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Bayes Without Bernoulli: Simple Conditions for Probabilistically Sophisticated Choice*

by

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and
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BAYES WITHOUT BERNOULLI: SIMPLE CONDITIONS FOR PROBABILISTICALLY SOPHISTICATED CHOICE

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In 1963, Anscombe and Aumann demonstrated that the introduction of an objective randomizing device into the Savage setting of subjective uncertainty considerably simplified the derivation of subjective probability from a decision maker's preferences over uncertain bets. The purpose of this paper is to present a more general derivation of classical subjective probability in such a framework, which neither assumes nor implies that the individual's risk preferences necessarily conform to the expected utility principle. We argue that the essence of "Bayesian rationality" is the assignment, correct manipulation, and proper updating of subjective event probabilities when evaluating and comparing uncertain prospects, regardless of whether attitudes toward risk satisfy the expected utility property.

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1. INTRODUCTION

The modern or "choice-theoretic" theory of subjective probability, as pioneered by Ramsey (1931) and de Finetti (1937), achieved its modern form in the characterizations of Savage (1954) and Anscombe and Aumann (1963). These two characterizations have a common goal, namely, the derivation of an individual's subjective probabilities of uncertain events, based on their ranking of "bets" defined on these events. However, they adopt very different types of uncertain settings.

The Savage characterization adopts a setting of **purely subjective uncertainty**. In this situation, the uncertainty facing the individual is represented by a set $\mathcal{S} = \{...,s,...\}$ of **states of nature** and a set $\mathcal{E} = \{...,E,...\}$ of **events** (subsets of $\mathcal{S}$). The objects of choice in this framework are bets or **acts** $f(\cdot)$, $g(\cdot)$, etc., which assign an outcome to each state of nature – in other words, functions from the state space $\mathcal{S}$ to an outcome space $\mathcal{X} = \{...,x,...\}$. Savage's axioms imply the existence of a finitely-additive, non-atomic subjective probability measure $\mu(\cdot)$ over events, and a von Neumann-Morgenstern utility function $U(\cdot)$ over outcomes, which together represent the individual's preference ranking $\succsim$ over acts, in the sense that

$$f(\cdot) \succsim g(\cdot) \quad \Leftrightarrow \quad \int_{\mathcal{S}} U(f(s))\cdot d\mu(s) \geq \int_{\mathcal{S}} U(g(s))\cdot d\mu(s)$$

for any pair of acts $f(\cdot)$ and $g(\cdot)$. Since such an individual's *beliefs* can be represented by means of classical (additive) probabilities, and their *attitudes toward risk* can be represented by the expectation of a von Neumann-Morgenstern utility function, we can describe the individual as a **probabilistically sophisticated expected utility maximizer**.

In contrast, the Anscombe-Aumann characterization adopts a setting of **mixed subjective/ objective uncertainty**. In addition to a state space $\mathcal{S} = \{s_1,...,s_n\}$, the individual faces additional probabilistic or **objective** uncertainty in the form of a randomization device, capable of generating any well-specified probability distribution $\mathbf{R}$ over outcomes.[1] In Anscombe and Aumann's evocative and widely adopted terminology, the states $\{s_1,...,s_n\}$ refer a set of n "horses," exactly one of which will win an upcoming race, and the objective randomizing device is termed a "roulette wheel." The objects of choice in this framework are bets of the form $[\mathbf{R}_1 \text{ if } s_1;...;\mathbf{R}_n \text{ if } s_n]$, which assign an objective roulette lottery $\mathbf{R}_i$ to each state of nature. Anscombe and Aumann's axioms imply the existence of a set $\{\pi_1,...,\pi_n\}$ of subjective probabilities for the states $\{s_1,...,s_n\}$, and a von Neumann-Morgenstern utility function $U(\cdot)$, which together represent the preference ranking $\succsim$ over such bets, in the sense that

$$[\mathbf{R}_1^* \text{ if } s_1;...;\mathbf{R}_n^* \text{ if } s_n] \succsim [\mathbf{R}_1 \text{ if } s_1;...;\mathbf{R}_n \text{ if } s_n] \quad \Leftrightarrow \quad \sum_{i=1}^{n} \pi_i \cdot E[U(\mathbf{R}_i^*)] \geq \sum_{i=1}^{n} \pi_i \cdot E[U(\mathbf{R}_i)]$$

for any pair of bets $[\mathbf{R}_1^* \text{ if } s_1;...;\mathbf{R}_n^* \text{ if } s_n]$ and $[\mathbf{R}_1 \text{ if } s_1;...;\mathbf{R}_n \text{ if } s_n]$, where $E[U(\mathbf{R}_i)]$ denotes the expected utility of the roulette lottery $\mathbf{R}_i$. Since such an individual again represents subjective uncertainty by the assignment of probabilities to events, and ranks bets by their overall expected utility, we may again describe them as a probabilistically sophisticated expected utility maximizer.

Anscombe and Aumann's formulation accordingly differs from Savage's in two respects. The first is that their state space is finite, as opposed to infinite. Second, and more significantly, they alter Savage's purely subjective setting by introducing an objective randomizing device.

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[1] Thus by "objective" we mean "already in a probabilistic form." For purposes of our analysis it is not necessary to address the issue of whether "objective" also denotes "scientifically verifiable," "universally agreed upon," etc.

Since one of the main goals of these analyses is to derive the existence of subjective probabilities for uncertain events, Anscombe and Aumann's inclusion of an objective – that is to say, an explicitly probabilistic – randomizing device into Savage's purely subjective framework might seem like stuffing at least part of the rabbit into the hat. Since their article came out seven years *after* Savage's book, and was not even the first one to consider the mixed subjective/objective case (see footnote 5 below), one might ask why it enjoys the reputation is does.[2]

The answer is the elegance and simplicity of the Anscombe-Aumann characterization. In comparison with Savage's derivation, which requires dozens of pages and several preliminary theorems, Anscombe and Aumann show that, in the presence of an objective randomizing device,

> "The addition of only two plausible assumptions to those of [expected] utility theory permits a simple and natural definition of [subjective] probabilities having the appropriate properties." (p. 200)

In their comparison with earlier subjective/objective characterizations, since Anscombe and Aumann took less than a single page to prove their result, they were quite justified in stating

> "We believe that our presentation of subjective probability is the simplest so far given, for anyone who accepts the [existence of an objective randomization device]." (p. 200)

Although the Savage and Anscombe-Aumann characterizations are milestones in the behavioral theory of probability, both are restricted to individuals whose risk preferences conform to the expected utility hypothesis. While there has been widespread normative adoption of the expected utility principle (due in large part to Savage's own work), there has also been a growing challenge to the *descriptive validity* of this hypothesis, with an increasing number of experimental violations being catalogued, and an increasing number of alternative models of risk preferences being proposed. These developments have recently led some to argue that expected utility maximization may not be the only form of "rational" decision-making under uncertainty.[3]

In a recent paper (Machina and Schmeidler (1992)), we presented an adaptation of the Savage approach which continues to characterize *beliefs* by means of *subjective probabilities*, but neither assumes or implies that *risk preferences* are necessarily *expected utility*. This adaptation consists of dropping the Savage axiom that is primarily responsible for implying the expected utility property (the "Sure-Thing Principle"), and strengthening one of his remaining axioms.[4] Our modified axiom set implies that the individual still assigns subjective probabilities to events and judges each act solely on the basis of its implied probability distribution over outcomes, but does not necessarily rank these probability distributions according to the expected utility principle. We call such an individual a **probabilistically sophisticated *non*-expected utility maximizer**.

[2] Thus, while Kreps (1988) refers to Savage (1954) as the "crowning glory of choice theory" (p.120), he also refers to Anscombe and Aumann (1963) as a "classic paper" (p.100).

[3] See Camerer and Weber (1992), Epstein (1992), Fishburn (1988), Karni and Schmeidler (1991), Machina (1987, 1989), Munier (1989), Schmeidler (1982, 1989), Sugden (1991) and Weber and Camerer (1987) for surveys of these empirical findings, new models, and normative arguments.

[4] See also the extension of our analysis provided by Grant (1994). In addition, characterizations of probabilistic sophistication in a setting where risk preferences are not necessarily transitive have been provided by Fishburn (1988, Ch.9) and Sugden (1993) (see our 1992 paper (Sect. 6.1) for a discussion and additional references).

Our 1992 paper shared with Savage the advantage of that it applied to arbitrary situations of purely subjective uncertainty. However, it shared the disadvantage of a lengthy set of axioms and a long proof. The purpose of this paper is to present an alternative analysis that parallels the contribution of Anscombe and Aumann – in other words, to show how the introduction of an objective randomizing device drastically simplifies the choice-theoretic derivation of subjective probability, even in the absence of the expected utility principle.

There is an additional, although related, motivation for our approach. In comparing their own contribution with similar characterizations[5], Anscombe and Aumann wrote

> "The novelty of our presentation, if any, lies in the double use of [expected] utility theory, permitting the very simple and plausible assumptions and the simple construction and proof" (p. 203).

We want to argue that this assessment is not quite correct – that the novelty of the Anscombe-Aumann approach is *not* tied to the expected utility axioms, but rather, is due to simpler conditions which, while hidden in and logically implied by the expected utility assumptions, can be separated from them and are sufficient to obtain *beliefs* that are characterized by subjective probabilities without the requirement that *risk preferences* are necessarily represented by *expected utility*. In this paper we identify and formalize these simpler conditions.

In the following section we present (slightly modified versions of) the Anscombe-Aumann setting and their expected utility-based axiom set. In Section 3 we describe what it means to act on the basis of well-defined subjective probabilities but with risk preferences that are not necessarily expected utility, and show that in the absence of the expected utility hypothesis, consistent responses to the standard probability calibration procedure are *not* sufficient to ensure such well-defined probabilistic beliefs. In Section 4 present a set of axioms which *are* sufficient to imply probabilistic sophistication without either assuming or implying expected utility risk preferences, and a formal characterization theorem. Section 5 gives a discussion of conditional probability and updating, as well as our proposal that the term "Bayesian rationality" be taken to refer the rational formulation and manipulation of beliefs, independent of attitudes toward risk.

2. SETTING AND BACKGROUND

2.1 Setting

Our formal setting consists of the following concepts:

$\mathcal{X} = \{\ldots,x,\ldots\}$	an arbitrary nonempty set of **outcomes** (finite or infinite)
$\mathcal{S} = \{\ldots,s,\ldots\}$	an arbitrary nonempty set of **states** (finite or infinite)
$\mathbf{R} = (x_1,p_1;\ldots;x_m,p_m)$	a **pure roulette lottery**, yielding outcome $x_i \in \mathcal{X}$ with probability $p_i \in [0,1]$ $(1 \le m < \infty,\ p_1+\ldots+p_m = 1)$
$\mathcal{R} = \{\ldots,\mathbf{R},\ldots\}$	the set of all pure roulette lotteries
$\mathbf{H} = [x_1 \text{ on } E_1;\ldots;x_n \text{ on } E_n]$	a **pure horse lottery**, yielding outcome x_i in event E_i, for some partition $\{E_1,\ldots,E_n\}$ of $\mathcal{S}$ $(1 \le n < \infty)$

[5] Specifically, Blackwell and Girshick (1954), Chernoff (1954), and Raiffa and Schlaifer (1961).

$\mathcal{H} = \{\ldots,\mathbf{H},\ldots\}$	the set of all pure horse lotteries
$\mathbf{R}^{\mathrm{H}} = (\mathbf{H}_1,p_1;\ldots;\mathbf{H}_m,p_m)$	a **roulette/horse lottery**, yielding pure horse lottery $\mathbf{H}_i$ with probability $p_i \in [0,1]$ $(1 \le m < \infty,\ p_1+\ldots+p_m = 1)$
$\mathcal{R}^{\mathcal{H}} = \{\ldots,\mathbf{R}^{\mathrm{H}},\ldots\}$	the set of all roulette/horse lotteries[6]
$\mathbf{H}^{\mathrm{R}} = [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n]$	a **horse/roulette lottery**, yielding pure roulette lottery $\mathbf{R}_i$ in event E_i, for some partition $\{E_1,\ldots,E_n\}$ of $\mathcal{S}$ $(1 \le n < \infty)$
$\mathcal{H}^{\mathcal{R}} = \{\ldots,\mathbf{H}^{\mathrm{R}},\ldots\}$	the set of all horse/roulette lotteries
$\mathcal{L} = \{\ldots,\mathbf{L},\ldots\} = \mathcal{R}\cup\mathcal{H}\cup\mathcal{R}^{\mathcal{H}}\cup\mathcal{H}^{\mathcal{R}}$	the combined set of all lotteries, with generic element $\mathbf{L}$
$\succsim$	the individual's weak preference relation on $\mathcal{L}$
$\delta_x = (x,1) \in \mathcal{R}$	the pure roulette lottery yielding $x \in \mathcal{X}$ with certainty
$0, M \in \mathcal{X}$	the "best" and "worst" outcomes in $\mathcal{X}$, in the sense that $M \succ 0$ and $M \succsim x \succsim 0$ for all $x \in \mathcal{X}$ (where for any outcomes $x, y \in \mathcal{X}$, we define $x \succsim y$ iff $\delta_x \succsim \delta_y$)[7]

Throughout our analysis, we shall identify pure roulette lotteries such as $(\ldots;x,p_i;x,p_{i+1};\ldots)$ and $(\ldots;x,p_i+p_{i+1};\ldots)$ that imply identical probability distributions over outcomes, as well as pure horse lotteries such as $[\ldots; x \text{ on } E_i; x \text{ on } E_{i+1};\ldots]$ and $[\ldots; x \text{ on } E_i\cup E_{i+1};\ldots]$ that imply identical assignments of outcomes to states, and similarly for roulette/horse lotteries of the form $(\ldots;\mathbf{H},p_i;\mathbf{H},p_{i+1};\ldots)$ and $(\ldots;\mathbf{H},p_i+p_{i+1};\ldots)$, and horse/roulette lotteries of the form $[\ldots;\mathbf{R} \text{ on } E_i; \mathbf{R} \text{ on } E_{i+1};\ldots]$ and $[\ldots;\mathbf{R} \text{ on } E_i\cup E_{i+1};\ldots]$. By "identify" we mean "assume that the individual is indifferent between." These identifications imply that any pair of pure horse lotteries $\mathbf{H}^* = [x_1^* \text{ on } E_1^*;\ldots;x_{n^*}^* \text{ on } E_{n^*}^*]$ and $\mathbf{H}^{**} = [x_1^{**} \text{ on } E_1^{**};\ldots;x_{n^{**}}^{**} \text{ on } E_{n^{**}}^{**}]$ can be alternatively written as:

$$\mathbf{H}^* = [\hat{x}_1^* \text{ on } E_1;\ldots;\hat{x}_n^* \text{ on } E_n] \qquad \text{and} \qquad \mathbf{H}^{**} = [\hat{x}_1^{**} \text{ on } E_1;\ldots;\hat{x}_n^{**} \text{ on } E_n]$$

where $\{E_1,\ldots,E_n\}$ is the common refinement of the partitions $\{E_1^*,\ldots,E_{n^*}^*\}$ and $\{E_1^{**},\ldots,E_{n^{**}}^{**}\}$, each $\hat{x}_i^*$ is an element of $\{x_1^*,\ldots,x_{n^*}^*\}$, and each $\hat{x}_i^{**}$ is an element of $\{x_1^{**},\ldots,x_{n^{**}}^{**}\}$. These identifications also imply that any pair of horse/roulette lotteries $\mathbf{H}^{\mathrm{R}^*}$ and $\mathbf{H}^{\mathrm{R}^{**}}$ can be written as:

$$\mathbf{H}^{\mathrm{R}^*} = [\hat{\mathbf{R}}_1^* \text{ on } E_1;\ldots;\hat{\mathbf{R}}_n^* \text{ on } E_n] \qquad \text{and} \qquad \mathbf{H}^{\mathrm{R}^{**}} = [\hat{\mathbf{R}}_1^{**} \text{ on } E_1;\ldots;\hat{\mathbf{R}}_n^{**} \text{ on } E_n]$$

over their common refinement $\{E_1,\ldots,E_n\}$, and also that an arbitrary roulette/horse lottery $\mathbf{R}^{\mathrm{H}} = (\mathbf{H}_1,p_1;\ldots;\mathbf{H}_m,p_m)$ can be expressed in the form:

$$\mathbf{R}^{\mathrm{H}} = ([\hat{x}_{1,1} \text{ on } E_1;\ldots;\hat{x}_{n,1} \text{ on } E_n],p_1;\ldots;[\hat{x}_{1,m} \text{ on } E_1;\ldots;\hat{x}_{n,m} \text{ on } E_n],p_m)$$

where $\{E_1,\ldots,E_n\}$ is the common refinement of the partitions for $\mathbf{H}_1,\ldots,\mathbf{H}_m$. Finally, we identify each pure roulette lottery $\mathbf{R}$ with the constant horse/roulette lottery $\{\mathbf{R} \text{ on } \mathcal{S}\} \in \mathcal{H}^{\mathcal{R}}$, and identify each pure horse lottery $\mathbf{H} = [x_1 \text{ on } E_1;\ldots;x_n \text{ on } E_n]$ with both the degenerate roulette/horse lottery $(\mathbf{H},1) \in \mathcal{R}^{\mathcal{H}}$ and the eventwise-degenerate horse/roulette lottery $[\delta_{x_1} \text{ on } E_1;\ldots;\delta_{x_n} \text{ on } E_n] \in \mathcal{H}^{\mathcal{R}}$.

[6] Note that $\mathcal{R}^{\mathcal{H}}$ does *not* denote the set of functions from $\mathcal{H}$ to $\mathcal{R}$, but rather, the set of all roulette/horse lotteries. Similarly, $\mathcal{H}^{\mathcal{R}}$ does not denote the set of functions from $\mathcal{R}$ to $\mathcal{H}$, but rather, the set of horse/roulette lotteries.

[7] We follow Anscombe and Aumann in assuming a best and worst outcome for simplicity only. Our results can be extended to unbounded outcomes by an argument like that in Machina and Schmeidler (1992, p.775, Step 5).

This setting differs from Anscombe and Aumann's in three respects: First, our state space S is not required to be finite.[8] Second, and partly as a result of this, our notation is somewhat different. Finally, we do not explicitly consider lotteries with more than one stage of objective uncertainty. The reader wishing to formally extend our framework to include such "roulette/roulette" (or "roulette/roulette/horse") lotteries can do so by the obvious identifications based upon the standard compounding of the (objective) probabilities in successive roulette stages.[9]

2.2 Axioms of the Expected Utility-Based Characterization

The assumptions invoked in Anscombe and Aumann's formal proof[10] appear in three forms. Some are implicit but clearly implied by their exposition, others are explicitly stated, and a couple are formally named and numbered. Adjusting for the differences in notation and setting, we can present these assumptions in terms of five formal axioms. The first of these axioms is standard:

AXIOM 1 (Ordering): The relation $\succsim$ on $\mathcal{L}$ is complete, reflexive and transitive.

As always, this implies that the associated indifference relation $\sim$ and strict preference relation $\succ$ are also transitive. Our second axiom, Anscombe and Aumann's "Reversal of order in compound lotteries" assumption, links preferences between the set of roulette/horse lotteries and the set of horse/roulette lotteries. Recall that we can represent an arbitrary roulette/horse lottery as

$$\mathbf{R}^{\mathrm{H}} = (\mathbf{H}_1,p_1;\ldots;\mathbf{H}_m,p_m) = ([x_{1,1} \text{ on } E_1;\ldots;x_{n,1} \text{ on } E_n],p_1;\ldots;[x_{1,m} \text{ on } E_1;\ldots;x_{n,m} \text{ on } E_n],p_m)$$

for some common refinement $\{E_1,\ldots,E_n\}$. This lottery consists of first spinning a roulette wheel with probabilities $p_1,\ldots,p_m$ to determine which assignment $\mathbf{H}_1,\ldots,\mathbf{H}_m$ of prizes to states will apply, and then determining the state, and hence prize, that obtains. If we simply change the order of these two random processes, i.e., first determine the state and *then* spin the roulette wheel, we obtain that, for each state in the event E_i, the individual faces probabilities $p_1,\ldots,p_m$ of obtaining the respective prizes $\{x_{i,1},\ldots,x_{i,m}\}$. But this is precisely the horse/roulette lottery

$$\mathbf{H}^{\mathrm{R}} = [(x_{1,1},p_1;\ldots;x_{1,m},p_m) \text{ on } E_1;\ldots;(x_{n,1},p_1;\ldots;x_{n,m},p_m) \text{ on } E_n]$$

The assumption that such a reversal of order does not matter can be stated as:

AXIOM 2 (Independence of Order of Resolution): Any roulette/horse lottery of the form

$$\mathbf{R}^{\mathrm{H}} = ([x_{1,1} \text{ on } E_1;\ldots;x_{n,1} \text{ on } E_n],p_1;\ldots;[x_{1,m} \text{ on } E_1;\ldots;x_{n,m} \text{ on } E_n],p_m) \in \mathcal{R}^{\mathcal{H}}$$

is indifferent to the horse/roulette lottery

$$\mathbf{H}^{\mathrm{R}} = [(x_{1,1},p_1;\ldots;x_{1,m},p_m) \text{ on } E_1;\ldots;(x_{n,1},p_1;\ldots;x_{n,m},p_m) \text{ on } E_n] \in \mathcal{H}^{\mathcal{R}}$$

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[8] Although we do follow both Anscombe-Aumann and Savage in restricting attention to finite-outcome lotteries.

[9] Neither we nor Anscombe-Aumann consider multiple "stages" of subjective (i.e., horse) uncertainty.

[10] Besides the references in Footnote 5, expected utility-based characterizations of subjective probability under mixed subjective/objective uncertainty have also been offered by Davidson and Suppes (1956), Pratt, Raiffa and Schlaifer (1964), Fishburn (1967,1969,1970) and others. Of these, the closest to our Axioms 1-5 seems to be Fishburn (1970, p.179, Thm.13.3), who also considers an arbitrary outcome space and an arbitrary state space, and whose Axioms B1, B2, B3 and B5 correspond respectively to our Axioms 1, 4, 3 and 5 (our Axiom 2 is implicit in Fishburn's formulation, his Axiom B4 is implied by our "best" and "worst" outcomes assumption, and his Axiom B6 is not needed in our setting of finite-outcome lotteries).

Since each *roulette/horse* lottery can accordingly be identified with a unique horse/roulette lottery in this manner, since each *pure roulette* lottery **R** can be identified with the constant horse/roulette lottery [**R** on $\mathcal{S}$], and since each *pure horse* lottery **H** = [x_1 on E_1;...;x_n on E_n] can be identified with the eventwise-degenerate horse/roulette lottery [δ_{x_1} on E_1;...; δ_{x_n} on E_n], we can express each of our remaining axioms in terms of preferences over horse/roulette lotteries.

Axioms 3, 4 and 5 represent Anscombe and Aumann's use of the expected utility postulates in their characterization of probabilistic sophistication. Although Axioms 3 and 4 are not formally spelled out in their exposition, they are definitely implied by their assumptions that

> "you have a preference ordering on [the set of pure roulette lotteries] satisfying the axioms of [expected] utility theory", and
>
> "you also have a preference ordering on [the set of horse/roulette lotteries] satisfying the axioms of [expected] utility theory."[11]

In the standard setting of purely objective uncertainty (e.g., Herstein and Milnor (1953), Fishburn (1982, Sect. 2.2)), the axioms of expected utility consist of an ordering axiom (such as our Axiom 1), a "mixture continuity" axiom, and an "independence" axiom. These latter two involve the notion of a **probability mixture** of a pair of roulette lotteries $\mathbf{R} = (x_1,p_1;\ldots;x_m,p_m)$ and $\mathbf{R}^* = (x_1^*,p_1^*;\ldots;x_{m^*}^*,p_{m^*}^*)$, which we define in the standard way as:

$$\alpha\cdot\mathbf{R} + (1-\alpha)\cdot\mathbf{R}^* = (x_1,\alpha\cdot p_1;\ldots;x_m,\alpha\cdot p_m\,;\,x_1^*,(1-\alpha)\cdot p_1^*;\ldots;x_{m^*}^*,(1-\alpha)\cdot p_{m^*}^*) \qquad \text{for } \alpha\in[0,1]$$

Given this, we have

AXIOM 3 (Mixture Continuity): For any partition $\{E_1,\ldots,E_n\}$, if

$$[\mathbf{R}_1^{**} \text{ on } E_1;\ldots;\mathbf{R}_n^{**} \text{ on } E_n] \succ [\mathbf{R}_1^{*} \text{ on } E_1;\ldots;\mathbf{R}_n^{*} \text{ on } E_n] \succ [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n]$$

then there exists a probability $\alpha\in(0,1)$ such that

$$[\mathbf{R}_1^{*} \text{ on } E_1;\ldots;\mathbf{R}_n^{*} \text{ on } E_n] \sim [\alpha\cdot\mathbf{R}_1^{**}+(1-\alpha)\cdot\mathbf{R}_1 \text{ on } E_1;\ldots;\alpha\cdot\mathbf{R}_n^{**}+(1-\alpha)\cdot\mathbf{R}_n \text{ on } E_n]$$

as well as

AXIOM 4 (Independence Axiom): For any partition $\{E_1,\ldots,E_n\}$ and roulette lotteries $\{\mathbf{R}_1^*,\ldots,\mathbf{R}_n^*\}$ and $\{\mathbf{R}_1,\ldots,\mathbf{R}_n\}$:

$$[\mathbf{R}_1^{*} \text{ on } E_1;\ldots;\mathbf{R}_n^{*} \text{ on } E_n] \succsim [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n] \quad\Rightarrow$$

$$[\alpha\cdot\mathbf{R}_1^{*}+(1-\alpha)\cdot\mathbf{R}_1^{**} \text{ on } E_1;\ldots;\alpha\cdot\mathbf{R}_n^{*}+(1-\alpha)\cdot\mathbf{R}_n^{**} \text{ on } E_n]$$

$$\succsim [\alpha\cdot\mathbf{R}_1+(1-\alpha)\cdot\mathbf{R}_1^{**} \text{ on } E_1;\ldots;\alpha\cdot\mathbf{R}_n+(1-\alpha)\cdot\mathbf{R}_n^{**} \text{ on } E_n]$$

for all probabilities $\alpha\in(0,1]$ and all roulette lotteries $\{\mathbf{R}_1^{**},\ldots,\mathbf{R}_n^{**}\}$.

Our fifth axiom, Anscombe and Aumann's "Monotonicity in the prizes" assumption, is:

[11] Anscombe and Aumann (1963, p.201). Our justification for interpreting their phrase "axioms of utility theory" as the axioms of *expected* utility theory is that Anscombe and Aumann cite Luce and Raiffa's (1957, Sect. 2.5) characterization of expected utility as an example of such axioms, and also that they go on to say that these axioms imply the existence of a utility function with the standard expected utility property (p.201, Property (ii)).

AXIOM 5 (Substitution Axiom): For any pair of pure roulette lotteries $\mathbf{R}_i^*$ and $\mathbf{R}_i$:

$$\mathbf{R}_i^* \succsim \mathbf{R}_i \Rightarrow [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_i^* \text{ on } E_i;\ldots;\mathbf{R}_n \text{ on } E_n] \succsim [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_i \text{ on } E_i;\ldots;\mathbf{R}_n \text{ on } E_n]$$

for all partitions $\{E_1,\ldots,E_n\}$ and all roulette lotteries $\{\mathbf{R}_1,\ldots,\mathbf{R}_{i-1},\mathbf{R}_{i+1},\ldots,\mathbf{R}_n\}$.

These three "expected utility-based" axioms play the same role in our mixed setting as they do in their standard settings of purely objective or purely subjective uncertainty. Mixture Continuity ensures that the distinct indifference classes of lotteries can be mapped one-to-one and continuously to the continuum. This guarantees the existence of a real-valued representation of preferences, although it places no restrictions on the "shape" or functional form of the preference functional. On the other hand, the Independence Axiom and Substitution Axiom do force the shape of the preference functional, by implying the expected utility properties of "linearity in the probabilities" in the case of objective uncertainty, and "separability across states/events" in the case of subjective uncertainty. We shall show that it is precisely these latter two axioms that can be weakened, so that we can obtain a characterization of probabilistically sophisticated beliefs that neither assumes nor implies the expected utility hypothesis on risk preferences.

3. PROBABILISTIC SOPHISTICATION WITHOUT EXPECTED UTILITY

3.1 Description and Empirical Implications

As mentioned above, our goal is to obtain conditions on an individual's preferences over subjective and/or objective lotteries which imply that they *do* represent subjective uncertainty by means of classical additive probabilities, and *do* evaluate bets (objective, subjective or mixed) solely on the basis of their implied probability distributions over outcomes, but *do not* necessarily rank these derived probability distributions by means of the expected utility criterion. The purpose of this section is to formalize this notion of probabilistic sophistication in the absence of the expected utility hypothesis, and explore some of the properties of such preferences.

Given a subjective probability measure $\mu(\cdot)$ over *events*, the probability distribution over *outcomes* implied by each bet $\mathbf{L} \in \mathcal{L}$, denoted by the pure roulette lottery $\mathbf{P}_\mu(\mathbf{L})$, is given by:

1. For each pure roulette lottery $\mathbf{R} = (x_1,p_1;\ldots;x_m,p_m)$: $\mathbf{P}_\mu(\mathbf{R}) = \mathbf{R} = (x_1,p_1;\ldots;x_m,p_m)$
2. For each pure horse lottery $\mathbf{H} = [x_1 \text{ on } E_1;\ldots;x_n \text{ on } E_n]$: $\mathbf{P}_\mu(\mathbf{H}) = (x_1,\mu(E_1);\ldots;x_n,\mu(E_n))$
3. For each roulette/horse lottery $\mathbf{R}^\mathbf{H} = (\mathbf{H}_1,p_1;\ldots;\mathbf{H}_m,p_m)$: $\mathbf{P}_\mu(\mathbf{R}^\mathbf{H}) = \mathbf{P}_\mu(\mathbf{H}_1)\cdot p_1 +\ldots+ \mathbf{P}_\mu(\mathbf{H}_m)\cdot p_m$
4. For each horse/roulette lottery $\mathbf{H}^\mathbf{R} = [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n]$: $\mathbf{P}_\mu(\mathbf{H}^\mathbf{R}) = \mu(E_1)\cdot\mathbf{R}_1 +\ldots+ \mu(E_n)\cdot\mathbf{R}_n$

Given this, we can formalize the notion that the individual represents subjective uncertainty by means of classical probabilities, and ranks all lotteries on the basis of their implied probability distributions over outcomes, as follows:

DEFINITION: An individual is said to be **probabilistically sophisticated** if there exists a finitely additive probability measure $\mu(\cdot)$ over events in $\mathcal{S}$, and a preference functional $V(\cdot)$ over probability distributions on outcomes, such that for all lotteries $\mathbf{L}$ and $\mathbf{L}^*$ in $\mathcal{L}$:

$$\mathbf{L^*} \succsim \mathbf{L} \quad \Leftrightarrow \quad V(\mathbf{P}_\mu(\mathbf{L^*})) \geq V(\mathbf{P}_\mu(\mathbf{L}))$$

Since all lotteries in the set $\mathcal{L} = \mathcal{R}\cup\mathcal{H}\cup\mathcal{R}^{\mathcal{H}}\cup\mathcal{H}^{\mathcal{R}}$ can be identified with horse/roulette lotteries, we can without loss of generality work exclusively in terms of the last of the above four formulas for $\mathbf{P}_\mu(\cdot)$, in which case our definition of probabilistic sophitication is equivalent to the condition that for all pairs of horse/roulette lotteries $[\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n]$ and $[\mathbf{R}_1^* \text{ on } E_1^*;\ldots;\mathbf{R}_{n*}^* \text{ on } E_{n*}^*]$:

$$[\mathbf{R}_1^* \text{ on } E_1^*;\ldots;\mathbf{R}_{n*}^* \text{ on } E_{n*}^*] \succsim [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n] \quad \Leftrightarrow \quad V(\Sigma_{i=1}^{n*}\mu(E_i^*)\cdot\mathbf{R}_i^*) \geq V(\Sigma_{i=1}^{n}\mu(E_i)\cdot\mathbf{R}_i)$$

The following table illustrates a simple example of probabilistically sophisticated non-expected utility preferences. Consider drawing a ball from an urn which contains 100 balls, each one colored either red, black or yellow. The table lists four subjective acts (pure horse lotteries) $\{\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3,\mathbf{H}_4\}$ defined over the three events {red, black yellow} (where M = one million):

	red	black	yellow
$\mathbf{H}_1$	\$1M	\$1M	\$1M
$\mathbf{H}_2$	\$5M	\$0	\$1M
$\mathbf{H}_3$	\$5M	\$0	\$0
$\mathbf{H}_4$	\$1M	\$1M	\$0

Say the individual exhibits the preferences:

$$\mathbf{H}_1 \succ \mathbf{H}_2 \quad \text{and} \quad \mathbf{H}_3 \succ \mathbf{H}_4$$

Since there is no set of subjective probabilities $\{p_{red},p_{black},p_{yellow}\}$ and von Neumann-Morgenstern utilities {U(\$0),U(\$1M),U(\$5M)} that can generate this ranking,[12] such preferences violate the expected utility hypothesis. However, they are completely consistent with the hypothesis of probabilistic sophistication, since they are implied by subjective probabilities $\{p_{red}, p_{black}, p_{yellow}\}$ = {.10, .01, .89} and the oft-observed ("Allais-type") *non*-expected utility risk preferences:

$$(\$1M,1.00) \succ (\$5M,.10;\$1M,.89;\$0,.01) \quad \text{and} \quad (\$5M,.10;\$0,0.90) \succ (\$1,.11;\$0,.89)$$

In other words, while the ranking $\mathbf{H}_1 \succ \mathbf{H}_2$ and $\mathbf{H}_3 \succ \mathbf{H}_4$ does violate the expected utility hypothesis, it is perfectly consistent with the preferences of a *non*-expected utility maximizer who possesses well-defined subjective probabilities, and ranks acts on the basis of their implied probability distributions over consequences (or in other words, is probabilistically sophisticated).

Perhaps the best way to illustrate the refutable implications of the hypothesis of probabilistic sophistication is to examine examples of preferences which *violate* this property.[13] The following table illustrates the well-known "Ellsberg Paradox" (Ellsberg (1961)). This time, the urn is known to contain 90 balls, exactly 30 of which are red, with each of the other 60 either black or yellow. The table gives four acts $\{\mathbf{H}_1^*, \mathbf{H}_2^*, \mathbf{H}_3^*, \mathbf{H}_4^*\}$ defined over the events {red, black, yellow}.

[12] If there were, these rankings would imply the inconsistent pair of inequalities $U(\$1M) > p_{red}\cdot U(\$5M) + p_{black}\cdot U(\$0) + p_{yellow}\cdot U(\$1M)$ and $p_{red}\cdot U(\$5M) + (p_{black}+p_{yellow})\cdot U(\$0) > (p_{red}+p_{black})\cdot U(\$1M) + p_{yellow}\cdot U(\$0)$.

[13] More formally, the following two examples will violate the joint hypothesis of probabilistic sophistication and first order stochastic dominance preference over two-point probability distributions on outcomes.

	30 balls	60 balls	
	red	black	yellow
$\mathbf{H}_1^*$	\$0	\$100	\$100
$\mathbf{H}_2^*$	\$100	\$0	\$100
$\mathbf{H}_3^*$	\$0	\$100	\$0
$\mathbf{H}_4^*$	\$100	\$0	\$0

The typical preferences of subjects in such a situation is $\mathbf{H}_1^* \succ \mathbf{H}_2^*$ and $\mathbf{H}_4^* \succ \mathbf{H}_3^*$. Now, if such an individual *did* assign probabilities $\{p_{red}, p_{black}, p_{yellow}\}$ to these three events, *and used them to rank the acts* $\{\mathbf{H}_1^*, \mathbf{H}_2^*, \mathbf{H}_3^*, \mathbf{H}_4^*\}$, then the ranking $\mathbf{H}_1^* \succ \mathbf{H}_2^*$ clearly implies that $p_{black} + p_{yellow} > p_{red} + p_{yellow}$, but the ranking $\mathbf{H}_4^* \succ \mathbf{H}_3^*$ implies $p_{red} > p_{black}$, which leads to a contradiction.

Our second example highlights another empirical implication of the hypothesis of probabilistic sophistication by illustrating another potential violation of this property, namely the phenomenon of "state-dependent" preferences. Say that umbrellas are available for sale on all street corners for \$10 each during rainstorms, and consider the following four acts:

	rain	no rain
$\mathbf{H}_1^{**}$	indoors with \$10	indoors with \$0
$\mathbf{H}_2^{**}$	indoors with \$0	indoors with \$10
$\mathbf{H}_3^{**}$	outdoors with \$10	outdoors with \$0
$\mathbf{H}_4^{**}$	outdoors with \$0	outdoors with \$10

This time, say the individual's preferences are $\mathbf{H}_1^{**} \sim \mathbf{H}_2^{**}$ and $\mathbf{H}_3^{**} \succ \mathbf{H}_4^{**}$. Since \$10 is strictly preferred to \$0, the ranking $\mathbf{H}_1^{**} \sim \mathbf{H}_2^{**}$ would seem to reveal that the individual's subjective probabilities satisfy $p_{rain} = p_{no\ rain}$. On the other hand, the ranking $\mathbf{H}_3^{**} \succ \mathbf{H}_4^{**}$ would seem to reveal that $p_{rain} > p_{no\ rain}$. Thus, these preferences violate our definition of probabilistic sophistication.

However, they are perfectly reasonable for someone whose subjective probabilities of the events {rain, no rain} are exactly {½,½}, and who would strictly prefer to be dry than be wet. If such preferences are consistent with – indeed, are generated by – such well-defined numerical subjective probabilities, how could they violate our definition of "probabilistic sophistication"?

The answer is that our notion of probabilistic sophistication is not merely that preferences are generated by means of an underlying subjective probability measure, but also the stronger property that the individual's evaluation of an act depends upon the assignment of outcomes to events *solely* through the probability distribution over outcomes generated by this assignment. As is well known, this latter condition is not satisfied when preferences are state-dependent: not all acts that imply, say, a ½:½ chance of the outcomes {outdoors with \$10, outdoors with \$0} are indifferent, since it matters how the money is correlated with the precipitation.[14]

[14] As noted by Karni, Schmeidler and Vind (1983) and others, this feature can be avoided by incorporating the relevant features of the state into the definition of the outcomes, so that the outcome set would become {outdoors with \$10 and rain, outdoors with \$10 and no rain,...} in which case our individual's ranking of acts over this

3.2 Insufficiency of the Standard Calibration Procedure

A widely used method for eliciting subjective probabilities is the standard "calibration" procedure, by which an individual is asked to compare the likelihood of a subjective event with the likelihood of an objectively generated event. Such a comparison may derive from verbally elicited beliefs, such as the question "Which do you think is more likely: that it will rain in Boston today or that the next spin of this roulette wheel will land an even number?" Or, it may be choice-theoretic, in which case the individual may be asked to choose among two bets: one of which pays off if it rains in Boston today, and the other of which pays the same prize if the roulette wheel lands even.

It is clear that a necessary condition for probabilistic sophistication is that such "revealed likelihood beliefs" be well-defined and consistent, in the sense that this means of ranking event likelihoods be independent of the prizes used. This property can be formalized as:

> **Consistent Calibration Condition:** For any event E, probability p, and outcomes $x \succ y$ and $x^* \succ y^*$:
>
> $$[x \text{ on } E\,;\, y \text{ on} \sim\!E] \succsim (x,p;y,(1\text{-}p)) \quad \Leftrightarrow \quad [x^* \text{ on } E\,;\, y^* \text{ on} \sim\!E] \succsim (x^*,p;y^*,(1\text{-}p))$$

If such a condition is satisfied, then by a sequence of questions, we could (in the limit) obtain a probability, call it p(E), such that for all outcomes x and y:

$$[\,x \text{ on } E\,;\, y \text{ on} \sim\!E\,] \;\sim\; (\,x\,,p(E)\,;\,y,(1\text{-}p(E))\,)$$

The above procedure is nothing new, and versions of it has been used for some time for the elicitation of subjective probability measures.[15] However, it turns out that the above consistency condition, even if it turns out to generate a well-defined probability measure p(·) over events, is *not* sufficient to imply probabilistic sophistication. To see this, consider the following example:[16]

> **Example**: Let $\mathcal{S} = [0,1]$, $\mathcal{X} = \{1,2,3\}$, and consider the preference functional V(·) defined over the set $\mathcal{R}\cup\mathcal{H}$ of pure roulette and pure horse lotteries, where:[17]
>
> $$V(\mathbf{R}) \;=\; V(\,1,p_1\,;2,p_2\,;3,p_3\,) \;\equiv\; \sum_{i=1}^{3} i\cdot p_i$$
>
> $$V(\mathbf{H}) \;=\; V(1 \text{ on } E_1\,;\,2 \text{ on } E_2\,;\,3 \text{ on } E_3\,) \;\equiv\; \sum_{i=1}^{3} i\cdot\lambda(E_i) \;+\; \left[\prod_{i=1}^{3}\lambda(E_i)\cdot\sum_{j=1}^{3} j\cdot\psi(E_j)\right]$$
>
> where $\lambda(E) \equiv \int_E 1\cdot ds$ is uniform Lebesgue measure on the interval [0,1], and the measure ψ(·) is defined by $\psi(E) \equiv \int_E s\cdot ds$. Over pure roulette lotteries, V(·) is simply expected value. For pure horse lotteries, it satisfies the standard "statewise monotonicity" property that if **H*** yields at least as great a payoff as **H** in each

---

enriched outcome set will not only be state-independent, also but probabilistically sophisticated. Our point, then, is twofold: (i) the phenomenon of state-dependence contradicts the hypothesis of probabilistic sophistication, and (ii) in such cases, both the properties of state-dependence/state-independence and probabilistic sophistication must be defined relative to a specific outcome set, at a specific level of description.

[15] See for example Pratt, Raiffa and Schlaifer (1964) or DeGroot (1970, Ch.6).

[16] This example is based on a similar one from Machina and Schmeidler (1992, pp.759-760).

[17] Note that one or two of the probabilities $\{p_1,p_2,p_3\}$ in the upper formula may be zero, and one or two of the events $\{E_1,E_2,E_3\}$ in the lower formula may be empty or have zero Lebesgue measure.

state, then $V(\mathbf{H}^*) \geq V(\mathbf{H})$, with strict equality if $\mathbf{H}^*$ yields a greater payoff over any event with positive Lebesgue measure.[18] It also satisfies $V(\delta_x) \equiv V(x \text{ on } \mathcal{S}) \equiv x$ for each $x \in \mathcal{X}$. Now, for any event E and outcomes $x \neq y$ from $\{1,2,3\}$, we have

$$V(x \text{ on } E\,;\, y \text{ on } {\sim}E) \;=\; x{\cdot}\lambda(E) + y{\cdot}\lambda({\sim}E) \;=\; V(x, \lambda(E)\,; y, (1-\lambda(E)))$$

so that $V(\cdot)$ satisfies the Consistent Calibration Condition, with a unique subjective probability measure corresponding to the uniform measure $\lambda(\cdot)$ over the state space $\mathcal{S} = [0,1]$. Thus, $V(\cdot)$ should be indifferent between the pure horse lotteries:

$$\mathbf{H}_1 = [1 \text{ on } [0,\tfrac{1}{3});2 \text{ on } [\tfrac{1}{3},\tfrac{2}{3});3 \text{ on } [\tfrac{2}{3},1]], \quad \mathbf{H}_2 = [1 \text{ on } [\tfrac{1}{3},\tfrac{2}{3});2 \text{ on } [0,\tfrac{1}{3});3 \text{ on } [\tfrac{2}{3},1]]$$

But since $V(\mathbf{H}_1) = 2\frac{11}{243}$ and $V(\mathbf{H}_2) = 2\frac{10}{243}$, $V(\cdot)$ is not probabilistically sophisticated.

This example shows that consistency in simple calibration schemes is *not* sufficient to imply probabilistically sophisticated choice. The reason is that the Consistent Calibration Condition only applies to preferences over *two-outcome* roulette and/or horse lotteries. However, the property of probabilistic sophistication applies to general, many-outcome bets. Our example does satisfy probabilistic sophistication over two-outcome lotteries, but not over more general lotteries.

4. A MINIMALIST CHARACTERIZATION OF SUBJECTIVE PROBABILITY

4.1 Two Weaker Axioms

As mentioned above, our goal is to replace the expected utility-based Independence and Substitution Axioms by two weaker axioms, which suffice to imply probabilistic sophistication even though they no longer imply expected utility risk preferences. The first of these axioms requires some standard definitions: An event E is said to be **null** if for any partition $\{E, E_2, \ldots, E_n\}$

$$[\mathbf{R}^* \text{ on } E\,;\, \mathbf{R}_2 \text{ on } E_2\,;\, \ldots\,;\, \mathbf{R}_n \text{ on } E_n] \;\sim\; [\mathbf{R} \text{ on } E\,;\, \mathbf{R}_2 \text{ on } E_2\,;\, \ldots\,;\, \mathbf{R}_n \text{ on } E_n]$$

for all roulette lotteries $\mathbf{R}^*$, $\mathbf{R}$, $\mathbf{R}_2$, ... $\mathbf{R}_n$ (otherwise E is termed **nonnull**). The roulette lottery $\mathbf{R}^* = (x_1^*, p_1^*; \ldots; x_{m^*}^*, p_{m^*}^*)$ is said to **first order stochastically dominate** $\mathbf{R} = (x_1, p_1; \ldots; x_m, p_m)$ if

$$\sum_{\{i \mid x_i^* \succsim x\}} p_i^* \;\geq\; \sum_{\{i \mid x_i \succsim x\}} p_i \qquad \text{for all } x \in \mathcal{X}$$

$\mathbf{R}^*$ strictly first order stochastically dominates $\mathbf{R}$ if, in addition, strict inequality holds for some $x \in \mathcal{X}$.

Given this, we have:

AXIOM 6 (First Order Stochastic Dominance Preference): For any pair of pure roulette lotteries $\mathbf{R}_i^*$ and $\mathbf{R}_i$, if $\mathbf{R}_i^*$ first order stochastically dominates $\mathbf{R}_i$, then

$$[\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_i^* \text{ on } E_i;\ldots;\mathbf{R}_n \text{ on } E_n] \;\succsim\; [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_i \text{ on } E_i;\ldots;\mathbf{R}_n \text{ on } E_n]$$

for all partitions $\{E_1, \ldots, E_n\}$ and all roulette lotteries $\{\mathbf{R}_1, \ldots, \mathbf{R}_{i-1}, \mathbf{R}_{i+1}, \ldots, \mathbf{R}_n\}$, with strict preference if $\mathbf{R}_i^*$ strictly stochastically dominates $\mathbf{R}_i$ and E_i is non-null.

[18] To see this, take an act $\mathbf{H} = [1 \text{ on } E_1;\, 2 \text{ on } E_2;\, 3 \text{ on } E_3]$ and raise its payoff by one on any subset A of E_1 or E_2. The sum $\sum_{i=1}^{3} i \cdot \lambda(E_i)$ rises by $\lambda(A)$, and if the term in brackets drops, it does so by less than $\lambda(A)$, so $V(\cdot)$ rises.

This axiom is similar to the Substitution Axiom, but it is weaker in the sense that it only implies the right-hand ranking of horse/roulette lotteries in the specific case when $\mathbf{R}_i^*$ stochastically dominates $\mathbf{R}_i$, and not necessarily in the more general case when $\mathbf{R}_i^* \succsim \mathbf{R}_i$.

Our final axiom provides the key to our characterization:

AXIOM 7 (Horse/Roulette Replacement Axiom): For any partition $\{E_1,\ldots,E_n\}$, if

$$\begin{bmatrix} \delta_M & \text{on } E_i \\ \delta_0 & \text{on } E_j \\ \delta_0 & \text{on } E_k,\ k \neq i,j \end{bmatrix} \sim \begin{bmatrix} \alpha\cdot\delta_M + (1-\alpha)\cdot\delta_0 & \text{on } E_i \\ \alpha\cdot\delta_M + (1-\alpha)\cdot\delta_0 & \text{on } E_j \\ \delta_0 & \text{on } E_k,\ k \neq i,j \end{bmatrix}$$

for some probability $\alpha \in [0,1]$ and pair of events E_i and E_j, then

$$\begin{bmatrix} \mathbf{R}_i & \text{on } E_i \\ \mathbf{R}_j & \text{on } E_j \\ \mathbf{R}_k & \text{on } E_k,\ k \neq i,j \end{bmatrix} \sim \begin{bmatrix} \alpha\cdot\mathbf{R}_i + (1-\alpha)\cdot\mathbf{R}_j & \text{on } E_i \\ \alpha\cdot\mathbf{R}_i + (1-\alpha)\cdot\mathbf{R}_j & \text{on } E_j \\ \mathbf{R}_k & \text{on } E_k,\ k \neq i,j \end{bmatrix}$$

for all roulette lotteries $\{\mathbf{R}_1,\ldots,\mathbf{R}_n\}$.

This axiom gets its name because it states that the rate at which the individual is willing to "replace" subjective uncertainty across the events E_i and E_j (as in the left-hand lotteries) with objective uncertainty in the event $E_i \cup E_j$ (as in the right-hand lotteries) does not depend upon the prizes (be they outcomes or roulette lotteries) in the events E_i and E_j, or in any other event E_k.

Note that this axiom does not, in and of itself, assert either the existence or uniqueness of a probability α that satisfies the upper indifference relation. However, as long as neither E_i nor E_j is null, First Order Stochastic Dominance Preference implies

$$\begin{bmatrix} \delta_M & \text{on } E_i \\ \delta_M & \text{on } E_j \\ \delta_0 & \text{on } E_k,\ k \neq i,j \end{bmatrix} \succ \begin{bmatrix} \delta_M & \text{on } E_i \\ \delta_0 & \text{on } E_j \\ \delta_0 & \text{on } E_k,\ k \neq i,j \end{bmatrix} \succ \begin{bmatrix} \delta_0 & \text{on } E_i \\ \delta_0 & \text{on } E_j \\ \delta_0 & \text{on } E_k,\ k \neq i,j \end{bmatrix}$$

so that, by Mixture Continuity, there does exist some $\alpha \in (0,1)$ satisfying the upper indifference relation. If E_i (resp. E_j) is null, then it will be satisfied for $\alpha = 0$ (resp. $\alpha = 1$). If E_i and E_j are *both* null, then both the upper and lower indifference relations will be satisfied for all $\alpha \in [0,1]$. Finally, as long as at least one of E_i or E_j is nonnull, First Order Stochastic Dominance Preference ensures that the value of α that satisfies the upper indifference relation will be unique.

4.2 How does the Replacement Axiom differ from the Independence and Substitution Axioms?

The distinction between the expected utility-based Independence and Substitution Axioms and the Replacement Axiom is best understood by examining their common structure. Each axiom has an "if-then" form, in which knowledge of a single preference ranking allows us to infer preferences over a class of horse/roulette lotteries. Since an individual's ranking of horse/roulette lotteries involves both their *risk attitudes* (preferences over probability distributions on outcomes) as well as their *beliefs* (subjective probabilities of events), each axiom can be interpreted as a "consistency condition" on the individual's risk attitudes and/or beliefs across a class of horse/roulette lotteries.

Of the three axioms, the Substitution Axiom is the simplest. It states that the ranking of the pure roulette lotteries $\mathbf{R}_i^*$ versus $\mathbf{R}_i$, which only reveals information about the individual's attitudes toward risk, is sufficient to infer their ranking of any pair of horse/roulette lotteries of the form

$$[\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_i^* \text{ on } E_i;\ldots;\mathbf{R}_n \text{ on } E_n] \;\succsim\; [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_i \text{ on } E_i;\ldots;\mathbf{R}_n \text{ on } E_n]$$

This axiom accordingly imposes a global consistency condition on the individual's risk attitudes. But it also imposes at least something of a consistency condition on beliefs, since it implies that preferences are separable across events, so that the individual's attitudes toward "betting" on event E_1 versus E_2 will not depend upon the outcomes received in events $E_3,\ldots,E_n$.

The Independence Axiom states that the individual's ranking of the horse/roulette lotteries

$$[\mathbf{R}_1^* \text{ on } E_1;\ldots;\mathbf{R}_n^* \text{ on } E_n] \quad \text{versus} \quad [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n],$$

which reveals information both about their attitudes toward risk and their likelihood beliefs, is sufficient to infer their rankings over all pairs of "mixed" horse/roulette lotteries of the form

$$\begin{bmatrix} \alpha\cdot\mathbf{R}_1^* + (1-\alpha)\cdot\mathbf{R}_1^{**} \text{ on } E_1 \\ \vdots \\ \alpha\cdot\mathbf{R}_n^* + (1-\alpha)\cdot\mathbf{R}_n^{**} \text{ on } E_n \end{bmatrix} \quad \text{versus} \quad \begin{bmatrix} \alpha\cdot\mathbf{R}_1 + (1-\alpha)\cdot\mathbf{R}_1^{**} \text{ on } E_1 \\ \vdots \\ \alpha\cdot\mathbf{R}_n + (1-\alpha)\cdot\mathbf{R}_n^{**} \text{ on } E_n \end{bmatrix}$$

Thus, this axiom also imposes a consistency condition on both risk attitudes and subjective beliefs.

Unlike the Substitution and Independence Axioms, however, the Replacement Axiom is *only* a condition on the individual's *beliefs*.[19] This can be seen from the structure of the two indifference rankings in the axiom. Consider in particular the initial ranking, namely

$$\begin{bmatrix} \delta_M & \text{on } E_i \\ \delta_0 & \text{on } E_j \\ \delta_0 & \text{on } E_k,\, k\neq i,j \end{bmatrix} \sim \begin{bmatrix} \alpha\cdot\delta_M + (1-\alpha)\cdot\delta_0 & \text{on } E_i \\ \alpha\cdot\delta_M + (1-\alpha)\cdot\delta_0 & \text{on } E_j \\ \delta_0 & \text{on } E_k,\, k\neq i,j \end{bmatrix}$$

Note that since it involves just the two outcomes 0 and M, the *only* value of α that can satisfy this indifference condition is the value at which the two lotteries imply identical probability distributions over outcomes, i.e., the value of α that satisfies

$$(M, \mu(E_i)\,;\, 0,\, 1\text{-}\mu(E_i)) \;=\; (M, \alpha\cdot\mu(E_i)+\alpha\cdot\mu(E_j)\,;\, 0,\, 1 - \alpha\cdot\mu(E_i) - \alpha\cdot\mu(E_j))$$

which implies that α and $(1\text{-}\alpha)$ equal the subjective odds ratios:

$$\alpha \;=\; \mu(E_i)/(\mu(E_i)+\mu(E_j)) \qquad \text{and} \qquad (1\text{-}\alpha) \;=\; \mu(E_j)/(\mu(E_i)+\mu(E_j))$$

In other words, knowledge of the initial indifference ranking *only* yields information about the individual's beliefs over the relative likelihoods of E_i and E_j, and no information whatsoever about their risk preferences. The axiom does, however, impose a consistency condition on such beliefs, since this knowledge on α is enough to imply that any two horse/roulette lotteries of the form

$$\begin{bmatrix} \mathbf{R}_i & \text{on } E_i \\ \mathbf{R}_j & \text{on } E_j \\ \mathbf{R}_k & \text{on } E_k,\, k\neq i,j \end{bmatrix} \sim \begin{bmatrix} \alpha\cdot\mathbf{R}_i + (1-\alpha)\cdot\mathbf{R}_j & \text{on } E_i \\ \alpha\cdot\mathbf{R}_i + (1-\alpha)\cdot\mathbf{R}_j & \text{on } E_j \\ \mathbf{R}_k & \text{on } E_k,\, k\neq i,j \end{bmatrix}$$

[19] The discussion of this paragraph assumes that at least one of the events E_i or E_j is not null.

must imply *identical probability distributions over outcomes*,[20] and hence must be indifferent, regardless of the individual's risk preferences. Thus, the Replacement Axiom is only a consistency condition on beliefs, and neither assumes nor imposes any specific properties of risk preferences.

4.3 Theorem

By removing the expected utility-based Independence and Substitution Axioms from the standard characterization and replacing them with First Order Stochastic Dominance Preference and the Horse/Roulette Replacement Axiom, we obtain the following characterization of probabilistic sophistication[21] in the absence of the expected utility hypothesis:

> **THEOREM**: Given the setting of Section 2.1, the following conditions on a preference relation $\succsim$ over $\mathcal{L}$ are equivalent:
>
> (i) $\succsim$ satisfies the following axioms:
>
> Axiom 1 (Ordering)
>
> Axiom 2 (Independence of Order of Resolution)
>
> Axiom 3 (Mixture Continuity)
>
> Axiom 6 (First Order Stochastic Dominance Preference)
>
> Axiom 7 (Horse/Roulette Replacement Axiom)
>
> (ii) There exists a unique finitely-additive probability measure $\mu(\cdot)$ on $\mathcal{S}$ and a mixture continuous, strictly monotonic[22] preference functional $V(\cdot)$ over $\mathcal{R}$, such that for any pair of horse/roulette lotteries $[\mathbf{R}_1^* \text{ on } E_1^*;\ldots;\mathbf{R}_{n^*}^* \text{ on } E_{n^*}^*]$ and $[\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n]$:
>
> $$[\mathbf{R}_1^* \text{ on } E_1^*;\ldots;\mathbf{R}_{n^*}^* \text{ on } E_{n^*}^*] \succsim [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n]$$
>
> $$\Leftrightarrow \quad V(\textstyle\sum_{i=1}^{n^*} \mu(E_i^*)\cdot \mathbf{R}_i^*) \geq V(\sum_{i=1}^{n} \mu(E_i)\cdot \mathbf{R}_i)$$
>
> *Proof in Appendix*

5. CONCLUDING REMARKS

As stated in the Introduction, the current setting, axioms, theorem and proof constitute a very simple framework for deriving standard subjective probabilities from choice behavior over lotteries. Axioms 1, 3 and 6 – Ordering, Mixture Continuity and First Order Stochastic Dominance Preference – are the analogs of the standard ordering, continuity and monotonicity assumptions used throughout regular consumer theory. We view Axiom 2 – Independence of Order of Resolution – as necessary for "equal treatment" of subjective and objective uncertainty,

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[20] Namely, the common distribution $\mu(E_i)\cdot\mathbf{R}_i + \mu(E_j)\cdot\mathbf{R}_j + \sum_{k\neq i,j}\mu(E_k)\cdot\mathbf{R}_k$.

[21] More properly, our theorem offers a joint characterization of probabilistic sophistication along with the properties of "continuity" and "monotonicity" of risk preferences, as defined in the following footnote.

[22] Adopting the standard definitions for preference functionals, we say that $V(\cdot)$ is **mixture continuous** if $V(\alpha\cdot\mathbf{R}^*+(1-\alpha)\cdot\mathbf{R})$ is continuous in α for all $\mathbf{R}^*$ and $\mathbf{R}$, and $V(\cdot)$ is **strictly monotonic** if $V(\mathbf{R}^*) \geq V(\mathbf{R})$ whenever $\mathbf{R}^*$ first order stochastically dominates $\mathbf{R}$, with strict inequality in the case of strict dominance.

independent of their relative timing. Finally, we view Axiom 7 – our Horse/Roulette Replacement Axiom – as the minimal condition for the "consistent linking" of subjective events with objective probabilities.[23] Other than ordering, continuity and monotonicity, we make no assumptions, nor imply any restrictions, on the nature of risk preferences.

We conclude with brief remarks on the notion of conditional probability and updating in this setting, as well as a proposal concerning the interpretation of the term "Bayesian Rationality."

5.1 Conditional Probability and Bayesian Updating

In our 1992 paper (Sect. 5, Thm. 3), we showed that a probabilistically sophisticated non-expected utility maximizer would update purely subjective uncertainty according to Bayes' Law. Here we show that, as with our general characterization, the introduction of objective uncertainty and the Horse/Roulette Replacement Axiom considerably simply the derivation of this result.[24]

Intuitively, it is clear that the value of α in the Replacement Axiom gives the conditional probability of the event E_i given the event $E_i \cup E_j$. This suggests the following definition:

DEFINITION: Given a pair of events E_a and E_b with E_b non-null, the **conditional probability of E_a given E_b**, written $\mu(E_a|E_b)$, is the[25] value of α that solves:

$$\begin{bmatrix} \delta_M & \text{on } E_a \cap E_b \\ \delta_0 & \text{on } E_b - E_a \\ \delta_0 & \text{on } \sim E_b \end{bmatrix} \sim \begin{bmatrix} \alpha\cdot\delta_M + (1-\alpha)\cdot\delta_0 & \text{on } E_a \cap E_b \\ \alpha\cdot\delta_M + (1-\alpha)\cdot\delta_0 & \text{on } E_b - E_a \\ \delta_0 & \text{on } \sim E_b \end{bmatrix}$$

From Step 2 of the proof of our theorem, we have that the *unconditional* subjective probability of any event E is the value $\mu(E)$ that solves $[\delta_M \text{ on } E;\ \delta_0 \text{ on } {\sim}E] \sim [\mu(E)\cdot\delta_M + (1-\mu(E))\cdot\delta_0 \text{ on } \mathcal{S}]$. This fact, along with the above definition and our Replacement Axiom, imply:

$$[\mu(E_a \cap E_b)\cdot\delta_M + (1-\mu(E_a \cap E_b))\cdot\delta_0 \text{ on } \mathcal{S}] \sim [\delta_M \text{ on } E_a \cap E_b;\ \delta_0 \text{ on } {\sim}(E_a \cap E_b)]$$

$$= \begin{bmatrix} \delta_M & \text{on } E_a \cap E_b \\ \delta_0 & \text{on } E_b - E_a \\ \delta_0 & \text{on } \sim E_b \end{bmatrix} \sim \begin{bmatrix} \mu(E_a|E_b)\cdot\delta_M + (1-\mu(E_a|E_b))\cdot\delta_0 & \text{on } E_a \cap E_b \\ \mu(E_a|E_b)\cdot\delta_M + (1-\mu(E_a|E_b))\cdot\delta_0 & \text{on } E_b - E_a \\ \delta_0 & \text{on } \sim E_b \end{bmatrix}$$

$$= [\mu(E_a|E_b)\cdot\delta_M + (1-\mu(E_a|E_b))\cdot\delta_0 \text{ on } E_b;\ \delta_0 \text{ on } {\sim}E_b]$$

$$\sim [\mu(E_b)\cdot(\mu(E_a|E_b)\cdot\delta_M + (1-\mu(E_a|E_b))\cdot\delta_0) + (1-\mu(E_b))\cdot\delta_0 \text{ on } \mathcal{S}]$$

$$= [\mu(E_b)\cdot\mu(E_a|E_b)\cdot\delta_M + (1-\mu(E_b)\cdot\mu(E_a|E_b))\cdot\delta_0 \text{ on } \mathcal{S}]$$

which by Stochastic Dominance Preference, implies $\mu(E_a \cap E_b) = \mu(E_b)\cdot\mu(E_a|E_b)$. Thus, we have

$$\mu(E_a|E_b) = \mu(E_a \cap E_b)/\mu(E_b)$$

---

[23] Recall from Section 3.2 that, in the absence of the expected utility hypothesis, the standard "calibration" property is no longer sufficient for this purpose.

[24] For brevity, we do not repeat our earlier discussion of conditional risk preferences (1992, Sect.5.1).

[25] Uniqueness of α is ensured by First Order Stochastic Dominance Preference and nonnullness of E_b.

which implies all of the standard Bayesian updating formulas.[26] Thus, someone whose preferences satisfy the Horse/Roulette Replacement Axiom will "reveal" conditional and unconditional event probabilities that are linked according to the standard formula. Note that none of the expected utility-based axioms – neither Independence nor Substitution – are required for this result.

5.2 The Meaning of "Bayesian Rationality"

In this paper, we have presented axioms that imply the following three features:

the existence of unique, additive subjective probabilities,

preferences among subjective (or mixed subjective/objective) lotteries that depend only upon their implied probability distributions over outcomes, and

behavioral definitions of conditional probability and of unconditional probability that are related according to the standard formula.

We have done so without either assuming or implying that risk preferences necessarily conform to the expected utility principle.

We have already given a *descriptive* term for the above set of properties, namely "probabilistic sophistication." In this section, we want to argue that the proper *normative* term for them is *"Bayesian rationality."*

In proposing this use of the term, we are taking issue with some of the most highly respected researchers in the field, who use "Bayesian rationality" and "the Bayesian approach" to denote the *joint hypothesis* of probabilistic sophistication *and* expected utility risk preferences.[27] It is our strong view, however, that the terms "Bayesian" and "Bayesian rationality" should refer solely to how the individual represents, manipulates, and updates *beliefs* – namely, by well-defined subjective probabilities that satisfy the classical additivity and updating formulas – as opposed to the nature of their *risk preferences* – whether or not they rank probability distributions over outcomes by the expected utility criterion.

To be sure, debates over terminology are not the deepest of intellectual endeavors. However, we feel that:

1. The results of this paper and our previous one drive a clear conceptual wedge between the properties of probabilistically sophisticated beliefs and expected utility risk preferences. Accordingly, *some* terminological distinction between the two properties is warranted.
2. There is much more widespread agreement with the idea that probability theory is the "normatively correct" manner of representing uncertain beliefs than with the idea that expected utility is the "normatively correct" manner of ranking probability distributions.
3. To the audience of statisticians and physical scientists familiar with probability theory but not models of risk preference, the term "Bayesian" suggests Bayes' Law for updating

[26] Recall that non-nullness of E_b implies that $\mu(E_b) > 0$.

[27] Prominent writers who have taken "Bayesian rationality," "Bayesian approach," etc. to imply the expected utility principle include Cyert and DeGroot (1987, Ch.2), Gibbard and Harper (1978, Intro.), Harsanyi (1975, Sect.1; 1977, Sect.4), Jeffrey (1965/1983, pp.5-6), Raiffa (1968, pp.xi,127,278) and Skyrms (1991, Sect.1).

probabilistic beliefs, and trying to latch the expected utility principle onto this term is much like what the apple pie lobby has attempted to do with the term "Motherhood."

4. If "Bayesian" is taken to mean "what Bayes 'really' meant," we can detect no evidence in the Reverend's famous essay that he subscribed to – or even knew about – the expected utility principle of Bernoulli (1738). Indeed, the closest Bayes comes to any statement regarding the valuation of lotteries suggests the notion of *expected value* maximization:

   > "The *probability of any event* is the ratio between the value at which an expectation depending upon the happening of the event ought to be computed, and the value of the thing expected upon it's happening." (Bayes (1763, Section I, Definition 5)).[28]

   Thus, those who insist that "Bayesian rationality" refers to risk preferences would seem to be compelled to label *risk averse* expected utility maximizers as "Bayesian irrational"!

Although it is weaker than the above-mentioned joint hypothesis, we still do not claim that our notion of Bayesian rationality is universally exhibited in practice. Indeed, in Section 3.1 we have given what we consider to be two quite robust types of violations of probabilistic sophistication, namely Ellsberg-type behavior and state-dependent preferences. Experimental violations of Bayesian updating and other of classic probability formulas (additivity, compounding, etc.) have also been uncovered by Peterson, Schneider and Miller (1965), Pitz, Downing and Reinhold (1967), Slovic (1969), Bar-Hillel (1973), Grether (1980), Tversky and Kahneman (1983) and others,[29] and violations of what we have termed the Consistent Calibration Condition have been uncovered by Heath and Tversky (1991). However, we agree with those normative proponents of expected utility who argue that the "rationality" of a property and its empirical prevalence are separate issues.

In concentrating on the property of probabilistic sophistication and arguing that it constitutes the essence of "Bayesian rationality," it is not our intention to deny the separate normative appeal of expected utility risk preferences, or to deny a normative term to the joint hypothesis of probabilistic sophistication and expected utility maximization. In light of their pathbreaking work in characterizing this joint hypothesis, we suggest that it be given the name "Ramsey/Savage rationality." Our purpose in separately characterizing – and separately naming – the specific property of probabilistic sophistication has been to focus attention on the key role of beliefs as a component in modeling choice under subjective, or subjective/objective, uncertainty.

[28] Thus, the prominent Bayesian D.V. Lindley writes "An interesting feature of Bayes' approach is that he defines probability in terms of expectation. The amount you would pay for the expectation of one unit of currency were B to occur is p(B)." (Lindley (1988, p.208)).

[29] See also the papers in the December 1970 issue of *Acta Psychologica*, in Kahneman, Slovic and Tversky (1982), and in Arkes and Hammond (1986).

APPENDIX – PROOF OF THEOREM

The implication (ii) $\Rightarrow$ (i) is straightforward. Our proof of (i) $\Rightarrow$ (ii) consists of three steps. Step 1 shows that preferences over roulette lotteries $\mathbf{R} \in \mathcal{R}$ can be represented by a mixture continuous, strictly monotonic preference functional $V(\mathbf{R})$. Step 2 derives the subjective probability measure $\mu(\cdot)$ and shows that the individual is indifferent between an arbitrary horse/roulette lottery $[\mathbf{R}_1$ on $E_1;\ldots;\mathbf{R}_n$ on $E_n]$ and the roulette lottery $\sum_{i=1}^{n}\mu(E_i)\cdot\mathbf{R}_i$, so that we can represent preferences on horse/roulette lotteries by the preference function $V(\sum_{i=1}^{n}\mu(E_i)\cdot\mathbf{R}_i)$. Finally, Step 3 shows that the subjective probability measure $\mu(\cdot)$ is unique.

STEP 1: Recall that $M \succsim x \succsim 0$ for every $x \in \mathcal{X}$, and that $M \succ 0$. For each roulette lottery $\mathbf{R}$, our identification conventions, Mixture Continuity, and First Order Stochastic Dominance Preference imply that there exists a unique probability $\nu_{\mathbf{R}}$ such that $\mathbf{R} \sim [\mathbf{R}$ on $\mathcal{S}] \sim [\nu_{\mathbf{R}}\cdot\delta_M+(1-\nu_{\mathbf{R}})\cdot\delta_0$ on $\mathcal{S}] \sim \nu_{\mathbf{R}}\cdot\delta_M + (1-\nu_{\mathbf{R}})\cdot\delta_0$. Define $V(\mathbf{R}) \equiv \nu_{\mathbf{R}}$. Thus, for any roulette lotteries $\mathbf{R}^*$ and $\mathbf{R}$:

$$\mathbf{R}^* \succsim \mathbf{R} \quad\Leftrightarrow\quad \nu_{\mathbf{R}^*}\cdot\delta_M + (1-\nu_{\mathbf{R}^*})\cdot\delta_0 \succsim \nu_{\mathbf{R}}\cdot\delta_M + (1-\nu_{\mathbf{R}})\cdot\delta_0 \quad\Leftrightarrow\quad \nu_{\mathbf{R}^*} \geq \nu_{\mathbf{R}} \quad\Leftrightarrow\quad V(\mathbf{R}^*) \geq V(\mathbf{R})$$

It is clear that $V(\cdot)$ inherits the properties of mixture continuity and strict monotonicity from $\succsim$.

STEP 2: For each event $E \subseteq \mathcal{S}$, define $\mu(E)$ as the mixture probability that satisfies

$$[\,\delta_M \text{ on } E\,;\,\delta_0 \text{ on } \sim E\,] \sim [\,\mu(E)\cdot\delta_M + (1-\mu(E))\cdot\delta_0 \text{ on } \mathcal{S}\,]$$

From the discussion following the Horse/Roulette Replacement Axiom, we have that $\mu(E)$ will exist and be unique, with $\mu(E) = 0$ if E is null, and $\mu(E) = 1$ if $\sim E$ is null (so that $\mu(\mathcal{S}) = 1$).

Picking an arbitrary non-constant horse/roulette lottery is equivalent to picking an arbitrary partition $\{E_1,\ldots,E_n\}$ with $n \geq 2$ and assigning arbitrary roulette lotteries $\{\mathbf{R}_1,\ldots,\mathbf{R}_n\}$ to these events. Order the events so that E_1 is nonnull. For $i = 1,\ldots,n-1$, let α_i be the mixture probability from the Replacement Axiom for the events $A_i = E_1\cup\ldots\cup E_i$ and $B_i = E_{i+1}$. Since none of the events $A_1,\ldots,A_{n-1}$ are null, it follows from the discussion following the Replacement Axiom that each α_i is well-defined, unique, and independent of the roulette lotteries assigned to the events A_i and B_i. Define

$$\begin{aligned}
\tau_1 &= \alpha_{n-1}\cdot\alpha_{n-2}\cdot\ldots\cdot\alpha_3\cdot\alpha_2\cdot\alpha_1\\
\tau_2 &= \alpha_{n-1}\cdot\alpha_{n-2}\cdot\ldots\cdot\alpha_3\cdot\alpha_2\cdot(1-\alpha_1)\\
\tau_3 &= \alpha_{n-1}\cdot\alpha_{n-2}\cdot\ldots\cdot\alpha_3\cdot(1-\alpha_2)\\
\tau_4 &= \alpha_{n-1}\cdot\alpha_{n-2}\cdot\ldots\cdot(1-\alpha_3)\\
&\vdots\\
\tau_{n-1} &= \alpha_{n-1}\cdot(1-\alpha_{n-2})\\
\tau_n &= (1-\alpha_{n-1})
\end{aligned}$$

(note that $\tau_1+\ldots+\tau_n = 1$).

For any horse/roulette lottery $[\mathbf{R}_1$ on $E_1;\ldots;\mathbf{R}_n$ on $E_n]$ over the partition $\{E_1,\ldots,E_n\}$, repeated application of the Replacement Axiom yields:

$$\begin{bmatrix} \mathbf{R}_1 \text{ on } E_1 \\ \mathbf{R}_2 \text{ on } E_2 \\ \mathbf{R}_3 \text{ on } E_3 \\ \mathbf{R}_4 \text{ on } E_4 \\ \vdots \\ \mathbf{R}_n \text{ on } E_n \end{bmatrix} \sim \begin{bmatrix} \alpha_1 \cdot \mathbf{R}_1 + (1-\alpha_1)\cdot \mathbf{R}_2 & \text{on } E_1 \\ \alpha_1 \cdot \mathbf{R}_1 + (1-\alpha_1)\cdot \mathbf{R}_2 & \text{on } E_2 \\ \mathbf{R}_3 & \text{on } E_3 \\ \mathbf{R}_4 & \text{on } E_4 \\ \vdots & \vdots \\ \mathbf{R}_n & \text{on } E_n \end{bmatrix} \sim \begin{bmatrix} \alpha_2 \cdot [\alpha_1 \cdot \mathbf{R}_1 + (1-\alpha_1)\cdot \mathbf{R}_2] + (1-\alpha_2)\cdot \mathbf{R}_3 & \text{on } E_1 \\ \alpha_2 \cdot [\alpha_1 \cdot \mathbf{R}_1 + (1-\alpha_1)\cdot \mathbf{R}_2] + (1-\alpha_2)\cdot \mathbf{R}_3 & \text{on } E_2 \\ \alpha_2 \cdot [\alpha_1 \cdot \mathbf{R}_1 + (1-\alpha_1)\cdot \mathbf{R}_2] + (1-\alpha_2)\cdot \mathbf{R}_3 & \text{on } E_3 \\ \mathbf{R}_4 & \text{on } E_4 \\ \vdots & \vdots \\ \mathbf{R}_n & \text{on } E_n \end{bmatrix}$$

$$= \begin{bmatrix} \tau_1 \cdot \mathbf{R}_1 + \ldots + \tau_n \cdot \mathbf{R}_n & \text{on } E_1 \\ \vdots & \vdots \\ \tau_1 \cdot \mathbf{R}_1 + \ldots + \tau_n \cdot \mathbf{R}_n & \text{on } E_n \end{bmatrix} = [\tau_1 \cdot \mathbf{R}_1 + \ldots + \tau_n \cdot \mathbf{R}_n \text{ on } \mathcal{S}] \sim \textstyle\sum_{i=1}^{n} \tau_i \cdot \mathbf{R}_i$$

Thus, any horse/roulette lottery $[\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n]$ over $\{E_1,\ldots,E_n\}$ is indifferent to the roulette lottery $\sum_{i=1}^{n} \tau_i \cdot \mathbf{R}_i$. For $i = 1,\ldots,n$: we can set $\mathbf{R}_i = \delta_M$ and $\mathbf{R}_j = \delta_0$ for $j \neq i$ to obtain

$$[\,\delta_M \text{ on } E_i\,;\, \delta_0 \text{ on } \sim E_i\,] \;\sim\; \tau_i \cdot \delta_M + (1-\tau_i)\cdot \delta_0 \qquad \text{for } i = 1,\ldots,n$$

This implies $\tau_i = \mu(E_i)$ for each i. But since the partition $\{E_1,\ldots,E_n\}$ was arbitrary, we have that any horse/roulette lottery $[\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n]$ is indifferent to the roulette lottery $\sum_{i=1}^{n} \mu(E_i)\cdot \mathbf{R}_i$. Thus, for any two horse/roulette lotteries $[\mathbf{R}_1^* \text{ on } E_1^*;\ldots;\mathbf{R}_{n^*}^* \text{ on } E_{n^*}^*]$ and $[\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n]$:

$$[\mathbf{R}_1^* \text{ on } E_1^*;\ldots;\mathbf{R}_{n^*}^* \text{ on } E_{n^*}^*] \succsim [\mathbf{R}_1 \text{ on } E_1;\ldots;\mathbf{R}_n \text{ on } E_n] \quad \Leftrightarrow$$

$$\textstyle\sum_{i=1}^{n^*} \mu(E_i^*)\cdot \mathbf{R}_i^* \succsim \sum_{i=1}^{n} \mu(E_i)\cdot \mathbf{R}_i \quad \Leftrightarrow \quad V(\sum_{i=1}^{n^*} \mu(E_i^*)\cdot \mathbf{R}_i^*) \geq V(\sum_{i=1}^{n} \mu(E_i)\cdot \mathbf{R}_i)$$

Finally, for any disjoint events E and E*, we have

$$\mu(E\cup E^*)\cdot \delta_M + (1-\mu(E\cup E^*))\cdot \delta_0 \;\sim\; [\,\delta_M \text{ on } E\cup E^*\,;\, \delta_0 \text{ on } \sim(E\cup E^*)\,] \;=$$

$$[\,\delta_M \text{ on } E\,;\, \delta_M \text{ on } E^*\,;\, \delta_0 \text{ on } \sim(E\cup E^*)\,] \;\sim\; (\mu(E)+\mu(E^*))\cdot \delta_M + (1-\mu(E)-\mu(E^*))\cdot \delta_0$$

from which it follows that $\mu(\cdot)$ is finitely additive.

STEP 3: Assume there exist some other subjective probability measure $\mu^*(\cdot)$ and mixture continuous strictly monotonic preference functional $V^*(\cdot)$ which together also represent $\succsim$, where $\mu^*(\cdot)$ is distinct from $\mu(\cdot)$ ($V^*(\cdot)$ may or may not be distinct from $V(\cdot)$). This implies that there exist some event E and probability ρ such that $\mu^*(E) > \rho > \mu(E)$. Define

$$\mathbf{H}^R = [\,\delta_M \text{ on } E\,;\, \delta_0 \text{ on } \sim E\,] \quad \text{and} \quad \mathbf{R}^* = \rho\cdot\delta_M + (1-\rho)\cdot\delta_0$$

Since the roulette lottery $\mu^*(E)\cdot\delta_M+(1-\mu^*(E))\cdot\delta_0$ strictly first order stochastically dominates the roulette lottery $\rho\cdot\delta_M+(1-\rho)\cdot\delta_0$, and since $V^*(\cdot)$ is strictly monotonic, we have

$$V^*(\mathbf{H}^R) = V^*(\mu^*(E)\cdot\delta_M + (1-\mu^*(E))\cdot\delta_0) > V^*(\rho\cdot\delta_M+(1-\rho)\cdot\delta_0) = V^*(\mathbf{R}^*)$$

which implies $\mathbf{H}^R \succ \mathbf{R}^*$. However, since $\rho\cdot\delta_M+(1-\rho)\cdot\delta_0$ strictly first order stochastically dominates the roulette lottery $\mu(E)\cdot\delta_M+(1-\mu(E))\cdot\delta_0$, and since $V(\cdot)$ is strictly monotonic, we have

$$V(\mathbf{H}^R) = V(\mu(E)\cdot\delta_M + (1-\mu(E))\cdot\delta_0) < V(\rho\cdot\delta_M+(1-\rho)\cdot\delta_0) = V(\mathbf{R}^*)$$

which implies $\mathbf{H}^R \prec \mathbf{R}^*$, which is a contradiction. **Q.E.D.**

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