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ORDER INDEPENDENCE FOR ITERATED WEAK DOMINANCE

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ABSTRACT

In general, the result of the elimination of weakly dominated strategies depends on order. We find a condition, satisfied by the normal form of any generic extensive form, and by some important games which do not admit generic extensive forms, under which any two games resulting from the elimination of weakly dominated strategies (subject to no more eliminations being possible) are equivalent. We also extend our condition and result to the case of elimination by mixed strategies. The result strengthens the intuitive connection between backward induction and weak dominance. And, under our condition, some computational problems relating to weak dominance, which are generally complex, become simple. *JEL Classification Number: C72*

I. INTRODUCTION.

As is well known, the result of the iterative removal of weakly dominated strategies can depend on the order of removal.[1] In this paper, we define the *Transitivity of Decision Maker Indifference* (TDI) condition, which is satisfied by the normal form of any generic extensive form, and by some important games which do not admit generic extensive forms, including discretized versions of first price auctions. We show that under TDI any two games resulting from the iterative elimination of weakly dominated strategies (subject to no more eliminations being possible) are strategically equivalent. That is, the two games differ only by the addition or removal of redundant strategies and a renaming of strategies.

Because TDI is satisfied for the normal form of an extensive form game for generic assignment of payoffs to terminal nodes, for almost all such games, the order of removal by weak dominance is irrelevant. We will argue that this result strengthens the intuitive connection between backward induction in the extensive form and weak dominance in the normal form.

Finally, we use our result to make a comment on the work of Gilboa, Kalai, and Zemel [1991] concerning the complexity of iterative weak dominance.

II. THE LITERATURE.

A number of previous papers have explored this issue. Gilboa, Kalai, and Zemel [1990] give conditions on a dominance operator that are sufficient for the order of elimination not to matter. These conditions are satisfied by strong dominance, but not by weak dominance.[2]

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[1]Let s_i and r_i be strategies for player i. Given opponents' strategy set W_{-i}, s_i weakly dominates r_i if $\pi_i(s_i,t_{-i}) \geq \pi_i(r_i,t_{-i})$ $\forall t_{-i} \in W_{-i}$ and $\pi_i(s_i,t_{-i}) > \pi_i(r_i,t_{-i})$ for some $t_{-i} \in W_{-i}$, where π_i is player i's payoff function.

[2]Gilboa et al. also consider a version of weak dominance which does not require the strict inequality. We shall refer to this as *very weak dominance*. They claim that, with some additional conditions (which are satisfied by both weak and very weak dominance), order of removal under a

Rochet [1980] considers the following condition:

(1) $\pi_i(s) = \pi_i(t) \Rightarrow \pi_j(s) = \pi_j(t)$ for all $i,j \in N$, $s,t \in \Pi_{i \in N} S_i$,

where N is the set of players, S_i is the set of strategies for player i, and π_i is player i's payoff function (see Section IV for formal definitions).

Rochet shows that if a game satisfying (1) is dominance solvable (when one eliminates all weakly dominated strategies at every stage), then the same outcome is obtained regardless of the order of elimination of weakly dominated strategies. Furthermore, Rochet shows that any normal form game derived from an extensive form of perfect information satisfying the extensive form analogy to (1) (i.e., satisfying that one player is indifferent over two terminal nodes only if all players are) is dominance solvable with the same outcome as determined by backward induction on the extensive form. Rochet shows by example that this need not be the case when the extensive form does not satisfy this condition.[3] Our results do not depend on dominance solvability, and TDI is weaker than (1). In Section III we give an example of a game satisfying TDI but not (1). We consider the relationship between weak dominance and backward induction in Section VI.

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relationship dom will not matter if

$$[x \text{ dom } y \text{ and } y \text{ dom } x] \Rightarrow [x \text{ and } y \text{ are payoff equivalent for all players}].$$

This claim is false as stated, because for weak dominance the antecedent never holds (since it can never be the case that x and y each weakly dominate the other) and so the condition is vacuously satisfied, but order can clearly matter. Their claim is correct for very weak dominance. The analysis of very weak dominance can be used as the basis for results similar to ours, although Gilboa et al. do not actually do this. Gilboa et al. note that their condition is satisfied for zero sum games under very weak dominance.

[3]See also Moulin [1986, Chapter 4.2] on Rochet's robustness result, dominance solvability (using weak dominance), and the relationship between (1) and extensive-form games. Moulin [1984] gives conditions for a game to be dominance solvable (using weak dominance) and shows that for a certain class of games, dominance solvability (using weak dominance) implies Cournot stability.

Gretlein [1983] works with games in which each player's preferences over the set of possible outcomes (i.e. payoff vectors) is strict. In such games, Gretlein shows that the set of outcomes that results from iterated elimination of weakly dominated strategies (subject to no more eliminations being possible) is the same regardless of the order of eliminations. This condition is stronger than (1), and so *a fortiori* stronger than TDI.[4]

A final point distinguishing our work from Rochet's and Gretlein's is that their results only consider domination by pure strategies. In Section V we extend our analysis to removal of strategies that are weakly dominated by mixtures of other strategies. Once again, we find that for generic extensive form games, and some important games which do not have a generic extensive form, order does not matter.

Finally, we believe that our proof of this result is particularly straightforward.

III. Transitivity of Decision Maker Indifference.

The normal form of a generic extensive form game always satisfies condition (1): for $\pi_i(s)=\pi_i(t)$ to hold, it must be that s and t reach the same terminal node. But then $\pi_j(s)=\pi_j(t)$ for all $j \in N$. So, even under condition (1), establishing that the result of iterative elimination under weak dominance does not depend on order is of considerable use. However, there are important games

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[4]Also, note that Gretlein's result states only that the set of outcomes is the same, not that the games that remain after iterated removal are in any sense equivalent. The following two games have the same set of outcomes, but are certainly not strategically equivalent.

	2	
1	6,2	2,6
	1,1	4,0

	2	
1	6,2	1,1
	4,0	2,6

In situations where iterated weak dominance is the only condition of interest, the difference between these two games is not very important. If however, one is interested, for example, in Nash equilibria of the game which results from the iterative removal, then it is important to know that different orders cannot yield games which differ as these two do.

not satisfying (1) to which we would also like our results to apply.

As an example, consider a first price auction. To make this a finite strategy game, assume that players receive signals about the value of the object which are drawn from a finite set Ω (the analysis to follow does not depend on the manner in which signals are related across players), and that players are restricted to make bids which are integer multiples of a penny up to some (large) maximum. Then, each player's strategy space is the set of all maps from signals in Ω to allowable bids, and so is finite. Let r_i and s_i be two strategies for some player i which differ only in that for some signal $\omega \in \Omega$, $r_i(\omega) < s_i(\omega)$, i.e., let r_i and s_i differ only in that for some signal, r_i specifies a smaller bid than s_i. Consider any strategy profile t_{-i} for the other players such that the largest bid possible under t_{-i} is less than $r_i(\omega)$. Then, $\pi_{-i}(r_i,t_{-i}) = \pi_{-i}(s_i,t_{-i})$. To see this, note that when i's signal is not ω, his behavior is the same under r_i and s_i. When i's signal is ω, players other than i do not receive the object, and so receive a payoff of 0 in either case. However, $\pi_i(r_i,t_{-i}) \neq \pi_i(s_i,t_{-i})$. Thus, this game does not satisfy (1).

The condition failed in this example because, once a player has lost, he is indifferent over the amount by which he loses. For almost all specifications of how signals map to valuations, this is the only way in which i can be indifferent between pure strategy profiles (r_i,t_{-i}) and (s_i,t_{-i}) -- by having those two strategy profiles differ only in how much i loses by when i loses. We formalize this in the Appendix. So, while a unilateral change of pure strategy by player i can change his payoff while leaving his opponents indifferent, the opposite cannot occur: i cannot be indifferent about this unilateral change while affecting his opponents' payoffs. Thus, while the game does not satisfy (1), it does satisfy the following weaker condition:

(2) $\pi_i(r_i,s_{-i}) = \pi_i(t_i,s_{-i}) \Rightarrow \pi_j(r_i,s_{-i}) = \pi_j(t_i,s_{-i})$ for all $i,j \in N$, $r_i,t_i \in S_i$, $s_{-i} \in S_{-i}$.

The agreement of player i's payoffs across two different strategy profiles only implies agreement for the other players if the strategy profiles differ only by the action of player i.

We shall refer to (2) as the *Transitivity of Decision Maker Indifference* (TDI) condition and will show for any finite player game satisfying TDI, any two games which are achieved by the iterative removal of weakly dominated strategies (subject to no more removals being possible) are equivalent.

IV. FORMALITIES AND THE MAIN RESULT.

We work with finite strategy, finite player, normal form games. Players $i \in N \equiv \{1,...,n\}$ have finite strategy spaces S_i. Payoffs are given by $\pi: \Pi_{i\in N} S_i \to \mathbf{R}^n$. The payoff function π is extended to mixed strategies in the standard way. We assume, without loss of generality, that $S_i \cap S_j = \emptyset$ for all $i,j \in N$, $i \neq j$. So, without ambiguity, we can drop the player subscripts on the strategy names. Let $S \equiv \cup_{i\in N} S_i$. For $W \subseteq S$, let the strategies in W that belong to i be denoted by $W_i \equiv W \cap S_i$. Say that $W \subseteq S$ is a restriction of S if $\forall i$, $W_i \neq \emptyset$. Note that any restriction W of S generates a unique game given by strategy spaces W_i and the restriction of π to $\Pi_{i\in N} W_i$. We will denote this game by (W,π). We similarly define $W_{-i} \equiv \Pi_{j\neq i} W_j$. A typical element $x_{-i} \in W_{-i}$ thus specifies a strategy $x_j \in W_j$ for each $j \neq i$.

Definition: Let W be a restriction of S, and let $r_i, s_i \in S_i$. Then r_i <u>weakly dominates</u> s_i on W, written $r_i \ WD_W \ s_i$, if $\pi_i(r_i,x_{-i}) \geq \pi_i(s_i,x_{-i})$ $\forall x_{-i} \in W_{-i}$ and $\pi_i(r_i,z_{-i}) > \pi_i(s_i,z_{-i})$ for some $z_{-i} \in W_{-i}$. s_i is <u>weakly dominated</u> on W if there exists $r_i \in W_i$ with $r_i \ WD_W \ s_i$.

Note that under this definition we do not allow weak dominance by mixtures of other strategies. We extend our results to dominance by mixtures in Section V.

The next definition defines "W is a reduction of S" to mean that W can be reached from S by iteratively eliminating weakly dominated strategies.

Definition: Let W be a restriction of S. Then W is <u>reduction</u> of S if $W=S\backslash X^1,...,X^m$ where $\forall k$, $X^k \subset S$ and $\forall x \in X^k$, $\exists z \in S\backslash X^1,...,X^k$ such that z $WD_{S\backslash X^1,...,X^{k-1}}$ x. W is a <u>full reduction</u> of S if W is a reduction of S and no strategies in W are weakly dominated on W.

Definition: Let V and W be restrictions of S. V is <u>equivalent to a subset of</u> W if there exist one-to-one maps $m_i:V_i \to W_i$, $i \in N$, such that $\pi(x)=\pi(m_1(x_1),...,m_n(x_n))$ $\forall x \in \Pi_{i \in N} V_i$.

Observation 1: The relation "equivalent to a subset of" is transitive.

Definition: Let W be a restriction of S, and let $r_i, s_i \in S_i$. Then r_i is <u>redundant to</u> s_i on W, written $r_i \ RE_W \ s_i$, if $\pi(r_i,x_{-i})=\pi(s_i,x_{-i})$ $\forall x_{-i} \in W_{-i}$.

The next definition defines "$\hat{W}$ is obtainable from W" to mean $\hat{W}$ can be reached from W by iteratively eliminating strategies that are either weakly dominated or redundant. In this elimination process, only one strategy may be removed at each iteration. Formally, we have:

Definition: Let W be a restriction of S. Then $\hat{W} \subseteq W$ is <u>obtainable from</u> W if $\hat{W}=W\backslash x^1,...,x^m$ where $\forall k$, $x^k \in W$ and $\exists z^k \in W\backslash x^1,...,x^k$ such that either z^k $WD_{W\backslash x^1,...,x^{k-1}}$ x^k or z^k $RE_{W\backslash x^1,...,x^{k-1}}$ x^k. $\hat{W}$ is <u>one-step obtainable from</u> W if the above holds with m=1.

Lemma A: Let W be obtainable from S, let $i \in N$, and let $s_i \in S_i \backslash W_i$. Then there exists $t_i \in W_i$ with $\pi_i(t_i,x_{-i}) \geq \pi_i(s_i,x_{-i})$ $\forall x_{-i} \in W_{-i}$.

Proof: Since $s_i \notin W_i$, s_i was eliminated by some $r_i \in S_i$. At that time, at least W was left and so $\pi_i(r_i,x_{-i}) \geq \pi_i(s_i,x_{-i})$ $\forall x_{-i} \in W_{-i}$. If r_i is a not a member of W_i, then it itself was eliminated along the path to W. Using the finiteness of S_i and the transitivity of $\geq$ we are done. ∎

Lemma B: Let (S,π) satisfy TDI, and let W be a reduction of S. Then W is obtainable from S.

Proof: We show that W^1, the set of strategies remaining after one round of elimination of weakly dominated strategies, is obtainable from S. Repeated application of the argument yields the result. Label the elements of $S \backslash W^1$ in some arbitrary order $x^1,\ldots,x^q$. For each $k=1,\ldots,q$, there is $z^k \in W^1$ for which z^k WD_S x^k. Let i be the player to which z^k and x^k belong. Since z^k WD_S x^k, $\forall y_{-i} \in (S \backslash x^1,\ldots,x^{k-1})_{-i}$, $\pi_i(z^k,y_{-i}) \geq \pi_i(x^k,y_{-i})$. Using TDI, either z^k $\mathbf{WD}_{S \backslash x^1,\ldots,x^{k-1}}$ x^k or z^k $\mathbf{RE}_{S \backslash x^1,\ldots,x^{k-1}}$ x^k. Since $z^k \in W^1 \subseteq S \backslash x^1,\ldots,x^k$, $S \backslash x^1,\ldots,x^k$ is one step obtainable from $S \backslash x^1,\ldots,x^{k-1}$, and we are done. ∎

Lemma C: Let W be a restriction of S, let $\hat{W}$ be obtainable from W, and let $V = W \backslash y$ be one-step obtainable from W. Then there exists $\hat{V}$ obtainable from V with $\hat{V}$ equivalent to a subset of $\hat{W}$.

Proof: Since $\hat{W}$ is obtainable from W, we can write $\hat{W} = W \backslash x^1,\ldots,x^m$, where $W \backslash x^1,\ldots,x^k$ is one-step obtainable from $W \backslash x^1,\ldots,x^{k-1}$ for all k. Proceed to remove strategies $x^1,\ldots,x^m$, in order, from V as long as they are weakly dominated or redundant on the set remaining. Let x^k be the last valid such removal. If $k = m$, we have arrived at $\hat{W} \backslash y$ and so we are done. Otherwise, x^{k+1} is either weakly dominated or redundant in $W \backslash x^1,\ldots,x^k$, but x^{k+1} is neither weakly dominated nor redundant in $W \backslash y,x^1,\ldots,x^k$.

Case (i): If x^{k+1} is redundant in $W \backslash x^1,\ldots,x^k$, then it must be that y is redundant to x^{k+1} in

$W\backslash x^1,...,x^k$. Thus $W\backslash x^1,...,x^k,x^{k+1}$ and $W\backslash y,x^1,...,x^k$ are equivalent, and the remaining eliminations, $x^{k+2},...,x^m$, are valid, subject to a renaming of strategies. In this way we obtain from V a set $\hat{V}$ which is, up to a renaming of strategies, $\hat{W}$.

Case (ii): If x^{k+1} is weakly dominated on $W\backslash x^1,...,x^k$, then it must be that y $\mathbf{WD}_{W\backslash x^1,...,x^k}$ x^{k+1}. By Lemma A, there is $z_i \in W_i\backslash y,x^1,...,x^k$ such that $\pi_i(z_i,x_{-i}) \geq \pi_i(y,x_{-i})$ $\forall x_{-i} \in (W\backslash x^1,...,x^k)_{-i}$. But, then z_i $\mathbf{WD}_{W\backslash x^1,...,x^k}$ x^{k+1}, and therefore z_i $\mathbf{WD}_{W\backslash y,x^1,...,x^k}$ x^{k+1}, a contradiction. ∎

Theorem: Let (S,π) satisfy TDI, and let X and Y be full reductions of S. Then, X and Y are the same up to the addition or removal of redundant strategies and a renaming of strategies.

Proof of Theorem: By Lemma B, we can write $Y=S\backslash y^1,...,y^m$, where $S\backslash y^1,...,y^k$ is one-step obtainable from $S\backslash y^1,...,y^{k-1}$, $\forall k=1,..,m$. We proceed by induction. By Lemma B, X is obtainable from S. Assume that a set equivalent to a subset of X is obtainable from $S\backslash y^1,...,y^{k-1}$ where $k \in \{1,..,m\}$. Since $S\backslash y^1,...,y^k$ is one-step obtainable from $S\backslash y^1,...,y^{k-1}$, Lemma C and observation 1 apply to show that a set equivalent to a subset of X is obtainable from $S\backslash y^1,...,y^k$. Thus, a set equivalent to a subset of X is obtainable from Y.

Since Y was a full reduction, any set obtainable from Y differs from Y only by the removal of redundant strategies. So, after the removal of some redundant strategies, Y differs from a subset of X only by a renaming of strategies. Reversing the roles of X and Y, we are done. ∎

V. Mixed Strategies

The results of the previous section dealt solely with eliminations by pure strategies. In this section we extend our analysis to mixed strategies. If W is a restriction of S, let $r_i \in \Delta(W_i)$ indicate that r_i is a mixed strategy using only pure strategies in W_i.

Definition: Let W be a restriction of S, let $s_i \in S_i$, and let $r_i \in \Delta(S_i \backslash s_i)$. Then r_i weakly dominates* s_i on W, written r_i WD*$_W$ s_i, if $\pi_i(r_i,x_{-i}) \geq \pi_i(s_i,x_{-i})$ $\forall x_{-i} \in W_{-i}$ and $\pi_i(r_i,z_{-i}) > \pi_i(s_i,z_{-i})$ for some $z_{-i} \in W_{-i}$. $s_i \in W_i$ is weakly dominated* on W if there is $r_i \in \Delta(W_i \backslash s_i)$ with r_i WD*$_W$ s_i

Definition: Let W be a restriction of S, let $s_i \in S_i$, and let $r_i \in \Delta(S_i \backslash s_i)$. Then s_i is redundant* to r_i on W, written s_i RE*$_W$ r_i, if $\pi(r_i,x_{-i}) = \pi(s_i,x_{-i})$ $\forall x_{-i} \in W_{-i}$.

Consider the following game:

		2		
		L	C	R
	T	2,1	2,3	0,2
1	M	0,3	3,1	0,2
	M'	1,4	1,4	1,4
	B	1,4	0,3	0,2

R is weakly dominated by ½L+½C. The game satisfies TDI, but clearly order of elimination under WD* can matter: If one first removes B, then S\B is a full reduction, while if one first removes R, then B and M' can also be removed. But, {T,M,M'}×{L,C,R} and {T,M}×{L,C} are clearly not equivalent games. So, to get an order independence result, we need to strengthen TDI.

The game illustrated has a highly non-generic feature: R is weakly dominated by ½L+½C, but by no other mixture. Consider different assignments of payoffs in this game, subject to existing ties in payoffs for each player being maintained (each player receives 4 different payoffs in this game, and so these assignments correspond to elements of $\mathbf{R}^8$).

For all such assignments, the equality of payoffs in the third row will remain, and so in particular, given M', any mixture of L and C will give the same payoff as R. However, for almost all such assignments, one of two things will occur: either there will be a mixture of L and C which strictly dominates R on {T,M,B}, or there will be no mixture which even weakly dominates R on {T,M,B}. In particular, the set of payoff assignments yielding a mixture of L and C that weakly dominates R on {T,M,B} but not yielding any mixture that strictly dominates R on {T,M,B} is a lower dimensional subspace of $\mathbf{R}^8$. If one allows perturbations which do not respect some of the existing ties, then the situation in this game becomes even less likely. Generalizing this argument to general games is notationally tedious, but straightforward.

So, even allowing for some "structural" ties in payoffs, for almost all games, if there is $r_i \in \Delta(W_i \backslash s_i)$ with r_i WE^*_W s_i, then there is another strategy $t_i \in \Delta(W_i \backslash s_i)$ which strictly dominates s_i except versus those opposition strategy profiles s_{-i} on which all elements in the support of t_i give the same payoff to i as does s_i. As long as such a t_i exists, TDI guarantees that we will not run into problems. Our extended TDI condition imposes the condition that such a t_i always exists:

Definition: (S,π) satisfies <u>TDI</u>* if for all restrictions W and for all $s_i \in W_i$, $r_i \in \Delta(W_i \backslash s_i)$, we have

r_i WD^*_W $s_i \Rightarrow \exists t_i \in \Delta(W_i \backslash s_i)$ s.t. (i) t_i WD^*_W s_i and (ii) $\forall s_{-i} \in W_{-i}$, $\pi_i(t_i,s_{-i}) = \pi_i(s_i,s_{-i}) \Rightarrow \pi(q_i,s_{-i})$ $= \pi(s_i,s_{-i})$ $\forall q_i \in W_i \backslash s_i$ which are in the support of t_i.

Fix a normal form and a set of payoff ties. Arguing along the lines above, for almost all assignments of payoffs satisfying the set of ties, TDI* will be satisfied. Thus, in particular, TDI* will be generically satisfied for the normal form of any given extensive form and for the discrete first price auction.

Under TDI^*, the analysis of section IV goes through fairly directly. One simply rereads section IV, replacing the original definitions of weak dominance, redundance, and TDI by their starred counterparts and allowing mixed strategies where appropriate.[5]

VI. Weak Dominance and Backward Induction.

In its purest form, backward induction consists of working from the end to the beginning of a game, iteratively removing actions which are strictly dominated given the information available when that action is taken.

A normal form strategy which is consistent with reaching a particular information set and which takes a strictly dominated action at that information set is weakly dominated by one which differs only by taking the dominating action at that information set. So, any sequence of removals of actions by backward induction in the extensive form corresponds to a sequence of sets of removals of strategies by weak dominance in the normal form.

This relationship between backward induction in the extensive form and weak dominance in the normal form extends to their respective motivations. Backward induction is basically about making decisions as if they matter: since the choice of action at an information set doesn't matter if the information set is not reached, the decision must make sense given that the information set is actually reached. Weak dominance has very much the same flavor: it is not enough to justify a choice between two strategies on the basis that the two strategies yield the player the same payoff given the

[5] In particular, in Lemma A, allow t_i to be a mixed strategy, i.e. replace "$t_i \epsilon W_i$" with "$t_i \epsilon \Delta(W_i)$". The statement and proof of Lemma B should be read with the starred definitions and with the extended TDI condition. In addition, the third sentence of the proof should be changed to read, "For each $k=1,\ldots,q$, there is $z^k \epsilon \Delta(W^1)$ for which z^k WD^*_S x^k, and thus by the extended TDI condition, there is $\hat{z}^k \epsilon \Delta(W^1)$ for which $\hat{z}^k$ WD^*_S x^k and $\pi_i(\hat{z}^k, y_{-i}) = \pi_i(x^k, y_{-i}) \Rightarrow \pi_i(t, y_{-i}) = \pi_i(x^k, y_{-i})$ $\forall t \epsilon supp(\hat{z}^k)$." Then $\hat{z}^k$ should be used in place of z^k in the remainder of the proof. Lemma C and the Theorem need no changes other than a substitution of the starred definitions.

predicted play of the opponents, rather the choice must also make sense given play by the opponents such that it actually matters.[6]

Thus, at an intuitive level, there seems to be an intimate relationship between backward induction and weak dominance. However, a problem with this relationship is that the order of removals under weak dominance can matter, while backward induction is deterministic. What is it about these two concepts that leads one to be deterministic and the other not?[7]

Consider a pair of strategies r_i and s_i which are consistent with reaching information set h and which differ only in that the action that s_i specifies at h is dominated at h by the action that r_i specifies at h. Under weak dominance, when i compares r_i and s_i, she excludes from consideration all strategy profiles for her opponents under which she is indifferent between r_i and s_i. Under backward induction, she excludes from attention those strategy profiles for the opponents under which h is not reachable, and for these strategy profiles, not only is i indifferent between r_i and s_i, but her opponents are indifferent as well. Weak dominance and backward induction thus differ in that under weak dominance a player might exclude from consideration an opposition strategy profile which leaves her indifferent between r_i and s_i but which does not leave her opponents indifferent.

We claim that this distinction between weak dominance and backward induction is the reason that weak dominance is sensitive to order while backward induction yields a deterministic result (even in the more powerful form suggested in footnote 7). To see this note that the force of TDI is precisely to rule out opposition strategy profiles which weak dominance has the player exclude from consideration but backward induction does not. And, under TDI, the order of removal under weak

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[6]On the general relation between normal and extensive form motivations and implementations of solution ideas, see Mailath, Samuelson and Swinkels (1993).

[7]The set of outcomes that survive backward induction remains deterministic even if we allow iterative elimination in any order of dominated actions at information sets, i.e. even if the elimination process does not start at the end of the tree and move up.

domination becomes irrelevant. The robustness of backward induction to order and the nonrobustness of weak domination is not evidence that these two concepts are fundamentally different, but rather the result of a fairly simple difference in the type of strategy profile for the opponent which the two concepts force a player to exclude when making a choice between strategies.

VII. WEAK DOMINANCE AND COMPLEXITY.

We close with a simple comment on how our results interact with those of Gilboa et al. [1991]. They point out that, in general, computational problems involving weak dominance are hard. In particular, given a full reduction of a game, the question remains whether different choices earlier in the sequence of weak dominance removals might have led to a strategically different result. To figure out all the strategic implications of weak dominance, one must thus check all possible orders of removal, which cannot be done in polynomial time. Gilboa et al. interpret this result as casting additional doubt on the use of weak dominance as a solution idea.

Consider, however, games which satisfy TDI, or some other condition such that order does not matter. Then, once one has arrived at a full reduction one knows that no other order could have resulted in a strategically different game. Since finding a full reduction is a polynomial problem, for an important class of games weak dominance becomes less suspect.[8]

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[8]See Gilboa et al. [1991] for a formal development of this material, in particular, see Section 4.3 for a proof that a full reduction can be computed in polynomial time.

Appendix

In this section we formalize the argument that the discrete first price auction generically satisfies TDI. Think of the auction as being generated by first fixing a set of players 1,...,n, a set of names of signals $\Omega=\{\omega^1,...,\omega^m\}$, a measure ρ on Ω^n, and map $V_i:\Omega^{n-1}x\Omega\rightarrow\mathbf{R}$ giving the value of the object to player i when the other bidders receive signals $\omega_{-i}\in\Omega^{n-1}$ and i receives signal $\omega_i\in\Omega$ (notation aside, having different sets Ω for different players presents no difficulties). Since V_i assigns a value to each of m^n different signal profiles, the functions V_i can be associated in the obvious way with an element of $\mathbf{R}^{mn}$. Assume that the V_i are chosen according to some Lebesgue measurable distribution on $\mathbf{R}^{mn}$. Now, fix a pure strategy profile s_{-i} for the players other than i. Given this behavior of the other players and any pure strategy s_i for i, let $p_i(\omega,s(\omega))$ be i's probability of winning when signals are ω and bids are according to s. The expected payoff to i from following s_i is thus

$$\pi_i(s_i,s_{-i})=\sum_{\omega\in\Omega^n}\rho(\omega)p_i(\omega,s(\omega))\big(V_i(\omega)-s_i(\omega_i)\big).$$

This expression depends on V_i only directly (ρ and p_i do not depend on the value assignment), and so it is a linear function of V_i. Consider two pure strategies s_i and t_i for i and a pure strategy profile s_{-i} for i's opponents. Suppose that $\pi_i(s_i,s_{-i})=\pi_i(t_i,s_{-i})$.

Case (i): $\exists\omega$ such that $\rho(\omega)>0$ and $p_i(\omega,s(\omega))\neq p_i(\omega,t_i(\omega_i),s_{-i}(\omega_{-i}))$. Then the equation $\pi_i(s_i,s_{-i})-\pi_i(t_i,s_{-i})=0$ has non-zero coefficient on $V_i(\omega)$ and so is satisfied for a set of V_i which is a lower-dimensional subspace of $\mathbf{R}^{mn}$.

Case (ii): $\forall\omega$ such that $\rho(\omega)>0$, $p_i(\omega,s(\omega))=p_i(\omega,t_i(\omega_i),s_{-i}(\omega_{-i}))$. If $s_i(\omega_i)=t_i(\omega_i)$ for all ω_i, then clearly $\pi(s_i,s_{-i})=\pi(t_i,s_{-i})$. Suppose $s_i(\hat{\omega}_i)\neq t_i(\hat{\omega}_i)$ for some $\hat{\omega}$. W.l.o.g., assume $s_i(\hat{\omega}_i)>t_i(\hat{\omega}_i)$. Let ω_{-i} be such that $\rho(\hat{\omega}_i,\omega_{-i})>0$. By assumption, i's probability of winning facing $s_{-i}(\omega_{-i})$ is the same with $s_i(\hat{\omega}_i)$ and $t_i(\hat{\omega}_i)$. But, then the highest bid for the opponents under ω_{-i} must be either less than $t_i(\hat{\omega}_i)$ or greater than $s_i(\hat{\omega}_i)$. In either case, player i's change from $s_i(\hat{\omega}_i)$ to

$t_i(\hat{\omega}_i)$ does not affect the payoff of players other than i.

In the situation of Case (i), TDI need not hold, but this is only possible for a zero measure set of values V_i. To see this note that there are a finite number of such s_i, t_i, s_{-i} combinations, and so the set of V_i for which a payoff tie is possible is a finite union of zero-measure sets, and so zero measure. In the situation of Case (ii), TDI holds. Therefore, TDI holds in this game for all but a measure-zero set of possible assignments of values to signals.

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