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INFORMATION STRUCTURE ON MAXIMAL CONSISTENT SETS*

by

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ABSTRACT. In the information structure model, introduced by Aumann (1976), each state ω of the world has been interepreted as a complete set of descriptions of the situation under study. In order to make this idea explicit, we construct states of the world as maximal consistent sets of a theory in an underlying formal language with knowledge operator $\Box_i$. In this framework, the following *expressibility* problem becomes well-defined:

> Given an event $A \subseteq \Omega$, is there any sentence α in the underlying language such that, for every $\omega \in \Omega$, i knows A at ω if and only if $\Box_i\alpha \in \omega$?

We examine the expressibility of the possibility sets $P^i(\omega)$. The expressibility of possibility sets is interepreted as a necessary condition for each state ω of the world to be a complete set of descriptions. Working in non-partitional setup, several characterizations for the expressibility condition are given. It is shown that if the possibility sets are expressible then minimal possibility sets exist and the number of them is finite. We also offer two characterizations of partitional information structures in terms of expressibility. One of them states that an information structure is partitional if and only if the meaning of 'possibility' that we attach to possibility sets is expressible. The other implies that the common knowledge assumption of an information structure is a characterizing condition for a partitional information structure.

1. Introduction

In a path breaking paper, Aumann (1976) introduced a model, often called an *information structure*, that consists of following objects[1]:

> The *state space* Ω with generic element $\omega \in \Omega$, called a state of the world.
> The *possibility correspondence* $P^i : \Omega \to 2^\Omega$, one for each decision maker.

where 2^Ω is the power set of Ω. Once we have an information structure, we can talk about a decision maker's *knowledge of an event*[2] $A \in 2^\Omega$: we say that i knows A at $\omega \in \Omega$ if and only if $P^i(\omega) \subseteq A$. Using this definition of knowledge of an event, Aumann (1976) defined the notion of common knowledge, and, subsequently, its implications have been derived by many authors.

This paper considers several conceptual issues that naturally arise regarding information structure models. Specifically, the main issue is summarized as a simple question: *What is a state of the world ω?*

The question has a standard answer, which states that a state of the world is a complete set of descriptions under study (Savage (1954), Aumann (1976), (1987)). In the literature, there are two approaches to formalize this rather informal notion of a state of the world. The first approach, starting with a given set of 'parameters', constructs a state of the world as an infinite hierarchy of beliefs on the parameter set (Mertens and Zamir (1985), Brandenburger and Dekel (1993)). Alternatively, the second approach starts with a theory in a formal language that is supposed to describe the decision making situation under study and then constructs states of the world by applying methods similar to those in mathematical logic (Samet (1990), Shin (1993)). The first approach, the *belief approach*, is preferable from the Bayesian point of view in that it follows the principle that every relevant question in decision making situation can be studied in terms of beliefs. On the other hand, the second approach, the *logical approach*, is attractive since it gives a faithful formalization of the standard answer mentioned above in the sense that it will be literally the case in this approach that a state of the world is a set of descriptions. That is to say, a state of the world will be a set of sentences in a formal sense. Following the logical approach, the paper studies the completeness of a state of the world as a set of descriptions. We formulate a condition called *expressibility* for an information structure. The expressibility condition can be regarded as a necessary condition for the completeness of a state of the world. The paper gives characterizations for the condition, and explores its consequences.

Roughly speaking, a state ω of the world can be regarded as a complete set of descriptions if for every relevant question about the situation under study, ω contains a description that specifies whether the answer for the question is Yes or No. In other words, ω is said to be complete if for every relevant description it contains either the description itself or its negation. Also, we want ω to be consistent in the sense that it does not contain, for example, both 'White Sox will win tomorrow' and 'White Sox will lose tomorrow'. In addition, there might be descriptions, such as 'White Sox will have one, and only one, ballgame tomorrow', that are supposed to be true in every state of the world.

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[1] No probabilistic aspects of information structure models are discussed in this paper.

[2] Throughout the paper, an event is just a subset of the state space.

The advantage of the logical approach to information structure models lies in the fact we can make all of these notions[3] precise. Assume that it has been determined that what kind of descriptions are relevant to the situation under study. We formulate the set of all relevant descriptions as the set of sentences (well-formed formulas) on a sentential language with an operator '$\Box_i$' that is assumed to mean 'i knows that'. From the set of all sentences, some sentences are designated as *theorems*. Theorems correspond to those descriptions that bear no uncertainty. That is to say, theorems are going to be the sentences that will be true in every state of the world. The set of all theorems is called a theory (in the given language). In this setup, the notion of a complete set of descriptions, ω, is faithfully formulated as that of a *maximal consistent set* of the theory. Roughly, a maximal consistent set is a set of sentences that is closed under deductive consequences, that does not contain mutually inconsistent sentences, and, for each sentence in the language, that contains either that sentence itself or its negation. We then consider the set of all maximal consistent sets and take this set as our state space Ω. On this state space, we construct a possibility correspondence $P^i(\cdot)$ to get an information structure, the one we will call the *canonical information structure* for the theory. We construct each possibility set $P^i(\omega)$ in terms of the *accessibility*[4] relation, where we say that a state ω' is accessible from ω if, for every sentence α, 'α' is true at ω' whenever 'i knows that α' is true at ω. Apart from details, our approach to construct the state space is the one that has been adopted by Aumann (1989), Samet (1990), and Shin (1993).[5]

In this setup, an important problem of *expressibility* arises in the following manner.[6] Each sentence α in the given language may well be true[7] at some states of the world and may not somewhere else. Thus each sentence α in the language corresponds to an event in Ω at which the sentence is true. If we denote this set by $\|\alpha\|$ then each sentence α is associated with an event

$$\|\alpha\| = \{\omega \in \Omega \mid \alpha \in \omega\}.$$

We do not have, however, the converse direction in general. That is, there might be events in the state space that cannot be characterized by any sentence in the language. In other words, there might be a set that is not describable by the language in the way described above. Given this observation, let us say that an event $A \subset \Omega$ is *expressible* if there is a sentence α in the language such that $\|\alpha\| = A$.

The following question is the focus of the paper. Consider the possibility set $P^i(\omega)$ for some $\omega \in \Omega$. Is $P^i(\omega)$ expressible? If it *is* then we have an exact parallelism between formal sentences that each state contains and traditional definition of knowledge. According to the traditional definition,

$$i \text{ knows that } P^i(\omega) \text{ at } \omega.$$

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[3]Except that what kind of descriptions are relevant for the decision making situation under study.

[4]See Chellas (1980) and Samet (1990).

[5]Of these, Aumann's (1989) setup is the one that is closest to ours.

[6]Both Samet (1990) and Shin (1993) recognize that there are problems in their setups that are similar to our expressibility. In particular, Samet (1990, p.204) mentions the expressibility of possibility sets. These issues are not their main concern, however.

[7]That a sentence α is true at ω just means that $\alpha \in \omega$.

since trivially $P^i(\omega) \subset P^i(\omega)$. Corresponding to this statement we have the formal language counterpart

$$\Box_i \alpha_\omega \in \omega,$$

where α_ω is an expressing sentence for $P^i(\omega)$. More generally, the analyses in the main body of the paper shall show that, for every $\omega \in \Omega$ and expressible $A \subset \Omega$,

$$P^i(\omega) \subset A \qquad \text{if and only if} \qquad \Box_i \alpha \in \omega,$$

where α is an expressing sentence for A. In short, for each claim we make about knowledge at state ω according to the traditional definition we can find a corresponding sentence in ω *provided* $P^i(\omega)$ *is expressible.*

Alternatively, suppose that $P^i(\omega)$ is *not* expressible. This means that there is no sentence α such that

$$\|\alpha\| = \{\omega' \in \Omega \mid P^i(\omega') = P^i(\omega)\}$$

holds. Thus, although it is trivially the case that i knows that $P^i(\omega)$ at ω according to the traditional definition, there is no formal language counterpart that corresponds to this i's knowledge about $P^i(\omega)$. This means that the formal language we adopted to construct ω is not powerful enough to capture the traditional definition of knowledge. There is a discrepancy between the two formulations of knowledge. Actually, one might take this discrepancy as an indication of the *incompleteness* of ω as a set of descriptions since statements made by the traditional definition of knowledge *are* relevant statements to our study. In particular, the nonexistence of a sentence that corresponds to *i's knowledge about one of his own possibility sets* $P^i(\omega)$ seems to imply that the formal language under discussion is so weak that it fails to incorporate an informal (but relevant) assumption that i knows his own information structure. Thus it is desirable to have $P^i(\omega)$ to be expressible. In fact, our intention is that the expressibility of possibility sets is a *necessary* condition for the canonical information structure with states of the world interpreted as complete sets of descriptions. This is the basic motivation of the current study. Our objective is to show what does the canonical information structure look like if it satisfies that conceptual necessary condition.

We assume that the formal theory out of which we construct the state space is a version of sentential logic that is assumed to contain sentences of the form ' if i know that A then A' and 'if i knows that A then i knows that i knows that A' as its theorems. The former (the latter, respectively) has been called 'axiom of knowledge (axiom of transparency)' by Binmore (1992), or 'non-delusion (knowing that you know)' by Geanakoplos (1990), or '**T** (**4**)' in modal logic literature. Besides these, the conditions we impose on the part of the formal theory are that it has all tautologies of the classical sentencial logic as its theorems and that it is equipped with some mild inference rules governing knowledge sentences. In short, we assume that our theory is at least as strong as the logic known as S4 in modal logic literature (Chellas (1980)). This means that, any *strengthening* of S4, such as allowing uncountably many sentences or adding infinitely logical operations, would not alter our conclusions.

We say that the canonical information structure is *expressible* if each possibility set $P^i(\omega)$ of the canonical possibility correspondence is expressible. Let us say that an expressible set $\|\alpha\|$ is *self-evident* (for i) if $\|\alpha\| = \|\Box_i \alpha\|$. Recall that a partially ordered set is *well-founded*[8] if every nonempty subset of it has a minimal element. The first result gives a characterization of expressibility (Theorem 3.7):

(i) The canonical information structure is expressible if and only if the set of all self-evident sets is well-founded with respect to set inclusion.

A condition similar to (i) has appeared in Samet (1990). He used the condition to get a generalization of Aumann's (1976) theorem. Thus (i) offers a justification of his assumption. Roughly speaking, the well-foundedness condition is satisfied if and only if a decision maker's knowledge at a state ω is based on a single basic sentence. In other words, the condition is satisfied if and only if i's knowledge does not show an infinite regress. We will discuss this point in Section 3.

We shall show that if a possibility set $P^i(\omega)$ is expressible then it is self-evident. Thus (i) implies that if the information structure is expressible then there are minimal possibility sets. The second results states that (Theorem 4.2):

(ii) The number of minimal possibility sets is finite.

Geanakoplos and Polemarchakis (1982) and Bacharach (1985) proved their convergence results under the finite partition assumption. Alternatively, Shin (1993) has used a similar condition to (ii) to get a characterization of common knowledge events in his setup. Thus (ii) shows that in information structures where states of the world are in fact complete their asumptions are automatically satisfied.

It should be noted that both of the well-foundedness of self-evident sets and the finiteness of minimal possibility sets appear as consequences of the expressibility, which is meant to be a formalization of the intuitive notion of a state of the world as a complete set of descriptions, which, in turn, does not involve any apparent finiteness flavor.

The third result is about a characterization of partitional information structure in terms of expressibility. Let us say that the canonical information structure is ***strongly expressible*** if there is a sentence α_ω such that $P^i(\omega) = \|\Diamond_i\alpha_\omega\|$ for every $\omega \in \Omega$, where '$\Diamond_i$' is a derived operator in the language, which is supposed to mean 'it is possible that', defined by '$\neg\Box_i\neg$'. Strong expressibility requires that the meaning of 'possibility' that we attach to possibility sets $P^i(\omega)$ should be expressible in the underlying language. We shall show that (Theorem 4.3):

(iii) Assume that the canonical information structure is expressible. Then the structure is strongly expressible if and only if it is partitional.

If the canonical information structure is strongly expressible (i.e., partitional) then there is a one-to-one correspondence between intersection relationship for possibility sets (i.e., whether $P^i(\omega) \cap \|\alpha\| \neq \varnothing$ or not for a given expressible set $\|\alpha\|$) and sentences in the underlying theory of the form

$$\alpha_\omega \rightarrow \Diamond_i\alpha,$$

where $P^i(\omega) = \|\alpha_\omega\|$. We shall argue in Section 5 that this correspondence gives us a formalization of the claim that the information structure itself is common knowledge

[8]See, for example, Johnstone (1987, p.61).

among decision makers. Moreover, Proposition 5.2 will show that the existence of this correspondence *characterizes* partitional canonical information structures. Thus, in our setup, the common knowledge assumption of an information structure turns out to be a characterizing condition for a partitional information structure. In other words, the common knowledge assumption and partitional information structures go together.

From a mathematical point of view, we will work in lattice theoretic framework. In particular, the main tool of the paper is the Stone representation theorem, the central result in the duality theory for Boolean algebras. Stone's theorem not only allows us to translate syntactical structure of the language into simple set theoretic operations on the state space but also gives us a topological characterization of expressibility of an event. More importantly, it gives us compactness of the state space. As we will see, our results are consequences of the compactness of the state space.[9]

The rest of the paper is organized as follows. Section 2 introduces our basic construction and formulates the expressibility condition for the constructed possibility correspondence. Section 3 gives complete characterizations of the expressibility condition. In Section 4, we derive consequences of the expressiblity condition stated above. Section 5 discusses issues concerning the notion of common knowledge. The paper ends with concluding remarks in Section 6. There are two appendices, one for rudimentary definitions and results from lattice theory, another for the proofs of the formal statements we are going to make.

2. Construction of Canonical Information Structure

In this section, we introduce our basic framework and then construct a state space and an information structure, which are the main object of the present study. We are going to use basic facts and results from lattice theory. Necessary definitions, as well as some results that we are going to invoke, are collected in the Appendix.[10] It should be noted that our construction in this section is a version of the canonical construction in the logic literature (Chellas (1980)). Also, similar constructions have been done by Aumann (1989), Samet (1990), and Shin (1993).

Consider a situation where a decision theorist is given a many-person decision problem to study. For simplicity, we assume that there are two decision makers, i and j. Assume that the theorist have specified all relevant descriptions of the decision making situation. These relevant descriptions are devided into two classes. One of which consists of descriptions that are supposed to hold under every contingencies. The other consists of descriptions that are not supposed to be so. We assume that each description of the theorist is written in a formalized language. We will call the first class of description a theory in the language. Each description in the theory is called a theorem (in the theory). With this background interpretation in mind, let us begin our formal analysis.

[9]By compactness, here we mean the most genaral version: A topological space is *compact* if every open cover has a finite subcover. A compactness argument that is similar to those we shall invoke repeatedly can be found in Aumann (1989, Section 11).

[10]A relevant reference is Davey and Priestley (1990).

A language is a tuple $\mathcal{L} = \langle \mathbb{S}, \bot, \to, \Box_i, \Box_j \rangle$, where $\bot \in \mathbb{S}$, $\to$ is a binary operation on $\mathbb{S}$ (i.e., $\to$ is a mapping from $\mathbb{S} \times \mathbb{S}$ to $\mathbb{S}$), $\Box_k$ $(k = i, j)$ is an unary operation on $\mathbb{S}$ (i.e., $\Box_k$ is a mapping from $\mathbb{S}$ to $\mathbb{S}$). We adopt the most readable notation system: we write $\alpha \to \beta$ for $\to(\alpha, \beta)$ and $\Box_k \alpha$ for $\Box_k(\alpha)$. An element of $\mathbb{S}$ is called a *sentence* of $\mathcal{L}$. $\bot$ is a constant sentence, which is supposed to express 'logical inconsistency'. We read $\alpha \to \beta$ as 'if α then β' and $\Box_k \alpha$ as 'k knows that α'. The usual logical connectives such as 'and', 'or', and 'not' are defined as follows. For every $\alpha, \beta \in \mathbb{S}$ define

$$\neg\alpha = \alpha \to \bot, \qquad \alpha \vee \beta = \neg\alpha \to \beta,$$
$$\alpha \wedge \beta = \neg(\neg\alpha \vee \neg\beta), \qquad \alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha).$$

In addition, following modal logic literature, we define

$$\Diamond_k \alpha = \neg\Box_k \neg\alpha,$$

which we read as 'k thinks it is possible that α'.

Formally speaking, we presented the language $\mathcal{L}$ as an algebraic structure. As a consequence, here a sentence $\alpha \in \mathbb{S}$ in $\mathcal{L}$ is not necessarily a finite string of symbols, contrary to the standard practice in mathematical logic. It can be, however. Thus our framework enjoys some generality. For example, there is no a priori restriction on the cardinality of $\mathbb{S}$. Also one can add another operation on $\mathbb{S}$, which might corresponds to, for example, infinite conjunctions.

We next introduce the notion of a *theory* in $\mathcal{L}$, as a formalization of the class of descriptions that are true under every contingencies. In other words, a theory should be a collection of descriptions that our decision theorist assume to hold in the model. Given these interpretation, we define a theory to be a set of sentences in $\mathcal{L}$ that satisfies several properties. A *theory* Σ in $\mathcal{L}$ is a subset of $\mathbb{S}$ that satisfies following conditions for every $\alpha, \beta, \gamma \in \mathbb{S}$:

(Con) $\bot \notin \Sigma$.

(PL) $\alpha \to (\beta \to \gamma) \in \Sigma$,
$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)) \in \Sigma$,
$\neg\neg\alpha \to \alpha \in \Sigma$.

(K) $\Box_k(\alpha \to \beta) \to (\Box_k \alpha \to \Box_k \beta) \in \Sigma$.

(MP) if $\alpha \to \beta \in \Sigma$ and $\alpha \in \Sigma$ then $\beta \in \Sigma$.

(N) if $\alpha \in \Sigma$ then $\Box_k \alpha \in \Sigma$.

where $k = i, j$. We call a sentence $\alpha \in \Sigma$ a *theorem* in Σ. These conditions together express our (and our decision theorist's) basic assumptions as in the following sense. First, by **(Con)**, **(PL)**, and **(MP)**, Σ contains all tautologies[11] of the classical sentential logic (see Lemma 2.1 (1) below). As for sentences involving knowledge operators, **(N)** expresses the assumption that k knows all sentences in the theory Σ. In other words, k knows all theorems in Σ or k simply knows the theory Σ. **(K)** governs the inferences involving sentences that are known by k. This is a regularity condition, whose intuitive meaning

[11] In our setup, the notion of tautologies requires a precise definition. This is given in the beginning of Appendix 2.

is that k knows all sentences that follow deductively (in the sense given by (**MP**)) from sentences known by k (see Lemma 2.1 (2) below). Thus one can regard this condition as a rationality assumption with respect to logical deduction. Let us summarize some consequences of these assumption.

2.1 Lemma. *Let Σ be a theory in $\mathcal{L}$. For every $\alpha_1, \ldots, \alpha_n, \alpha \in \mathbb{S}$ and $k = i, j$.*

(1) *Σ contains all tautologies in classical sentential logic.*
(2) *If $\alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \alpha \in \Sigma$ then $\Box_k\alpha_1 \wedge \cdots \wedge \Box_k\alpha_n \rightarrow \Box_k\alpha \in \Sigma$.*
(3) *$\Box_k\alpha_1 \wedge \cdots \wedge \Box_k\alpha_n \leftrightarrow \Box_k(\alpha_1 \wedge \cdots \wedge \alpha_n) \in \Sigma$.*
(4) *$\Diamond_k\alpha_1 \vee \cdots \vee \Diamond_k\alpha_n \leftrightarrow \Diamond_k(\alpha_1 \vee \cdots \vee \alpha_n) \in \Sigma$.*
(5) *$\Box_k\alpha_1 \vee \cdots \vee \Box_k\alpha_n \rightarrow \Box_k(\alpha_1 \vee \cdots \vee \alpha_n) \in \Sigma$.*

Now let us start to construct our state space. Given a theory Σ, define a binary relation $\preceq$ on $\mathbb{S}$ by

$$\alpha \preceq \beta \qquad \text{if and only if} \qquad \alpha \rightarrow \beta \in \Sigma.$$

By Lemma 2.1 (1), $\preceq$ is reflexive and transitive. Denote by $\simeq$ the equivalence relation defined by $\alpha \simeq \beta$ if and only if $\alpha \preceq \beta$ and $\beta \preceq \alpha$. By definition, $\alpha \simeq \beta$ if and only if $\alpha \leftrightarrow \beta \in \Sigma$. Denote the quotient set of $\mathbb{S}$ with respect to $\sim$ by $\mathbb{S}/\Sigma$. Namely, $\mathbb{S}/\Sigma = \{|\alpha| \mid \alpha \in \mathbb{S}\}$. In words, we identify those sentences that are deductively equivalent in Σ. In particular, $|\top|$, defined by $|\neg\bot|$, is the set of all theorems in Σ. That is, $|\top| = \Sigma$. In what follows, with an abuse of terminology, we may call an element of $\mathbb{S}/\Sigma$ a sentence.

We notice that $\mathbb{S}/\Sigma$ inherits $\preceq$ from $\mathbb{S}$:

$$|\alpha| \preceq |\beta| \qquad \text{if and only if} \qquad \alpha \rightarrow \beta \in \Sigma.$$

The following fact has a primal importance to our study.

2.2 Proposition. *$\langle \mathbb{S}/\Sigma, \preceq \rangle$ is a Boolean algebra, with following meets* (greatest lower bound), joins (least upper bound), and complements: for every $\alpha, \beta \in \mathbb{S}$,

$$|\alpha| \wedge |\beta| = |\alpha \wedge \beta|, \; |\alpha| \vee |\beta| = |\alpha \vee \beta|, \; \neg|\alpha| = |\neg\alpha|.$$

In addition, $|\top|$ ($\bot$) is the top (bottom, respectively) element of $\langle \mathbb{S}/\Sigma, \preceq \rangle$.

In logic literature, the Boolean algebra $\langle \mathbb{S}/\Sigma, \preceq \rangle$ is called the *Lindenbaum algebra* for the theory Σ.[12] Notice that, in the statement of 2.2, we have used same symbols for different operations.

As discussed in the Introduction, the traditional interpretation of a state of the world is that it is a complete set of descriptions of the decision problem. In our setup, we can faithfully formalize this idea by the notion of maximal filter in a Boolean algebra.

2.3 Lemma. *A subset ω of $\mathbb{S}/\Sigma$ is a maximal filter in $\mathbb{S}/\Sigma$ if and only if ω satisfies the following conditions for every $|\alpha|, |\beta| \in \mathbb{S}/\Sigma$:*

(1) $|\bot| \notin \omega$.
(2) If $|\alpha|, |\beta| \in \omega$ then $|\alpha \wedge \beta| \in \omega$.
(3) If $|\alpha| \in \omega$ and $\alpha \rightarrow \beta \in \Sigma$ then $|\beta| \in \omega$.
(4) Either $|\alpha| \in \omega$ or $|\neg\alpha| \in \omega$.

[12] See, for example, Bell and Slomson (1969).

Conditions (1) and (2) together express consistency. (3) expresses closednesss with respect to deduction. (4) is the notion of maximality. Notice that, by (1) and Lemma 2.1 (1), 'or' in (4) is exclusive. In what follows, we call a maximal filter in $\mathbb{S}/\Sigma$ a *state of the world*. Accordingly, our state space is defined as

$$\Omega_\Sigma = \{\omega \in 2^{\mathbb{S}/\Sigma} \mid \omega \text{ is a maximal filter in } \mathbb{S}/\Sigma\}.$$

For every $\alpha \in \mathbb{S}$, define

$$\|\alpha\| = \{\omega \in \Omega_\Sigma \mid |\alpha| \in \omega\}.$$

In words, $\|\alpha\|$ is the set of states of the world at which the sentence α is *true*. Notice that $\|\alpha\|$ is a subset of Ω_Σ while $|\alpha|$ is an element of $\mathbb{S}/\Sigma$. Also, by construction,

$$\omega \in \|\alpha\| \qquad \text{if and only if} \qquad |\alpha| \in \omega.$$

Moreover, conjunctions (disjunctions, negations, respectively) in $\mathcal{L}$ translate into intersections (unions, complements, respectively) in Ω_Σ. We denote by A^c the complement of $A \subset \Omega_\Sigma$ with respect to Ω_Σ.

2.4 Proposition. *For every $\alpha, \beta \in \mathbb{S}$,*

(1) $\|\alpha\| = \Omega_\Sigma$ *if and only if α is a theorem in Σ* (i.e., $\alpha \in \Sigma$).
(2) $\|\alpha\| = \varnothing$ if and only if $\neg\alpha \in \Sigma$.
(3) $\|\alpha\| \subset \|\beta\|$ if and only if $\alpha \to \beta \in \Sigma$.
(4) $\|\alpha\| \cap \|\beta\| = \|\alpha \wedge \beta\|$.
(5) $\|\alpha\|^c = \|\neg\alpha\|$.
(6) $\|\alpha\| \cup \|\beta\| = \|\alpha \vee \beta\|$.

So far we have observed that for each sentence α in $\mathbb{S}$ there is a unique subset $\|\alpha\|$ of Ω_Σ, and that syntactic relationships between sentences are faithfully translated into set theoretic operations on Ω_Σ. Now let us turn to the 'converse' direction. Namely, given a subset A of Ω_Σ, can we find a sentence α in $\mathbb{S}$ such that $A = \|\alpha\|$? As stated in the Introduction, this is the notion of expressibility, which is the central theme of our study. Formally we define that $A \subset \Omega_\Sigma$ is *expressible* if there is $\alpha \in \mathbb{S}$ such that $A = \|\alpha\|$. The key mathematical result to deal with this problem is the *Stone representation theorem* for Boolean algebras. Define

$$EX(\Sigma) = \{\|\alpha\| \in 2^{\Omega_\Sigma} \mid \alpha \in \mathbb{S}\},$$

$$\tau_\Sigma = \text{the topology on } \Omega_\Sigma \text{ generated by } EX(\Sigma) \text{ as its open base.}$$

Notice that by Proposition 2.4 (4), $EX(\Sigma)$ is closed under finite intersections. Consequently, τ_Σ is well-defined. In words, $EX(\Sigma)$ is the set of all expressible sets. By Lemma 2.4 (5), it is immediate that if A is expressible then A is *clopen*, that is, both closed and open, with respect to the Stone topology τ_Σ. An application of Stone's theorem gives us further properties of the topological space $\langle \Omega_\Sigma, \tau_\Sigma \rangle$.

2.5 Theorem.

(1) *The topological space $\langle \Omega_\Sigma, \tau_\Sigma \rangle$ is compact and Hausdorff.*
(2) *$\langle EX(\Sigma), \subset \rangle$ is a Boolean algebra, which is isomorphic to $\mathbb{S}/\Sigma$. In other words, there exists a Boolean isomorphism $\eta : EX(\Sigma) \to \mathbb{S}/\Sigma$.*
(3) *For every $A \subset \Omega_\Sigma$, $A \in EX(\Sigma)$ if and only if A is clopen.*

The compactness of $\langle \Omega_{\Sigma}, \tau_{\Sigma}\rangle$ will play an essential role in following sections. What (2) says is that the algebra of sets $\langle EX(\Sigma), \subset\rangle$, which is a subalgebra (or a sublattice) of the power set algebra (or lattice) $\langle 2^{\Omega_{\Sigma}}, \subset\rangle$, is an exact copy of the Lindenbaum algebra $\mathbb{S}/\Sigma$ of the theory Σ. On the other hand, (3) gives us a topological characterization of the expressibility: An event $A \subset \Omega_{\Sigma}$ is expressible if and only if A is clopen with respect to the Stone topology τ_{Σ}.

Finally we are ready to define the canonical information structure for the theory Σ. Define for $k = i, j$,

$$\kappa(k)_{\Sigma}^{-1}(\omega) = \{|\alpha| \in \mathbb{S}/\Sigma \mid |\Box_k \alpha| \in \omega\},$$
$$P^k(\omega) = \{\omega' \in \Omega_{\Sigma} \mid \kappa(k)_{\Sigma}^{-1}(\omega) \subset \omega'\}.$$

In words, $\kappa(k)_{\Sigma}^{-1}(\omega)$ is the set of sentences that are known by k at ω. $P^k(\omega)$ is the set of states of the world with which $\kappa(k)_{\Sigma}^{-1}(\omega)$ is consistent. In other words, $P^k(\omega)$ is the set of all states of the world that are *accessible* from ω (Chellas (1980), Samet (1990)). Henceforth we call $P^k(\omega)$ a *possibility set* at $\omega \in \Omega_{\Sigma}$. The canonical possibility correspondence for k is a mapping

$$P^k : \Omega_{\Sigma} \to 2^{\Omega_{\Sigma}}$$

defined by $P^k(\omega)$ for every $\omega \in \Omega_{\Sigma}$. We call $\langle \Omega_{\Sigma}, P^i(\cdot), P^j(\cdot)\rangle$ the *canonical information structure* for Σ. The basic properties of possibility sets are as follows.

2.6 Proposition. *For every $\omega \in \Omega_{\Sigma}$ and $k = i, j$,*

(1) *$\omega \in \|\Box_k \alpha\|$ if and only if $P^k(\omega) \subset \|\alpha\|$,*

(2) *$P^k(\omega) = \bigcap\{\|\alpha\| \mid |\alpha| \in \kappa(k)_{\Sigma}^{-1}(\omega)\} = \bigcap\{\|\alpha\| \mid \omega \in \|\Box_k \alpha\|\}$,*

(3) *$P^k(\omega)$ is closed, i.e., $P^k(\omega)^c \in \tau_{\Sigma}$.*

Proposition 2.6 (1) gives us the relationship between the traditional definition of knowledge and knowledge sentences in Σ. *If $A \subset \Omega_{\Sigma}$ is expressible* then, for every $\omega \in \Omega_{\Sigma}$,

$$k \text{ knows } A \text{ at } \omega \qquad \text{if and only if} \qquad \omega \in \|\Box_k \alpha\|$$

where $\alpha \in \mathbb{S}$ is an expressing sentence for A. If we define, as has usually been done, the set $K^k(A)$ by $K^k(A) = \{\omega \in \Omega_{\Sigma} \mid P^k(\omega) \subset A\}$ then the above equivalence turns into

$$K^k(A) = \|\Box_k \alpha\| = \{\omega \in \Omega_{\Sigma} \mid |\Box_k \alpha| \in \omega\}.$$

Thus, as we claimed in the Introduction, there is an exact parallelism between statements about knowledge that are made by traditional definition of knowledge and knowledge sentences in $\mathbb{S}$, *provided if the set in question is expressible.*

Proposition 2.6 (2) simply states that $P^k(\omega)$ is the infimum of known sentences at ω. Notice, however, $P^k(\omega)$ is the infimum of $\{\|\alpha\| \mid \omega \in \|\Box_k \alpha\|\}$ *in* $2^{\Omega_{\Sigma}}$, which is complete as a Boolean algebra with respect to set inclusion. The point here is that it need not be the case that $\kappa(k)_{\Sigma}^{-1}(\omega)$, the set of known sentences at ω, has an infimum in $\mathbb{S}/\Sigma$, *which need not be complete.* In fact, it can be proved that $P^k(\omega)$ is expressible if and only if the infimum of $\kappa(k)_{\Sigma}^{-1}(\omega)$ exists in $\mathbb{S}/\Sigma$. From this observation, we know that the expressibility of possibility sets requires the Boolean algebra $\mathbb{S}/\Sigma$ to have a version of completeness in lattice theoretic sense. At this level of generality, however, we do not know much about the structure of the set $\kappa(k)_{\Sigma}^{-1}(\omega)$. Consequently, the observation itself is not so informative. Alternatively, if we know some properties of $\kappa(k)_{\Sigma}^{-1}(\omega)$ then we can derive more intuitive characterization of expressibility. In other words, in order to get more informative characterization of expressibility, we need stronger theory than the one we have in this section. In next section, we will strengthen our theory Σ and derive sharper characterizations of expressibility. Finally, we notice that by 2.6 (3) and 2.5 (3) $P^k(\omega)$ is expressible if and only if $P^k(\omega)$ is open with respect to the Stone topology τ_{Σ}.

3. Expressibility of Information Structure: Characterization

Let us say that the canonical information structure $\langle \Omega_{\Sigma}, P^i(\cdot), P^j(\cdot)\rangle$ associated with a theory Σ is *expressible* if $P^k(\omega)$ is expressible for every $\omega \in \Omega_{\Sigma}$ and $k = i, j$. As we observed in the previous section, each possibility set $P^k(\omega)$ is expressible if and only if the set of known sentences at ω, $\kappa(k)_{\Sigma}^{-1}(\omega)$, has a greatest lower bound in the Boolean algebra $\mathbb{S}/\Sigma$ of sentences in $\mathcal{L}$. In this section, by strengthening our theory Σ, we derive sharper characterizations of the expressibility condition. One of the key notions in this section is that of self-evident sets, which have already played an important role in the literature.

We additionally impose the following two conditions to our theory Σ. First, what is known by a decision maker k is true, and second, if k knows something then k knows that k knows it. Formally, we assume, throughout the rest of the paper, that

(T) $\quad \Box_k\alpha \to \alpha \in \Sigma$,

(4) $\quad \Box_k\alpha \to \Box_k\Box_k\alpha \in \Sigma$,

for every $\alpha \in \mathbb{S}$ and $k = i, j$. The main goal of this section is to show that the canonical information structure is expressible if and only if the *set of self-evident sets is well-founded*, i.e., every subset of it has a minimal element (see, for example, Johnstone (1987, p.61)).

Before starting formal development, we discuss a possible interpretation to the characterizing condition above at this point. We consider the set of sentences that are known by the decision maker k at a state of the world $\omega \in \Omega_{\Sigma}$. Let us pick such a sentence α_1 and ask 'How does k know α_1?' This question might have an answer that 'k knows another sentence α_2 at ω and that α_1 is one of a deductive consequence of α_2.' Then let us ask further that 'How does k know α_2?' Again there might be α_3 that have α_2 as one of its deductive consequences. Let us repeat this process again and again. There are two possible outcomes. The repeated process terminates in finite steps or else shows an *infinite regress*. It turns out that the notion of well-foundedness is precisely the one that excludes an infinite regress in the repeated process. We shall show that the expressibility of the canonical information structure is characterized by the well-foundedness condition. Therefore, under the expressibility, there must be a single self-evident sentence that is known by k at ω and all sentences known by k at ω are deductive consequences of that sentence. Namely, the canonical information structure is expressible if and only if we can give it the following 'behavioral scenario': 'If state ω occurs then k is informed that a sentence α_ω is true. k derives all deductive consequences of α_ω and only those are the sentences that are known by k at ω.' Apparently, this is a version of 'provability interpretation of knowledge', which was proposed by Shin (1993). Also, a condition similar to the 'no infinite regress' condition above has appeared in Samet (1990).

In this section and the next, we fix $k = i, j$. Accordingly, we shall drop superscripts and subscripts that specify a decision maker for notational convenience.

Let us start with a lemma.

3.1 Lemma. *For every $\omega, \omega' \in \Omega_\Sigma$,*

(1) $\omega \in P(\omega)$,

(2) *If $\omega' \in P(\omega)$ then $P(\omega') \subset P(\omega)$.*

Given the fact that a logic with schemata (**T**) and (**4**) is known as an *S4* system, we call an information structure that satisfies (1) and (2) above an *S4-information structure*. Several authors have already studied S4-information structures extensively (e.g., Samet (1990), Geanakoplos (1990), Brandenburger *et al.* (1992)). Also we refer (1) ((2), respectively) of Lemma 3.1 as (**T**) ((**4**), respectively) in what follows.

One remarkable fact about S4-information structures is that we can justifiably restrict our attention to so called self-evident sets. Let us say that $\|\alpha\| \in EX(\Sigma)$ is *self-evident* if $\|\alpha\| = \|\Box\alpha\|$. Accordingly, we say that $\alpha \in \mathbb{S}$ is self-evident if $\|\alpha\|$ is. Now define

$$SE(\Sigma) = \{\|\alpha\| \in EX(\Sigma) \mid \|\alpha\| = \|\Box\alpha\|\}.$$

$$SE(\omega) = \{\|\alpha\| \in SE(\Sigma) \mid \omega \in \|\alpha\|\}.$$

Thus $SE(\Sigma)$ is the set of all self-evident sets. On the other hand, $SE(\omega)$ is the set of all self-evident sets that are true at $\omega \in \Omega_\Sigma$. Notice that

$$\|\alpha\| \in SE(\Sigma) \qquad \text{if and only if} \qquad \alpha \leftrightarrow \Box\alpha \in \Sigma.$$

In words, a self-evident sentence is the one that cannot be true without being known. Self-evident sets play an important role in the literature in that the notion of common knowledge is frequently defined in terms of self-evident sets (see Section 5). In fact, in S4 information structures self-evident sets possesses a lot of tractable properties.

3.2 Lemma. *For every $\omega \in \Omega_\Sigma$,*

(1) $P(\omega) = \bigcap SE(\omega)$.

(2) *If $P(\omega)$ is expressible then $P(\omega) \in SE(\Sigma)$.*

(3) *For every $\|\alpha\| \in SE(\Sigma)$, $\|\alpha\| = \bigcup_{\omega \in \|\alpha\|} P(\omega)$.*

(4) *$\langle SE(\Sigma), \subset\rangle$ is a distributive lattice.*

(5) *$SE(\omega)$ is a prime filter in $\langle SE(\Sigma), \subset\rangle$.*

Let us comment on these properties. First, (1) simplifies the analysis of the expressibility significantly, for it allows us to restrict our attention to the set of self-evident sets at ω, $SE(\omega)$, instead of the set of known sentences at ω, $\kappa_\Sigma^{-1}(\omega)$. (2) says that if $P(\omega)$ is expressible then it has to be self-evident. Thus (3), together with (2), tells us that the possibility sets form a 'base' for the self-evident sets if possibility sets are expressible. Notice that, in a traditional information structure model, every possibility set is self-evident without any qualification. We have defined, however, self-evidence property in such a way that it applies to only expressible sets. We claim that this restriction is conceptually natural because our framework started from a theory on a formal language and the information structure is a derived construct rather than a primitive one. Finally, (4) and (5) set the environment in which we derive our characterizations of expressibility.

Here we should notice that $\langle SE(\Sigma), \subset\rangle$ need not be a Boolean algebra. In other words, the set of self-evident sets need not be closed under complementation.

By Lemma 3.2, we have local characterization of the expressibility.

3.3 Lemma. *For every $\omega \in \Omega_\Sigma$, $P(\omega)$ is expressible if and only if $SE(\omega)$ is a principal filter.*

We are also interested in 'global' case, i.e., the expressibility of the canonical information structure itself. By Lemma 3.2 (5) and 3.3, it is necessary for the information structure to be expressible that every prime filter in $\langle SE(\Sigma), \subset\rangle$ that appears as $SE(\omega)$ for some $\omega \in \Omega_\Sigma$ is principal. Thus we want to know which prime filter has the form $SE(\omega)$. Actually, it turns out that all prime filter in $\langle SE(\Sigma), \subset\rangle$ have the form $SE(\omega)$.

3.4 Lemma. *For every prime filter F in $\langle SE(\Sigma), \subset\rangle$, there exists $\omega \in \Omega_\Sigma$ such that $F = SE(\omega)$.*

By Lemma 3.2 (5), 3.3, and 3.4 we have that

3.5 Corollary. *The canonical information structure $\langle \Omega_\Sigma, P(\cdot)\rangle$ is expressible if and only if every prime filter in $\langle SE(\Sigma), \subset\rangle$ is principal.*

Let us say that a partially ordered set $\langle L, \preceq\rangle$ is *well-founded* if there is a minimal (with respect to $\preceq$) element $a \in A$ for every nonempty subset $A \subset L$. An *infinitely decreasing chain* in $\langle L, \preceq\rangle$ is a subset $C = \{c_n : n \in \mathbb{N}\}$ of L such that if $n < m$ then $c_m \preceq c_n$ and $c_n \neq c_m$ for every $n, m \in \mathbb{N}$ with the usual order $<$ on $\mathbb{N}$. Now we are ready to state the main result of this section.

3.6 Lemma. *For a distributive lattice $\langle L, \preceq\rangle$, the following conditions are equivalent.*

(1) *$\langle L, \preceq\rangle$ is well-founded.*
(2) *There is no infinitely decreasing chain in $\langle L, \preceq\rangle$.*
(3) *Every prime filter in $\langle L, \preceq\rangle$ is principal.*
(4) *Every filter in $\langle L, \preceq\rangle$ is principal.*

3.7 Theorem. *The canonical information structure $\langle \Omega_\Sigma, P(\cdot)\rangle$ is expressible if and only if $\langle SE(\Sigma), \subset\rangle$ is well-founded.*

It follows from (**K**), (**N**), (**T**), (**4**), and Proposition 2.4 (3) that if $P(\omega) = \|\alpha\|$ and $\|\alpha\| \subset \|\beta\|$ then $\alpha \to \beta \in \Sigma$ and $\Box\alpha \to \Box\beta \in \Sigma$, independent of whether β is self-evident or not. Thus we can justifiably say that $P(\omega)$ is expressible if and only if there is a single sentence α such that all known sentences at ω are precisely those that are derived from α in the theory Σ.

4. Expressibility of Information Structure: Consequences

By Theorem 3.7 from the last section, the canonical possibility correspondence is expressible if and only if $SE(\Sigma)$, the set of self-evident sets, is well-founded. Given this characterization, we shall derive several consequences of the expressibility of the canonical information structure. First, we will focus on minimal possibility sets, whose existence are guaranteed by the well-foundedness of $SE(\Sigma)$. The first result of this section states that the number of minimal possibility set is finite. In other words, there are finitely many sentences $\mu_1, \ldots, \mu_n \in \mathbb{S}$, such that each μ_h $(h = 1, \ldots, n)$ is an expressing sentence of a minimal possibility set. This follows from the compactness of the state space Ω_Σ. Since there are only finitely many such μ_h, we can consider the sentence $\mu_1 \vee \cdots \vee \mu_n$ and the corresponding self-evident set $\|\mu_1 \vee \cdots \vee \mu_n\|$. It is immediate (with Lemma 4.1 below) that the canonical information structure is partitional if and only if the sentence $\mu_1 \vee \cdots \vee \mu_n$ is a theorem in the theory Σ, that is, $\|\mu_1 \vee \cdots \vee \mu_n\| = \Omega_\Sigma$. The second result gives a less obvious characterization of partitional information structures in terms of expressibility.

Throughout this section we assume that the canonical information structure is expressible. We start with a lemma, which states, as one might expect, that minimal nonempty self-evident sets are actually possibility sets and are mutually disjoint.

4.1 Lemma.

(1) *If $\|\alpha\|$ is minimal in $SE(\Sigma) - \{\varnothing\}$ then there is $\omega \in \Omega_\Sigma$ such that $\|\alpha\| = P(\omega)$.*
(2) *Minimal elements in $SE(\Sigma) - \{\varnothing\}$ are mutually disjoint.*

Since our state space Ω_Σ is compact with respect to the Stone topology τ_Σ, $\bigcap\{\bar{A} \mid A \in \mathcal{A}\} \neq \varnothing$ for every $\mathcal{A} \subset 2^{\Omega_\Sigma}$ with the finite intersection property, where $\bar{A}$ is the closure of a set A. The finiteness of the following result comes from this fact. For details, see the proof in the Appendix.

4.2 Theorem. *Assume that $\langle \Omega_\Sigma, P(\cdot) \rangle$ is expressible. Then the number of minimal possibility sets is finite.*

This result has nothing to do with the cardinality of the set $\mathbb{S}$ of sentences in $\mathcal{L}$, since we have imposed no a priori restriction on the cardinality of $\mathbb{S}$. It is solely the expressibility condition that is responsible for Theorem 4.2.

In the common knowledge literature, various kinds of assumptions or conditions that ensure finiteness of minimal possiblity sets have appeared (e.g., Geanakoplos and Polemarchakis (1982), Bacharach (1985), Shin (1993)). Compared to these previous studies, Theorem 4.2 is unique in that it gives us the finiteness as a consequence of the expressibility, which does not involve any apparent finiteness flavor. In this regard, we recall that our setup is general enough to accomodate sentences of 'infinite length' (see Section 2). Thus it is misleading to think that the finiteness in Theorem 4.2 comes from something like the finiteness of the 'length of a sentence', a statement that would have been meaningful if we had stick with the standard framework of mathematical logic literature. In other words, as long as the theory Σ in the underlying language has the structure of Boolean algebra, the expressibility implies the finiteness of minimal possibility sets. This is the main message of Theorem 4.2.

By Lemma 4.1 and Theorem 4.2, the expressible canonical information structure is partitional if and only if the sentence $\mu_1 \vee \cdots \vee \mu_n$ is a theorem in the theory Σ. There is a less obvious characterization of expressible partitional information structures. This is done by a stronger version of the expressibility condition.

To motivate the stronger condition, let us recall the traditional interpretation that we associate to information structure models. A possibility set $P(\omega)$ is thought to be the information that the decision maker k receives when a state $\omega \in \Omega_\Sigma$ occurs. The crucial point here is that what the meaning of the information $P(\omega)$ is supposed to be. The standard interpretation is that k thinks that all states and only those states in $P(\omega)$ are possibly to have occurred. It is worth noticing here that the intended interpretation involves a modal qualifier 'possibly'. Given the spirit of expressibility, we should ask the following question. Is this 'possibility' is expressible in the language $\mathcal{L}$?

Let us say that the canonical information structure $\langle \Omega_\Sigma, P(\cdot)\rangle$ associated with a theory Σ is *strongly expressible* if for every $\omega \in \Omega_\Sigma$ there is a sentence $\alpha_\omega \in \mathbb{S}$ such that $P(\omega) = \|\Diamond\alpha_\omega\|$. It is clear that if $\langle \Omega_\Sigma, P(\cdot)\rangle$ is strongly expressible then it is expressible. We say that $\langle \Omega_\Sigma, P(\cdot)\rangle$ is *partitional* if $\{P(\omega) \mid \omega \in \Omega_\Sigma\}$ forms a partition of Ω_Σ.

4.3 Theorem. *Assume that $\langle \Omega_\Sigma, P(\cdot)\rangle$ is expressible. Then $\langle \Omega_\Sigma, P(\cdot)\rangle$ is strongly expressible if and only if it is partitional.*

Theorem 4.3 says that we have a partitional information structure if and only if the meaning of possibility that is associated with possibility sets is in fact expressible in the underlying formal language.

As stated in the Introduction, the main idea of expressibility condition is a stipulation that there should be an exact parallelism between traditional definition of knowledge and descriptions that are included in states ω of the world. The strong expressibility further demands that that parallelism should extend to traditional interpretation of possibility. We claim that the strong expressibility is a natural condition to consider, as long as we take the intended interpretation to possibility sets seriously. Theorem 4.3 then guarantees that, if we do so, what we are going to have is a partitional information structure. In this sense, the idea of expressibility offers a justification for partitional information structures.

By noticing the fact that $\|\Diamond\alpha\| = \|\neg\Box\neg\alpha\|$, we get a corollary.

4.4 Corollary. *Assume that $\langle \Omega_\Sigma, P(\cdot)\rangle$ is expressible. Then $\langle \Omega_\Sigma, P(\cdot)\rangle$ is partitional if and only if $\|\Box\alpha\| = \|\alpha\| = \|\Diamond\alpha\|$ holds for every $\|\alpha\| \in SE(\Sigma)$.*

The corollary shows that we have a partitional information structure if and only if all the three notions of knowledge, truth, and possibility coincide for every self-evident set. In other words, we have *non*-partitional information structure if and only if there is a state $\omega \in \Omega_\Sigma$ and a self-evident set $\|\alpha\|$ such that k thinks $\|\alpha\|$ is possibly to be true at ω while it is actually false.

5. Common Knowledge and Common Theory

In this section, we discuss issues concerning the notion of common knowledge. There are two kinds of common knowledge in the literature. One is the formal notion of common knowledge of *events*, defined by Aumann (1976). The other is 'informal', which is the one we refer when we speak of, for example, common knowledge of information structures. It is to be noted that observations similar in sprit to those given in this section are found in Aumann (1989) and Shin (1993).

As for the formal notion of common knowledge, the well-foundedness condition has some straightforward implications. Let us say that an expressible set $\|\alpha\|$ is *common* *knowledge* at $\omega \in \Omega_\Sigma$ if $\omega \in \|\Box^{(n)}\alpha\|$ for every $\Box^{(n)} \in \Box^*$, where $\Box^*$ is the set of all finite strings made up from $\Box_i$ and $\Box_j$.

5.1 Proposition. *For every $\|\alpha\| \in EX(\Sigma)$ and $\omega \in \Omega_\Sigma$, $\|\alpha\|$ is common knowledge at ω if and only if there is $\|\alpha^*\|$ such that $\omega \in \|\alpha^*\| \subset \|\alpha\|$ and $\|\alpha^*\| \in SE_i(\Sigma) \cap SE_j(\Sigma)$. Moreover, the set $\|\alpha^*\|$ has the property that $\|\alpha\|$ is common knowledge at ω if and only if $\omega \in \|\alpha^*\|$.*

The first half is the familiar characterization that the iterate definition of common knowledge and the definition in terms of self-evident sets coincide. The second half says that the statement 'it is common knowledge that α' is expressed by α^*. That is, the expressibility of possibility sets carries over that of common knowledge. In the proof of 5.1 in the Appendix, one finds, as usual, that $\|\alpha^*\|$ is the intersection of all $\|\Box^{(n)}\alpha\|$. It is worth noticing that, by Lemma 3.6 (2), every common knowledge is *n-th mutual knowledge*, i.e., the iteration of $\Box_i$ and $\Box_j$ must terminate in some $n \in \mathbb{N}$ steps.

Now let us turn to the 'informal' notion of common knowledge. We shall verify that, in our setup, the common knowledge of an information structure is a characterizing condition for a partitional information structure.

We first observe that there is a close relationship between the interpretation of a state of the world as a complete set of descriptions and the conceptual assumption that the information structure is itself common knowledge. This is exemplified in the following passage:

> Included in the full description of a state ω of the world is the manner in which information is imparted to the two persons (Aumann (1976, p.1237)).

Since our state ω of the world is indeed a set of description, we can find some formal statements that correspond to the quotation. First of all, by successive application of (**N**), we have that, for every $\alpha \in \Sigma$ and $n \in \mathbb{N}$,

$$\Box_{h_1}\Box_{h_2}\cdots\Box_{h_n}\alpha \in \Sigma,$$

where $h_1, \ldots, h_n \in \{i, j\}$. That is, the theory Σ in fact can be regarded as 'common theory' between i and j (Aumann (1989), Shin (1993)). In other words, each theorem in Σ is a common theorem between i and j.

Now it follows from Proposition 2.6 and Theorem 4.3 that

5.2 Proposition.

(1) *For every $\omega \in \Omega_\Sigma$, $\|\alpha\| \in EX(\Sigma)$, and $k = i, j$, $\omega \in \|\Diamond_k\alpha\|$ if and only if $P^k(\omega) \cap \|\alpha\| \neq \varnothing$.*

(2) *Assume that $\langle\Omega_\Sigma, P^i(\cdot), P^j(\cdot)\rangle$ is expressible. Then it is strongly expressible if and only if*

$$\|\Diamond_k\alpha\| = \bigcup_{\omega\in\Lambda_k(\alpha)} P^k(\omega)$$

for every $\|\alpha\| \in EX(\Sigma)$, where $\Lambda_k(\alpha) = \{\omega \in \Omega_\Sigma \mid P^k(\omega) \cap \|\alpha\| \neq \varnothing\}$ and $k = i, j$.

Proposition 5.2 (1) states that the possibility operator $\Diamond_k$ in the language $\mathcal{L}$ corresponds to the intersection operation on the canonical state space Ω_Σ. Moreover, it follows from 5.2 (2) that, *if* the information structure is strongly expressible, then

$$P^k(\omega) \cap \|\alpha\| \neq \varnothing \qquad \text{if and only if} \qquad \|\alpha^k_\omega\| \subset \|\Diamond_k \alpha\|,$$

where $P^k(\omega) = \|\alpha^k_\omega\|$. In other words, by Lemma 2.4 (3),

$$P^k(\omega) \cap \|\alpha\| \neq \varnothing \qquad \text{if and only if} \qquad \alpha^k_\omega \to \Diamond_k \alpha \in \Sigma.$$

We remark that this equivalence is available only if we have a strongly expressible information structure. Also, the set $\|\alpha\|$ is not restricted to be a self-evident set. In particular, it can be a possibility set of the other decision maker. From our perspective, this is the most interesting case. Let $P^i(\omega) = \|\alpha^i_\omega\|$ and $P^j(\omega) = \|\alpha^j_\omega\|$. Then the equivalence above turns into

$$P^i(\omega) \cap P^j(\omega) \neq \varnothing \qquad \text{if and only if} \qquad \alpha^i_\omega \to \Diamond_i \alpha^j_\omega \in \Sigma \quad \text{and} \quad \alpha^j_\omega \to \Diamond_j \alpha^i_\omega \in \Sigma.$$

This means that the reachability relation of Aumann (1976) can be defined in terms of common theorems in Σ.[13]

In summary, in a strongly expressible information structure, the (nonempty) intersection relationship and the inclusion relationship among expressible sets are recorded as theorems in the common theory Σ in the form of sentences involving $\Diamond_k$ and $\Box_k$, respectively. In this sense, Aumann's (1976) claim is formally justifiable in a strongly expressible information structure. Not only that, we have seen that his claim is actually a characterizing condition for a strongly expressible (i.e., partitional) information structure. That is, the common knowledge assumption and partitional information structures are tied together.

6. Concluding Remarks

We started with an assumption that there is a set of descriptions that is closed with respect to 'relevance' to the decision making situation under study. We proposed the expressibility condition as a necessary condition for the relevance-closedness. Our results tell us what relevance-closed, or 'all-inclusive', information structures look like. In this sense, our line of reasoning is in the reverse direction to that of the belief approach (Mertens and Zamir (1985), Brandenburger and Dekel (1993)). Given a set of 'parameters', they started to construct hierarchies of beliefs and showed that this construction has a fixed point. That is, we can close the model. In contrast, by imposing the expressibility condition, we have started from a closed model and then derived its properties in the logical approach environment.

Within the logical approach itself, the conditions similar to Theorem 3.7 and 4.2 have already appeared as assumptions (Samet (1990), Shin (1993)). In addition, the finite partition assumption has played key roles in several convergence results (Geanakoplos and Polemarchakis (1982), Bacharach (1985)). Our study shows that their assumptions are bound to be satisfied in information structures where states of the world are in fact all-inclusive.

It seems difficult to say that there is a general agreement about the status of partitional information structures. While some authors (e.g., Brandenburger, Dekel, and Geanakoplos (1992)) have described non-partitional information structures as 'deviations

[13] Notice that, by Lemma 2.1, $\alpha \to \Diamond_i \beta \in \Sigma$ and $\beta \to \Diamond_j \gamma \in \Sigma$ imply $\alpha \to \Diamond_i \Diamond_j \gamma \in \Sigma$.

from rationality', thereby implying that the partitional case is the standard one, several views that question the validity of the partitional case have also been addressed (e.g., Binmore (1992), Shin (1993)). Our two characterizations of partitional information structures, Theorem 4.3 and Proposition 5.2, shed another light on the status of partitional information structures. If we require not only the traditional definition of knowledge but also the intended interpretation for information structure models to be captured by the underlying system of descriptions then we must have a patitional structure. Alternatively, as long as we stick with the common knowledge assumption, there is a good reason to believe that decision makers do have information partitions. Putting together, we have shown that partitional structures are formally justifiable as models of idealized situations where states of the world are complete and the common knowledge assumption is satisfied.

In the common knowledge literature, one often finds issues concerning the cardinality of state spaces (e.g., Hart, Heifetz and Samet (1993)). Theorem 4.2 states, on the other hand, that minimal *possibility sets* are going to be finitely many. We do not know whether there is a nontrivial expressible canonical information structure that has infinitely many states of the world.[14]

The formal development of our study shows that our results continue to hold as long as the theory Σ in a formal language $\mathcal{L}$ from which we started has the structure of Boolean algebra. In particular, as long as we restrict ourselves to sentential (as opposed to quantificational) setup, any kinds of strengthening of the theory and/or language, such as allowing uncountably many sentences or adding infinitely operations, would not alter the conclusion. In this sense, our results are robust.

Appendix

1. Definitions and Results from Lattice Theory

The following concepts and standard results from lattice theory are used in the body of the paper. For details, see, for example, Davey and Priestry (1990).

Let L be a nonempty set. A *partial order* on L is a binary relation $\preceq$ on L that is reflexive, transitive, and anti-symmetric (i.e., $a \preceq b$ and $b \preceq a$ imply $a = b$ for every $a, b \in L$). We read $a \preceq b$ as 'a is less than or equal to b'. A partially ordered set is a pair $\langle L, \preceq \rangle$, where $\preceq$ is a partial order on L. A *chain* in a partially ordered set $\langle L, \preceq \rangle$ is a nonempty subset C of L such that either $a \preceq b$ or $b \preceq a$ for every $a, b \in C$. A partially ordered set $\langle L, \preceq \rangle$ is a *lattice* if for every $a, b \in L$ the greatest lower bound (*infimum* or *meet*) of $\{a, b\}$, denoted by $a \wedge b$, and the least upper bound (*supremum* or *join*) of $\{a, b\}$,

[14] A (trivially) expressible canonical information structure with continuum of states can be constructed as follows. Consider the set of well-formed formulas of non-modal sentencial logic with countably infinite atomic sentences. For each formula φ define recursively that

$$\Box\varphi = \begin{cases} \top, & \text{if } \varphi \text{ is a tautology} \\ \bot, & \text{otherwise.} \end{cases}$$

Consider the set $\Sigma = \{\varphi \mid \Box\varphi = \top\}$. It is easily verified that Σ is in fact an S5 system and that $P(\omega) = \Omega_\Sigma$ for every $\omega \in \Omega_\Sigma$ in the associated information structure. On the other hand, Ω_Σ has the cardinality of $2^{\aleph_0}$ since there are countably infinite atomic sentences.

denoted by $a \vee b$, exist. A *sublattice* of a lattice $\langle L, \preceq \rangle$ is a subset A of L such that $a \wedge b \in A$ and $a \vee b \in A$ for every $a, b \in A$. A lattice $\langle L, \preceq \rangle$ is *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ hold for every $a, b, c \in L$. A lattice $\langle L, \preceq \rangle$ is *bounded* if there are $\bot, \top \in L$ such that $\bot$ ($\top$, respectively) is a least (greatest, respectively) element in L. A *complement* of an element $a \in L$ of a bounded lattice $\langle L, \preceq \rangle$ is $b \in L$ such that $a \wedge b = \bot$ and $a \vee b = \top$. A complement of $a \in L$ is unique if it exists and in this case we denote it by $\neg a$. A bounded distributive lattice $\langle L, \preceq \rangle$ is a *Boolean algebra* if every element of L has a complement.

In what follows let $\langle L, \preceq \rangle$ be a bounded distributive lattice. A *filter* in L is a nonempty subset F of L such that

(1) For every $a \in F$ and $b \in L$, if $a \preceq b$ then $b \in F$.
(2) For every $a, b \in F$, $a \wedge b \in F$.

Note that, for every $a \in L$, the set $\uparrow a = \{b \in L \mid a \preceq b\}$ is a filter in L. In particular, $L = \uparrow \bot$ is a filter in L. A filter F in L is *principal* if there is $a \in L$ such that $F = \uparrow a$. Any filter that is not principal is called *non-principal*. Note that a filter F is principal if and only if F has a minimum element. A filter F in L is *prime* if $a \vee b \in F$ implies either $a \in F$ or $b \in F$ for every $a, b \in L$. A filter F is *maximal* if L is the only filter that strictly includes (with respect to set inclusion) F.

A.1.1 Theorem. *Let $\langle L, \preceq \rangle$ be a Boolean algebra and F be a filter in L. Then the following statements are equivalent.*

(1) *F is a maximal filter.*
(2) *F is a prime filter.*
(3) *For every $a \in L$ either $a \in F$ or $\neg a \in F$ and not both.*

A.1.2 Theorem. *Let $\langle L, \preceq \rangle$ be a Boolean algebra and F be a filter in L such that $F \neq L$. Then there is a maximal filter F' in L such that $F \subset F'$.*

An *ideal* in L is a nonempty subset I of L such that

(1) For every $a \in I$ and $b \in L$, if $b \preceq a$ then $b \in I$.
(2) For every $a, b \in I$, $a \vee b \in I$.

A.1.3 Theorem. *Let F (I, respectively) be a filter (ideal, respectively) in $\langle L, \preceq \rangle$ such that $F \cap I = \varnothing$. Then there is a prime filter F' in L such that $F \subset F'$ and $F' \cap I = \varnothing$.*

2. Proofs

Definition. Here we give a precise definition of *tautology* in our language $\mathcal{L}$. Let $\{p_n \mid n \in \mathbb{N}\}$ be sentential variables. Denote by Γ the smallest set satisfying: (1) $p_n \in \Gamma$ for every $n \in \mathbb{N}$; (2) $F \in \Gamma$; (3) If $\varphi, \psi \in \Gamma$ then $(\varphi \Rightarrow \psi) \in \Gamma$; (4) If $\varphi \in \Gamma$ then $L_k \varphi \in \Gamma$ where $k = i, j$. An element of Γ is called a *formula*. We omit parentheses occasinally. Define $\neg \varphi \equiv \varphi \Rightarrow F$, where $\equiv$ denotes syntactic equality. We take following axiom schemata: (1) $\varphi \Rightarrow (\psi \Rightarrow \varphi)$; (2) $(\varphi \Rightarrow (\psi \Rightarrow \eta)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\psi \Rightarrow \eta))$; (3) $\neg\neg\varphi \Rightarrow \varphi$, where $\varphi, \psi, \eta \in \Gamma$.

A finite sequence $\langle \varphi_1, \dots, \varphi_n \rangle$ of formulas is a *deduction* if either (D1) φ_h is an axiom; or (D2) There are $r, s < h$ such that $\varphi_r \equiv \varphi_s \Rightarrow \varphi_h$ holds for $h = 1, \dots, n$. A formula φ is a *syntactic tautology* if there is a deduction $\langle \varphi_1, \dots, \varphi_n \rangle$ such that $\varphi_n \equiv \varphi$. Notice that if $\langle \varphi_1, \dots, \varphi_n \rangle$ is a deduction then φ_h is a syntactic tautology for every $h = 1, \dots, n$.

Let $r : \{p_n \mid n \in \mathbb{N}\} \to \mathbb{S}$ be *any* mapping. We extend r to $R : \Gamma \to \mathbb{S}$ as follows: (1) If $\varphi \in \{p_n \mid n \in \mathbb{N}\}$ then $R(\varphi) = r(\varphi)$; (2) $R(F) = \bot$; (3) If $\varphi, \psi \in \Gamma$ then $R(\varphi \Rightarrow \psi) = R(\varphi) \to R(\psi)$; (4) If $\varphi \in \Gamma$ then $R(L_k \varphi) = \Box_k R(\varphi)$ where $k = i, j$. By the recursive nature of Γ, R is uniquely detrmined by r. We call any mapping thus constructed a *realization*. We finally define: A sentence $\alpha \in \mathbb{S}$ is a *tautology* if there is a realization $R : \Gamma \to \mathbb{S}$ and a syntactic tautology $\varphi \in \Gamma$ such that $R(\varphi) = \alpha$.

It is easy to show that if $\alpha \in \mathbb{S}$ is a tautology then $\alpha \in \Sigma$. This is done by induction on the length of a deduction $\langle \varphi_1, \dots, \varphi_n \rangle$, where $R(\varphi_n) = \alpha$, together with (**PL**) and (**MP**). This proves Lemma 2.1 (1).

Proof of 2.1: For (1), see the last paragraph of above definition. For others, see Chellas (1980, Chapter 4). □

Proof of 2.2: See Bell and Slomson (1969, Chapter 2). □

Remark: Although our setup is identical to that of neither Chellas (1980) nor Bell and Slomson (1969), essentially same proofs as theirs work for our case as well.

Proof of 2.3: By Theorem A.1.1. □

Proof of 2.4: (1) (thus (2)) follows from Theorem A.1.2. The rest follows from involved definitions and Proposition 2.3. □

Proof of 2.5: An application of the *Stone representation theorem for Boolean algebras*. See Davey and Priestley (1990, Theorem 10.8). □

Remark: Until Proof of 5.1, we fix $k = i, j$. Accordingly, in order to have notational convenience, we omit subscripts (superscripts, respectively) for $\Box$ ($P(\cdot)$, respectively) and related objects.

Proof of 2.6: Clearly, (1) implies (2), which, in turn, implies (3). The following three claims prove (1).

Claim 1: If $\omega \in \|\Box\alpha\|$ then $P(\omega) \subset \|\alpha\|$.

Proof: Suppose $\omega \in \|\Box\alpha\|$ and pick $\omega' \in P(\omega)$. Since $\omega \in \|\Box\alpha\|$, $[\alpha] \in \kappa_{\mathbb{Y}}^{-1}(\omega)$. Since $\omega' \in P(\omega)$, $\kappa_{\mathbb{Y}}^{-1}(\omega) \subset \omega'$. Thus $[\alpha] \in \omega'$ or $\omega' \in \|\alpha\|$.

Claim 2: $P(\omega) = \bigcap\{\|\alpha\| \mid \omega \in \|\Box\alpha\|\}$.

Proof: $P(\omega) \subset \bigcap\{\|\alpha\| \mid \omega \in \|\Box\alpha\|\}$ follows from Claim 1. The converse follows from involved definitions.

Claim 3: If $P(\omega) \subset \|\alpha\|$ then $\omega \in \|\Box\alpha\|$.

Proof: Assume $\omega \notin \|\Box\alpha\|$. We want to show that there is $\omega' \in P(\omega)$ such that $\omega' \in \|\neg\alpha\|$. By Claim 2 and the fact that $\Omega_{\mathbb{Y}}$ is compact, it suffices to show that the set

$$\{\|\gamma\| \mid \omega \in \|\Box\gamma\|\} \cup \{\|\neg\alpha\|\}$$

has the finite intersection property. For a contradiction, assume that there exist $\|\gamma_1\|, \ldots, \|\gamma_n\| \in \{\|\gamma\| \mid \omega \in \|\Box\gamma\|\}$ such that $\|\gamma_1\| \cap \cdots \cap \|\gamma_n\| \cap \|\neg\alpha\| = \varnothing$. Then $\|\gamma_1\| \cap \cdots \cap \|\gamma_n\| \subset \|\alpha\|$. Thus by Proposition 2.4 (3), $\gamma_1 \wedge \cdots \wedge \gamma_n \to \alpha \in \Sigma$. Since $\|\gamma_1\|, \ldots, \|\gamma_n\| \in \{\|\gamma\| \mid \omega \in \|\Box\gamma\|\}$, it follows from Lemma 2.3 (2) that $|\Box\gamma_1 \wedge \cdots \wedge \Box\gamma_n| \in \omega$. Therefore, by Lemma 2.1 (2) and 2.3 (3), $\omega \in \|\Box\alpha\|$. □

A.2.1 Lemma. *In the setup in Section 3, the following statements hold:*

(1) *For every $\omega \in \Omega_\Sigma$, if $\omega \in \|\Box\alpha\|$ then $P(\omega) \subset \|\Box\alpha\|$.*

(2) *For every $\|\alpha\| \in EX(\Sigma)$, $\|\Box\alpha\| \in SE(\Sigma)$.*

(3) *$P(\omega) = \bigcap\{\|\Box\alpha\| \mid \omega \in \|\Box\alpha\|\}$.*

Proof of A.2.1:

(1) $\omega \in \|\Box\alpha\|$ and (4) together imply that $\omega \in \|\Box\Box\alpha\|$. Thus $|\alpha| \in \kappa_\Sigma^{-1}(\omega)$. Therefore by Proposition 2.6 (1), $P(\omega) \subset \|\Box\alpha\|$.

(2) By (**T**) and (**4**).

(3) By Proposition 2.6 (2) and (**T**), $\bigcap\{\|\Box\alpha\| \mid \omega \in \|\Box\alpha\|\} \subset P(\omega)$. Conversely, pick $\omega' \in P(\omega)$. Let $\omega \in \|\Box\alpha\|$. By (1) above, $P(\omega) \subset \|\Box\alpha\|$. Thus $\omega' \in \|\Box\alpha\|$. This shows that $\omega' \in \bigcap\{\|\Box\alpha\| \mid \omega \in \|\Box\alpha\|\}$. Therefore $P(\omega) \subset \bigcap\{\|\Box\alpha\| \mid \omega \in \|\Box\alpha\|\}$. □

Proof of 3.1:

(1) By Lemma 2.3 (3) and (**T**), if $|\Box\alpha| \in \omega$ then $|\alpha| \in \omega$. Thus $\kappa_\Sigma^{-1}(\omega) \subset \omega$, which means that $\omega \in P(\omega)$ by the definition of $P(\omega)$.

(2) Let ω'. Then $\kappa_\Sigma^{-1}(\omega) \subset \omega'$. Pick $\omega'' \in P(\omega')$. It suffices to show that $\omega'' \in P(\omega)$, or equivalently, $\kappa_\Sigma^{-1}(\omega) \subset \omega''$. By (**4**) and Lemma 2.3 (3), if $|\alpha| \in \kappa_\Sigma^{-1}(\omega)$ then $|\Box\alpha| \subset \kappa_\Sigma^{-1}(\omega)$. Since $\kappa_\Sigma^{-1}(\omega) \subset \omega'$, if $|\alpha| \in \kappa_\Sigma^{-1}(\omega)$ then $|\Box\alpha| \subset \omega'$. Thus by definition of $\kappa_\Sigma^{-1}(\cdot)$, if $|\alpha| \in \kappa_\Sigma^{-1}(\omega)$ then $|\alpha| \in \kappa_\Sigma^{-1}(\omega')$, which means that $\kappa_\Sigma^{-1}(\omega) \subset \kappa_\Sigma^{-1}(\omega')$. On the other hand, $\kappa_(^{-1}\Sigma)(\omega') \subset \omega''$ since $\omega'' \in P(\omega')$. Therefore $\kappa_\Sigma^{-1}(\omega) \subset \omega''$. □

Proof of 3.2:

(1) Let $A = \{\|\Box\alpha\| \mid \omega \in \|\Box\alpha\|\}$. In view of A.2.1 (3), it suffices to show that $A = SE(\omega)$. Pick $\|\Box\beta\| \in A$. Then $\omega \in \|\Box\beta\|$. By A.2.1 (2), $\|\Box\beta\| \in SE(\omega)$. Thus $A \subset SE(\omega)$. Conversely, Pick $\|\beta\| \in SE(\omega)$. Then $\omega \in \|\beta\|$ and $\|\beta\| = \|\Box\beta\|$. Thus $\omega \in \|\Box\beta\|$. Therefore, $\|\Box\beta\| \in A$. Since $\|\beta\| = \|\Box\beta\|$, $\|\beta\| \in A$. This shows that $SE(\omega) \subset A$. Consequently, $A = SE(\omega)$.

(2) Assume that $P(\omega) = \|\alpha\|$ for some $\alpha \in \mathbb{S}$. Then Then by Proposition 2.6 (1), $\omega \in \|\Box\alpha\|$. By Lemma A.2.1 (2), $\|\Box\alpha\| \in SE(\Sigma)$. By Lemma 3.2 (3), $P(\omega) \subset \|\Box\alpha\|$. Since $P(\omega) = \|\alpha\|$ by assumption, $\|\alpha\|\|\Box\alpha\|$. Thus by (**T**), $\|\alpha\| = \|\Box\alpha\| \in SE(\Sigma)$.

(3) $\|\alpha\| \subset \bigcup_{\omega \in \|\alpha\|} P(\omega)$ follows from (**T**). The other direction follows from (**4**), $\|\alpha\| = \|\Box\alpha\|$, and A.2.1 (1).

(4) Since $\langle EX(\Sigma), \subset\rangle$ is a Boolean algebra, it is distributive by definition. Since every sublattice of a distributive lattice is distributive, it suffices to show that $\langle SE(\Sigma), \subset\rangle$ is a sublattice of $\langle EX(\Sigma), \subset\rangle$. By Lemma 2.1 (3), $\langle SE(\Sigma), \subset\rangle$ is closed under intersections (i.e., meets). Let us consider unions (i.e., joins). Pick $\|\alpha\|, \|\beta\| \in SE(\Sigma)$. We want to show $\|\alpha\| \cup \|\beta\| = \|\Box(\alpha \vee \beta)\|$. By (**T**), we have $\|\Box(\alpha \vee \beta)\| \subset \|\alpha\| \cup \|\beta\|$. On the other hand, $\|\alpha\| \cup \|\beta\| = \|\Box\alpha\| \cup \|\Box\beta\|$ and Lemma 2.1 (5) give the converse inclusion.

(5) By construction, ω is a maximal filter in $\mathbb{S}/\Sigma$. Since $\mathbb{S}/\Sigma$ and $EX(\Sigma)$ are isomorphic,

$$EX(\omega) = \{\|\alpha\| \in EX(\Sigma) \mid \omega \in \|\alpha\|\}$$

is a maximal filter in $EX(\Sigma)$. In general, every maximal filter in a distributive lattice is prime (Davey and Priestley (1990, Chapter 9)). Thus $EX(\omega)$ is a prime filter in $EX(\Sigma)$. On the other hand, by definition, $SE(\omega) \subset EX(\omega)$. Therefore $SE(\omega)$ is a filter in $SE(\Sigma)$ since $SE(\Sigma)$ is closed under intersection. Let us check the primeness. Pick $\|\alpha\|, \|\beta\| \in SE(\Sigma)$ and assume that $\|\alpha\| \cup \|\beta\| \in SE(\omega)$. Then $\|\alpha\| \cup \|\beta\| \in EX(\omega)$. Since $EX(\omega)$ is prime, either $\|\alpha\| \in EX(\omega)$ or $\|\beta\| \in EX(\omega)$. Thus, by the definition of $SE(\omega)$ and the assumption that $\|\alpha\|, \|\beta\| \in SE(\Sigma)$, either $\|\alpha\| \in SE(\omega)$ or $\|\beta\| \in SE(\omega)$, which means that $SE(\omega)$ is prime. □

Proof of 3.3:

($\Rightarrow$) Let $P(\omega) = \|\alpha\|$. Then by Lemma 3.2 (2), $P(\omega) \in SE(\Sigma)$. Therefore by Lemma 3.2 (1), $SE(\omega)$ is principal.

($\Leftarrow$) If $SE(\omega)$ is principal then there is $\|\alpha\| \in SE(\omega)$ such that $\|\alpha\| = \bigcap SE(\omega)$. Thus by Lemma 3.2 (1), $P(\omega) = \bigcap SE(\omega) = \|\alpha\| \in EX(\Sigma)$. □

Proof of 3.4: Let F be a prime filter in $SE(\Sigma)$. Consider the set

$$F' = \{\|\neg\beta\| \mid \|\beta\| \in SE(\Sigma) \quad \text{and} \quad \|\beta\| \notin F\}.$$

Notice that $F' \subset EX(\Sigma)$ but it need not be the case that $F' \subset SE(\Sigma)$.

Claim: $F \cup F'$ has the finite intersection property.

Proof: Since F is a filter (in $SE(\Sigma)$) it suffices to show that

$$\|\alpha\| \cap \|\neg\beta_1\| \cap \cdots \cap \|\neg\beta_n\| \neq \varnothing$$

where $\|\alpha\| \in F$ and $\|\neg\beta_h\| \in F'$ for $h = 1, \ldots, n$. For a contradiction, assume that

$$\|\alpha\| \cap \|\neg\beta_1\| \cap \cdots \cap \|\neg\beta_n\| = \varnothing.$$

Since $\|\neg\beta_1\| \cap \cdots \cap \|\neg\beta_n\| = \|\neg(\beta_1 \vee \cdots \vee \beta_n)\|$, $\|\alpha\| \subset \|\neg(\beta_1 \vee \cdots \vee \beta_n)\|^c$.

Thus, $\|\alpha\| \subset \|\beta_1 \vee \cdots \vee \beta_n\|$.

Since $\|\beta_h\| \in SE(\Sigma)$ for every $h = 1, \ldots, n$, Lemma 3.2 (4) implies $\|\beta_1 \vee \cdots \vee \beta_n\| \in SE(\Sigma)$.

Since F is a filter in $SE(\Sigma)$ and $\|\alpha\| \in F$, $\|\beta_1 \vee \cdots \vee \beta_n\| \in F$.

However, since $\|\neg\beta_h\| \in F'$, $\|\beta_h\| \notin F$

for every $h = 1, \ldots, n$, which means that F is not prime.

Since $\langle \Omega_\Sigma, \tau_\Sigma \rangle$ is compact, the Claim implies that there exists $\omega \in \Omega_\Sigma$ such that $\omega \in \bigcap(F \cup F')$. It is clear that $F \subset SE(\omega)$. Conversely, pick $\|\beta\| \in SE(\omega)$ and assume that $\|\beta\| \notin F$. Then $\|\neg\beta\| \in F'$. Thus $\omega \in F'$. Since Σ is assumed to be consistent, $\omega \notin \|\Box\beta\|$ by (**T**). On the other hand, since $\|\beta\| \in SE(\Sigma)$, $\|\beta\| = \|\Box\beta\|$. Therefore $\omega \notin \|\beta\|$. Thus $SE(\omega) \subset F$. □

Proof of 3.6: (4)$\Rightarrow$(3) is trivial. (2)$\Rightarrow$(1) follows from the Axiom of (Dependent) Choice (see Johnstone (1987, p.85)). (1)$\Rightarrow$(4) follows from a simple observation that a minimal element of a filter has to be unique thus it is actually a minimum element. Thus it suffices to show that:

Claim: If there is an infinitely decreasing chain in $\langle L, \preceq\rangle$ then there exists a non-principal prime filter in $\langle L, \preceq\rangle$.

Proof: Let $C = \{c_n \mid n \in \mathbb{N}\}$ be an infinitely decreasing chain in $\langle L, \preceq\rangle$. Let

$$A = \{a \in L \mid c_n \preceq a \text{ for some } n \in \mathbb{N}\}$$
$$B = \{b \in L \mid b \preceq c_n \text{ for every } n \in \mathbb{N}\}.$$

It follows from the definition of infinitely decreasing chain that $A \cap B = \varnothing$. On the other hand, it is immediate that A (B, respectively) is a filter (ideal, respectively) in $\langle L, \preceq\rangle$. Therefore, by A.1.3, there is a prime filter F in $\langle L, \preceq\rangle$ such that $F \cap B = \varnothing$. The filter F must be non-principal since $C \subset A \subset F$ and $F \cap B = \varnothing$. □

Proof of 4.1:

(1) Assume $\|\alpha\| \in SE(\Sigma) - \{\varnothing\}$ is minimal in $SE(\Sigma) - \{\varnothing\}$. Pick $\omega \in \|\alpha\|$. By Lemma 3.2 (3), $P(\omega) \subset \|\alpha\|$. Since $P(\omega) \in SE(\Sigma) - \{\varnothing\}$, minimality of $\|\alpha\|$ implies that $\|\alpha\| = P(\omega)$.

(2) By the definition of minimality and the fact that $SE(\Sigma)$ is closed under intersection. □

Proof of 4.2: We first observe that:

Claim 1: Let $\|\alpha\| \in EX(\Sigma)$ and $\|\beta\| \in SE(\Sigma)$. If $\|\alpha\| \cap \|\beta\| = \varnothing$ then $\|\beta\| \subset \|\Box\neg\alpha\|$.

Proof: Assume $\|\alpha\| \cap \|\beta\| = \varnothing$. Then $\|\beta\| \subset \|\alpha\|^c = \|\neg\alpha\|$. Thus by (**N**), (**MP**) and (**K**), $\|\Box\beta\| \subset \|\Box\neg\alpha\|$. Since $\|\beta\| \in SE(\Sigma)$, $\|\beta\| \subset \|\Box\neg\alpha\|$.

Let $\{\mu_\lambda \mid \lambda \in \Lambda\}$ be the set of all minimal nonempty self-evident sets. It suffices to show that the index set Λ is finite. For a contradiction, we assume that Λ is infinite. Consider the set

$$F = \{\|\Box\neg\mu_\lambda\| \mid \lambda \in \Lambda\}.$$

Claim 2: F has the finite intersection property.

Proof: Pick $\|\Box\neg\mu_{\lambda_1}\|, \dots, \|\Box\neg\mu_{\lambda_n}\| \in F$. We can pick also $\lambda \in \Lambda$ such that $\lambda \neq \lambda_h$ for $h = 1, \dots, n$ since Λ is infinite. By Lemma 4.1 (2), $\|\mu_{\lambda_1} \vee \cdots \vee \mu_{\lambda_n}\| \cap \|\mu_\lambda\| = \varnothing$. Since $\|\mu_\lambda\| \in SE(\Sigma)$, Claim 1 implies that $\varnothing \neq \|\mu_\lambda\| \subset \|\Box\neg(\mu_{\lambda_1} \vee \cdots \vee \mu_{\lambda_n})\|$. Thus,

$$\|\Box\neg\mu_{\lambda_1}\| \cap \cdots \cap \|\Box\neg\mu_{\lambda_n}\| = \|\Box\neg(\mu_{\lambda_1} \vee \cdots \vee \mu_{\lambda_n})\| \neq \varnothing,$$

which shows that F has the finite intersection property.

Since Ω_Σ is compact, Claim 2 implies that there is $\omega \in \Omega_\Sigma$ such that $\omega \in \|\Box\neg\mu_\lambda\|$ for every $\lambda \in \Lambda$. Therefore $P(\omega) \subset \|\Box\neg\mu_\lambda\| \subset \|\neg\mu_\lambda\|$ by Lemma 3.2 (3) and (**T**). Therefore, $P(\omega) \cap \|\mu_\lambda\| = \varnothing$ for every $\lambda \in \Lambda$. By well-foundedness of $SE(\Sigma)$, there is $\|\mu\| \subset P(\omega)$ such that $\|\mu\|$ is a minimal nonempty self-evident set. This contradicts the fact that the set $\{\mu_\lambda \mid \lambda \in \Lambda\}$ consists of all minimal nonempty set in $SE(\Sigma)$. □

Proof of 4.3: Let $\|\mu_1\|, \ldots, \|\mu_n\|$ be the set of all minimal possibility sets. By Lemma 4.1, $\langle \Omega_\Sigma, P(\cdot)\rangle$ is partitional if and only if $\|\mu_1\| \cup \cdots \cup \|\mu_n\| = \Omega_\Sigma$.

($\Rightarrow$) By the strong expressibility, there are $\eta_1, \ldots, \eta_n$ such that $\|\mu_h\| = \|\Diamond\eta_h\|$ for $h = 1, \ldots, n$. Thus by Lemma 2.1, $\|\neg(\mu_1 \vee \cdots \vee \mu_n)\| = \|\neg(\Diamond(\eta_1 \vee \cdots \vee \eta_n))\| = \|\Box\neg(\eta_1 \vee \cdots \vee \eta_n)\|$. Therefore $\|\neg(\mu_1 \vee \cdots \vee \mu_n)\| \in SE(\Sigma)$ by (**T**) and (**4**). However, since $\|\mu_1\|, \ldots, \|\mu_n\|$ are the set of all minimal nonempty self-evident sets, it follows that $\|\neg(\mu_1 \vee \cdots \vee \mu_n)\| = \varnothing$.

($\Leftarrow$) Consider $\|\mu_1\|$, one of minimal possibility sets. Since $\|\mu_1\| \cup \cdots \cup \|\mu_n\| = \Omega_\Sigma$, $\|\neg\mu_1\| = \|\mu_2\| \cup \cdots \cup \|\mu_n\|$. Thus $\|\neg\mu_1\| \in SE(\Sigma)$ or $\|\neg\mu_1\| = \|\Box\neg\mu_1\|$ by Lemma 3.2 (4). Taking complementation gives that $\|\mu_1\| = \|\Diamond\mu_1\|$. □

Proof of 4.4: The proof of ($\Leftarrow$) part of Therem 4.3, Lemma 3.2 (3), and Lemma 2.1 (4) gives ($\Rightarrow$) direction. The other direction can be proved similarly to the proof of ($\Rightarrow$) part of the proof of Theorem 4.3. □

Proof of 5.1: One direction in the first statement is trivial. Consider the other. By (**T**) and (**4**), it suffices to consider only those finite strings of $\Box_i$ and $\Box_j$ such that $\Box_i$ and $\Box_j$ appear alternatively. We redifine $\Box^*$ as the set of all such strings. Clearly, $\Box^*$ is partitioned into $\Box_i^*$ and $\Box_j^*$, where $\Box_k^*$ is the set of all finite alternating strings whose left most symbol is $\Box_k$ ($k = i, j$). Consider the set $IK_i(\alpha) = \{\|\Box^{(n)}\alpha\| \mid \Box^{(n)} \in \Box_i^*\}$. By (**T**) and (**4**), $IK_i(\alpha)$ is a chain in $SE_i(\Sigma)$. Thus there is a minimum, denoted by $\|\alpha_i^*\|$, in $IK_i(\alpha)$. Similarly, there is a minimum $\|\alpha_j^*\|$ in $SE_j(\alpha)$. We want to show that $\|\alpha_i^*\| = \|\alpha_j^*\|$. Consider $\|\Box_j\alpha_i^*\|$. Since $\|\Box_j\alpha_i^*\| \subset IK_j(\alpha)$, $\|\alpha_j^*\| \subset \|\Box_j\alpha_i^*\|$. On the other hand, by (**T**), $\|\Box_j\alpha_i^*\| \subset \|\alpha_i^*\|$. Thus $\|\alpha_j^*\| \subset \|\alpha_i^*\|$. Similarly, $\|\alpha_i^*\| \subset \|\alpha_j^*\|$. Therefore $\|\alpha_i^*\| = \|\alpha_j^*\|$.

The second statement is clear from the construction above. □

Proof of 5.2:

(1) It follows from $\|\Diamond_k\alpha\| = \|\neg\Box_k\neg\alpha\|$, Lemma 2.3 (4), and Proposition 2.6 (1).

(2) ($\Rightarrow$) $\|\Diamond_k\alpha\| \subset \bigcup_{\omega\in\Lambda(\alpha)} P(\omega)$ follows from (1) and (**T**). Conversely, pick $\omega' \in \bigcup_{\omega\in\Lambda(\alpha)} P(\omega)$. Since information structure is partitional, there is $\omega \in \Lambda(\alpha)$ such that $P^k(\omega') = P^k(\omega)$. By $\omega \in \Lambda(\alpha)$, $P^k(\omega') \cap \|\alpha\| \neq \varnothing$. Thus, by (1), $\omega' \in \|\Diamond_k\alpha\|$.

($\Leftarrow$) Assume that $\|\Diamond_k\alpha\| = \bigcup_{\omega\in\Lambda(\alpha)} P(\omega)$ for every $\|\alpha\| \in EX(\Sigma)$. Since the right hand side is self-evident, so is the left hand side. Thus $SE_k(\Sigma)$ is closed under complementation. Now the ($\Rightarrow$) part of the proof of 4.3 shows that $\langle \Omega_\Sigma, P^k(\cdot)\rangle$ is partitional. □

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