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Sequential Equilibria and Cheap Talk in Infinite Signaling Games. Part 1: Sequential Equilibria*

by

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Abstract

An example shows that there are well-behaved infinite signaling games with no sequential equilibria. We explore the relationship between equilibrium outcomes of the infinite game and those of approximating games. Consider a sequence of signaling games approaching a limit game. A (sub)sequence of equilibrium outcomes of the approximating games will converge to a limit distribution. That limit distribution will be an equilibrium outcome of the limit game if it can be realized by strategies of the limit game. As a result of this general convergence result, we prove the existance of sequential and weak-best-response equilibria for strongly monotonic games. In a companion article we explore the role of cheap talk in solving the non-existence problem.

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1 Introduction

Most aspects of the theory of extensive-form games require that players' choice variables have only a finite number of possible values (e.g., Kreps and Wilson 1982). In many economic applications of game theory, however, choices are more conveniently modeled as continuous variables. As a result, there is a growing literature extending the theory to infinite choice sets.[1] At present, there is no satisfactory definition of sequential equilibrium for general extensive-form games with infinite information sets and action spaces. Furthermore, there are some recent examples (Harris (1990), Reny and Robson (1991) and Seidmann (1992)) of well behaved infinite games that have no equilibrium. As a first step, this paper concerns the special case of infinite signaling games.

In a signaling game, player 1 first learns some private information, called his type, and then sends a signal to player 2. Player 2 observes this signal, makes an inference about player 1's probable type, and then responds with an action. The game ends with the players receiving payoffs that in general depend on player 1's type and signal and player 2's action. Signaling games have generated considerable interest of their own. They have been extensively applied in economics and finance.[2] Several authors have used these games to analyze refinements of the sequential equilibrium concept.[3] There is also a developing literature on signaling games with cheap talk (costless signaling).[4]

We first extend the definition of Sequential Equilibrium (SE) to signaling games with infinite type and action spaces. We then prove general theorems concerning convergence of SE for continuous games (in which the action and type spaces are compact metric spaces and the payoff functions are continuous), and existence of SE for a class of continuous signaling games. A general existence theorem is not possible; we discuss an example (due to Eric van Damme (1987)) of a simple continuous signaling game that has no sequential equilibria.

Given a game, we consider a sequence of games that approximates it. The outcome of a SE is the probability distribution on the types and signals of player 1 and the responses of player 2 resulting from playing the SE strategies. Our convergence theorem states that a limit of SE outcomes for the

[1] See, for example, Börgers (1989, 1991), Chakrabarti (1989a, b), Cotter (1991), Dasgupta and Maskin (1986), Fudenberg and Levine (1986), Harris (1985a, 1985b, 1990), Hellwig and Leininger (1987), Hellwig, Leininger, Reny and Robson (1991), Milgrom and Weber (1985), Simon and Stinchcombe (1989, 1991), Simon and Zame (1990), and Stinchcombe (1992a, b).

[2] Papers using signaling games (or variants) include Bhattacharya (1978), Leland and Pyle (1977), Milgrom and Roberts (1982), Myers and Majluff (1984), Riley (1979), and Spence (1974).

[3] See Banks and Sobel (1987), Cho and Kreps (1987), and Cho and Sobel (1987).

[4] See, for example, Farrell and Gibbons (1986,1989), Matthews, Okuno-Fujiwara and Postlewaite (1990), Seidmann (1990), and Stein (1989).

sequence of approximating games is a SE outcome (for the limit game) provided it can be realized by a pair of strategies of the limit game, in which player 2's strategy is continuous. As part of the convergence theorem, we show how to obtain equilibrium strategies supporting the outcome for the limit game and how to derive player 2's equilibrium beliefs from her beliefs in the approximating games.

Since finite games have SE, approximating an infinite game with a sequence of finite games leads (through the convergence theorem) to existence results: SE exist for strongly monotonic, continuous signaling games. In a companion essay (I-M 1993), we apply the convergence results to explore the question of existence in signaling games by using cheap-talk, player 1's ability to send a payoff irrelevant message to player 2.

In many finite games, a plethora of SE exist. To pare down the number of equilibria, a number of authors have proposed and analyzed refinements of the SE concept. We extend our convergence and existence theorems using one of the strongest of these refinements, the Weak Best Response test of Kohlberg and Mertens (1986).

As we said, we prove our existence results using a sequence of finite approximating games and a convergence theorem on outcomes. The antecedents of this convergence method probably begin with Harris (1985), who in proving the existence of pure strategy subgame perfect equilibria for a class of infinite games shows that the the map from histories into continuation paths is upper hemi-continuous.[5] Hellwig, Leininger, Reny and Robson (1990) explicitly use the convergence of subgame perfect equilibrium paths to prove existence in Harris' games.[6] Börgers (1991) defines a notion of approximation for a class of games and proves that the map from games to *pure* subgame perfect equilibrium outcomes is upper hemi-continuous. Börgers' result, as the previous ones, only applies to outcomes generated by pure strategies. Since finite games may not have pure strategy equilibria, the convergence result is not enough to yield existence in Börgers' games. Our method incorporates mixed outcomes, the result of randomization by the players.[7]

One can envision at least three other approaches to proving existence of equilibrium. First, one could try using a fixed-point argument as Milgrom and Weber (1985) do for one-shot, simultaneous-move games with incomplete information. This does not seem possible with signaling games because even though payoffs are continuous, expected payoffs are not with respect to any common topology on strategies that makes the space of strategies compact.

[5] Hellwig and Leininger (1987) provide an alternative proof of Harris result.

[6] Börgers (1989) obtains a related upper hemi-continuity result for truncations of infinite-horizon games.

[7] Harris (1991) provides a counter example to the existence of equilibria in Börgers' games, and proves the existence of stage correlated equilibria. We further discuss his result in I-M (1993).

A second alternative is to use a sequence of finite approximating games and a convergence theorem on strategies (instead of outcomes). One has to impose severe restrictions on a game to guarantee that a sequence of strategies will converge. We prove a convergence and existence result along these lines as a particular case of our convergence theorem. We have abandoned this approach because of the additional assumptions needed and because it does not seem to be applicable to more general extensive-form games.

A third alternative is to construct SE strategies using the equilibrium conditions of the game. Cho and Sobel (1987), Crawford and Sobel (1982), and Riley (1979) use this approach. All these papers show the existence of SE for certain classes of signaling games. All impose strong assumptions on the players' payoff functions, e.g., differentiability and restrictions on the cross derivatives. Because of their strong assumptions, all can analyze the properties of equilibria, such as how much information player 1 will reveal. Also assuming differentiability, Mailath (1987) shows that in a class of signaling games, separating equilibria imply differentiable equilibrium strategies. He provides necessary and sufficient conditions for the existence of separating equilibria in that class of games. This type of approach will also be difficult to apply to more general games.

The paper is organized as follows. Section II introduces the game, notation and definitions. Section III discusses the cause of the non-existence problem with four examples. Section IV presents and proves the convergence result on SE outcomes. Section V derives an existence theorem for a class of games. Section VI extends our results to equilibria satisfying the Weak Best Response test. An Appendix contains some of the proofs.

2 Notation and the Game

We summarize a game by $\Gamma = [(T, \rho), X, Y, U^1, U^2]$. In this game, player 1 first privately observes his type t from the set T of possible types and then sends a signal x from the set X. Player 2 observes this signal, infers player 1's probable types, and then selects an action y from the set Y. The game ends and each player i receives payoff $U^i(t, x, y)$. To complete the specification of the game, we assume that player 2 has prior beliefs ρ about the possible types t of player 1; ρ is a Borel probability distribution on T that is common knowledge. We denote the support of ρ by $\text{supp}[\rho]$ and assume for convenience that $\text{supp}[\rho] = T$.

We shall be concerned in this paper with games that are continuous in the following sense.

Definition 1 $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ *is* continuous *if T, X and Y are compact metric spaces and U^1 and U^2 are continuous.*

For the remainder of this section, fix a particular game $\Gamma = [(T, \rho), X, Y, U^1, U^2]$.

For any metric space A, let $M(A)$ denote the set of Borel probability distributions on A. Define an *event* of A to be a Borel-measurable subset. Given another metric space B and $\mu \in M(A \times B)$, let μ_A and μ_B denote the marginal distributions.

Kreps and Wilson (1982) defined sequential equilibrium for finite games. We adapt their definition to infinite signaling games as follows. A Sequential Equilibrium (SE) for game Γ is a triple $(\hat{\alpha}, \hat{\eta}, \beta)$, where $\hat{\alpha}$ denotes a strategy for player 1, η denotes a strategy for player 2, and β denotes a system of beliefs of player 2 about the type of player 1. To be a SE, the strategies and beliefs must satisfy three sets of requirements: the strategies must be *admissible*, the beliefs of player 2 must be *consistent* with the strategy of player 1, and the strategies of both players must be *sequentially rational*. We next discuss each of these elements individually and then summarize the discussion with a formal definition of sequential equilibrium. The admissible strategies for the players are behavior strategies conditioned on their information. For player 1, a behavior strategy is a function α from types in T to probability distributions in $M(X)$. Given a type $t \in T$ and an event $E \subset X$, $\alpha(t)(E)$ denotes the probability assigned to E by the distribution $\alpha(t)$.

Admissible strategies must be measurable so that they will induce well-defined joint probability distributions on the endpoints (t, x, y) of the game and thereby allow the players' expected payoffs to be computed. Let $\alpha \bullet \rho$ denote the joint distribution on $T \times X$ induced by the behavior strategy α when combined with the distribution ρ on T. [8] It is defined for all event rectangles $D \times E \subset T \times X$ as

$$(\alpha \bullet \rho)(D \times E) = \int_D \alpha(t)(E)\, \rho(dt).$$

We define the set of admissible strategies for player 1 in the game Γ as

$$\Sigma^1(\Gamma) = \{\alpha : T \longrightarrow M(X) \mid \alpha \text{ is measurable }\},$$

and shall require for a SE $(\hat{\alpha}, \hat{\eta}, \beta)$ that $\hat{\alpha} \in \Sigma^1(\Gamma)$.

The set of admissible strategies for player 2 in Γ is denoted by

$$\Sigma^2(\Gamma) = \{\eta : X \longrightarrow M(Y) \mid \eta \text{ is measurable}\}.$$

The form of a strategy for player 2 will make it convenient to work with the space $M(Y)$ instead of with the space Y directly. We extend the players' payoff functions U^i from Y to $M(Y)$ by taking

[8] The distribution $\alpha \bullet \rho$ will be well-defined if we require the function $t \longrightarrow \alpha(t)(E)$ to be measurable for each fixed event $E \subset X$ (Billingsley 1979, p. 394). This requirement is equivalent to putting the topology of weak convergence of measures on $M(X)$ and requiring that $\alpha : T \longrightarrow M(X)$ be a measurable function (Bertsekas and Shreve 1978, Proposition 7.26).

expected values: for each $(t, x, \eta) \in T \times X \times M(Y)$, we let

$$U^i(t, x, \eta) = \int_Y U^i(t, x, y)\, \eta(dy). \tag{1}$$

U^i is a continuous function on $T \times X \times Y$ if and only if the extension of U^i to $T \times X \times M(Y)$ is continuous (Lemma 4 below proves the non-trivial half of this statement).

We represent the beliefs of player 2 about player 1's type conditional on player 1's signal as a function $\beta : X \longrightarrow M(T)$; $\beta(x)$ is a probability distribution on player 1's types T given that player 1 has sent signal x.

For finite signaling games, Kreps and Wilson's consistency requirement only implies that the beliefs β must be derived using Bayes' Rule from player 1's strategy α and the prior distribution ρ. That is, β must be a conditional probability distribution of t given x derived from the joint distribution $\alpha \bullet \rho$. From the general definition of conditional distribution, this means

(B1) β is measurable, and

(B2) $\beta \bullet \mu_X = \mu$, where $\mu = \alpha \bullet \rho$.

Define

$$\Sigma^3(\Gamma) = \{(\beta, \alpha) \mid (\beta, \alpha) \text{ satisfy B1 and B2}\}.$$

Our consistency requirement on beliefs for a SE $(\hat\alpha, \hat\eta, \hat\beta)$ will be that $(\hat\beta, \hat\alpha) \in \Sigma^3(\Gamma)$.

The last two requirements for a sequential equilibrium are sequential rationality of strategies. Let $\hat\eta$ be player 2's strategy. The first requirement is that player 1 must only send signals x that maximize his expected payoff $U^1(t, x, \hat\eta(x))$ given that his type is t. Let $\hat\beta$ be player 2's beliefs. The second requirement is that player 2's response $\eta \in M(Y)$ to a signal x must maximize her expected payoff $\int_T U^2(t, x, \eta)\, \hat\beta(x)(dt)$. These conditions are stated as S3 and S4 below.

We summarize this discussion with the following

Definition 2 *A* Sequential Equilibrium (SE) *for* $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ *is a triple* $(\hat\alpha, \hat\eta, \hat\beta)$ *satisfying*

(S1) $\hat\alpha \in \Sigma^1(\Gamma)$, $\hat\eta \in \Sigma^2(\Gamma)$;

(S2) $(\hat\beta, \hat\alpha) \in \Sigma^3(\Gamma)$;

(S3) $\forall t \in T,\ \int_X U^1(t, x, \hat\eta(x))\, \hat\alpha(t)(dx) \ \geq\ U^1(t, x', \hat\eta(x'))\ \ \forall\, x' \in X$;

(S4) $\forall x \in X,\ \int_T U^2(t, x, \hat\eta(x))\, \hat\beta(x)(dt) \ \geq\ \int_T U^2(t, x, \eta)\, \hat\beta(x)(dt)\ \ \forall\, \eta \in M(Y)$.

3 Examples

Any infinite game may be approximated by a sequence of finite games obtained through increasingly finer discretizations. Any sequence of SE outcomes of the finite games will converge (in a subsequence) to a limit distribution. The following examples, variations of an example due to Eric van Damme[9] (1987), illustrate why the limit distribution may fail to be a SE outcome of the limit game. This, in turn, explains why SE need not exist in infinite games.

Example 1 shows that SE may not exist because the limit of SE outcomes cannot be realized by strategies of the limit game. Define the game Γ by

$$T = \{-1, 1\},\ \rho(-1) = \rho(1) = 1/2,\ X = Y = [-1, 1],$$

$$U^1(t, x, y) = -x^2 + ty,\ U^2(x, y) = xy.$$

To simplify notation, we write $\rho(-1)$ when we mean $\rho(\{-1\})$ since ρ is a measure. For simplicity, we will consider in this section mainly pure strategies, although all the claims are valid for behavior strategies as well. We will denote pure strategies by real-valued functions.

The game Γ is continuous, but it has no sequential equilibrium. To see this, observe that since player 2's payoff function is independent of t, her best responses do not depend on her beliefs concerning t. Letting the function $\tilde{y}$ denote player 2's strategy, in any potential sequential equilibrium we must have

$$\tilde{y}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases} \tag{2}$$

Now player 1 will choose x given t to maximize his payoff $U^1(t, x, \tilde{y}(x))$. This amounts to player 1 choosing a maximizer for U^1 from the graph of the function $\tilde{y}$. On this graph, the payoff for either type of player 1 is strictly increasing as $x \to 0$ from either above or below. But no matter how $\tilde{y}(0)$ is chosen (even randomly), the graph of $\tilde{y}$ will not be closed at $x = 0$, so that a maximizer will not exist for at least one type of player 1. Hence, a sequential equilibrium cannot exist.

Let $\langle X^n \rangle$ be a sequence of finite subsets of X that are increasingly finer approximations to X, say

$$X^n = \{\ (-1)^l k/n \mid l = 1,\ 2\ and\ k = 1, ..., n\}.$$

Let $\langle Y^n \rangle$ be an analogous sequence for Y, say $Y^n = X^n$. The game $\Gamma^n = [(T, \rho), X^n, Y^n, U^1, U^2]$ is finite, so it has a SE. SE strategies are $\hat{x}^n(-1) = -1/n$, $\hat{x}^n(1) = 1/n$, and $\hat{y}^n(x) = \tilde{y}(x)$ given by (2).

[9] We thank Subir Chakrabarti for bringing an example of van Damme's to our attention.

Now look at the sequence of outcomes for these SE for Γ^n. This is a sequence of distributions λ^n on $T \times X^n \times Y^n$ given by $\lambda^n(-1, -1/n, -1) = \lambda^n(1, 1/n, 1) = 1/2$ (again simplifying the notation for measures). [10] This sequence will converge weakly to the distribution λ on $T \times X \times Y$ given by

$$\lambda(-1, 0, -1) = \lambda(1, 0, 1) = 1/2. \tag{3}$$

This limit distribution cannot be a SE outcome because it cannot be implemented by a pair of strategies. We can construct a pure strategy for player 1 out of λ: it is $x(-1) = x(1) = 0$. We cannot, however, construct a response for player 2 because there is no unique conditional value of y given x: to be consistent with the limit distribution λ player 2 must respond to $x = 0$ with $y = -1$ (when $t = -1$) and with $y = 1$ (when $t = 1$). In each game Γ^n in this example, there is coordination between player 1 and player 2–player 2 plays high or low depending on whether player 1's type is high or low. But this coordination is lost in the limit game Γ.

Example 2 shows that SE may not exist even though the limit distribution is implementable in the infinite game. Define the game Γ by

$$T = X = Y = [-1, 1], \ \rho \text{ is uniform on } T,$$

$$U^1(t, x, y) = -(t - x + y)^2, \ U^2(x, y) = xy.$$

As in the first example, a SE does not exist. In any potential SE player 2's strategy must follow (2). Type $t = 1$ of player 1 prefers to send a negative signal x to any positive one, and his payoff strictly increases as $x \uparrow 0$. Similarly, $t = -1$ prefers to send a positive signal to any negative one, and his payoff strictly increases as $x \downarrow 0$. Since the graph of $\tilde{y}$ is not closed at 0, independent of how $\tilde{y}(0)$ is defined, no SE can exist. Intuitively, types $t = 1$ and $t = -1$ would like to separate themselves. Although any signal $x < 0$ for $t = 1$ and $x > 0$ for $t = -1$ would allow them to separate, none can be used in equilibrium because a less expensive one (for instance $x/2$) is always available. Therefore, separation in equilibrium must be costless. Since there is a unique costless signal for both players ($x = 0$), they cannot differentiate themselves: no equilibrium exists.

Define X^n and Y^n as in Example 1, and T^n similarly. Let ρ^n assign equal probability $1/2n$ to all elements in T^n. Thus, $\langle \rho^n \rangle \Rightarrow \rho$. Consider $\Gamma^n = [(T^n, \rho^n), X^n, Y^n, U^1, U^2]$. Player 2's equilibrium strategy is given by $\tilde{y}^n = \tilde{y}$ as defined in (2). Let

$$\tilde{x}(t) = \begin{cases} t - 1 & \text{if } 1 > t \geq 0 \\ 1 - t & \text{if } -1 < t < 0 \end{cases} \tag{4}$$

[10] In the next section, we use a slightly different definition of outcome to accommodate behavior strategies.

Player 1's SE strategy $\hat{x}^n$ in Γ^n is $\hat{x}^n(1) = -1/n$, $\hat{x}^n(-1) = 1/n$, and $\hat{x}^n(t) = \tilde{x}(t)$ everywhere else.

These strategies implement the SE outcome $\hat{\lambda}^n$ that assigns equal probability $1/2n$ to all elements of

$$\{(t, \tilde{x}(t), \tilde{y}(t)) : t \in T^n\} \cup \{(1, -1/n, -1), (-1, 1/n, 1)\}.$$

As $n \to \infty$, $\langle \hat{\lambda}^n \rangle \Rightarrow \hat{\lambda}$, the uniform distribution on

$$\{(t, t-1, -1) : 1 \geq t \geq 0\} \cup \{(t, 1-t, 1) : 1 \leq t < 0\}$$

The limit distribution $\hat{\lambda}$ is implemented in Γ by any strategy pair $(\hat{x}, \hat{y})$ in which $\hat{x}(t)$ respects (4), and $\hat{y}$ (2). Regardless of how $\hat{x}(1)$, $\hat{x}(-1)$, and $\hat{y}(0)$ are defined, types $t = 1$ or $t = -1$ of player 1 will not satisfy sequential rationality S3. Any strategy for player 2 that implements $\hat{\lambda}$ must be discontinuous at $x = 0$, and S3 requires that both $t = 1$ and $t = -1$ play $x = 0$.

Implementability of the limit distribution is not enough to guarantee that that distribution will be a SE outcome: Altering a strategy on a set of measure zero will not change the outcome of playing that strategy. The definition of SE, however, requires that sequential rationality (S3 and S4) hold everywhere.

We prove in Theorem 1 that the limit distribution is a SE outcome of the limit game if it is implementable with a continuous strategy for player 2. Given that player 2's strategy must respect (2), types $t = 1$ and $t = -1$ of player 1 always prefer to send smaller signals. In turn, for any definition of $\hat{y}(0)$, either type $t = 1$ or $t = -1$ of player 1 will be better off by sending a slightly different signal ($x < 0$ or $x > 0$ respectively.) This occurs because small changes in signals (at $x = 0$) generate very different responses from player 2 and, therefore, considerable changes in payoffs for player 1. If the limit distribution were implementable by a continuous strategy for player 2, small variations in the signal would effect small variations in responses (2 could not hold). This argument highlights that a weaker condition than continuity would also yield existence: It suffices that the payoffs to types $t = 1$ and $t = -1$ of sending similar signals be similar, given the responses that those signals elicit. We formulate this weaker condition after introducing Theorem 1.

Finally, note that the failure of S3 in the example occurs for a set of types of measure zero. If the failure occurred over a set of positive measure, the outcome could not be implemented by strategies as is the case in the first example.

Example 3 shows that it is not possible to find necessary and sufficient conditions (based solely on the limit distribution) to determine whether that limit distribution is a SE outcome of the limit game. Modify Γ in Example 2 by adding one point to the signaling space X so that $X = [-1, 1] \cup \{\bar{x}\}$. $U^1(t, \bar{x}, y) = -(1-t)^2$ and $U^2(t, \bar{x}, y) = 0$.

A sequence of approximating games $\langle\Gamma^n\rangle$ will yield the same limit distribution λ found in Example 2. Although the limit outcome $\hat{\lambda}$ is not implementable by a continuous strategy for player 2. it is, nonetheless, a SE outcome of the modified limit game. The SE strategies are $\hat{y} = \tilde{y}$ if $x \neq 0$, $y(0) = -1$, and $\hat{y}(\bar{x}) = 1$; $\hat{x}(t) = \tilde{x}(t)$ if $|t| \neq 1$, $\hat{x}(1) = \bar{x}$, and $\hat{x}(-1) = 0$. There are now two costless signals ($x = 0$ and $x = \bar{x}$) that types $t = 1$ and $t = -1$ can use to differentiate themselves. S3 is satisfied.

The same limit distribution λ is obtained in Examples 2 and 3. In the latter λ is a SE outcome; in the former it is not. Thus, necessary and sufficient conditions (based solely on λ) to determine whether the limit distribution is a SE outcome are not forthcoming.

Example 4 shows that the limit distribution $\hat{\lambda}$ may fail to be a SE outcome of the limit game because S4 is not satisfied. Player 2 chooses an inferior response to some signal.

Consider the game in Example 2 with $X = [-1, 1] \cup \{\bar{x}\}$, $U^1(t, \bar{x}, y) = -(1-t)^2 - (y+1)$, and $U^2(t, \bar{x}, y) = -(y^2)$. The same strategies that implemented λ in Example 3 implement λ here. S3 is satisfied but not S4 because at $x = \bar{x}$ player 2 chooses an inferior response.

4 Convergence of SE Outcomes

Standard usage defines the outcome of a SE to be the distribution generated by the equilibrium strategies on the end-points of the game. For a SE (α, η, β) for Γ, this outcome, which we shall call the ***standard outcome***, is a distribution $\hat{\nu}$ on $T \times X \times Y$ defined by $\hat{\nu} = \eta \bullet (\alpha \bullet \rho)$. We will need a more refined concept of outcome.

We will conserve notation by using Ψ in place of $M(Y)$. An $\eta \in \Psi$ will denote a generic mixed response by player 2. We define an outcome analogously to the standard outcome, but as a distribution on $T \times X \times \Psi$ instead of $T \times X \times Y$.

Definition 3 *The distribution λ on $T \times X \times \Psi$ is an outcome of Γ if there is a strategy pair (α, η) in Γ such that $\hat{\lambda} = (\hat{\alpha} \bullet \rho) \circ f_{\hat{\eta}}^{-1}$, where the function $f_{\hat{\eta}} : X \longrightarrow T \times X \times \Psi$ is defined by $f_{\hat{\eta}}(t, x) = (t, x, \hat{\eta}(x))$. We say that the strategy pair $(\hat{\alpha}, \hat{\eta})$ implements or generates the outcome λ. $SE(\Gamma)$ denotes the set of SE for Γ and $SEO(\Gamma)$ the set of SE outcomes.*

To clarify the concept of outcome, consider the game $\Gamma' = [(T, \rho), X, M(Y), U^1, U^2]$, an equivalent representation of Γ.[11] Player 2's response in Γ' to a signal x is an element of $M(M(Y)) = M(\Psi)$. We intend Γ and Γ' to be two different formulations of the same signaling game. Thus, for any strategy

[11] The payoff functions in Γ' are extended according to (1).

$\hat{\eta}$ of player 2 in Γ, there is a corresponding strategy in Γ', the function $g_{\hat{\eta}} : X \longrightarrow M(\Psi)$ such that $g_{\hat{\eta}}(x) = \delta_{\hat{\eta}(x)}$ where $\delta_{\hat{\eta}(x)}$ is the degenerate distribution at $\hat{\eta}(x)$. For any strategy pair (α, η) in Γ there is an equivalent strategy pair $(\hat{\alpha}, g_{\hat{\eta}})$ in Γ'. Its "standard outcome" in Γ' is the distribution λ on $T \times X \times \Psi$, given by $\hat{\lambda} = g_{\hat{\eta}} \bullet (\hat{\alpha} \bullet \rho)$. This distribution is precisely the "outcome" of Γ. Thus,

$$(\hat{\alpha} \bullet \rho) \circ f_{\hat{\eta}}^{-1} = \hat{\lambda} = g_{\hat{\eta}} \bullet (\hat{\alpha} \bullet \rho).$$

It is not difficult to see that any outcome generates a unique standard outcome and vice versa.[12] Although our results are valid when stated and interpreted in terms of the standard outcome, we need the concept of outcome in the proofs. We hope that its role will become clear in Section 4.2 where we discuss the proof of Theorem 1.

We will consider a sequence of games, say $\Gamma^n = [(T^n, \rho^n), X^n, Y^n, U^{1n}, U^{2n}]$ for $n = 1, 2, \ldots$, and a limit game $\Gamma = [(T, \rho), X, Y, U^1, U^2]$. We assume that all the type and action spaces are compact subspaces of ambient compact metric spaces $\tilde{T}$, $\tilde{X}$ and $\tilde{Y}$. Convergence of any sequence is always relative to the relevant ambient space.

Convergence for probability distributions on a sequence $\langle A^n \rangle$ of metric spaces means weak convergence on the ambient space $\tilde{A}$. Given a sequence of distributions $\langle \mu^n \rangle$, we write $\langle \mu^n \rangle \Rightarrow \mu$ when the sequence converges weakly to μ.

Convergence for the type and action spaces means closed convergence, or equivalently, convergence using the Hausdorff metric on sets (see Hildenbrand 1974, p. 15). Given a sequence of sets $\langle A^n \rangle$, define $\text{Li}\, \langle A^n \rangle$ as the set of limits of sequences $\langle x^n \rangle$ with $x^n \in A^n$ for all n; define $\text{Ls}\, \langle A^n \rangle$ as the set of limits of subsequences. We write $\langle A^n \rangle \longrightarrow A$ if and only if $\text{Li}\, \langle A^n \rangle = A = \text{Ls}\, \langle A^n \rangle$, and we say A is the *closed limit* of the sequence $\langle A^n \rangle$. One can show $\langle A^n \rangle \longrightarrow A$ if and only if $\langle M(A^n) \rangle \Rightarrow M(A)$.

Convergence for the payoff functions U^{in} means continuous convergence. We say that $\langle U^{in} \rangle \longrightarrow U^i$ continuously for $(t, x, y) \in T \times X \times Y$ when for all such (t, x, y), $\langle t^n, x^n, y^n \rangle \longrightarrow (t, x, y)$ implies $\langle U^{in}(t^n, x^n, y^n) \rangle \longrightarrow U^i(t, x, y)$.

Definition 4 *A sequence of continuous games $\langle \Gamma^n \rangle$ converges to a continuous game Γ, $\langle \Gamma^n \rangle \longrightarrow \Gamma$, if*

(H1) $\langle X^n \rangle \longrightarrow X$, $\langle Y^n \rangle \longrightarrow Y$, $\langle T^n \rangle \longrightarrow T$, $\langle \rho^n \rangle \Rightarrow \rho$;

(H2) $\langle U^{in} \rangle \longrightarrow U^i$ *continuously for* $(t, x, y) \in T \times X \times Y$, $i = 1, 2$.

Theorem 1 *Consider a sequence of continuous games $\Gamma^n = [(T^n, \rho^n), X^n, Y^n, U^{1n}, U^{2n}]$, $n = 1, 2, \ldots$, and a continuous game $\Gamma = [(T, \rho), X, Y, U^1, U^2]$. Suppose $\langle \Gamma^n \rangle \longrightarrow \Gamma$, and*

[12] In a I-M (1993), we discuss in more detail the relationship between both notions of outcome.

(H3) *there exists* $\langle \hat{\lambda}^n \rangle \Rightarrow \hat{\lambda}$ *with* $\hat{\lambda}^n \in SEO(\Gamma^n)$, *and*

(H4) *there exists a continuous* $\eta^1 : supp[\lambda_X] \longrightarrow \Psi$ *such that* $\lambda = \lambda_{T \times X} \circ f_{\eta^1}{}^{-1}$.

Then $\hat{\lambda} \in SEO(\Gamma)$.

H3 requires that a sequence of SE outcomes of the approximating games converge to a limit distribution $\hat{\lambda}$. Since the space of distributions is compact, given any sequence of SE outcomes it is always possible to find a converging subsequence. H4 requires that there be a pair of strategies of the limit game Γ that generate the limit distribution λ, and that player 2's strategy be continuous: In our setting, conditional probabilities exist (for instance Theorem V.8.1 in Parthasarathy (1967)). Thus, there is a strategy α^1 for player 1 such that $\alpha^1 \bullet \rho = \lambda_{T \times X}$. We could then restate H4 as.

$$\exists \alpha^1 \in \Sigma^1,\ \exists \eta^1 \in \Sigma^2 \text{ such that } \lambda = (\alpha^1 \bullet \rho) \circ f_{\eta^1}{}^{-1}, \text{ and } \eta^1 \text{ is continuous.}$$

Requiring that the limit distribution be implementable means that it must be an outcome of the limit game; it must be feasible for the players in Γ to obtain that limit distribution. Example 1 shows that the limit distribution $\hat{\lambda}$ may be infeasible in the limit game. Example 2 and 4 show that feasibility does not suffice; the limit outcome may fail to be a SE outcome because it is not possible to satisfy sequential rationality (S3 and S4).

Theorem 1 proves that given a sequence of games converging to a limit game, and a sequence of SE outcomes converging to a limit distribution λ, if $\hat{\lambda}$ is continuously implementable (H4), then it is a SE outcome of the limit game.

Remark: We can replace the continuity assumption in H4 by the requirement: $(t, x, \eta) \in$ supp$[\hat{\lambda}]$ *and* $(t', x, \eta') \in$ supp$[\lambda] \Rightarrow U^1(t, x, \eta) = U^1(t, x, \eta')$.

4.1 General Lemmas

We now summarize several technical results that we use in our proofs. The first three lemmas are simple measure theoretic facts.

Lemma 1 *If* $\mu \in M(A \times B)$ *has a compact support, then* $supp[\mu_A] = Proj_A\ supp[\mu]$, *where* $Proj_A$ *denotes the projection onto the* A *space.*

Lemma 2 *: If* $\langle \mu^n \rangle \Rightarrow \mu$ *and* $x \in supp[\mu]$, *then there exists a sequence* $\langle x^n \rangle \Rightarrow x$ *with* $x^n \in supp[\mu^n]$ *for each* n.

Proof: This follows from $\liminf \mu^n(G) \geq \mu(G)$ for all open sets G (Billingsley, 1968, Theorem 2.1). **QED**

Lemma 3 *: Let $\mu \in M(S)$ have a compact support. Let $g : S \longrightarrow Z$ be measurable, and let $\lambda = \mu \circ g^{-1}$. Let the event $B \subset S$ have μ-measure 1 and let $A = g(B)$. Then (i) $\lambda(A) = 1$, and (ii) $supp[\lambda] \subset \bar{A}$. Furthermore, if g is continuous, then (iii) $supp[\lambda] = g(\ supp[\mu])$.*

Proof: $B \subset g^{-1}(g(B)) = g^{-1}(A)$, so $\lambda(A) = \mu(g^{-1}(A)) \geq \mu(B) = 1$. This shows (i). (ii) follows since (i) implies the closed set $\bar{A}$ has λ-measure 1 and supp$[\lambda]$ is the smallest such set.

To show (iii), take $B =$ supp$[\mu]$. Then $A = g(B)$ is closed since g is continuous and B is compact. From part (ii), supp$[\lambda] \subset g(B)$. To show the reverse, let $z \in$ supp$[\lambda]$. Then there exists an open set G containing z such that $\lambda(G) = 0$ and thus $\mu(g^{-1}(G)) = 0$. This implies $g1(G) \cap B = \emptyset$ since $g1(G)$ is open by the continuity of g. Then $G \cap g(B) = g(g^{-1}(G) \cap B) = \emptyset$ also, and since $z \in G$, $z \in g(B)$. Thus $z \in$ supp$[\lambda]$ implies $z \in g(B)$, which means $g(B) \subset$ supp$[\lambda]$. **QED**

The next two lemmas are continuity results.

Lemma 4 *Let $U : X \times Y \longrightarrow \Re$ be a measurable function. For $n = 1, 2, \ldots$, let $\eta^n \in M(Y)$, let $x^n \in X$ and let $U^n : XY \longrightarrow \Re$ be measurable. Assume $\langle \eta^n \rangle \Rightarrow \eta$, $\langle x^n \rangle \Rightarrow x$ and $\langle U^n \rangle \Rightarrow U$ continuously for $(x, y) \in X \times Y$. Then $\langle \int_Y U^n(x^n, y)\ \eta^n(dy) \rangle \longrightarrow \int_Y U(x, y)\ \eta(dy)$.*

Proof: Follows from Theorem 5.5 of Billingsley (1968). **QED**

Define the mixed best-response correspondence for player 2 as

$$MBR(x, Y, T, U) = \{(\eta, \beta) \in M(Y) \times M(T) \mid \int U(t, x, \eta)\ \beta(dt) \geq \int U(t, x, \eta')\ \beta(dt) \forall\ \eta' \in M(Y)\ \}.$$

Given that player 1 signals x, $MBR(x, Y, T, U)$ is the set of pairs of a response η from $M(Y)$ and a belief β on T where η is a best response given the belief β and given that player 2's payoff function is U. This definition differs from the standard one in the literature in that we associate to each best response the belief that supports it and we allow the set Y and the function U to vary.

Lemma 5 *The MBR correspondence is upper hemi-continuous in the following sense. For $n = 1, 2, \ldots$, let $(\eta^n, \beta^n) \in MBR(x^n, Y^n, T^n, U^n)$. Let $\langle x^n \rangle \Rightarrow x$, $\langle Y^n \rangle \longrightarrow Y$, $\langle T^n \rangle \longrightarrow T$, and $\langle U^n \rangle \Rightarrow U$ continuously for $(t, x, y) \in T \times X \times Y$. Then $\langle \eta^n, \beta^n \rangle \Rightarrow (\eta, \beta)$ implies that $(\eta, \beta) \in MBR(x, Y, T, U)$.*

Proof: First $\langle Y^n \rangle \longrightarrow Y$ and $\langle T^n \rangle \longrightarrow T$ imply $\langle M(Y^n) \times M(T^n) \rangle \longrightarrow M(Y) \times M(T)$. This with $(\eta^n, \beta^n) \in M(Y^n) \times M(T^n)$ and $\langle \eta^n, \beta^n \rangle \Rightarrow (\eta, \beta)$ imply $(\eta, \beta) \in M(Y) \times M(T)$ by the definition of closed convergence.

The proof of upper hemi-continuity is the standard one of the Theorem of the Maximum (Hildenbrand 1974, Theorem B.III.3). It uses the continuity of the integral $\int U^n(t, x^n, \eta)\, \beta^n(dt)$ as a function of n and η (Lemma 4). **QED**

The next lemma establishes the existence of a measurable function and the last one is a generalization of Lusin's Theorem. We will use it to approximate player 2's strategies with continuous functions.

Lemma 6 *Let $B \subset X \times Y$ and let $\hat{X} = Proj_X B$. If B is compact, then there exists a measurable function $\beta : \hat{X} \longrightarrow Y$ such that $(x, \beta(x)) \in B$ for all $x \in \hat{X}$.*

Proof: The set B is the graph of a correspondence $\hat{B} : \hat{X} \longrightarrow Y$ defined by $\hat{B}(x) = \{y \in Y \mid (x, y) \in B\}$. Since B is compact, $\hat{B}$ is a closed correspondence with a compact range and thus is upper hemi-continuous. Therefore by Proposition B.III.1 in Hildenbrand (1974), for each closed set $F \subset Y$ the set $\{\, x \mid \hat{B}(x) \cap F = \emptyset\}$ is closed, hence measurable. It then follows from Hildenbrand's Lemma D.II.2.1 that $\hat{B}$ has a measurable selection, i.e., a measurable function $\beta : \hat{X} \longrightarrow Y$ such that for every x, $\beta(x) \in \hat{B}(x)$ and thus $(x, \beta(x)) \in B$. This proves the lemma. **QED**

We will need to approximate player 2's strategies with continuous functions, which will require the following variation on Lusin's Theorem for real-valued functions.

Lemma 7 : *Let X and Y be compact metric spaces, let $\mu \in M(X)$ and let $\hat{\eta} : X \longrightarrow M(Y)$ be measurable. Then given $\epsilon > 0$, there exists a continuous function $\eta' : X \longrightarrow M(Y)$ and a compact set $C \subset X$ such that $\hat{\eta} = \eta'$ on C and $\mu(C) > 1 - \epsilon$.*

Proof: Since $M(Y)$ is a compact metric space, there exists a sequence of simple functions converging uniformly to $\hat{\eta}$ on X. Then the standard proof of Lusin's Theorem (e.g., Parthasarathy 1967, Lemma II.4.1) shows there exists a compact set C with $\mu(C) > 1 - \epsilon$ such that the restriction $\eta|C$ is continuous. To complete the proof, $\hat{\eta}|C$ is extended to a continuous function η' on X using a version of Tietze's Extension Theorem generalized to the range space $M(Y)$. The proof of the latter theorem is tedious, so we omit it, but it hinges on the following convexity property of the Prohorov metric p on $M(Y)$: let c_i be a set of positive numbers summing to 1, and let η_i and ζ come from $M(Y)$. Then $p(\sum c_i \eta_i, \zeta) \leq \sum c_i p(\eta_i, \zeta)$. **QED**

4.2 Two Key Propositions

To prove Theorem 1 we must find admissible strategies (S1) that implement the limit outcome λ, and satisfy sequential rationality (S3 and S4) with respect to some consistent beliefs (S2). For this endeavor, we have a SE, $(\hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n)$, with outcome $\hat{\lambda}^n$ for each of the approximating games Γ^n.

Consider first the sequential rationality of player 1 (S3), roughly, that any type t of player 1 will only randomize among actions ($\text{supp}[\alpha(t)]$) that maximize his payoff given the strategy of player 2. With our definition of outcome, this equilibrium property of $(\hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n)$ extends to all elements in the support of $\hat{\lambda}^n$ (Proposition 1).[13] No type t of player 1 is worse off by sending a signal x when the response is η, provided (t, x, η) belongs to the support of $\hat{\lambda}^n$.

Weak convergence of $(\hat{\lambda}^n)$ implies that any element in the support of the limit outcome λ is approximated, by a sequence of elements in the the supports of the converging outcomes $\hat{\lambda}^n$. Then, a continuity argument shows that the support of the limit outcome λ inherits the equilibrium properties of the approximating outcomes (Lemma 10).

We must also show that the procedure can be reversed: the equilibrium conditions of the support of the limit outcome can be transferred to implementing strategies of the limit game. We do so by constructing implementing strategies in which both players select their actions (in the equilibrium path,) within the support of λ. We make this approach precise by defining as a working tool, a *regular* SE.

Second, we must verify that the implementing strategies satisfy sequential rationality (S2 and S4) for player 2; player 2's response $\hat{\eta}(x)$ to the signal x must maximize her expected payoff given her beliefs $\hat{\beta}(x)$ about what types of player 1 may have sent that signal. Proposition 2 addresses this question. Its two main hypotheses are the continuous implementability of λ (H4), and the existence of beliefs justifying player 2's behavior when she observes a signal x not prescribed by player 1's strategy (i.e. x is off the equilibrium path). If player 2's strategy (implementing λ) is the best one player 2 has among all continuous strategies, then it is possible to construct beliefs that would make player 2 sequentially rational. The importance of Proposition 2 is twofold. It provides the appropriate beliefs, and permits to verify S4 in Γ by considering the sequence $\hat{\lambda}^n$: We only need to compare $\eta(\cdot)$ against continuous strategies $\eta'(\cdot)$. Since any $\eta'(\cdot)$ can be (continuously) approximated by continuous strategies $(\eta'^n \in \Sigma^2(\Gamma^n))$, S4 in the limit game follows from S4 in the approximating games.

Finally, to complete the skeleton of the proof we must describe the construction of beliefs and strategies off the equilibrium path. Let x be any signal not sent in equilibrium (x is not in $\text{supp}[\lambda_X]$). Any limit of responses and beliefs $(\hat{\eta}^n(x^n), \hat{\beta}^n(x^n))$ in the approximating games (with $(x^n) \longrightarrow x$) constitutes a valid response $(\hat{\eta}(x), \hat{\beta}(x))$ for player 2 in Γ (Lemma 9).

[13] Note that (t^n, x^n, η^n) may belong to $\text{supp}[\hat{\lambda}^n]$ even though x^n is not in the support of $\hat{\alpha}^n(t^n)$ or $\eta^n \neq \hat{\eta}^n(x)$.

Given a SE $(\hat{\alpha}, \hat{\eta}, \hat{\beta})$ for Γ, we define the equilibrium payoffs to player 1 of type t to be

$$V^1(t) = \sup_{x \in X} U^1(t, x, \eta(x)).$$

Condition S3 implies

$$V^1(t) = \int_X U^1(t, x, \eta(x))\hat{\alpha}(t)(dx).$$

The next proposition derives analogous conditions from the outcome.

Proposition 1 *Let $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ be continuous. Let (α, η, β) be a SE for Γ with outcome $\hat{\lambda}$ and equilibrium payoff function V^1 for player 1. Then*

(i) $\forall t \in T, \exists (x, \eta)$ *such that* $(t, x, \eta) \in$ *supp*$[\lambda]$;

(ii) $(t, x, \eta) \in$ *supp*$[\hat{\lambda}] \Rightarrow V^1(t) = U^1(t, x, \eta)$;

(iii) $(t', x', \eta') \in$ *supp*$[\hat{\lambda}] \Rightarrow V^1(t) \geq U^1(t, x', \eta')\ \forall t \in T$.

Proof: Part (i) follows from

$$\mathrm{Proj}_T\ \mathrm{supp}[\lambda] = \mathrm{supp}[\hat{\lambda}_T] = \mathrm{supp}[\rho] = T.$$

which in turn follows from Lemma 1, the construction of λ, and an assumption on ρ and T.

Begin the proof of part (ii) by defining the set

$$E = \{(t, x) \mid V^1(t) = U^1(t, x, \hat{\eta}(x))\}.$$

By the Theorem of the Maximum, V^1 is continuous. The continuity of V^1 and U^1 and the measurability of $\hat{\eta}$ imply that the set E is measurable. Let $\hat{\mu} = \alpha \bullet \rho$. S3 implies that $\hat{\mu}(E) = 1$.

To prove part (ii), consider $(t, x, \eta) \in$ supp$[\hat{\lambda}]$. By definition, $\lambda = \mu \circ f_\eta{}^{-1}$, where $f_\eta(t, x) = (t, x, \hat{\eta}(x))$. Since $\hat{\mu}(E) = 1$, Lemma 3 asserts $(t, x, \eta) \in$ supp$[\lambda]$ is in the closure of the set $f_\eta(E)$. Therefore there exists a sequence $\langle t^n, x^n, \eta(x^n)\rangle \longrightarrow (t, x, \eta)$ with $(t^n, x^n) \in E$ for all n. By definition of E, $V^1(t^n) = U^1(t^n, x^n, \hat{\eta}(x^n))$. Taking limits using the continuity of V^1 and U^1, $V^1(t) = U^1(t, x, \eta)$. This proves part (ii).

To prove part (iii), let $(t', x', \eta') \in$ supp$[\hat{\lambda}]$. As before, there exists a sequence $\langle t^n, x^n, \eta(x^n)\rangle \longrightarrow (t', x', \eta')$ with $(t^n, x^n) \in E$ for all n. By S3,

$$V^1(t) \geq U^1(t, x^n, \eta(x^n)).$$

and part (iii) follows taking the limit. QED

Conclusion (i) is a simple technical fact. (ii) states that any element (t, x, η) in the support of a SE outcome gives type t of player 1 his equilibrium payoff. (iii) shows that any type t prefers his equilibrium payoff to that achieved with any signal-response pair (x', η') in the equilibrium path $((x', \eta') \in \text{supp}[\hat{\lambda}])$. (iii) has no consequence for pairs (x', η') off the equilibrium path: η' need not be part of an optimal response by player 2. Thus, if some type t prefers a pair (x', η') to his equilibrium payoff, that alternative is not feasible; player 2 would not respond to the signal x' with η'.

As we mentioned, we define a *regular* SE as a tool. SE in finite games are always regular. Our results hold for regular SE as well.

Definition 5 *Given a SE $(\hat{\alpha}, \hat{\eta}, \hat{\beta})$, let $\hat{\lambda} = (\hat{\alpha} \bullet \rho) \circ f_{\hat{\eta}}^{-1}$ be the outcome. The SE (α, η, β) is regular if*

(R1) $(t, x) \in \text{supp}[\hat{\lambda}_{T \times X}] \Rightarrow V^1(t) = U^1(t, x, \eta(x))$.

(R2) $t \in T$ *and* $x \in \text{supp}[\hat{\alpha}(t)] \Rightarrow (t, x) \in \text{supp}[\hat{\lambda}_{T \times X}]$.

(R3) $x \in \text{supp}[\hat{\lambda}_X]$ *and* $t \in \text{supp}[\hat{\beta}(x)] \Rightarrow (t, x) \in \text{supp}[\hat{\lambda}_{T \times X}]$.

A SE outcome is regular if there is a regular SE that supports it.

Conditions R1 and R2 together imply S3 and are used to show S3 holds. Example 3 in Section III gives a game whose unique SE outcome is not regular because condition R1 cannot be satisfied.

Conditions R2 and R3 are symmetrical in T and X since $\text{supp}[\hat{\lambda}_T] = \text{supp}[\rho] = T$ by assumption. Condition R3 says that when player 2 receives a signal x on the equilibrium path ($x \in \text{supp}[\hat{\lambda}_X]$), she should concentrate her beliefs on the types t of player 1 that may have sent this signal given that he plays the strategy $\hat{\alpha}$. R3 is only needed in Section VI on WBR equilibria. R1 will be used there also.

To show R3 and by symmetry R2, we will need the following result.

Lemma 8 *Let $\hat{\lambda}_{T \times X} \in M(T \times X)$. Then there exists a measurable function $\hat{\beta} : X \longrightarrow M(T)$ such that $\hat{\lambda}_{T \times X} = \hat{\beta} \bullet \hat{\lambda}_X$ and $\hat{\beta}$ and $\hat{\lambda}_{T \times X}$ satisfy R3.*

Proof: By Theorem V.8.1 of Parthasarathy (1967), there exists a measurable function $\beta^1 : X \longrightarrow M(T)$ such that $\hat{\lambda}_{T \times X} = \beta^1 \bullet \hat{\lambda}_X$ and a measurable set $X1$ with $\hat{\lambda}_X$-measure 1 such β^1 and $\hat{\lambda}_{T \times X}$ satisfy R3 on $X1$. Applying Lemma 6 with $B = \text{supp}[\hat{\lambda}_{T \times X}]$, there exists a measurable function $g : \text{supp}[\hat{\lambda}_X] \longrightarrow T$ such that $(x, g(x)) \in \text{supp}[\hat{\lambda}_{T \times X}]$ for all $x \in \text{supp}[\hat{\lambda}_X]$. Define $\beta^0 : \text{supp}[\hat{\lambda}_X] \longrightarrow M(T)$ by letting $\beta^0(x)$ put unit mass on $g(x)$. Then β^0 is measurable and satisfies R3. Finally, define $\hat{\beta}$ by setting $\hat{\beta}(x) = \beta^1(x)$ for all $x \in X^1$ and $\hat{\beta}(x) = \beta^0(x)$ otherwise. Then $\hat{\beta}$ is

measurable and satisfies R3, and $\lambda_{T \times X} = \beta^1 \bullet \lambda_X = \beta \bullet \lambda_X$ since $\beta^1 = \beta$ almost everywhere $[\lambda_X]$.
QED

Proposition 2 *: Let* $\Gamma = [(T,\rho), X, Y, U^1, U^2]$ *be continuous. Let* $\hat{\alpha} \in \Sigma^1(\Gamma)$, $\eta \in \Sigma^2(\Gamma)$, *and* $\hat{\lambda} \in M(T \times X \times \Psi)$. *Let* $\lambda_{T \times X} = \alpha \bullet \rho$, *let* $\hat{\eta}$ *be continuous on* $supp[\lambda_X]$, *and let* $\beta^0 : X \longrightarrow M(T)$ *be a measurable function satisfying* $(\hat{\eta}(x), \beta^0(x)) \in MBR(x, Y, T, U^2)$ *for all* $x \in X \setminus supp[\lambda_X]$. *Suppose that for all continuous functions* $\eta' : X \longrightarrow \Psi$,

$$\int_{T \times X} U^2(t, x, \hat{\eta}(x))\, \lambda_{T \times X}(dt \times dx) \geq \int_{T \times X} U^2(t, x, \eta'(x))\, \lambda_{T \times X}(dt \times dx).$$

Then there exists $\hat{\beta} : X \longrightarrow M(T)$ *satisfying R3 such that* $(\hat{\alpha}, \eta, \hat{\beta})$ *satisfies S2 and S4.*

Proof: For $k = 0,\ 1,\ 2$, we define measurable sets Xk that together partition X and measurable functions $\beta^k : X^k \longrightarrow M(T)$. We then set

$$\beta(x) = \sum_{k=0}^{2} \beta^k(x) \mathbf{1}_{X^k}(x),$$

where $\mathbf{1}_{X^k}$ is the indicator function for the set Xk.

We take β^0 as given in the statement of the theorem and define $X^0 = X \setminus \text{supp}[\lambda_X]$. Let β^1 be any version of a conditional distribution of t given x derived from $\lambda_{T \times X}$. By Lemma 8, we may assume that β^1 satisfies R3. Let

$$X^1 = \{x \in \text{supp}[\hat{\lambda}_X] \mid ((\eta(x), \beta^1(x)) \in MBRU2\}.$$

and $X^2 = \text{supp}[\hat{\lambda}_X] \setminus X1$. Assume that X^2 is measurable and that $\lambda_X(X^2) = 0$. We will show this below. Then X^1 is dense in $X^1 \bigcup X^2$, and given this we will use β^1 on $X1$ to define a measurable function β^2 on X^2 such that $(\hat{\eta}(x), \beta^2(x)) \in MBR(x, Y, T, U^2)$ for all $x \in X2$ and β^2 satisfies R3.

To define β^2, let $C = \{(x, \hat{\eta}(x), \beta^1(x)) \mid x \in X^1\}$. Applying Lemma 6 to the closure $\bar{C}$, there exists a measurable pair (η^2, β^2) such that $(x, \eta^2(x), \beta^2(x)) \in \bar{C}$ for all $x \in \text{Proj}_X \bar{C}$. Since X^1 is dense in $X^1 \bigcup X^2$, we have $\text{Proj}_X \bar{C} = X^1 \cup X^2$. By hypothesis, $\hat{\eta}$ is continuous on $X^1 \bigcup X2$, so we have $\eta^2 = \eta$. Since $(\hat{\eta}(x), \beta^1(x)) \in MBR(x, Y, T, U^2)$ for $x \in X1$, we have $(\hat{\eta}(x), \beta^2(x)) \in MBR(x, Y, T, U^2)$ for $x \in X^2$ by the upper hemi- continuity of MBR (Lemma 5).

To show β^2 satisfies R3, let $x \in X^2$ and $t \in \text{supp}[\beta^2(x)]$. We must show that $(t, x) \in \text{supp}[\lambda_{T \times X}]$. Using the definition of β^2 and Lemma 2, there exists a sequence $\langle t^n, x^n, \beta^1(x^n)\rangle \longrightarrow (t, x, \beta^2(x))$ with $x^n \in X1$ and $t^n \in \text{supp}[\beta^1(x^n)]$ for all n. Since β^1 satisfies R3, $(t^n, x^n) \in \text{supp}[\lambda_{T \times X}]$ for all n. This implies $(t, x) \in \text{supp}[\lambda_{T \times X}]$ since this set is closed. Hence β^2 satisfies R3.

Thus, under our assumptions that X^2 is measurable and $\hat{\lambda}_X(X^2) = 0$, we can define a measurable function $\hat{\beta}(x) = \sum_k \beta^k(x)\mathbf{1}_{X^k}(x)$. $\hat{\beta}$ satisfies R3 because both β^1 and β^2 do. $(\hat{\eta}(x), \hat{\beta}(x)) \in MBR(x, Y, T, U^2)$ for all $x \in X$ because for each k, $(\hat{\eta}(x), \beta^k(x)) \in MBR2$ for all $x \in X^k$. Thus η and $\hat{\beta}$ satisfy S4. $\hat{\beta}$ is a conditional distribution derived from $\lambda_{T \times X}$ because $\hat{\beta}$ equals the conditional distribution β^1 on $X1$ and $\hat{\lambda}_X(X^1) = 1$. Since $\lambda_{T \times X} = \hat{\alpha} \bullet \rho$, we have $(\hat{\beta}, \hat{\alpha}) \in \Sigma^3(\Gamma)$ and so they satisfy S2. Hence, the triple $(\alpha, \hat{\eta}, \hat{\beta})$ satisfies all the conclusions of the lemma.

To complete the proof, we now have to show that X^2 is measurable and $\lambda_X(X^2) = 0$. We sketch the proof of this. To show X^2 is measurable, define a function

$$h_\eta(x) = \int_T \left[U^2(t, x, \hat{\eta}(x)) - U^2(t, x, \eta)\right] \beta^1(x)(dt),$$

and let $\hat{\Psi}$ be a countable, dense subset of Ψ. The function h_η is a conditional expectation and thus is measurable. The function h_η is also continuous in η for fixed x by Lemma 4. For each $\eta \in \Psi$, define $A_\eta = \{x \mid h_\eta(x) < 0\}$. These are measurable sets. The continuity of h_η with respect to η implies

$$X^2 = \bigcup_{\eta \in \hat{\Psi}} A_\eta.$$

Since this is a countable union of measurable sets, X^2 is measurable.

To show $\hat{\lambda}_X(X^2) = 0$, we show all the sets A_η have λ_X-measure 0. Given $\eta \in \Psi$, define a function $\eta' \in \Sigma^2(\Gamma)$ by setting $\eta'(x) = \eta$ on A_η and $\eta'(x) = \hat{\eta}(x)$ on $X \setminus A_\eta$. Now if $\lambda_X(A_\eta) > 0$, we can use Lusin's Theorem (Lemma 7) to find a continuous function η'' that approximates η' closely enough so that η'' violates condition (i) of the lemma. This contradiction implies we must have $\lambda_X(A_\eta) = 0$ for each η, and the proof is complete. QED

Proof of Theorem 1: We will construct the promised SE for Γ from λ and from the sequence of SE $\langle \hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n \rangle$, supporting the outcomes $\langle \lambda^n \rangle$ for $\langle \Gamma^n \rangle$.

We first show how to construct player 2's strategy and beliefs off the equilibrium path. Let

$$A^n = \{(x, \hat{\eta}^n(x), \hat{\beta}^n(x)) \mid x \in X^n\}$$

and $B = \text{Ls}\,\langle A^n \rangle$. B is non-empty and compact. B is the set of all $(x, \eta, \beta) \in \tilde{X} \times M(\tilde{Y}) \times M(\tilde{T})$ such that there exists a subsequence $\langle x^{nk}, \hat{\eta}^{nk}(x^{nk}), \hat{\beta}^{nk}(x^{nk}) \rangle \Rightarrow (x, \eta, \beta)$. The strategy and beliefs for player 2 off the equilibrium path are provided by an arbitrarily chosen measurable selection (η^0, β^0) from the set B.

Lemma 9 *: There exists a measurable function* $(\eta^0, \beta^0) : X \longrightarrow \Psi \times M(T)$ *such that* $(x, \eta^0(x), \beta^0(x)) \in B$ *for all* $x \in X$*. Furthermore,* $(x, \eta, \beta) \in B$ *implies* $(\eta, \beta) \in MBR(x, Y, T, U^2)$.

Proof: B as a closed limit is compact. Let $X = \text{Proj}_{\tilde{X}} B$. By Lemma 6, there exists a measurable function $(\eta^0, \beta^0) : \tilde{X} \longrightarrow M(\tilde{Y}) \times M(\tilde{T})$ such that $(x, \eta^0(x), \beta^0(x)) \in B$ for all $x \in Xh$.

We must show that (η^0, β^0) is a function from X to $\Psi \times M(T)$. $\langle X^n \rangle \longrightarrow X$ implies $X == \text{Proj}_{\tilde{X}} B = X$, so (η^0, β^0) is a function on X. Given $x \in X$, let $(x, \eta^0(x), \beta^0(x))$ be the limit of a sequence $\langle x^n, \hat{\eta}^n(x^n), \hat{\beta}^n(x^n) \rangle$ with $x^n \in X^n$. Since $(\hat{\eta}^n(x^n), \hat{\beta}^n(x^n)) \in MBR(x^n, Y^n, T^n, U^{2n})$ on the sequence, the limit $(\eta, \beta) \in MBR(x, Y, T, U^2)$ by Lemma 5. This implies in particular that $(\eta^0(x), \beta^0(x)) \in \Psi \times M(T)$ for all $x \in X$ and also proves the last statement of the lemma. **QED**

The next lemma shows that the limit outcome inherits some equilibrium properties on and off the equilibrium path.

Lemma 10 *: Let $(t, x, \eta) \in$ supp$[\lambda]$. If either $(t', x', \eta') \in$ supp$[\lambda]$ or $(x', \eta', \beta') \in B$, then $U^1(t, x, \eta) \geq U^1(t, x', \eta')$.*

Proof: Let both (t, x, η) and (t', x', η') come from supp$[\lambda]$. Then $\langle \hat{\lambda}^n \rangle \Rightarrow \lambda$ and Lemma 2 imply there exist sequences $\langle t^n, x^n, \eta^n \rangle \Rightarrow (t, x, \eta)$ and $\langle t'^n, x'^n, \eta'^n \rangle \Rightarrow (t', x', \eta')$ with (t^n, x^n, η^n) and (t'^n, x'^n, η'^n) in supp$[\hat{\lambda}^n]$ for each n. By Proposition 1,

$$U^{1n}(t^n, x^n, \eta^n) = V^{1n}(t^n) \geq U^{1n}(t^n, x'^n, \eta'^n).$$

Taking limits using H2 and Lemma 4, we have

$$U^1(t, x, \eta) \geq U^1(t, x', \eta').$$

This proves the lemma when $(t', x', \eta') \in$ supp$[\lambda]$.

A similar argument establishes the case $(x', \eta', \beta') \in B$. By definition of B, there exists a sequence $\langle x'^n, \hat{\eta}^n(x'^n), \hat{\beta}^n(x'^n) \rangle \longrightarrow (x', \eta', \beta')$. By Proposition 1 and S3,

$$U^{1n}(t^n, x^n, \eta^n) = V^{1n}(t^n) \geq U^{1n}(t^n, x'^n, \hat{\eta}^n(x'^n)).$$

Taking limits as above completes the proof. QED

We now show that $\lambda \in SEO(\Gamma)$. We construct a SE (α, η, β) supporting λ in three steps. First, we define the strategy $\hat{\alpha}$ of player 1 and show it satisfies S1. Second, we define the strategy η for player 2 so that it satisfies S1 and then show that α and η result in outcome λ and satisfy S3. Lastly, we apply Proposition 2 to show there exist beliefs β such that $(\hat{\alpha}, \hat{\eta}, \beta)$ satisfies S2, S4, and R3. In the process, we will show that R1 and R2 also hold, so the SE $(\hat{\alpha}, \hat{\eta}, \beta)$ is regular.

To construct $\hat{\alpha}$, we must first show that $\lambda \in M(T \times X \times \Psi)$ and $\lambda_T = \rho$. Define $\Psi^n = M(Y^n)$. Now $\langle T^n \times X^n \times Y^n \rangle \longrightarrow T \times X \times Y$ only if $\langle M(T^n \times X^n \times \Psi^n) \rangle \longrightarrow M(T \times X \times \Psi)$. Therefore, H1 and $\langle \hat{\lambda}^n \rangle \Rightarrow \hat{\lambda}$ imply $\hat{\lambda} \in M(T \times X \times \Psi)$. Similarly, since $\langle \lambda^n \rangle \Rightarrow \lambda$ implies $\langle \lambda^n_{\tilde{T}} \rangle \Rightarrow \lambda_{\tilde{T}}$, we have $\langle \hat{\lambda}^n_{T^n} \rangle \Rightarrow \hat{\lambda}_T$. Since $\hat{\lambda}^n_{T^n} = \rho^n$ and $\langle \rho^n \rangle \Rightarrow \rho$, $\lambda_T = \rho$ as we set out to show.

Now let $\hat{\alpha}$ be a conditional distribution of x given t derived from $\lambda_{T \times X}$. Then since $\lambda \in M(T \times X \times \Psi)$, we have $\hat{\alpha} \in \Sigma^1(\Gamma)$, and since $\lambda_T = \rho$, we have $\alpha \bullet \rho = \lambda_{T \times X}$. By Lemma 8, we may assume that $\hat{\alpha}$ and $\hat{\lambda}_{T \times X}$ satisfy R2.

Define the strategy $\hat{\eta} : X \longrightarrow \Psi$ for player 2 using the function η^1 given in H4 and the function η^0 derived in Lemma 9:

$$\hat{\eta}(x) = \begin{cases} \eta^1(x) & \text{if } x \in \text{ supp}[\hat{\lambda}_X], \\ \eta^0(x) & \text{otherwise.} \end{cases}$$

The strategy $\hat{\eta}$ is a measurable since it is defined from measurable functions on measurable sets, so η satisfies S1. Because of H4 and that $\alpha \bullet \rho = \lambda_{T \times X}$ and $\eta = \eta^1$ on $\text{supp}[\lambda_X]$, we have

$$\lambda = \lambda_{T \times X} \circ f_{\eta^1}{}^{-1} = (\alpha \bullet \rho) \circ f_{\hat{\eta}}{}^{-1}.$$

Thus $\hat{\alpha}$ and $\hat{\eta}$ result in outcome λ.

We show that $\hat{\alpha}$ and $\hat{\eta}$ satisfy S3. Let $t \in T$, $x \in \text{ supp}[\hat{\alpha}(t)]$ and choose $x' \in X$ arbitrarily. By construction, $\hat{\alpha}$ satisfies R2, so $(t, x) \in \text{ supp}[\hat{\lambda}_{T \times X}]$. We will show that

$$U^1(t, x, \hat{\eta}(x)) \geq U^1(t, x', \hat{\eta}(x')), \tag{5}$$

which implies R1 and S3.

By Lemma 1, $(t, x) \in \text{ supp}[\lambda_{T \times X}]$ only if there exists an $\eta \in \Psi$ such that $(t, x, \eta) \in \text{ supp}[\lambda]$. Since $\hat{\lambda} = \hat{\lambda}_{T \times X} \circ f_{\hat{\eta}}{}^{-1}$ and $\hat{\eta}$ is continuous on $\text{supp}[\lambda_X]$, Lemma 3 asserts that $\text{supp}[\lambda] = f_{\hat{\eta}}(\text{ supp}[\hat{\lambda}_{T \times X}])$, and thus $(t, x, \eta) = (t, x, \hat{\eta}(x))$. Hence $(t, x) \in \text{ supp}[\lambda_{T \times X}]$ implies $(t, x, \eta(x)) \in \text{ supp}[\hat{\lambda}]$.

Take the case where $x' \in \text{ supp}[\lambda_X]$. By the same logic as above, there exists t' such that $(t', x', \hat{\eta}(x')) \in \text{ supp}[\hat{\lambda}]$. Applying Lemma 10 yields (5) for this case.

Now take the case where $x' \in \text{ supp}[\lambda_X]$. Then $(\hat{\eta}(x'), \beta(x')) = (\eta^0(x'), \beta^0(x'))$ and thus from Lemma 9, $(x', \hat{\eta}(x'), \hat{\beta}(x')) \in B$. Applying Lemma 10 in this case also yields (5). R1 and S3 follow.

We have shown that $\hat{\alpha}$, $\hat{\eta}$ and $\hat{\lambda}$ satisfy the first five requirements of Proposition 2, i.e., $\alpha \in \Sigma^1(\Gamma)$, $\hat{\eta} \in \Sigma^2(\Gamma)$, $\hat{\lambda} \in M(T \times X \times \Psi)$, $\lambda_{T \times X} = \alpha \bullet \rho$, and η is continuous on $\text{supp}[\lambda_X]$. The next requirement is that there exists a measurable function $\beta^0 : X \longrightarrow M(T)$ such that $(\hat{\eta}(x), \beta^0(x)) \in MBR(x, Y, T, U^2)$ for all $x \in X \setminus \text{ supp}[\lambda_X]$. This holds using β^0 from Lemma 9.

The final requirement of Proposition 2 is that for any continuous function $\eta' : X \longrightarrow \Psi$, we must have

$$\int_{T \times X} U^2(t, x, \hat{\eta}(x))\, \lambda_{T \times X}(dt \times dx) \geq \int_{T \times X} U^2(t, x, \eta'(x))\, \lambda_{T \times X}(dt \times dx).$$

To demonstrate this, choose a continuous function η' arbitrarily. Define $\Psi^n = M(Y^n)$. One can show that H1 and the continuity of η' imply there exists a sequence $\langle \eta^n \rangle$ of measurable functions $\eta^n : X^n \longrightarrow \Psi^n$ such that $\langle \eta^n \rangle \longrightarrow \eta'$ continuously for $x \in X$. Since $\langle \hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n \rangle$ is a SE for Γ^n, S4 implies

$$\textstyle\int_{T^n} U^{2n}(t, x, \hat{\eta}^n(x))\, \hat{\beta}^n(x)(dt) \geq \int_{T^n} U^{2n}(t, x, \eta^n(x))\, \hat{\beta}^n(x)(dt)$$

for all $x \in X$. Since $\hat{\beta}^n$ is a conditional distribution derived from $\hat{\lambda}^n_{T^n \times X^n}$, both sides of this inequality are conditional expectations derived from $\hat{\lambda}^n_{T^n \times X^n}$. Therefore, we may integrate both sides using $\hat{\lambda}^n_{X^n}$ to get

$$\textstyle\int_{X^n} \int_{T^n} U^{2n}(t, x, \hat{\eta}^n(x))\, \hat{\beta}^n(x)(dt) \hat{\lambda}^n_{X^n}(dx) \geq \int_{X^n} \int_{T^n} U^{2n}(t, x, \eta^n(x))\, \hat{\beta}^n(x)(dt) \hat{\lambda}^n_{X^n}(dx),$$

and thus by definition of conditional expectation,

$$\textstyle\int_{T^n \times X^n} U^{2n}(t, x, \hat{\eta}^n(x))\, \hat{\lambda}^n_{T^n \times X^n}(dt \times dx) \geq \int_{T^n \times X^n} U^{2n}(t, x, \eta^n(x))\, \hat{\lambda}^n_{T^n \times X^n}(dt \times dx).$$

Since $\hat{\lambda}^n$ is the outcome for $(\hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n)$ it follows that

$$\textstyle\int_{T^n \times X^n \times \Psi^n} U^{2n}(t, x, \eta)\, \hat{\lambda}^n(dt \times dx \times d\eta) \geq \int_{T^n \times X^n \times \Psi^n} U^{2n}(t, x, \eta^n(x))\, \hat{\lambda}^n(dt \times dx \times d\eta).$$

Taking limits using H2, H3, Lemma 4, and the assumption that $\langle \eta^n \rangle \longrightarrow \eta'$ continuously,

$$\textstyle\int_{T \times X \times \Psi} U^2(t, x, \eta)\, \hat{\lambda}(dt \times dx \times d\eta) \geq \int_{T \times X \times \Psi} U^2(t, x, \eta'(x))\, \lambda(dt \times dx \times d\eta).$$

Finally, since $\hat{\lambda}$ is the outcome of playing $\hat{\alpha}$ and $\hat{\eta}$, we have

$$\textstyle\int_{T \times X} U^2(t, x, \hat{\eta}(x))\, \lambda_{T \times X}(dt \times dx) \geq \int_{T \times X} U^2(t, x, \eta'(x))\, \lambda_{T \times X}(dt \times dx)$$

as required. Proposition 2 then asserts there exists beliefs $\hat{\beta}$ satisfying R3 such that $(\hat{\alpha}, \hat{\eta}, \hat{\beta})$ satisfies S2 and S4. This completes the proof. QED

In the process of proving Theorem 1 we have shown some intermediate results that do not require the continuous implementability (H4) of the limit outcome λ. We summarize them in the following proposition since they will be referenced when proving our existence results.

Proposition 3 *Consider a sequence of continuous games* $\Gamma^n = [(T^n, \rho^n), X^n, Y^n, U^{1n}, U^{2n}]$ *with outcomes* $\hat{\lambda}^n \in SEO(\Gamma^n)$, $n = 1, 2, \ldots$ *and a continuous game* $\Gamma = [(T, \rho), X, Y, U^1, U^2]$. *If hypotheses H1-H3 of Theorem 1 hold, then*

(C1) $\hat{\lambda} \in M(T \times X \times \Psi)$ *and* $\hat{\lambda}_T = \rho$ *;*

(C2) $(t,x,\eta) \in$ *supp*$[\hat{\lambda}]$ *and* $(t',x',\eta') \in$ *supp*$[\lambda] \Rightarrow U^1(t,x,\eta) \geq U^1(t,x',\eta')$.

Additionally if H4 holds, then

(C3) $\hat{\lambda}$ *is a regular SE outcome;*

(C4) *the equilibrium payoffs* $\langle V^{1n} \rangle \longrightarrow V^1$ *continuously for* $t \in T$.

Proof Conclusions C1-C3 were shown during the course of the proof of Theorem 1. In particular, C2 is a result of Lemma 10.

To show C4, let $\langle t^n \rangle \Rightarrow t$ with $t^n \in Tn$ and $t \in T$. We will show any subsequence of $\langle V^{1n}(t^n) \rangle$ has in turn a subsequence that converges to $V^1(t)$, which will imply $\langle V^1(t^n) \rangle \Rightarrow V^1(t)$ as required.

Let $\langle V^{1n}(t^n) \rangle$ now represent any subsequence of itself. By parts (i) and (ii) of Proposition 1, for each t^n there exists x^n and η^n such that $(t^n, x^n, \eta^n) \in$ supp$[\lambda^n]$ and $V^{1n}(t^n) = U^{1n}(t^n, x^n, \eta^n)$. Taking a further subsequence if necessary, let $\langle t^n, x^n, \eta^n \rangle \longrightarrow (t,x,\eta)$ and thus $\langle V^{1n}(t^n) \rangle \longrightarrow U^1(t,x,\eta)$ by H2. To complete the proof, we show that $U^1(t,x,\eta) = V^1(t)$.

By Proposition 1 again, there exists x' and η' such that $(t,x',\eta') \in$ supp$[\lambda]$ and $V^1(t) = U^1(t,x',\eta')$. By Lemma 2, there exists $\langle t'^n, x'^n, \eta'^n \rangle \Rightarrow (t,x',\eta')$ with $(t'^n, x'^n, \eta'^n) \in$ supp$[\lambda^n]$. Now from two applications of part (iii) of Proposition 1,

$$U^{1n}(t^n,x^n,\eta^n) \geq U^{1n}(t^n,x'^n,\eta'^n) \text{ and } U^1(t'^n,x'^n,\eta'^n) \geq U^{1n}(t'^n,x^n,\eta^n),$$

and thus in the limit

$$U^1(t,x,\eta) \geq U^1(t,x',\eta') \geq U^1(t,x,\eta).$$

Since $U^1(t,x',\eta') = V^1(t)$ by construction, we have $U^1(t,x,\eta) = V^1(t)$ as we were to show. QED

5 Existence of Sequential Equilibria for Strongly Monotonic Games

Any infinite game may be approximated by a sequence of finite games constructed as successively finer discretizations of the infinite game. Consider a sequence of SE outcomes of the finite games. Through a subsequence if necessary, the sequence of SE outcomes converges to a limit distribution. By Theorem 1, that limit distribution will be a SE outcome of the infinite game provided it can be implemented with a continuous strategy for player 2. Hence, we may prove existence of SE for a class of infinite games by identifying games for which the limit distribution is always implementable by a continuous strategy. We do so in this section. The examples in Section 3 show that a general existence result is not possible.

Cho and Sobel (1987) define a signaling game Γ to be *monotonic* if for all x in X, for all (η, β) and (η', β') in $MBR(x, Y, T, U^2)$, and for all t and t' in T,

$$U^1(t, x, \eta) \geq U^1(t, x, \eta') \Rightarrow U^1(t', x, \eta) \geq U^1(t', x, \eta').$$

Monotonicity holds in many applications of signaling games. For example, if y is player 2's choice of payment to player 1, monotonicity implies, among other things, that player 1 prefers more money to less independently of t and x.

Monotonicity holds if Y is an interval in $\Re$, U^1 is increasing in y, and U^2 is strictly concave in y. We will need a stronger property also implied by these conditions.

We define Γ to be *strongly monotonic* if for all x in X, for all (η, β) and (η', β') in $MBR(x, Y, T, U^2)$, and for all t and t' in T,

$$U^1(t, x, \eta) \geq U^1(t, x, \eta') \text{ and } U^1(t', x, \eta') \geq U^1(t', x, \eta) \quad \text{imply} \quad \eta = \eta'.$$

Strong monotonicity implies monotonicity. It also implies there is a total ordering of the best responses of player 2 that reflects the preferences of all types of player 1. We will show that a strongly monotonic signaling game has a SE.

We begin by providing an alternative (and equivalent) condition to implementability by a continuous strategy for player 2.

Proposition 4 *Let $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ be continuous and let $\hat{\lambda} \in M(T \times X \times \Psi)$. Then the following two statements are equivalent.*

(i) $(t, x, \eta) \in$ *supp*$[\hat{\lambda}]$ *and* $(t', x, \eta') \in$ *supp*$[\hat{\lambda}]$ *imply* $\eta = \eta'$.

(ii) *There exists a (unique) continuous η^1 : supp$[\lambda_X]$ —— Ψ, such that $\lambda = \lambda_{T \times X} \circ f_{\eta^1}{}^{-1}$.*

Proof: *(i)* $\Rightarrow$ *(ii)* Since $\text{supp}[\lambda_{T \times X}] = \text{Proj}_{T \times X} \text{supp}[\lambda]$, $(t, x) \in \text{supp}[\lambda_{T \times X}]$ if and only if there exists η such that $(t, x, \eta) \in \text{supp}[\lambda]$. By *(i)* there is only one such η for each x with $(t, x) \in \text{supp}[\hat{\lambda}_{T \times X}]$. Setting $\eta^1(x)$ equal to this η, we have that *(i)* implies

$$(t, x, \eta) \in \text{supp}[\lambda] \text{ if and only if } \eta = \eta^1(x) \text{ and } (t, x) \in \text{supp}[\lambda_{T \times X}]. \quad (6)$$

The function η^1 is clearly unique. To show that it is continuous, observe that (6) implies that $(x, \eta) \in \text{supp}[\hat{\lambda}_{X \times \Psi}]$ if and only if $\eta = \eta^1(x)$ and $x \in \text{supp}[\hat{\lambda}_X]$. This implies that the graph of η is the closed set $\text{supp}[\hat{\lambda}_{X \times \Psi}]$ and thus that η^1 is continuous.

To complete the proof that $(i) \Rightarrow (ii)$, we must show that for any event $E \subset T \times X \times \Psi$, $\hat{\lambda}(E) = \hat{\lambda}_{T \times X}({f_{\eta^1}}^{-1}(E))$. Let $R =$ supp$[\hat{\lambda}]$ and $S =$ supp$[\hat{\lambda}_{T \times X}]$. We can write (6) as

$$(t, x, \eta) \in R \text{ if and only if } f_{\eta^1}(t, x) = (t, x, \eta) \text{ and } (t, x) \in S.$$

This implies

$$E \cap R = ({f_{\eta^1}}^{-1}(E) \cap S) \times \Psi \cap R.$$

Then

$$\begin{aligned} \hat{\lambda}(E) &= \hat{\lambda}(E \cap R) = \hat{\lambda}(({f_{\eta^1}}^{-1}(E) \cap S) \times \Psi \cap R) = \hat{\lambda}(({f_{\eta^1}}^{-1}(E) \cap S) \times \Psi) \\ &= \hat{\lambda}_{T \times X}({f_{\eta^1}}^{-1}(E) \cap S) = \hat{\lambda}_{T \times X}({f_{\eta^1}}^{-1}(E)) \end{aligned}$$

as we were to show.

$(ii) \Rightarrow (i)$ Let (t, x, η) and (t', x, η') both come from supp$[\hat{\lambda}]$. From Lemma 3, (ii) implies supp$[\hat{\lambda}] = f_{\eta^1}($ supp$[\hat{\lambda}_{T \times X}])$. Therefore both η and η' equal $\eta^1(x)$, and (i) follows. **QED**

Condition (i) requires that there be a unique response to any given message in the equilibrium path. This condition is violated by the limit outcome λ in Example 1.

If a game is strongly monotonic, it will have a SE outcome (Proposition 5): Construct a sequence of finite games (through progressively finer discretizations) converging to the limit game. Let $\langle \lambda^n \rangle$ be a (sub)sequence of SE outcomes of the approximating games that converges to a limit distribution $\hat{\lambda}$. If $\hat{\lambda}$ were implementable by a continuous strategy for player 2 (H4), Theorem 1 would imply that $\hat{\lambda}$ is a SEO. To show this we use Proposition 4: Let $(t, x, \eta) \in$ supp$[\hat{\lambda}]$ and $(t', x, \eta') \in$ supp$[\hat{\lambda}]$. C2 of Proposition 3 implies that $U^1(t, x, \eta) \geq U^1(t, x, \eta')$ and $U^1(t', x, \eta') \geq U^1(t', x, \eta)$. By strong monotonicity, $\eta = \eta'$ and thus $\hat{\lambda}$ satisfies hypothesis (i) of Proposition 4. Hence, $\hat{\lambda}$ is implementable by a continuous strategy for player 2 (i.e. H4 holds).[14] This proves

Proposition 5 *: If $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ is continuous and strongly monotonic, then Γ has a SE.*

Intuitively, the sequential rationality of player 1 will be carried to the limit outcome. That is any type t of player 1 would prefer as a response to his signal x an action η of player 2 prescribed by the equilibrium path. Different types of player 1 that send in equilibrium the same signal may prefer different responses. This was the reason for the lack of existence in Example 1. Strong monotonicity, however, rules out this behavior since two types t and t' who send the same signal in equilibrium must prefer exactly the same response.

[14] Note that C1 of Proposition 3 implies $\hat{\lambda} \in M(T \times X \times \Psi)$ and therefore Propostion 4 may be applied.

We may identify another class of signaling games for which SE always exist. These are games where the signaling space X is finite (although the other relevant spaces may not be so). It is possible to show that for those games the limit outcome is always implementable by a continuous strategy for player 2 (H4). Let $\Gamma = [(T,\rho), X, Y, U^1, U^2]$ with X finite. Since all player 2's strategies in Γ are continuous, we only need to show that the limit outcome is implementable to apply Theorem 1. Let $\Gamma^n = [(T^n, \rho^n), X, Y^n, U^1, U^2]$ be a finite game with $\langle\Gamma^n\rangle \longrightarrow \Gamma$, let $\hat{\lambda}^n \in SEO(\Gamma^n)$ be supported by $(\hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n)$, and let $\langle\hat{\lambda}^n\rangle \longrightarrow \lambda$. Since X is finite, it is simple to show that (in a subsequence) $\langle\eta^n\rangle$ converges uniformly to a strategy $\hat{\eta}$ of player 2 in Γ. Uniform convergence implies that the limit distribution $\hat{\lambda}$ is implemented by $\hat{\eta}$. Thus, (H4) is satisfied and we obtain the following result.

Proposition 6 *If $\Gamma = [(T,\rho), X, Y, U^1, U^2]$ is continuous and X is finite, then Γ has a SE.*

The argument leading to the previous proposition shows, in addition to the existence of SE, that in games with finite signaling spaces X, it is possible to obtain a useful form of convergence of the strategies of the approximating games. We turn now to the question of whether the SE strategies of a sequence of approximating games will converge in a useful sense to strategies of the limit game. The examples of Section 3 give a negative answer to this question in general.

Proposition 7 *Let $\Gamma = [(T,\rho), X, Y, U^1, U^2]$ be continuous. Suppose that Y is a compact set in $\Re$, U^2 is strictly concave in y, and U^1satisfies differential monotonicity* (**DM**), *i.e. there exists $L \in \Re$ with $0 \leq L \leq \infty$ such that for all $t \in T$, x, $x' \in X$ and y, $y' \in Y$,*

$$U^1(t,x,y) \geq U^1(t,x',y') \Rightarrow (y - y') \geq -M\|x - x'\|.$$

In addition, let $\Gamma^n = [(T,\rho), X^n, Y, U^1, U^2]$ with X^n finite, and $\langle\Gamma^n\rangle \longrightarrow \Gamma$. Let $\hat{\lambda}^n \in SEO(\Gamma^n)$ be supported by $(\hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n)$, and $\langle\hat{\lambda}^n\rangle \longrightarrow \lambda$. Then $\lambda \in SEO(\Gamma)$ is supported by (α, η, β), and there exists a (sub)sequence such that $\hat{\eta}^n \longrightarrow \eta$ continuously.

Remark: Note that if DM holds then U^1 is strictly increasing in y. If U^1 is continuously differentiable and strictly increasing in y, then DM holds. The first implication follows directly from the definition of U^1, and the partial converse from the Mean-value Theorem.

DM requires that player 1's marginal rate of substitution of y for x be bounded. It implies strong monotonicity, and that U^1 must be strictly increasing in y. Hence, the existence result of Proposition 7 does not extend Proposition 5.

The additional assumptions, DM and the concavity of U^2 allow us to show that the strategies (of suitably chosen) approximating games converge uniformly to SE strategies of the limit game.

We sketch the proof. By the convexity assumption, there is no loss of generality in considering pure strategies for player 2. For simplicity we will write $\hat{y} : X \longrightarrow Y$ to denote a pure strategy instead of $\hat{\eta}(x) = \delta_y$. DM implies that the approximating SE strategies are Lipshcitz:[15] It follows from S3 for Γ^n, that there exists a set $A^n \subset X^n$ with $\hat{\lambda}^n_{X^n}(A^n) = 1$ such that for any $x \in A^n$, there exists $t \in T$ with[16]

$$U^1(t, x, \hat{y}^n(x)) \geq U^1(t, x', \hat{y}^n(x')),\ \forall x' \in A^n.$$

Then, DM implies that $\hat{y}^n(x) - \hat{y}^n(x')/\|x - x'\| \geq -L$. Similarly, for any $x' \in A^n$ there is a $t' \in T$ providing the bound on the other direction, $\hat{y}^n(x') - \hat{y}^n(x)/\|x - x'\| \geq -L$. This establishes that $\hat{y}^n$ is Lipshitz on a set of full measure.

The following theorem, which we prove in the Appendix, permits the extension the strategies to Lipschitz functions on the whole set X. Once this is accomplished, the Arzela-Ascoli Theorem implies the uniform convergence (in a subsequence) of the SE strategies.

Theorem 2 *(Lipschitz Extension Theorem) Let (X, d) be a metric space and $A \subset X$. Let the function $g : A \longrightarrow \Re$ be Lipschitz with constant L. Then there is an extension of g to X that is also Lipschitz with constant L.*

The essential idea for the proof of Theorem 2 is due to David Hartvigsen (1992).

6 Weak Best Response Equilibria

Many finite signaling games have a large number of sequential equilibria. Various authors have proposed refinements of the SE concept to reduce the number of equilibria by weeding out those that seem intuitively unreasonable (see Cho and Kreps 1987 and their references). We want to consider the strongest possible refinement criterion and show that SE exist that satisfy this criterion. The strongest criterion is Kohlberg and Mertens' (1986) strategic stability; it implies many of the weaker criteria that authors have proposed. It is not possible, however, to apply this criterion directly to infinite games.

Cho and Sobel (1987) show that for finite monotonic signaling games, the following criterion is generically equivalent to stability.

Define $V^1(t)$ as in Section IV as the equilibrium payoffs to player 1 when his type is t.

[15] Let (X, d) be a metric space and $A \subset X$. We say a function $g : A \longrightarrow \Re$ is *Lipschitz* with constant L if for all $x, y \in A$, $\|g(x) - g(y)\| \leq Ld(x, y)$.

[16] The existence of the required A^n is immediate if we take $(\hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n)$ to be regular. In that case, $A^n = \text{supp}[\hat{\lambda}^n_{X^n}]$. We have shown that regular SE exist in games with finite X.

Definition 6 $(\hat{\alpha}, \hat{\eta}, \hat{\beta})$ *satisfies the Weak Best Response (WBR) criterion if for any given* $x \in X$,

(W1) *there exists* $t \in T$ *and* $(\eta, \beta) \in MBR(x, Y, T, U^2)$ *such that* $U^1(t, x, \eta) \geq V^1(t)$

implies

(W2) $\forall t \in$ *supp*$[\hat{\beta}(x)]$, *there exists* $(\eta', \beta') \in MBRU2$ *such that* $U^1(t, x, \eta') = V^1(t)$, *and* $\forall t' \in T$, $U^1(t', x, \eta') \leq V^1(t')$.

Definition 7 *Denote by* $WBR(\Gamma)$ *the set of SE of* Γ *that satisfy the WBR criterion, and by* $WBRO(\Gamma)$ *the set of outcomes generated by WBR equilibria, elements of* $WBR(\Gamma)$.

If a SE $(\hat{\alpha}, \hat{\eta}, \hat{\beta})$ is regular, then R1, R3 and S4 together imply that W2 holds for $x \in$ supp$[\lambda_X]$ using $(\eta', \beta') = (\hat{\eta}(x), \hat{\beta}(x))$. Hence, the WBR criterion will not be effective for signals x that are observed on the equilibrium path of a regular SE. In finite games, all SE are regular and, therefore, the WBR criterion may be stated solely in terms of deviations.

W2 restricts player 2's beliefs given x to those t for which x is a weak best response in some SE (with belief $\beta(x) = \beta'$ and response $\eta(x) = \eta'$) with the same outcome as (α, η, β) (see Cho and Kreps 1987). W1 says such a restriction cannot be ruled out a priori. WBR is just a convenient restatement of Kohlberg and Mertens' Never a Weak Best Response test. It implies various other refinements of SE, e.g., the Intuitive Criterion of Cho and Kreps (1987), and the Universal Divinity test of Banks and Sobel (1987).

This section follows the general form of section 5. Theorem 3 is a convergence result for WBR equilibria, the analogue of Theorem 1. It states that the limit of WBR outcomes (of a sequence of approximating games) will be a WBR outcome of the limit game provided it can be implemented with a continuous strategy for player 2.

Theorem 3 *Consider a sequence of continuous games* $\Gamma^n = [(T^n, \rho^n), X^n, Y^n, U^{1n}, U^{2n}]$, $n = 1, 2, \ldots$ *and a continuous game* $\Gamma = [(T, \rho), X, Y, U^1, U^2]$. *Suppose* $\langle \Gamma^n \rangle \longrightarrow \Gamma$ *and*

(H3) *there exists* $\langle \hat{\lambda}^n \rangle \Rightarrow \hat{\lambda}$ *with* $\hat{\lambda}^n \in WBRO(\Gamma^n)$;

(H4) *there exists a continuous* $\eta^1 \in \Sigma^2(\Gamma)$ *such that* $\lambda = \lambda_{T \times X} \circ f_{\eta^1}{}^{-1}$.

Then $\hat{\lambda} \in WBRO(\Gamma)$.

Proof: From Theorem 1, λ is a SE outcome for Γ. To show that λ is a WBR outcome, we will support it using the Theorem 1 strategies for player 1 and the Theorem 1 strategies and beliefs for player 2 on the equilibrium path. We may need, however, to refine somewhat the strategies and beliefs of player 2 off the equilibrium path.

By Proposition 3, $\hat{\lambda}$ is regular and therefore as explained above, the WBR criterion holds automatically for x on the equilibrium path.

For Theorem 1, player 2's strategy/belief pair given signal x off the equilibrium path was $(\eta^0(x), \beta^0(x))$. We will define a new strategy/belief pair $(\tilde{\eta}, \tilde{\beta})$ to replace (η^0, β^0). We will verify that this new pair will satisfy the WBR criterion and the equilibrium conditions for player 2, and it will not upset the equilibrium strategy $\hat{\alpha}$ for player 1.

We define the pair $(\tilde{\eta}(x), \tilde{\beta}(x))$ for each $x \in X$ with the understanding that $(\tilde{\eta}(x), \tilde{\beta}(x))$ will only be used to replace $(\eta^0(x), \beta^0(x))$ off the equilibrium path. Let $\langle \hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n \rangle$ be a sequence of SE supporting the outcomes $\langle \hat{\lambda}^n \rangle$ of $\langle \Gamma^n \rangle$. For each $x \in X$, we consider a subsequence $\langle x^n, \hat{\eta}^n(x^n), \hat{\beta}^n(x^n) \rangle$ —— $(x, \eta^0(x), \beta^0(x))$ and use it to define $(\tilde{\eta}(x), \tilde{\beta}(x))$. This sequence exists by definition of (η^0, β^0). There are three cases to consider. These cases are not all mutually exclusive, but they do not need to be.

Case 1: W1 fails to hold for Γ at x. Then WBR holds vacuously using $(\tilde{\eta}(x), \tilde{\beta}(x)) = (\eta^0(x), \beta^0(x))$, and since $(\eta^0(x), \beta^0(x))$ is the SE strategy for player 2 off the equilibrium path, $(\tilde{\eta}(x), \tilde{\beta}(x))$ will automatically be in equilibrium for player 2 and will not upset the equilibrium strategy of player 1.

Case 2: W1 holds for Γ at x, and there is an infinite subsequence of $\langle x^n \rangle \Rightarrow x$ such that W1 holds for $(\hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n)$ at x^n. Restrict attention to this subsequence. By H3, each $(\hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n)$ satisfies WBR, so W2 holds on the subsequence. We set $(\tilde{\eta}(x), \tilde{\beta}(x)) = (\eta^0(x), \beta^0(x))$ and show that W2 holds for $\tilde{\beta}(x)$.

Pick any $t \in \text{supp}[\tilde{\beta}(x)]$. Since $\langle \hat{\beta}^n(x^n) \rangle \Rightarrow \tilde{\beta}(x)$, there exists $\langle t^n \rangle \Rightarrow t$ with $t^n \in \text{supp}[\hat{\beta}^n(x^n)]$. Let $(\eta^n, \beta^n) \in MBR(x^n, Y^n, T^n, U^{2n})$ satisfy W2 for t^n in the SE for Γ^n and let $\langle \eta^n, \beta^n \rangle \Rightarrow (\eta', \beta')$ on a subsequence. Then $(\eta', \beta') \in MBR(x, Y, T, U^2)$ by Lemma 5, and thus the first requirement of W2 holds for $\tilde{\beta}(x)$. By W2 for Γ^n, $U^{1n}(t^n, x^n, \eta^n) = V^{1n}(t^n)$, and taking limits yields $U^1(t, x, \eta') = V^1(t)$ since $\langle V^{1n} \rangle$ converges continuously to V^1 by Proposition 3. Thus the equation in W2 holds for $\tilde{\beta}(x)$. To show the inequality in W2 holds, pick any $t' \in T$ and take a sequence $\langle t'^n \rangle$ —— t'. By Σ^3 for Γ^n, we have $U^{1n}(t'^n, x^n, \hat{\eta}^n(x^n)) \leq V^{1n}(t'^n)$, so that in the limit $U^1(t', x, \eta') \leq V^1(t')$ as required. Hence the three requirements of W2 hold for $\tilde{\beta}(x)$.

As in Case 1, we have set $(\tilde{\eta}(x), \tilde{\beta}(x)) = (\eta^0(x), \beta^0(x))$, so $(\tilde{\eta}(x), \tilde{\beta}(x))$ will automatically be in equilibrium for player 2 and will not upset the equilibrium strategy of player 1.

Case 3: W1 holds for Γ at x, but there is an infinite subsequence of $\langle x^n \rangle \Rightarrow x$ such that W1 fails to hold with a strict inequality for Γ^n at x^n. Notice we require only that W1 fail to hold with a strict inequality at x^n, i.e., there does not exist $t \in Tn$ and $(\eta, \beta) \in MBR(x^n, Y^n, T^n, U^{2n})$ such that

$$U^{1n}(t, x^n, \eta) > V^{1n}(t).$$

This requirement is satisfied if W1 fails to hold with a weak inequality. The requirement guarantees that the set of x satisfying Case 3 is closed (and extends the case somewhat so it overlaps with Case 2).

Restrict attention to the given subsequence. We will show that there exists a $\tilde{t} \in T$ such that if $\tilde{\beta}(x)$ puts unit mass on $\tilde{t}$, then $\tilde{\beta}(x)$ will satisfy W2.

Since W1 holds for Γ, there exists t' and $(\eta', \beta') \in MBR(x, Y, T, U^2)$ satisfying

$$U^1(t', x, \eta') \geq V^1(t'). \tag{7}$$

Let $\langle \beta^n \rangle$ be any sequence converging to β' with $\beta^n \in M(T^n)$ for all n. Take any convergent subsequence $\langle \eta^n \rangle$ such that $(\eta^n, \beta^n) \in MBR(x^n, Y^n, T^n, U^{2n})$ and let $\eta'' = \lim_n \eta^n$. Since by hypothesis W1 fails to hold with strict inequality on the subsequences we are considering,

$$U^{1n}(t, x^n, \eta^n) \leq V^{1n}(t)$$

holds for every $t \in T^n$. Taking limits,

$$U^1(t, x, \eta'') \leq V^1(t) \tag{8}$$

holds for every $t \in T$.

For $s \in [0, 1]$, let $\eta^s = s\eta' + (1-s)\eta''$. By Lemma 5, (η'', β') is in $MBR(x, Y, T, U^2)$, as is (η', β'), and this implies so is (η^s, β'). Define a function $m : [0, 1] \longrightarrow \Re$ by

$$m(s) = max_{t \in T} \; [U^1(t, x, \eta^s) - V^1(t)].$$

The function m is well defined since V^1 and U^1 are continuous and T is compact. By the Theorem of the Maximum, m is continuous. Now by (7), $m(1) \geq 0$ and by (8), $m(0) \leq 0$, so there must be an $s \in [0, 1]$ with $m(s) = 0$ by the Intermediate-Value Theorem. By definition of m, for this s there must exist a $\tilde{t} \in T$ such that

$$U^1(\tilde{t}, x, \eta^s) = V^1(\tilde{t}) \tag{9}$$

and for all $t \in T$,

$$U^1(t, x, \eta^s) \leq V^1(t). \tag{10}$$

Let $\tilde{\beta}(x)$ put unit mass on $\tilde{t}$. This belief satisfies W2 because of (9) and (10) and because $(\eta^s, \beta') \in MBR(x, Y, T, U^2)$.

Let $\tilde{\eta}(x)$ be a best response by player 2 to x given the belief $\tilde{\beta}(x)$, so that $(\tilde{\eta}(x), \tilde{\beta}(x))$ will satisfy equilibrium condition S4 for player 2. The logic used to establish (8) implies we may assume that

$$U^1(t, x, \tilde{\eta}(x)) \leq V^1(t) \tag{11}$$

holds for all t. That is, some best responses given beliefs $\tilde{\beta}(x)$ may violate (11), but at least one will satisfy it. Given (11), playing $\tilde{\eta}(x)$ instead of $\eta^0(x)$ will not upset the equilibrium strategy of player 1.

We have defined $(\tilde{\eta}(x), \tilde{\beta}(x))$ so that it satisfies the WBR criterion and SE conditions S3 and S4. We must now refine $(\tilde{\eta}, \tilde{\beta})$ to make it a measurable function so that conditions S1 and S2 will be satisfied. Since $(\tilde{\eta}, \tilde{\beta})$ equals the measurable function (η^0, β^0) except in Case 3, we only need to show that the set C of x where Case 3 holds is measurable and that we can refine $(\tilde{\eta}, \tilde{\beta})$ so that it is measurable on C.

We defined Case 3 so that the set C is closed and thus measurable. Now define $D \subset C \times \Psi \times M(T)$ to be the set of triples (x, η, β) such that $(\eta, \beta) \in MBR(x, Y, T, U^2)$, x and η satisfy (11), and x and β satisfy W2.

We showed in Case 3 that there exists such a pair (η, β) for each $x \in C$. The triples in D will satisfy WBR, S2 and S3. D is compact, so by Lemma 6 it has a measurable selection. If we replace $(\tilde{\eta}(x), \tilde{\beta}(x))$ on C with this selection, then $(\tilde{\eta}, \tilde{\beta})$ will be measurable everywhere as required. **QED**

The next two propositions show, as applications of Theorem 3, the existence of equilibria satisfying the WBR criterion for continuous signaling games with finite X, and for strongly monotonic games.

Proposition 8 *If $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ is continuous and X is finite, then Γ has a SE satisfying the WBR criterion.*

Proof: For $n = 1, 2, \ldots$, define finite games $\Gamma^n = [(T^n, \rho^n), X, Y^n, U^{1n}, U^{2n}]$ as follows. Let Tn be a finite subset of T with $\langle T^n \rangle \Rightarrow T$. Let $\rho^n \in M(T^n)$ and let $\langle \rho^n \rangle \Rightarrow \rho$. Define finite $Y^n \subset Y$ analogously to Tn. For each $i = 1, 2$, let $\epsilon^{in} : T^n \times X \times Y^n \longrightarrow \Re$ be functions such that $\langle \epsilon^{in} \rangle \longrightarrow 0$ continuously. Define $U^{in} : T^n \times X \times Y^n \longrightarrow \Re$ by setting

$$U^{in}(t, x, y) = U^i(t, x, y) + \epsilon^{in}(t, x, y).$$

Choose each ϵ^{in} so that the game Γ^n has a SE $(\alpha^n, \eta^n, \beta^n)$ satisfying WBR. Such ϵ^{in} exist because WBR equilibria exist generically for finite games.

We now apply Theorem 3. We have chosen the games Γ^n so that H1 holds. H2 also holds because the way we chose ϵ^{in} implies $\langle U^{in} \rangle \Rightarrow Ui$ continuously. Let $\langle \lambda^n \rangle$ be the sequence of outcomes for

$\langle\hat{\alpha}^n, \hat{\eta}^n, \hat{\beta}^n\rangle$. Choose a subsequence if necessary so that $\langle\hat{\lambda}^n\rangle \Rightarrow \lambda$ and thus H3 holds. Choose a further subsequence so that the sequences $\langle\hat{\eta}^n(x)\rangle$ converge uniformly. As in Section V, this uniform convergence implies H4 holds. By Theorem 3, $\lambda \in WBRO(\Gamma)$. **QED**

It follows from Kohlberg and Mertens (1986) that WBR equilibria exist generically in finite games. Proposition 8 implies that WBR equilibria exist for all finite games.

Proposition 9 *If $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ is continuous and strongly monotonic, then Γ has a SE satisfying the WBR criterion.*

7 Appendix: Proofs of Lemmas

Lemma 11 *(Hartvigsen): Let (X, d) be a metric space, $A \subset X$, and $x \in X \setminus A$. Let $g : A \longrightarrow \Re$ be Lipschitz with constant L. Let*

$$r = \inf_{y \in A} g(y) + Ld(\hat{x}, y).$$

Then r is finite. Extend g to $B = A \cup \{\hat{x}\}$ by setting $g(\hat{x}) = r$. Then the extended g is Lipschitz with constant L.

Proof: It suffices to show that for all $y \in A$,

$$\|r - g(y)\| \leq Ld(x, y). \tag{12}$$

This implies r is finite.

Fix an arbitrary $y \in A$. Since g is Lipschitz and d satisfies the Triangle Inequality,

$$g(y) - g(z) \leq Ld(y, z) \leq Ld(x, y) + Ld(x, z)$$

holds for all z in A. Rearranging yields $g(y) - Ld(x, y) \leq g(z) + Ld(x, z)$. Using the definition of r, we have $g(y) - Ld(\hat{x}, y) \leq r \leq g(z) + Ld(x, z)$ for all z in A. Rearranging again and substituting y for z, we have

$$g(y) - r \leq Ld(x, y) \; and \; r - g(y) \leq Ld(x, y).$$

This implies (12). **QED**

Proof of Lipschitz Extension Theorem We use the Principle of Transfinite Construction (Dugundji 1966, p.40). Construct a well-ordering $<$ of X such that each element of A precedes each element of $X \setminus A$. (Do this by ordering A and $X \setminus A$ separately and then adjoining them.) Let $W(x)$

denote the predecessors of x, i.e., $W(x) = \{y \in X \mid y < x \text{ and } y \neq x\}$. Let $\bar{\Re}$ denote the extended real numbers. Given a function $\phi : W(x) \longrightarrow boldE$, define

$$R_x(\phi) = \begin{cases} g(x) & \text{if } x \in A, \\ \inf_{y \in W(x)} \phi(y) + Ld(x, y) & \text{otherwise.} \end{cases}$$

By the Principle of Transfinite Construction, there exists a function $G : X \longrightarrow \bar{\Re}$ such that $R_x(G|W(x)) = G(x)$, which is to say,

$$G(x) = \begin{cases} g(x) & \text{if } x \in A, \\ \inf_{y \in W(x)} G(y) + Ld(x, y) & \text{otherwise.} \end{cases}$$

By construction, the restriction of G to A is g. We must show that G is finite-valued and Lipschitz. To this end, let S be the set of $x \in X$ such that either (i) $G(x)$ is infinite, or (ii) x and some preceding y violate the Lipschitz condition $\|G(x) - G(y)\| \leq Ld(x, y)$. We must show that S is empty.

Suppose S is not empty, and let x be the first element of S. That x is the first element implies that $G(y)$ must be finite for all $y \in W(x)$ and that all pairs from $W(x)$ must satisfy the Lipschitz condition. This implies by Lemma 11 and the definition of G that $G(x)$ is finite and that x and every $y \in W(x)$ must also satisfy the Lipschitz condition. This contradicts the assumption that $x \in S$ and implies S must be empty as required. QED

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