_1)dF_{Y_1}dX_1+\delta\int_1^x\int_1^{X_1}nY_1f(X_1)dF_{Y_1}dX_1\end{aligned}$$

A necessary condition for x to be chosen optimally is that the derivative of this expression be zero or that $F(v_H) - F(x) - xf(x) = 0$ independent of δ and n. ■

Remark: Recall that in optimal static auctions with independent private values, the optimal reserve price is independent of the number of bidders. Theorem Five illustrates that this result does not extend to auctions in which the seller cannot commit to keeping the good off the market. There is a good intuition for this difference. With the possibility of future auctions, along any equilibrium path, the opportunity cost to a bidder of failing to trade in a given auction is determined by the continuation value from subsequent auctions. That is, in any period, a bidder's net value of trading is an induced value determined in part by the continuation path of the equilibrium. A bidder's expected utility from an auction is determined in part by the degree of competition. Thus rises in n increase the opportunity of a failure to trade. In the second to last period, this is the only effect at work since in the last period, the seller's reserve price is, by assumption, independent of n. In longer games, though, there is the additional effect that the seller alters her reserve price as well in response to changes in the profile of induced bidder valuations brought on by changes in n.

The next result illustrates that even though the seller's revenues approach those of an auction with no commitment, the reserve price remains bounded above the no-commitment reserve price.

Corollary Three: *For $v_t \geq z_l$ as δ approaches one, the equilibrium reserve price is bounded above one.*

Proof of Corollary Three: Let $r^\delta(v)$ denote the function determining the maximum reserve price for which a bidder with valuation v will submit a serious bid. Then

$$\lim_{\delta \to 1} r^\delta(v) = E[Y_1 | Y_1 \leq v]$$

To see this, note that for any equilibrium corresponding to δ, let $\{v_{t+1+i}\}$ denote the expected sequence of cutoff levels along the equilibrium path when a reserve price $R_t = r^\delta(v_{t+1})$ is posted. Using the proof of Theorem One, r^δ satisfies

$$r^{\delta}(v_{t+1})F_{Y_1}(v_{t+1}) = \sum_{i=0}^{N(\delta)} \delta^i((1-\delta)v_{t+1+i}F_{Y_1}(v_{t+1+i}) + \delta\int_{v_{t+2+i}}^{v_{t+1+i}} Y_1 dF_{Y_1})$$

By Theorem Three, since the number of terms in the summation term is bounded by N, this limit is computed by replacing N(δ) with N and letting δ go to one. Theorem Five (iii) illustrates that for $v_t \geq z_1$, the equilibrium cutoff type of the next serious bidder is bounded above 1 independent of δ. By the above argument, the minimal reserve price needed to induce his participation is also bounded above 1. ■

The reader acquainted with literature on mechanism design might wonder why an assumption on distributions commonly used in the analysis of reserve price auction, the so-called monotone likelihood ratio condition, is not needed here. There are two reasons. First, by construction, we restrict attention to the smaller class of mechanisms which is the class of reserve price auctions. Thus, in a full sequentially optimal mechanism game, where the strategy choice of the seller may range across the whole class of implementable mechanisms, the equilibrium path is likely to be different in the absence of this assumption. However, in a slightly different environment where many sellers compete in mechanisms, McAfee (1993) shows that, in fact, seller choices of reserve price auctions are a necessary feature of equilibria. Second, the assumption of a monotone likelihood ratio condition is often used to ensure the concavity of the seller's static optimization problem and thus the uniqueness of a solution. As is evident in the proof, we do not require the seller's best response correspondence to be singleton-valued in order to obtain uniqueness of the equilibrium path. In periods where the seller's best response correspondence may be multiple-valued, self-interest on the part of the seller ensures that actions are taken in early periods to ensure that the highest element of this set is selected. (See Lemma Three and the discussion in Footnote 5.)

4. A Linear Example

The proofs of the existence and uniqueness of equilibria in Section 2 also show how the equilibria can be constructed by iterating from informationally "small" games (games with the support limited to the bidders with low types) to larger games. There are no general closed form solutions to these games and even with assumption of uniform densities, the characteristics of the optimal solution become complicated quickly. If the assumption that a gap exists between the seller and lowest bidder type is dropped, however, it is possible to construct an example of an equilibrium of the sequentially optimal auction game with very simple properties.

Suppose there are n bidders each with valuations drawn independently and identically from the Uniform [0,1] distribution. This case does not fit the class examined in the previous section since the bottom of the support of the bidders is not positive. Thus, as in Ausubel and Deneckere (1989), we can not be sure of the uniqueness of stationary equilibria. However, we can examine the nature of the stationary equilibria.

Guess that seller best response cutoff function in any period with beliefs v is given by $\gamma^*(v) = \gamma v$ and that the reserve price function determining the maximal reserve price for which a bidder of type x submits a serious bid is $r(x,\gamma(x)) = kx$. By definition of r:

$$r(x,\gamma(x))F_{Y_1}(x)=kx^n=(1-\delta)x^n+\delta(r(\gamma(x),\gamma(\gamma(x)))F_{Y_1}(\gamma(x))+\int_{\gamma(x)}^{x}Y_1dF_{Y_1})$$
$$=(1-\delta)x^n+\delta(k(\gamma x)^n+\frac{n-1}{n}(1-\gamma^n)x^n)$$

which implies that

$$k = 1 - \frac{\delta}{n}\frac{1-\gamma^n}{1-\delta\gamma^n} \tag{7}$$

Now note that

$$g_i(v_t,x,v) = r_i(x,v)nF_{Y_1}(x)[F(v_t)-F(x)] + \int_x^{v_t}\int_x^{X_1} nY_1 f(X_1)dF_{Y_1}dX_1 + \delta\Pi_{i-1}(x)$$

and that

$$\frac{\partial\Pi(x)}{\partial x} = nr(\gamma(x),\gamma(\gamma(x)))F_{Y_1}(\gamma(x)) + n\int_{\gamma x}^{x}(n-1)Y_1^{n-1}dY_1$$

An optimal choice of cutoff level x given beliefs v must satisfy

$$\frac{\partial g(v,x,v)}{\partial x} = 0 = -n(r(x,\gamma(x))F_{Y_1}(x) + n(v_t-x)nkx^{n-1}$$
$$-n(n-1)(v_t-x)x^{n-1} + \delta(nr(\gamma(x),\gamma(\gamma(x)))F_{Y_1}(\gamma(x)) + n\int_{\gamma(x)}^{x}(n-1)Y_1^{n-1}dY_1)$$

Using (7), this simplifies to

$$0 = -(1-\delta)x + (v_t-x)nk - (n-1)(v_t-x)$$

Since we are assuming that $\gamma(v_t) = \gamma\ v_t = x$, then this implies γ and k must satisfy

$$k = 1 + ((1-\delta)\frac{\gamma}{1-\gamma} - 1)/n \tag{8}$$

Equations (8) and (7) together define the linear solution to the stationary equilibrium. They combine to yield

$$2\gamma - 1 = \delta\gamma^{n+1}$$

Comments: These equations imply

i) As δ rises, γ increases. Simulations indicate that k also increases in γ. The limit of these equations as δ approaches zero approaches the static solution

$$\lim_{\delta \to 0} \gamma = \frac{1}{2} \textit{ and } \lim_{\delta \to 0} k = 1$$

ii)As n rises, γ falls and the limit as n approaches infinity also approaches the static solution

$$\lim_{n \to \infty} \gamma = \frac{1}{2} \textit{ and } \lim_{n \to \infty} k = 1$$

iii) As δ approaches one, γ is the solution of $\gamma(2-\gamma^{n+1}) = 1$ (For n = 1, the unique solution is $\gamma = 1$, for n> 1, the correct solution is less than one.) and k approaches (n-1)/n.

iv) Simulation of the equations indicates that γ falls with n, and k increases with n and the reserve price, $k\gamma$ increases with n.

The US Forest Service uses a reserve price policy of a form that very closely matches that illustrated in the above example. If the tract fails to sell at a current reserve price, the property is re-auctioned at a reserve price that is ten percent below the previous reserve[7]. That is, the Forest Service has adopted a policy that involves a linearly decreasing reserve price. However, at a real interest rate of anywhere from three percent to ten percent, and assuming that the US Forestry Service reauctions tracts every six months, such a policy would be optimal only if the number of bidders is essentially one. While this is evidently counterfactual, the policy could be interpreted as a concern about collusive behaviour by bidders, a possibility ruled out exogenously in this analysis.

The closed form equilibrium strategies allows a more precise determination of the value

[7] We are grateful to Robert Marshall for drawing our attention to this fact.

of posting reserve prices with limited commitment. Assuming an annual interest rate of 5%, if the auctioneer can commit to keeping the object off the market for as long as a year each time it fails to sell, his gain is at most 10% of the increment earned in the case of full commitment The 10% gain is computed with 2 bidders, and falls to 4% in the case of 5 bidders. If the auctions are spaced only six months apart, the gain falls from 5% to around 3% of the extra revenues earned in the auction with full commitment. These results reinforce the conclusions of Bulow and Klemperer (1994) that the very small benefits from imposing reserve prices may often be swamped by other considerations.

5. Conclusion

The results of our analysis confirm natural conjectures about the ability of sellers to impose reserve prices. As in the case of sequential bargaining, the ability to impose a credible reserve price hinges on the seller's ability to commit to either destroy the product in the event of no sale or keep it for herself. Excess rents are derived from this commitment power. The paper also suggests testable implications of the theory of sequentially optimal auctions. Suppose data which tracks objects for sale at a sequence of auctions and records the number of bidders and\or the length of time between auctions were available, Theorem Four and the example in Section 4 provide predictions about the response of reserve prices to changes in interest rates, auction frequency and the number of bidders. A note of caution must be voiced though. The practice of many auctioneers may frustrate the attempt to gather such data. Ashenfelter (1989) remarks on the tendency of auctioneers to keep reserve prices secret. However, some auctioneers do post explicit reserve prices sometimes as a matter of policy and in other cases, effective reserve prices may be derivable from other data such as suggested minimum bids. Unless bidders are required to apply for eligibility before bidding (as happens in many government auctions), it may also be extremely difficult to extract exact information on the number of bidders. Thus, the positive applications of the theory of sequentially optimal auctions, are limited as are many results from the theory of auctions, by the availability of the appropriate data. Nevertheless, as the analysis of Section 4 illustrates, there remain normative applications of the theory that may be useful either in providing guidance to policymakers or deriving information about other, hidden, aspects of the auction environment.

Appendix

Proof of Lemma 0: i) Fix a reserve price R_t and any bidder and let dB_1 be the density of the highest bid of the other n-1 bidders. Conditional on submitting a serious bid, trade will occur in the current period with probability one. The expected return from any bid b is

$$(v-r_t)\int_0^{r_t} dB_1 + \int_{r_t}^{b}(v-B_1)dB_1$$

For any bidding behavior of the other bidders, a bid of b = v maximizes this expression.

ii) Observe that if a bidder bids seriously against R_t then by i) he bids $\beta(v) = v$ and will never bid if $v < R_t$. Let dB_1 be the density of the highest of the other n-1 bids in the current period and consider the expected utility from the equilibrium continuation to a bidder of type v when the history is h_t, the bidder has not submitted a bid in the current period and the game has continued to the next period. If v submits a bid then

$$(v-R_t)\int_0^{R_t} dB_1 + \int_{R_t}^{v}(v-B_1)dB_1 \geq \delta V_B(v,h_t) Prob[B_1 < R_t] \tag{9}$$

Suppose there is a type v' > v who does not submit a bid. Then

$$(v'-R_t)\int_0^{R_t} dB_1 + \int_{R_t}^{v}(v'-B_1)dB_1 + \int_{v}^{v'}(v'-B_1)dB_1 \leq \delta V_B(v',h_t) Prob[B_1 < R_t] \tag{10}$$

Subtracting (9) from (10) and rearranging yields

$$v'-v \leq \frac{Prob[B_1 \leq R_t]}{Prob[B_1 \leq v]}\delta(V_B(v',h_t) - V_B(v,h_t)) \tag{11}$$

Observe that a bidder of type v can always mimic the behaviour of bidder of type v'. Let $\alpha_{t+j}(v',h_t)$ be the probability a bidder who behaves as if he were v' obtains the object in period t+j in the pBe following history h_t (calculated from period t) and let $p_{t+j}(v',h_t)$ be the expected price paid conditional on obtaining the good. By definition

$$V_B(v,h_t) \geq \sum_{j=0}^{\infty} \delta^{t+j} \alpha_{t+j}(v',h_t)(v-p_{t+j}(v',h_t)) \tag{12}$$

while

$$V_B(v',h_t) = \sum_{j=0}^{\infty} \delta^{t+j} \alpha_{t+j}(v',h_t)(v'-p_{t+j}(v',h_t)) \tag{13}$$

Subtracting (12) from (13) and combining with (11) yields

$$v'-v \leq (v'-v)\delta \frac{Prob[B_1 \leq R_t]}{Prob[B_1 \leq v]} \sum_{j=0}^{\infty} \delta^j \alpha_{t+j}(v',h_t)$$

a contradiction since the sum of the α's must be one or less.

Proof of Lemma One: (**H3**) $r_i(x,w) < x$ for $x > w$ since it is a convex combination of x and values strictly less than x. To see how it changes with w,

$$\frac{\partial r_i(x,w)F_{Y_1}(x)}{\partial w} = \delta\left(\frac{d[r_{i-1}(v,\gamma^*_{i-2}(w))F_{Y_1}(w)]}{dw} - wf_{Y_1}(w)\right)$$

This term is positive by (**H3**) for i-1. Furthermore,

$$\begin{gathered}\frac{\partial[r_i(x,w)F_{Y_1}(x)]}{\partial x} = \frac{\partial r_i(x,w)}{\partial x}F_{Y_1}(x) + r_i(x,w)f_{Y_1}(x) \\ = xf_{Y_1}(x) + (1-\delta)F_{Y_1}(x) \\ or \\ \frac{\partial r_i(x,w)}{\partial x}F_{Y_1}(x) = (x - r_i(x,w))f_{Y_1}(x) + (1-\delta)F_{Y_1}(x)\end{gathered}$$

so $r_i(x,w)$ is increasing in x for $x \geq w$. Since r_i is also increasing in w and since γ^*_{i-1} is increasing, equation (1) is satisfied for i.

(H1) Since g_i is continuous in v_t and x and increasing and continuous in r_i, since $r_i(x,w)$ is increasing and upper semi-continuous, g_i is upper semi-continuous. A version of the theorem of the maximum (exploiting the fact that $w > \gamma_j(w)$) then implies that $\Pi_i(v_t)$ is continuous and $\gamma_i(v_t)$ is upper hemi-continuous.

(H2) To see that $\gamma_i(v_t)$ is increasing, let $y < y'$ and $x \in \gamma_i(y)$, $x' \in \gamma_i(y')$ and suppose that $x' < x$. To save on notation, let $m = r_i(x,w)F_{Y1}(x)$ for some $w \in \gamma_{i-1}(x)$and $m' = r_i(x',w')F_{Y1}(x')$ for some $w' \in \gamma_{i-1}(x')$. By the induction hypothesis, $w' < w$. By definition,

$$g_i(y,x,w)+\int_y^{y'}\int_x^{X_1} nY_1 dF_{Y_1}f(X_1)dX_1 \\ +m\,n[F(y')-F(y)]=g_i(y',x,w) \tag{14}$$

and

$$g_i(y,x',w')+\int_y^{y'}\int_{x'}^{X_1} nY_1 dF_{Y_1}f(X_1)dX_1 \\ +m'\,n[F(y')-F(y)]=g_i(y',x',w') \tag{15}$$

Subtracting equation (15) from (14) yields

$$n(F(y')-F(y))(m-m'-\int_{x'}^{x}Y_1 dF_{Y_1}) \\ =g_i(y',x,w)-g_i(y',x',w')-(g_i(y,x,w)-g_i(y,x',w')) \tag{16}$$

The right side of equation (16) is non-positive by definition of x,y, x' and y'. Since $w' \le w$ and we have shown that r_i is increasing in w, by definition of r_i,

$$\begin{aligned} m&=r_i(x,w)F_{Y_1}(x)\\ &\ge r_i(x,w')F_{Y_1}(x)\\ &=m'+(1-\delta)(xF_{Y_1}(x)-x'F_{Y_1}(x'))+\delta\int_{x'}^{x}Y_1 dF_{Y_1}\end{aligned}$$

We can rewrite the left side of (16) therefore as greater than

$$n(1-\delta)(F(y')-F(y))\{(x-x')F_{Y_1}(x')+ (F_{Y_1}(x)-F_{Y_1}(x'))(x-E[Y_1|x'\leq Y_1\leq x])\}$$

Since x > x' this expression is strictly positive -- a contradiction. Therefore, $\gamma_i(x)$ is increasing and $\gamma^*_i(x)$ is increasing and upper semi-continuous.

Proof of Lemma Two: Observe that

$$\frac{\partial g_1(v_t,x,1)}{\partial x}=n(1-\delta)[F(v_t)-F(x)-xf(x)]F_{Y_1}(x)$$

Since $f(x) > 0$, there is an $\epsilon > 1$ such that this expression is strictly less than zero for all $v_t \in [1,\epsilon)$. Fix i-1. By definition, for $x \epsilon \gamma_i(v_t)$, $x \leq z_{i-1}$ and therefore $\Pi_i(x) = \Pi_{i-1}(x)$. Since

$$\begin{aligned}\Pi_i(v_t)-\delta\Pi_i(x)&=n(F(v_t)-F(x))r_i(x,\gamma_{i-1}(x))F_{Y_1}(x)+ n\int_x^{v_t}\int_x^{X_1}Y_1f(X_1)dF_{Y_1}dX_1+\delta(\Pi_{i-1}(x)-\Pi_i(x))\\ &=n(F(v_t)-F(x))r_i(x,\gamma_{i-1}(x))F_{Y_1}(x)+n\int_x^{v_t}\int_x^{X_1}Y_1f(X_1)dF_{Y_1}dX_1\end{aligned}$$

and

$$\Pi_i(v_t)-\delta\Pi_i(x)\geq(1-\delta)\Pi_i(v_t)\geq n\int_1^{v_t}\int_1^{X_1}Y_1f(X_1)dF_{Y_1}dX_1(1-\delta)$$

we have that for all $x \in \gamma_i(v_t)$, there exists an $\nu > 0$ independent of i such that $x \leq v_t - \nu$. Since γ_i is increasing and upper hemicontinuous and satisfies $\gamma_i(x) \leq x - \nu$, the convex hull of γ_i has an inverse which is increasing and upper hemicontinuous defined over $[1,v_H]$ and lies above the line, $y - \nu = x$. Thus $z_i = \max\{v \mid \gamma_i(v) = z_{i-1}\}$ exists and satisfies $z_i \geq z_{i-1} + \nu$. This procedure extends the definition of z_i over the interval $[1,v_H]$.

Proof of Lemma Three: Let a denote the supremum such that C1 holds for j = i and a. Since for $v_t \leq z_{i-1}$, $r_{i-1} = r_i$ and $\gamma_{i-1} = \gamma_i$ for $v_t < z_{i-1}$, then $a \geq z_{i-1}$. Observe that for a > 1, since $\Pi(v_t)$ is bounded above zero, and f(v) is positive, there is always an $\epsilon > 0$ such that $(F(v)-F(a))v_{Hn} + \delta\Pi_i(a) < \Pi_i()$, for all v in $[a,a+\epsilon)$. Suppose $a < z_i$. Then there exists a σ and h_t and v_t in $(a,a+\epsilon)$ such that the seller's best response reserve price exceeds $r_i(z_{i-1},z_{i-2})$ and since a reserve price below that level generates payoffs determined by C2, the payoff from σ is bounded from above and below by

$$\Pi_i(v_t) \leq P^{\sigma}(v_t,h_t) \leq (F(v_t)-F(a))v_H + \delta\Pi_i(a) < \Pi_i(v_t)$$

Which is a contradiction. Therefore, $a \geq z_i$.

Now, suppose that $R_t = r(v_{t+1},y)$ for $y < \gamma^*(v_{t+1})$, $y \in \gamma(v_{t+1})$. Since r(x,v) is strictly increasing in v, for every $v' > v_{t+1}$, and every $y' \in \gamma(v')$, there is an $\epsilon > 0$ such that $r(v',y') \geq r(v_{t+1},y) + \epsilon$. A reserve price of r' instead of R_t, yields the seller an expected revenue of

$$g(v_t,v',y') \geq n(F(v_t)-F(v'))(r_t+\epsilon)F_{Y_1}(v') + \int_{v'}^{v_t}\int_{v'}^{v_t} Y_1 dF_{Y_f} f(X_1)dX_1 + \delta\Pi_i(v')$$

Since this function is continuous in v'

$$\lim_{v' \downarrow v_{t+1}} g(v_t,v',y') - g(v_t,v_{t+1},y) \geq n\epsilon(F(v_t)-F(v_{t+1}))F_{Y_1}(v_{t+1}) > 0$$

the seller could have improved on R_t by offering a slightly higher reserve price contradicting the assumption that R_t was an equilibrium reserve price.

Proof of Lemma Four: Since r(x,w) is strictly increasing in both its arguments and γ is an increasing correspondence, the correspondence, $r(x,\gamma^*(x))$, $x \epsilon \gamma(v)$, has a unique inverse, call it $\rho(r)$. By Lemma 0, for any reserve price, R_t, there is a v_{t+1} such that only bidder types above

v_{t+1} submit serious bids. Suppose $R_t > r(z_i,\gamma^*(z_i))$ and $v_{t+1} < z_i$. By C1, the next period reserve price is $R_{t+1} \leq r(\gamma_{*(v}t+1),\gamma^*(\gamma^*(v_{t+1}))$. By bidding in period t, v_{t+1} receives

$$(v_{t+1}-R_t)F_{Y_1}(v_{t+1}) \qquad (17)$$

while by waiting until the next period, he would get no worse than

$$\delta[(v_{t+1}-R_{t+1})F_{Y_1}(v_{t+2})+\int_{v_{t+2}}^{v_{t+1}}(v-Y_1)dF_{Y_1}] \qquad (18)$$

By definition of r, equation (18) strictly exceeds equation (17) so all types in the neighbourhood of v_{t+1} do better not to bid when the reserve price is R_t. Now suppose that $R_t < r(z_i,\min\gamma(z_i))$ and $v_{t+1} > \rho(R_t) = v$. Let τ be the smallest number such that along the equilibrium continuation path, $z_i > v_{t+\tau+1}$. (If the equilibrium involves mixed strategies, then the following argument can be made using distributions over continuation paths). If $v_{t+\tau} \geq v$, then $R_{t+\tau} > r(\gamma^*(v),\gamma^*\gamma^*(v))$ (by C1) and bidder type v would have done better to bid when the reserve price was R_t. If $v_{t+\tau} < v$, since $v_{t+\tau-1} \geq z_i$, $R_{t+\tau-1} \geq r(z_i,\min\gamma(z_i)) \geq r(z_{i-1},\gamma^*(z_{i-1}))$ and

$$R_{t+\tau-1}=(1-\delta)v_{t+\tau-1}+\frac{\delta}{F_{Y_1}(v_{t+\tau-1})}(R_{t+\tau}F_{Y_1}(v_{t+\tau-1})+\int_{v_{t+\tau}}^{v_{t+\tau-1}}Y_1dF_{Y_1})$$

we must have $R_{t+\tau} \geq r(z_{i-1},\gamma^*(z_{i-1}))$ and $v_{t+\tau} \geq z_{i-1}$. But this violates the optimality of type v's decision not to bid when the reserve price is R_t since

$$R_t=(1-\delta)v+\frac{\delta}{F_{Y_1}(v)}(r(y,\gamma^*(y))F_{Y_1}(v)+\int_{\gamma^*(v)}^{v}Y_1dF_{Y_1}),$$

for some $y \in$ Convexhull$\gamma(v)$.

References

Ashenfelter, Orley. How auctions work for wine and art. Journal of Economic Perspectives Vol. 3(3). Summer 1989. Pp. 23-36.

Ausubel. Larry and Ray Deneckere. Reputation in bargaining and durable goods monopoly. Econometrica, vol. 57, no. 3, 1989. Pp. 511-532.

Bulow. Jeremy and Paul Klemperer. Auctions versus negotiations. NBER Working Paper No. 4608, 1994.

Cramton, Peter, Sequential bargaining mechanisms. In Game Theoretic Models of Bargaining. Alvin Roth, ed.. London/New York. Cambridge University Press, 1986. Pp. 149-179.

Freixas, Xavier, Guesnerie, Roger, and Jean Tirole. Planning under incomplete information and the ratchet effect. Review of Economic Studies, (1985) LII, 173-191.

Fudenberg, Drew, Levine, David, and Jean Tirole, Infinite horizon models of bargaining with incomplete information. In Game Theoretic Models of Bargaining. Alvin Roth, ed.. London/New York, Cambridge University Press, 1986. Pp.73-98.

Gul, Faruk, Sonnenschein, Hugo and Robert Wilson. Foundations of dynamic monopoly and the Coase Conjecture. Journal of Economic Theory **39**, June, 1986. Pp.155-190.

McAfee, R. Preston. Mechanism design by competing sellers. Econometrica Vol. 61, no. 6, 1985. Pp. 1281 - 1312.

Myerson, Roger. Optimal auction design. Mathematics of Operations Research 6, 1981. Pp. 58-73.

Myerson, Roger and Mark Satterthwaite. Efficient mechanisms for bilateral trading. Journal of Economic Theory, 29, 1983. Pp. 265-281.

Milgrom, Paul and R. Weber. A theory of auctions and bidding. Econometrica 50. September 1982. Pp. 1089-1022.

Tirole, Jean. The Theory of Industrial Organization. MIT Press, Cambridge, MA, 1988.